ASSET ALLOCATION BY VARIANCE SENSITIVITY ANALYSIS*

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Abstract - This paper provides a solution to the curse of dimensionality associated to multivariate GARCH estimation. We work with univariate portfolio GARCH models and show how the multivariate dimension of the portfolio allocation problem may be recovered from the univariate approach. The main tool we use is "variance sensitivity analysis", the change in the portfolio variance induced by an infinitesimal change in the portfolio allocation. We suggest a computationally feasible method to find minimum variance portfolios and estimate full variance-covariance matrices. An application to real data portfolios implements our methodology and compares its performance against that of selected popular alternatives.

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Estimates of volatilities and correlations are used for pricing, asset allocation, hedging purposes and risk management in general. In today's fast changing financial world, it is essential that these measures are easy to understand and to implement. Since their introduction by Engle (1982), ARCH models have been used extensively both in academia and by practitioners to estimate the volatility of financial variables. Many papers have been written on the subject, extending the original ARCH model in many directions. The multivariate extension, however, has been met with many difficulties, mainly due to the fact that the number of parameters that need to be estimated increases exponentially, as one leaves the univariate domain.

This paper suggests looking at the multivariate problem from a different perspective. The key idea is to work with univariate portfolio models, and to develop tools to recover the multivariate dimension that is lost in the univariate estimation. This is accomplished by recognising that the estimated univariate portfolio variance is a function of the weights of the assets that form the portfolio. By taking the derivatives of the variance with respect to these weights, it is possible to obtain information about the local behaviour (around the portfolio weights) of the estimated variance.

Estimation of large multivariate GARCH models is notoriously challenging, requiring strong assumptions to make such estimation feasible. For instance, the most general multivariate GARCH model, the GARCH(1,1) vec representation introduced by Engle and Kroner (1995), requires the estimation of 21 parameters to obtain the variance-covariance matrix of just two assets. When the assets are five, there are 465 parameters to estimate and with ten assets the number of parameters raises to 6105! Moreover, restrictions need to be imposed on the variance-covariance matrix to ensure its positive definiteness. It is easy to argue that the high level of parameterisation and the assumptions on the structure of the variance-covariance matrix are likely to increase the dangers of misspecification and poor performance of the model.

On the other hand, the advantage of fitting variance models directly to the time series of portfolio returns is that they indirectly incorporate any time varying correlation among the assets. This makes it possible to estimate parsimonious models that summarise the relevant characteristics of the assets entering the portfolio. This is done, for example, by McNeil and Frey (2000) to calculate the Value at Risk of the portfolio. Leaving aside for the moment theoretical considerations, the main empirical drawback of this approach is that the multivariate dimension of the portfolio allocation problem is lost. Given the estimated variance of a portfolio, a risk manager would be unable to determine how this variance changes as the portfolio composition evolves, or to isolate the main sources of risk. It is not clear how to address these issues in an univariate framework. In the following pages, we suggest the use of sensitivity measures to overcome this problem.

Recently, measures of sensitivity to the weights of the portfolio allocation have been proposed for Value at Risk (VaR) models. Garman (1996) suggested computing the derivative of the VaR with respect to the individual components of the portfolio, to assess the potential impact of a trade on a firm's VaR. Gourieroux, Laurent and Scaillet (2000) study the theoretical

implication of this exercise on different VaR models. The same type of question can be asked with respect to the variance of a portfolio. When a full variance-covariance matrix is available, this is a straightforward exercise. But when univariate portfolio variances are estimated it is not obvious how to proceed.

The main contribution of this paper is to show how to perform variance sensitivity analysis in the context of univariate GARCH models. We derive the sensitivity of the univariate portfolio GARCH variance to the portfolio weights, by analytically computing the derivatives of the estimated GARCH variance with respect to these weights. It is important to recognise that not only the portfolio returns, but also the estimated parameters of the GARCH model are function of the weights. We show how a simple application of the Implicit Function Theorem to the first order conditions of the log-likelihood maximisation problem can be used to overcome this obstacle.

Our sensitivity measure has many interesting practical applications. To start with, risk managers might use the GARCH sensitivity analysis to test whether their actual portfolio has minimum variance. Indeed, the minimum variance portfolio will be characterised by having all first derivatives with respect to the portfolio weights equal to zero. The GARCH sensitivity analysis could also be used to evaluate the impact that each individual (or group of) asset has on the portfolio variance. This would help risk managers to find out the major sources of risk, or allow them to evaluate the impact on the portfolio variance of a certain transaction. A third application, proposed in this paper, is a new and computationally feasible method to find minimum variance portfolios. We show how an allocation problem with (n+1) assets can be solved by minimising a well-behaved function of n variables, whose first and second derivatives are related to the variance sensitivity of the portfolio. Moreover – by exploiting the analytical relationship among variances, covariances and the variance derivatives with respect to the

portfolio weights – we suggest a simple method to estimate full variance-covariance matrices of large portfolios, that are automatically guaranteed to be positive definite. The intuition is that, once the minimum variance portfolio has been found, estimation of the variance-covariance matrix is equivalent to finding the coefficients of a paraboloid with vertex at the minimum variance portfolio and curvature equal to the second derivatives of the portfolio variances.

The plan of the paper is the following. The next section illustrates our methodology. Section 3 shows how to employ variance sensitivity analysis to find minimum variance portfolios and estimate full variance-covariance matrices. Section 4 contains an empirical application and section 5 concludes.

1. VARIANCE SENSITIVITY ANALYSIS

In this section, we show how to compute the derivative of the univariate GARCH portfolio variance with respect to the portfolio weights. From a pure theoretical perspective, Nijman and Sentana (1996) have shown that GARCH processes are not closed under contemporaneous (or cross-sectional) aggregation. More precisely, they show that a linear combination of variables generated by a multivariate GARCH process will only be a weak GARCH process. Therefore, fitting GARCH processes directly to portfolio returns will generally result in misspecified models. In this paper, we take a more naïve approach and consider any GARCH model as only a rough approximation to the "true" relationship among the observed data. Our results can nevertheless still be interpreted in the "quasi-maximum likelihood" sense of White (1994). That is – for the chosen parameterisation – they provide the closest approximation (in terms of the Kullback-Leibler Information Criterion) to the true DGP. We believe this to be a reasonable working assumption (at least as reasonable as assuming that the univariate GARCH processes are

correctly specified and then deriving the theoretical true GARCH relationship of the aggregated portfolios).

Changing the portfolio weights changes the time series of portfolio returns, and thus changes the information set used in the estimation of the univariate GARCH model. As a consequence, the estimated variance is function of the portfolio weights, both through the vector of portfolio returns and through the estimated parameters (which obviously depend on the time series of portfolio returns used in estimation). Differentiation of the portfolio returns with respect to the portfolio weights is straightforward. To differentiate the estimated parameters we appeal to the Implicit Function Theorem. The idea is that, since the estimated parameters must satisfy the first order conditions of the log-likelihood maximisation problem, if certain continuity conditions are satisfied, the first-order conditions define an implicit function between the estimated parameters and the portfolio weights.

Let y_t be the return of the portfolio composed by n+1 assets and let $y_{t,i}$ be the ith asset return, for t = 1,...,T and i = 1,...,n+1. Indicating the weight of asset i by a_i, the portfolio return at time t is $y_t = \sum_{i=1}^{n+1} a_i y_{t,i}$. Note that since the weights a_i have to sum to one, we can write one weight as a function of the others, $a_{n+1} = 1 - \sum_{i=1}^{n} a_i$.

Assume that y_t is modelled as a zero-mean[1] process with a GARCH(p,q) conditional variance h_t:

(1) $$y_t = \sqrt{h_t} e_t \qquad e_t \mid \Omega_t \sim (0,1)$$

(2) $$h_t = z_t' q$$

where $\underset{mx1}{z_t} = (1, y_{t-1}^2, ..., y_{t-q}^2, h_{t-1}, ..., h_{t-p})'$, $\underset{mx1}{q} = (a_0, a_1, ..., a_q, b_1, ..., b_p)'$ and $m = p+q+1$.

The information set of this model is $\Omega_t = \left\{a, \left[y_{r,1}\right]_{r=1}^{t-1}, ..., \left[y_{r,n+1}\right]_{r=1}^{t-1}\right\}$, where a denotes the n-vector of portfolio weights.[2] Note that the information set includes the time series of the individual assets returns and that a change in the vector of portfolio weights implies a change in the information set. Therefore, to assess the potential impact of a trade on the estimated variance, one would have to re-estimate the whole model, given that a and hence the information set has changed. The problem is that such a procedure would quickly become cumbersome and impractical, as the number of assets increases.

The potential effect of any change in the portfolio weights on the estimated variance could be evaluated by simply computing the first derivative of the variance with respect to the weights. A positive derivative would indicate that the change will increase the variance of the portfolio and vice versa for a negative derivative. Let $\hat{h}_t = \hat{z}_t'\hat{q}$ be the estimated variance, where a hat (^) above a variable denotes that the variable is evaluated at the estimated parameter. In computing the derivative of $\hat{h}_t$ one must recognise that not only the vector $\hat{z}_t$, but also the vector of estimated coefficients $\hat{q}$ depends on a. By the chain rule, the derivative of $\hat{h}_t$ with respect to a is given by:

(3) $$\underset{n\times 1}{\frac{\partial \hat{h}_t}{\partial a}} = \underset{n\times m}{\frac{\partial \hat{z}_t'}{\partial a}} \underset{m\times 1}{\hat{q}} + \underset{n\times m}{\frac{\partial \hat{q}'}{\partial a}} \underset{m\times 1}{\hat{z}_t}$$

To achieve a clearer picture of the local behaviour of the estimated variance with respect to the portfolio allocation, one could determine its concavity by computing the second derivative:

$$(4)\qquad \underset{n\times n}{\frac{\partial^2 \hat{h}_t}{\partial a \partial a'}} = \underset{n\times m}{\frac{\partial \hat{z}_t'}{\partial a}} \underset{m\times n}{\frac{\partial \hat{q}}{\partial a'}} + \underset{n\times m}{\frac{\partial \hat{q}'}{\partial a}} \underset{m\times n}{\frac{\partial \hat{z}_t}{\partial a'}} + \underset{n\times mn}{(\underset{1\times m}{\hat{q}'} \otimes \underset{n\times n}{I_n})} \underset{mn\times n}{\frac{\partial}{\partial a'} vec\left(\frac{\partial \hat{z}_t'}{\partial a}\right)} + \underset{n\times mn}{(\underset{1\times m}{\hat{z}_t'} \otimes \underset{n\times n}{I_n})} \underset{mn\times n}{\frac{\partial}{\partial a'} vec\left(\frac{\partial \hat{q}'}{\partial a}\right)}$$

where $\otimes$ denotes the Kronecker product and I_n is an $(n\times n)$ identity matrix.

To evaluate (3) and (4), we need to compute $\frac{\partial \hat{q}'}{\partial a}$ and $\frac{\partial}{\partial a'} vec\left(\frac{\partial \hat{q}'}{\partial a}\right)$, the other terms being easily obtained. We compute these derivatives by applying the Implicit Function Theorem to the first order conditions of the log-likelihood maximisation problem. The first order conditions for model (1)-(2) are:

$$(5)\qquad T^{-1}\sum_{t=1}^{T} \left.\frac{\partial l_t(q)}{\partial q}\right|_{q=\hat{q}} = 0$$

where $l_t(\cdot)$ is the time t component of the log-likelihood function. The following theorem derives the analytical expressions for $\frac{\partial \hat{q}'}{\partial a}$ and $\frac{\partial}{\partial a'} vec\left(\frac{\partial \hat{q}'}{\partial a}\right)$.

<u>Theorem 1</u> – Assume that $l_t(\cdot)$ is continuously differentiable in a and q, and define $\hat{I}_{qq} \equiv T^{-1}\sum_{t=1}^{T} \left.\frac{\partial^2 l_t(q)}{\partial q\, \partial q'}\right|_{q=\hat{q}}$ and $\hat{I}_{qa} \equiv T^{-1}\sum_{t=1}^{T} \left.\frac{\partial^2 l_t(q)}{\partial q\, \partial a'}\right|_{q=\hat{q}}$. If $\hat{I}_{qq}$ is non-singular, then:

(6) $$\underset{n\times m}{\frac{\partial \hat{q}'}{\partial a}} = -(\hat{I}_{qa})'(\hat{I}_{qq})^{-1}$$

(7) $$\underset{mn\times n}{\frac{\partial}{\partial a'} vec\left(\frac{\partial \hat{q}'}{\partial a}\right)} = -\left[(\hat{I}_{qq})^{-1} \otimes \frac{\partial \hat{q}'}{\partial a}\right]\frac{\partial}{\partial a'} vec(\hat{I}_{qq}) - \left[(\hat{I}_{qq})^{-1} \otimes I_n\right]\frac{\partial}{\partial a'} vec(\hat{I}'_{qa})$$

Proof - Since the score is continuous and differentiable both in a and q, if $\hat{I}_{qq}$ is non singular it is possible to apply the implicit function theorem to the first order conditions. The result follows. _

In Appendix B, we provide the analytical derivation of (6) and (7), assuming that the standardised residuals are normally distributed – that is $l_t(q) = -\frac{1}{2}\ln(h_t) - \frac{1}{2} y_t^2 h_t^{-1}$. Both $\hat{I}_{qq}$ and $\hat{I}_{qa}$ can be easily derived analytically, although the algebra might be messy. Note that $\hat{I}_{qq}$ is just the estimated Hessian of the GARCH model. One may wonder how it is possible to compute the sensitivity of GARCH variances from the simple series of portfolio returns. In fact, formulae (3)-(4) and theorem 1 make use not only of portfolio returns, but also of the returns of the individual assets entering the information set. We illustrate this point with a simple example. Let $y_t = ay_{t,1} + (1-a)y_{t,2}$, where a and (1-a) are the weights associated to assets 1 and 2, respectively. Suppose that an ARCH(1) model is estimated, so that the parametric form of the estimated variance is $\hat{h}_t = \hat{q}y_{t-1}^2$.[3] Then one can show that:

(8) $$\hat{I}_{qa} \equiv \hat{h}_t^{-2}[y_{t-1}^2 y_t(y_{t,1} - y_{t,2}) + 2y_t^2 y_{t-1}(y_{t-1,1} - y_{t-1,2})]$$

Hence, both formulae (3)-(4) and theorem 1 exploit not only the information contained in $\{y_t\}_{t=1}^T$, but also that contained in the individual series $\{y_{t,1}\}_{t=1}^T$ and $\{y_{t,2}\}_{t=1}^T$.

2. SENSITIVITY ANALYSIS AND ASSET ALLOCATION

The sensitivity analysis idea developed in the previous section can be used for estimating large variance-covariance matrices and for optimal conditional portfolio allocation in a mean-variance context. In this section we discuss how this can be accomplished.

The typical portfolio allocation problem under Markovitz's mean-variance framework can be expressed as follows:

$$\max_a E_t[u(y_t)] = E_t[y_t] - k\left(\mathrm{var}_t(y_t) + E_t[y_t]^2\right) \tag{9}$$

where $y_t = \sum_{i=1}^{n+1} a_i y_{t,i}$ denotes the portfolio return at time t, a is the n-vector of weights, and E_t and var_t denote respectively the conditional expectation and conditional variance at time t, given the information set Ω_t.[4] This problem involves maximising a function of the conditional mean and the conditional variance, with respect to the portfolio weights. The first order conditions of this problem will obviously depend on the derivatives of the conditional mean and the conditional variance with respect to a. Any modelling choice of the conditional mean (including a possible GARCH in mean component) can be easily incorporated into the GARCH framework of equations (1)-(2). The first and second derivatives of the mean and the variance of the portfolio returns with respect to the weights can therefore be derived from the corresponding likelihood,

using exactly the same procedure described in section 2. This implies that the maximisation problem (9) can be solved by simply maximising a function of *n* variables (the portfolio weights), of which we know the first and second derivatives. This is a relatively straightforward task – provided that the function is sufficiently well-behaved – and does not involve the estimation of any variance-covariance matrix. If the portfolio GARCH models were correctly specified, the function would be quadratic in the weights, and the corresponding optimisation process would be trivial. In reality, as discussed at the beginning of section 2, univariate portfolio GARCH models are bound to be misspecified. The degree of misspecification – and its impact on the shape of our function – is impossible to tell on pure theoretical grounds. In the empirical application of section 4, we find the degree of misspecification to be relatively unimportant. Optimisation for a portfolio of 30 assets is extremely robust to the choice of the initial conditions, suggesting a sufficiently well-behaved objective function. Obviously, further research along these lines is needed to tell how generally our procedure can be applied.

Sensitivity analysis can be used to estimate large variance-covariance matrices as well. The procedure involves three simple steps:

Step 1 – Minimise the portfolio variance with respect to the weights. This is just a special case of problem (9), when we set the conditional mean equal to zero. For daily observations, this can be considered a reasonable approximation.

Step 2 – Compute the second derivatives of portfolio variance with respect to the weights. In theory, second derivatives should be constant and independent of the values of the weights at which they are computed. In practice, this will not be the case, because of the misspecification of the GARCH(1,1) model applied to portfolios. One possibility is to take averages of second

derivatives computed for different portfolio weights, or simply take the second derivative corresponding to the minimum variance portfolio computed in step 1.

<u>Step 3</u> – Define $w = [a', 1 - i'a]'$ as the (n+1)-vector of weights corresponding to each asset entering the portfolio, where i is an n-vector of ones. Compute the variance-covariance matrix (Σ_t) as the solution to the following system:

i) $w^{*\prime}\Sigma_t w^* = h_t(a^*)$

ii) $\left.\dfrac{\partial w'\Sigma_t w}{\partial a}\right|_{a=a^*} = 0$

iii) $\left.\left(\dfrac{\partial^2 w'\Sigma_t w}{\partial a \partial a'}\right)\right|_{a=a^*} = K_t$

where K_t is an ($n \times n$) matrix containing the estimated second derivatives and a^*, w^* are the optimal weights associated to the minimum variance portfolio found in step 1.

Note that Σ_t has a total of (n+2)(n+1)/2 parameters to be estimated. Equation i) gives one condition, equation ii) n conditions and equation iii) n(n+1)/2 conditions, for a total of exactly (n+2)(n+1)/2 conditions. Note also that the solution of the system i)-iii) gives the coefficients of a paraboloid with vertex corresponding to the minimum variance portfolio and curvature K_t. This implies – since the minimum variance is strictly positive – that the estimated variance-covariance matrix is automatically positive definite. To see this, consider the generic quadratic form $w'\hat{\Sigma}_t w$, where $\hat{\Sigma}_t$ has been estimated with the above procedure and $w'\begin{bmatrix} i \\ 1 \end{bmatrix} = 1$. By conditions ii) and iii),

this quadratic form achieves a minimum at $w = w^*$, and condition i) ensures that this minimum is strictly positive. It follows that $w'\hat{\Sigma}_t w > 0$ for any w.

It is straightforward to find an analytical solution to the above system. Partition Σ_t as follows: $\Sigma_t = \begin{bmatrix} A_t & b_t \\ b_t' & c_t \end{bmatrix}$, where A_t is ($n \times n$), b_t is ($n \times 1$) and c_t is a scalar. Then:

$$(10) \qquad \begin{aligned} w'\Sigma_t w &= [a', 1 - i'a] \begin{bmatrix} A_t & b_t \\ b_t' & c_t \end{bmatrix} \begin{bmatrix} a \\ 1 - i'a \end{bmatrix} \\ &= a'(A_t - ib_t' - b_t i' + c_t ii')a + 2(b_t' - c_t i')a + c_t \end{aligned}$$

Therefore, we have:

$$(11) \qquad \frac{\partial w'\Sigma_t w}{\partial a} = 2(A_t - ib_t' - b_t i' + c_t ii')a + 2(b_t - c_t i)$$

$$(12) \qquad \frac{\partial^2 w'\Sigma_t w}{\partial a \partial a'} = 2(A_t - ib_t' - b_t i' + c_t ii')$$

Combining these relationships with the system i)-iii), simple calculations give:

$$(13) \qquad A_t = 0.5K_t + h_t(a^*)ii' + 0.5a^{*\prime} K_t a^* ii' - 0.5ia^{*\prime} K_t - 0.5K_t a^* i'$$

$$(14) \qquad b_t = h_t(a^*)i + 0.5a^{*\prime} K_t a^* i - 0.5K_t a^*$$

$$(15) \qquad c_t = h_t(a^*) + 0.5a^{*\prime} K_t a^*$$

which completely determine the variance-covariance matrix at time t.

3. EMPIRICAL APPLICATION

In this section, we implement our methodology on a selected sample of stocks. We first estimate the sensitivity of GARCH variances on a two-stock portfolio, as described in section 2. Then we

find the minimum variance portfolios for 5 different portfolios, using the methodology outlined in section 3, and compare its performance to that of a few popular alternatives (namely, Dynamic Conditional Correlation, orthogonal GARCH and Exponentially Weighted Moving Average).

3.1 Sensitivity of GARCH variance

We estimate the first and second derivatives of GARCH variances as described in section 2, using a two-asset portfolio, composed by General Motors (GM) and IBM. Daily data are taken from Bloomberg and run from 2 January 1992 through 11 March 2002.

We estimate univariate GARCH(1,1) models for 31 portfolios constructed from these two assets, with the GM weight (a) ranging from –1 to 2, with increments of 0.1. For each estimated GARCH model, we compute the first and second derivatives of the estimated variance with respect to the weight a.

[PLEASE, INSERT FIGURE 1 ABOUT HERE]

In figure 1, we plot the estimated variances on 11 March 2002 for the 31 portfolios as a function of the weight, together with their first and second derivatives. Note that the variance corresponding to $a = 0$ is the variance of IBM, while the variance corresponding to $a = 1$ is the variance of GM. The portfolios with a weight greater than 1 or less than 0 are short on IBM or GM, respectively. The estimated variance plotted in figure 1 is a parabolic and convex function of the portfolio weights a, suggesting that diversification produces significant gains in terms of risk reduction. If the true variance-covariance matrix were available and one computes the portfolio variances as a weighted sum of the individual asset variances and covariances, this function would be exactly a parabola. The fact that fitting univariate GARCH models to the time

series of portfolios produces results very close to those one would expect in theory, indicates that these univariate GARCH models provide a reasonable approximation of the true (but unknown) model. This intuition is confirmed by the shape of the first and second derivatives. If the function were truly a parabola, than the first derivative would be a straight positively sloped line and the second derivative a flat line. The plots in figure 1 show that both the first and second derivatives are very close to their theoretical shape.

[PLEASE, INSERT FIGURE 2 ABOUT HERE]

In figure 2, we report the time series of the first derivatives of the estimated variance, $\frac{\partial \hat{h}_t(a)}{\partial a}$, for the two degenerate portfolios, i.e. for IBM (a = 0) and GM (a = 1). The picture indicates by how much the variance would decrease or increase over time, if one diversifies away from the portfolios composed of only GM or IBM asset. Similar pictures can be drawn for any portfolio weight, thus giving the risk manager a precise indication about the consequences – in terms of risk – of changing the composition of the current portfolio.

A second interesting feature of figure 2 is that the first derivative is always positive for GM and almost always negative for IBM. This implies that the minimum variance portfolio during the period considered in this analysis was formed by a convex combination of these two assets. The fact that for a few days towards the end of the sample both first derivatives were positive signals that during those days the risk manager would have had to short GM to construct the minimum variance portfolio.

Figure 2 provides also an insight about the major sources of risk of a portfolio. Indeed, the greater (in absolute value) the first derivative, the greater will be the risk reduction following a

portfolio reallocation. Figure 2 shows that the first derivative of the portfolio containing only the IBM asset is much higher on average (in absolute value) than the first derivative corresponding to the GM portfolio.[5] This implies that during the 1990's an investor could achieve greater variance reduction by diversifying away from the portfolio with only IBM (the "new economy" stock), than from the GM portfolio (the "old economy" stock). In the case of a portfolio with more than two assets, one could compute the variance sensitivity corresponding to each asset and gain in this way an insight about the major sources of risk of the portfolio. In order to reduce the risk, the risk manager should sell the assets with highest first derivative and buy those with the lowest one.

[PLEASE, INSERT FIGURE 3 ABOUT HERE]

In figure 3 we report the time series of the second derivatives for GM, together with its difference from the average second derivatives computed over all the 31 portfolios considered in this exercise. In theory, for a correctly specified model, the second derivative should not depend on the portfolio composition: when plotted against the portfolio weights at any given point in time, it should be a flat line. Therefore, the difference between the average second derivatives and that of GM should be zero. Consistently with the findings of figure 1, this difference is hovering around zero throughout the sample, thus providing further evidence that portfolio univariate GARCH models provide a good approximation of the true variance. The difference is not exactly zero because – as we noted in section 2 – univariate GARCH models are surely mis-specified.

The second derivative, being the slope of the first derivative, tells the risk manager by how much the variance sensitivity will change after a change in the portfolio allocation. The greater the magnitude of the second derivative, the greater will be the change in the variance sensitivity,

implying that a smaller portfolio reallocation will be necessary to achieve a given size of variance reduction. Figure 3 shows that in the last couple of years portfolio reallocations had much greater impact on the variance than during the 90's. The average of the second derivative was 12.36 between 1992 and 1999, and rose to 17.17 from 1999 to 2002. In other words, these results show that the concavity of the portfolio variance (as a function of the weight *a*) has increased dramatically over the past few years, for GM and IBM. This has obviously important consequences for managing the risk of a portfolio composed by these two assets.

3.2 Minimum Variance Portfolio Allocation

In this subsection we implement the methodology described in section 3 to find the allocation minimising the portfolio variance. We tested our methodology on different sub-samples of the Dow Jones index, with the same time span as before, i.e. from 2 January 1992 through 11 March 2002.

We compare the performance of our methodology with three alternative multivariate models: 1) Dynamic Conditional Correlation (DCC), 2) Orthogonal GARCH (OGARCH), 3) Exponentially Weighted Moving Average (EWMA). These are some of the most popular methods used in the industry to estimate large variance-covariance matrices.[6] The DCC model has been recently proposed by Engle (2002) and Engle and Sheppard (2001). This model can be seen as a generalisation of the Constant Conditional Correlation model, originally proposed by Bollerslev (1990). In the DCC model, conditional correlations are directly parameterised, rather than assumed constant. Engle (2002) shows that the estimation of the multivariate model can be drastically simplified, by using a two-step procedure. First the univariate GARCH models are estimated for each of the assets. Then the conditional correlation specification is fitted to the standardised residuals obtained in the first step. In the simplest mean-reverting model proposed

by Engle (2002) – which is the one we estimate in our empirical application – the same pair of parameters is estimated for all the correlations considered (implying that all correlations have the same degree of persistence). More parameterised models can be used, at the cost, however, of increasing the computational burden.

Alexander (2000) proposes the Orthogonal GARCH model, based on a principal component GARCH methodology. First, she constructs unconditionally uncorrelated factors, which are linear combinations of the original returns. Then she fits univariate GARCH models to the principal components. Since the conditional variance-covariance matrix of the principal component series is diagonal (i.e. conditional correlations are zero), it is possible to recover the original assets' variance-covariance matrix, through a fixed mapping matrix.

The last method we consider is the EWMA, popularised by RiskMetrics. With this method the variance-covariance matrix at time t is simply computed as a convex combination of the variance-covariance matrix in the previous period, t-1, and the matrix of squared and cross-product lagged returns. The weight is usually set equal to 0.94 or 0.97. In the following application, we set the decay coefficient for the EWMA equal to 0.94.

For these three methods, we first estimate the variance-covariance matrix as of 11 March 2002. Then we analytically compute the weights that give the minimum variance portfolio. If the variance-covariance matrix is partitioned as in (10), then the optimal weights can be found by setting (11) equal to zero, giving:

$$(16) \qquad a^* = (\hat{A}_t - i\hat{b}_t' - \hat{b}_t i' + \hat{c}_t i i')^{-1}(\hat{c}_t i - \hat{b}_t)$$

Finally, we estimate the univariate GARCH variance associated to this portfolio and report the annualised estimated volatility in Table 1.[7] We repeat this procedure for each model under consideration and for five different portfolios: with 2, 5, 10, 20 and 30 assets respectively.[8]

The Variance Sensitivity Analysis (VSA) model is estimated by minimising directly the univariate GARCH variance with respect to the portfolio weights. We use the function *fminunc* in Matlab, providing as input the first and second analytical derivatives computed in section 2. Convergence is very rapid and very robust to the choice of the initial conditions.[9] This suggests that the objective function is well-behaved even for high-dimensional problems. To produce the results of Table 1, we chose as initial conditions of the VSA model the optimal weights of EWMA.

[PLEASE, INSERT TABLE 1 ABOUT HERE]

Since the VSA method is designed to find the minimum variance portfolio as measured by univariate GARCH, it is not surprising that it outperforms all the other models. What is more interesting is that the outperformance (as measured by the percentage difference in annualised volatility) increases with the number of assets. While with a two-asset portfolio the differences in the minimum variances are negligible, these differences raise monotonically with the number of assets for the DCC and OGARCH methods. With 5 assets, DCC and OGARCH over-estimate the minimum variance portfolio by about 7%. With 10 assets, the difference raises to 10% and 18% respectively, while with 20 and 30 assets it ranges from 23% to almost 40%. Clearly, this is an indication that – as the number of assets increases – the restrictions imposed by the standard multivariate GARCH models become more and more binding.

A slightly different picture emerges from the results of the EWMA. Its performance does not seem to deteriorate as much and as fast as for the other two competing models. Albeit surprising at first sight, this result is due to the way EWMA estimates are constructed. Using the same decay coefficient l for all variance and covariance terms is equivalent to estimating directly the EWMA portfolio variance with coefficient l. If the estimated coefficients of the GARCH(1,1) model applied to the minimum EWMA variance portfolio are not too different from the chosen l, one would expect a good performance of this model – whenever performance is measured by minimum variance. Notice, however, that this does not imply that EWMA produces reasonable estimates of the variance-covariance matrix. As we illustrated in section 3, estimation of variance-covariance matrices is equivalent to finding the coefficients of a paraboloid with given vertex and curvature. Even if the vertex provided by EWMA is close to the true one, there is no guarantee at all that the same holds for the curvature.

A final aspect worth noticing is that the computation time of VSA does not seem to rise too much with the number of assets.[10] A 30-asset portfolio is optimised by VSA in less than 8 minutes, on a standard Pentium IV computer. Computational feasibility seems to be a further attractive feature of the methodology proposed in this paper.

4. CONCLUSIONS

Fitting variance models directly to the time series of portfolio returns has many advantages, such as the possibility of estimating parsimonious models and computational tractability. The problem of this strategy is that the multivariate dimension of the portfolio allocation is lost. This paper suggests a strategy to overcome this problem, working within a GARCH framework. We assess

the potential impact of a trade on the estimated variance by computing the sensitivity of the estimated variance with respect to the weight of the asset involved in the trade. This sensitivity measure is simply the derivative of the estimated variance with respect to the portfolio weights. As a by-product of this analysis, we propose a new and simple method to estimate full variance-covariance matrices, which exploits the analytical relationship among variances, covariances and sensitivity measures.

We illustrate the functioning and the performance of our methodology with two empirical applications. In the first one, we estimate the variance sensitivity for a portfolio of two assets. We document how this sensitivity has been changing over time and stress its implications for risk management. We also compute the second derivative of the estimated variance with respect to the portfolio weights. We argue that this measure gives an indication of the diversification opportunities at any given point in time: the higher this second derivative, the greater the gains (in terms of variance reduction) from a proper diversification strategy.

In the second application, we implement the suggested methodology to find minimum variance portfolios at any given point in time. We test the model with 5 sub-samples of the Dow Jones index. We evaluate the performance of our methodology against that of popular alternatives, including the Dynamic Conditional Correlation, the Orthogonal GARCH and the Exponentially Weighted Moving Average models. Our findings suggest that our methodology generally leads to considerable efficiency gains.

FIGURE LEGENDS

Figure 1: Plot of estimated variance, first and second derivative on 11 March 2002, for 31 portfolios constructed from GM and IBM. On the horizontal axis there is the portfolio weight for GM, which ranges from –1 to 2, with increments of 0.1. The variance is computed by re-estimating a GARCH(1,1) model for each of the 31 portfolios. The first and second derivatives are computed analytically, as described in section 2.

Figure 2: Plot of the first derivative of the estimated variance for the two degenerate portfolios GM and IBM. The first derivative indicates by how much the variance would increase or decrease over time, if one increases the weight of GM (a) in the portfolio. In the case the portfolio is composed of only GM (a=1), buying an extra share of GM and going short of IBM would increase the overall portfolio variance (upper line). Vice versa, diversifying away from a portfolio composed of only IBM stocks (a=0) would decrease the variance (bottom line).

Figure 3: Plot of the estimated second derivatives, computed from the degenerate GM portfolio (upper graph) and the its difference w.r.t. the average second derivatives of 31 different portfolios (lower graph). Under correct model specification, the second derivative should not depend on the portfolio weight. Hence the lower graph should be exactly equal to zero. The fact that the dotted line hovers mostly around zero confirms the results obtained in figure 1, i.e. that GARCH(1,1) model provides a reasonable approximation of the portfolio variance process.

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APPENDIX A: LIST OF STOCKS USED IN THE ANALYSIS

	Ticker	Name
1	AA	Alcoa Inc.
2	AXP	American Express Co.
3	T	AT&T Corp.
4	BA	Boeing Co.
5	CAT	Caterpillar Inc.
6	C	Citigroup Inc.
7	KO	Coca-Cola Co.
8	DIS	Walt Disney Co.
9	DD	E.I. DuPont de Nemours & Co.
10	EK	Eastman Kodak Co.
11	XOM	Exxon Mobil Corp.
12	GE	General Electric Co.
13	GM	General Motors Corp.
14	HPQ	Hewlett-Packard Co.
15	HD	Home Depot Inc.
16	HON	Honeywell International Inc.
17	INTC	Intel Corp.
18	IBM	International Business Machines Corp.
19	IP	International Paper Co.
20	JNJ	Johnson & Johnson
21	JPM	J.P. Morgan Chase & Co.
22	MCD	McDonald's Corp.
23	MRK	Merck & Co.
24	MSFT	Microsoft Corp.
25	MMM	3M Co.
26	MO	Altria Group Inc.
27	PG	Procter & Gamble
28	SBC	SBC Communications Inc.
29	UTX	United Technologies Corp.
30	WMT	Wal-Mart Stores Inc.

NOTE: These were the 30 stocks composing the Dow Jones Industrial Average index, as of 11 March 2002.

APPENDIX B: COMPUTATION OF FIRST AND SECOND DERIVATIVES UNDER NORMAL LIKELIHOOD FOR A GARCH(1,1) MODEL

We follow the conventions and rules on matrix differentiation, as in Appendix A.13 of Lütkepohl (1990). We indicate the explicit dependence of q on a by $q(a)$. Whenever this dependence is not made explicit, it means that we treat q as not depending on a. We derive the analytical expressions of (3) and (4), under the assumption of a normal likelihood. Rewrite (3), making explicit the dependence on a:

$$\text{(A1)} \qquad \frac{\partial h_t(q(a))}{\partial a} = \frac{\partial z_t(q(a))'}{\partial a} q + \frac{\partial q(a)'}{\partial a} z_t$$

where $\frac{\partial z_t(q(a))'}{\partial a} = \left[0_{(n\times 1)} \quad \frac{\partial Y_{t-1}^2}{\partial a} \quad \frac{\partial h_{t-1}(q(a))}{\partial a} \right]$, $\frac{\partial h_{t-1}(q(a))}{\partial a}$ can be computed recursively, $\frac{\partial Y_{t-1}^2}{\partial a} = 2Y_{t-1}(y_{t-1\backslash n+1} - i\, y_{t-1,n+1})$, i is an $(n,1)$-vector of ones and $y_{t\backslash n+1} = [y_{t,1},..., y_{t,n}]'$. To compute $\frac{\partial q(a)'}{\partial a}$ we apply theorem 1, under the assumption that $l_t = -0.5[\ln(h_t) + Y_t^2 h_t^{-1}]$. The score is

$$\text{(A2)} \qquad \frac{\partial l_t}{\partial q} = -0.5 \frac{\partial h_t}{\partial q} H$$

where $H \equiv (h_t^{-1} - Y_t^2 h_t^{-2})$, $\frac{\partial h_t}{\partial q} = z_t + \frac{\partial z_t'}{\partial q} q$ and $\frac{\partial z_t'}{\partial q} = \left[0_{(3\times 2)} \quad \frac{\partial h_{t-1}}{\partial q} \right]$. Therefore:

$$\text{(A3)} \qquad \frac{\partial^2 l_t}{\partial q \partial q'} = -0.5 \left[\frac{\partial h_t}{\partial q} \frac{\partial H}{\partial q'} + H \frac{\partial^2 h_t}{\partial q \partial q'} \right]$$

where $\frac{\partial H}{\partial q'} = \frac{\partial h_t}{\partial q'} \tilde{H}$, $\tilde{H} \equiv -h_t^{-2} + 2Y_t^2 h_t^{-3}$, $\frac{\partial^2 h_t}{\partial q \partial q'} = \frac{\partial z_t}{\partial q'} + \frac{\partial z_t'}{\partial q} + (q' \otimes I_3) \frac{\partial}{\partial q'} vec\left(\frac{\partial z_t'}{\partial q} \right)$ and $\frac{\partial}{\partial q'} vec\left(\frac{\partial z_t'}{\partial q} \right) = \begin{bmatrix} 0_{(6\times 3)} \\ \frac{\partial^2 h_{t-1}}{\partial q \partial q'} \end{bmatrix}$.

$$\text{(A4)} \qquad \frac{\partial^2 l_t}{\partial q \partial a'} = -0.5 \left[\frac{\partial h_t}{\partial q} \frac{\partial H}{\partial a'} + H \frac{\partial^2 h_t}{\partial q \partial a'} \right]$$

where $\frac{\partial H}{\partial a'} = \frac{\partial h_t}{\partial a'} \tilde{H} - h_t^{-2} \frac{\partial Y_t^2}{\partial a'}$, $\frac{\partial h_t}{\partial a'} = q' \frac{\partial z_t}{\partial a'}$, $\frac{\partial z_t}{\partial a'} = \begin{bmatrix} 0_{(1\times n)} \\ \frac{\partial Y_{t-1}^2}{\partial a'} \\ \frac{\partial h_{t-1}}{\partial a'} \end{bmatrix}$, $\frac{\partial^2 h_t}{\partial q \partial a'} = \frac{\partial z_t}{\partial a'} + (q' \otimes I_3) \frac{\partial}{\partial a'} vec\left(\frac{\partial z_t'}{\partial q} \right)$ and $\frac{\partial}{\partial a'} vec\left(\frac{\partial z_t'}{\partial q} \right) = \begin{bmatrix} 0_{(6\times n)} \\ \frac{\partial^2 h_{t-1}}{\partial q \partial a'} \end{bmatrix}$.

The second derivative is given by (4):

(A5)
$$\frac{\partial^2 h_t(q(a))}{\partial a \partial a'} = \frac{\partial z_t(q(a))'}{\partial a}\frac{\partial q(a)}{\partial a'} + \frac{\partial q(a)'}{\partial a}\frac{\partial z_t(q(a))}{\partial a'} + \\ + (q' \otimes I_n)\frac{\partial}{\partial a'} vec\left(\frac{\partial z_t(q(a))'}{\partial a}\right) + (z_t(q(a))' \otimes I_n)\frac{\partial}{\partial a'} vec\left(\frac{\partial q(a)'}{\partial a}\right)$$

where $\frac{\partial}{\partial a'} vec\left(\frac{\partial z_t(q(a))'}{\partial a}\right) = \begin{bmatrix} 0_{(n\times n)} \\ \frac{\partial^2 Y_{t-1}^2}{\partial a \partial a'} \\ \frac{\partial^2 h_{t-1}(q(a))}{\partial a \partial a'} \end{bmatrix}$, $\frac{\partial Y_{t-1}^2}{\partial a \partial a'} = 2(y_{t-1\backslash n+1} - i\, y_{t-1,n+1})(y_{t-1\backslash n+1} - i\, y_{t-1,n+1})'$ and $\frac{\partial^2 h_{t-1}(q(a))}{\partial a \partial a'}$ can be computed recursively. To compute $\frac{\partial}{\partial a'} vec\left(\frac{\partial q(a)'}{\partial a}\right)$, we apply again Theorem 1 and note that, if A is a (p,p) symmetric non-singular matrix, $\frac{\partial vec(A^{-1})}{\partial a'} = -(A^{-1} \otimes A^{-1})\frac{\partial vec(A)}{\partial a'}$:

(A6)
$$\frac{\partial}{\partial a'} vec\left(\frac{\partial q(a)'}{\partial a}\right) = -\left[(I_{qq})^{-1} \otimes \frac{\partial q(a)'}{\partial a}\right]\frac{\partial}{\partial a'} vec[I_{qq}(q(a))] - \left[(I_{qq})^{-1} \otimes I_n\right]\frac{\partial}{\partial a'} vec\left[I_{qa}'(q(a))\right]$$

We compute the individual components of $\frac{\partial}{\partial a'} vec[I_{qq}(q(a))]$ and $\frac{\partial}{\partial a'} vec\left[I_{qa}'(q(a))\right]$, in turn.

(A7)
$$\frac{\partial}{\partial a'} vec\left[\frac{\partial^2 l_t}{\partial q \partial q'}(q(a))\right] = -0.5\left\{\left(I_3 \otimes \frac{\partial h_t}{\partial q}\right)\frac{\partial}{\partial a'}\left(\frac{\partial H}{\partial q}(q(a))\right) + \left(\frac{\partial H}{\partial q} \otimes I_3\right)\frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial q}(q(a))\right) + \right. \\ \left. + vec\left(\frac{\partial^2 h_t}{\partial q \partial q'}\right)\frac{\partial H(q(a))}{\partial a'} + H\frac{\partial}{\partial a'} vec\left[\frac{\partial^2 h_t}{\partial q \partial q'}(q(a))\right]\right\}$$

where:

- $\frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial q}(q(a))\right) = \frac{\partial z_t(q(a))}{\partial a'} + \frac{\partial z'_t}{\partial q}\frac{\partial q(a)}{\partial a'} + (q' \otimes I_3)\frac{\partial}{\partial a'} vec\left(\frac{\partial z'_t}{\partial q}(q(a))\right)$
- $\frac{\partial}{\partial a'} vec\left(\frac{\partial z'_t}{\partial q}(q(a))\right) = \begin{bmatrix} 0_{(6\times n)} \\ \frac{\partial}{\partial a'}\left(\frac{\partial h_{t-1}}{\partial q}(q(a))\right) \end{bmatrix}$
- $\frac{\partial H(q(a))}{\partial a'} = \frac{\partial h_t(q(a))}{\partial a'}\tilde{H} - h_t^{-2}\frac{\partial Y_t^2}{\partial a'}$
- $\frac{\partial}{\partial a'}\left(\frac{\partial H}{\partial q}(q(a))\right) = \frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial q}(q(a))\right)\tilde{H} + \frac{\partial h_t}{\partial q}\frac{\partial \tilde{H}(q(a))}{\partial a'}$
- $\frac{\partial \tilde{H}(q(a))}{\partial a'} = (2h_t^{-3} - 6Y_t^2 h_t^{-4})\frac{\partial h_t(q(a))}{\partial a'} + 2h_t^{-3}\frac{\partial Y_t^2}{\partial a'}$
- $\frac{\partial}{\partial a'} vec\left[\frac{\partial^2 h_t}{\partial q \partial q'}(q(a))\right] = \frac{\partial}{\partial a'} vec\left[\frac{\partial z'_t}{\partial q}(q(a))\right] + \frac{\partial}{\partial a'} vec\left[\frac{\partial z_t}{\partial q'}(q(a))\right] + \frac{\partial}{\partial a'} vec\left[\frac{\partial}{\partial q} vec\left(\frac{\partial z_t}{\partial q'}\right)'(I_3 \otimes q)\right]$
- $\frac{\partial}{\partial a'} vec\left[\frac{\partial z_t}{\partial q'}(q(a))\right] = K_{3,3}\left[\frac{\partial}{\partial a'} vec\left[\frac{\partial z'_t}{\partial q}(q(a))\right]\right]$, where $K_{3,3}$ is the commutation matrix (see Lütkepohl 1990, p.466)
- $\frac{\partial}{\partial a'} vec\left[\frac{\partial}{\partial q} vec\left(\frac{\partial z_t}{\partial q'}\right)'(I_3 \otimes q)\right] = (I_3 \otimes G')\frac{\partial}{\partial a'} vec(I_3 \otimes q) + (I_3 \otimes q' \otimes I_3)\frac{\partial}{\partial a'} vec(G')$
- $G = \begin{bmatrix} 0_{(6\times 3)} \\ \frac{\partial^2 h_{t-1}}{\partial q \partial q'} \end{bmatrix}$

- $\frac{\partial}{\partial a'} vec(I_3 \otimes q) = (I_3 \otimes K_{1,3} \otimes I_3)\left(vec(I_3) \otimes \frac{\partial q(a)}{\partial a'} \right)$
- $\frac{\partial}{\partial a'} vec(G') = \begin{bmatrix} 0_{(18\times n)} \\ \frac{\partial}{\partial a'} vec\left[\frac{\partial^2 h_t}{\partial q \partial q'}(q(a)) \right] \end{bmatrix}$

It remains to compute $\frac{\partial}{\partial a'} vec\left[I'_{qa}(q(a)) \right]$ in (A6):

$$\frac{\partial}{\partial a'} vec\left[\frac{\partial^2 l_t}{\partial q' \partial a}(q(a)) \right] = -0.5\left\{ \left(I_3 \otimes \frac{\partial H}{\partial a} \right) \frac{\partial}{\partial a'} \left(\frac{\partial h_t}{\partial q}(q(a)) \right) + \left(\frac{\partial h_t}{\partial q} \otimes I_n \right) \frac{\partial}{\partial a'} \left(\frac{\partial H}{\partial a}(q(a)) \right) + \right.$$
$$\left. + vec\left(\frac{\partial^2 h_t}{\partial q' \partial a} \right) \frac{\partial H(q(a))}{\partial a'} + H \frac{\partial}{\partial a'} vec\left[\frac{\partial^2 h_t}{\partial q' \partial a}(q(a)) \right] \right\} \tag{A8}$$

where:

- $\frac{\partial}{\partial a'}\left(\frac{\partial H}{\partial a}(q(a)) \right) = \tilde{H} \frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial a}(q(a)) \right) + \frac{\partial h_t}{\partial a} \frac{\partial \tilde{H}(q(a))}{\partial a'} - h_t^{-2} \frac{\partial^2 Y_t^2}{\partial a \partial a'} + 2h_t^{-3} \frac{\partial Y_t^2}{\partial a} \frac{\partial h_t(q(a))}{\partial a'}$
- $\frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial a}(q(a)) \right) = \frac{\partial z'_t}{\partial a} \frac{\partial q(a)}{\partial a'} + (q' \otimes I_n) \frac{\partial}{\partial a'}\left(\frac{\partial z'_t}{\partial a}(q(a)) \right)$
- $\frac{\partial}{\partial a'} vec\left(\frac{\partial z'_t}{\partial a}(q(a)) \right) = \begin{bmatrix} 0_{(n\times n)} \\ \frac{\partial^2 Y_{t-1}^2}{\partial a \partial a'} \\ \frac{\partial}{\partial a'}\left[\frac{\partial h_{t-1}}{\partial a}(q(a)) \right] \end{bmatrix}$
- $\frac{\partial}{\partial a'} vec\left[\frac{\partial^2 h_t}{\partial q' \partial a}(q(a)) \right] = \frac{\partial}{\partial a'} vec\left[\frac{\partial z'_t}{\partial q}(q(a)) \right] + \frac{\partial}{\partial a'} vec\left[F(q(a))'(q(a) \otimes I_3) \right]$
- $F = \begin{bmatrix} 0_{(6\times n)} \\ \frac{\partial^2 h_{t-1}}{\partial q \partial a'} \end{bmatrix}$
- $\frac{\partial}{\partial a'} vec\left[F(q(a))'(q(a) \otimes I_3) \right] = (I_3 \otimes F') \frac{\partial}{\partial a'} vec(q(a) \otimes I_3) + (q' \otimes I_3 \otimes I_n) \frac{\partial}{\partial a'} vec\left[F(q(a))' \right]$
- $\frac{\partial}{\partial a'} vec(q(a) \otimes I_3) = (K_{3,3} \otimes I_3)\left(\frac{\partial q(a)}{\partial a'} \otimes vec(I_3) \right)$
- $\frac{\partial}{\partial a'} vec\left[F(q(a))' \right] = \begin{bmatrix} 0_{(6n\times n)} \\ \frac{\partial}{\partial a'} vec\left[\frac{\partial^2 h_{t-1}}{\partial q' \partial a}(q(a)) \right] \end{bmatrix}$

Figure 1

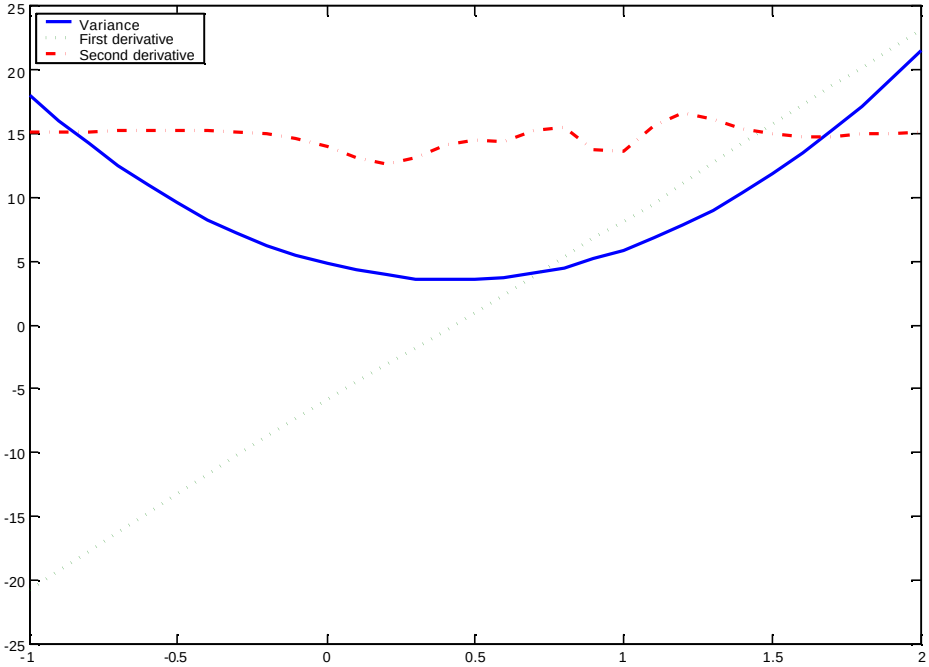

Variance
First derivative
Second derivative
25
20
15
10
5
0
-5
-10
-15
-20
-25
-1
-0.5
0
0.5
1
1.5
2

Figure 2

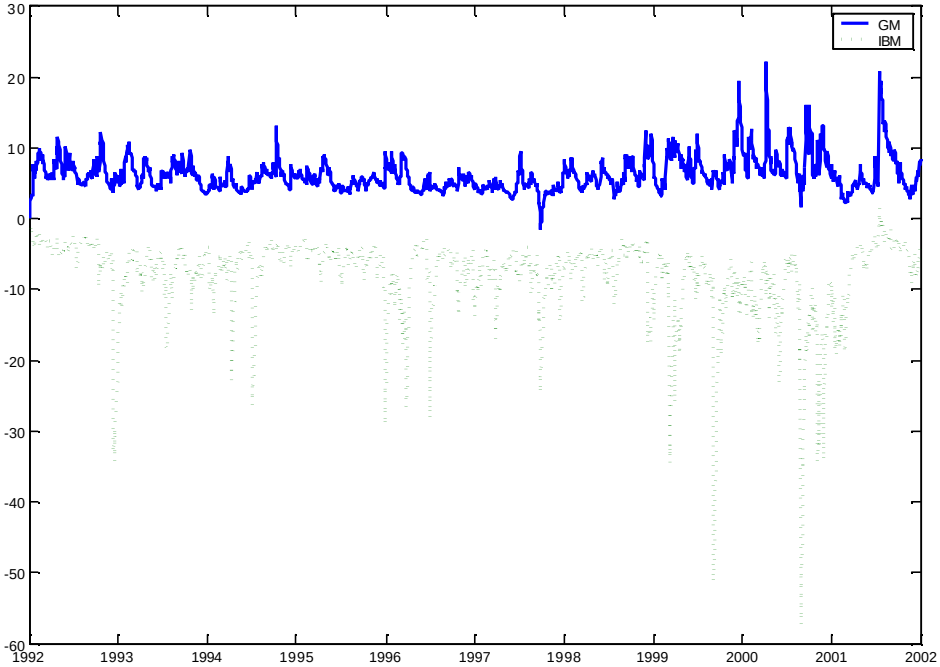
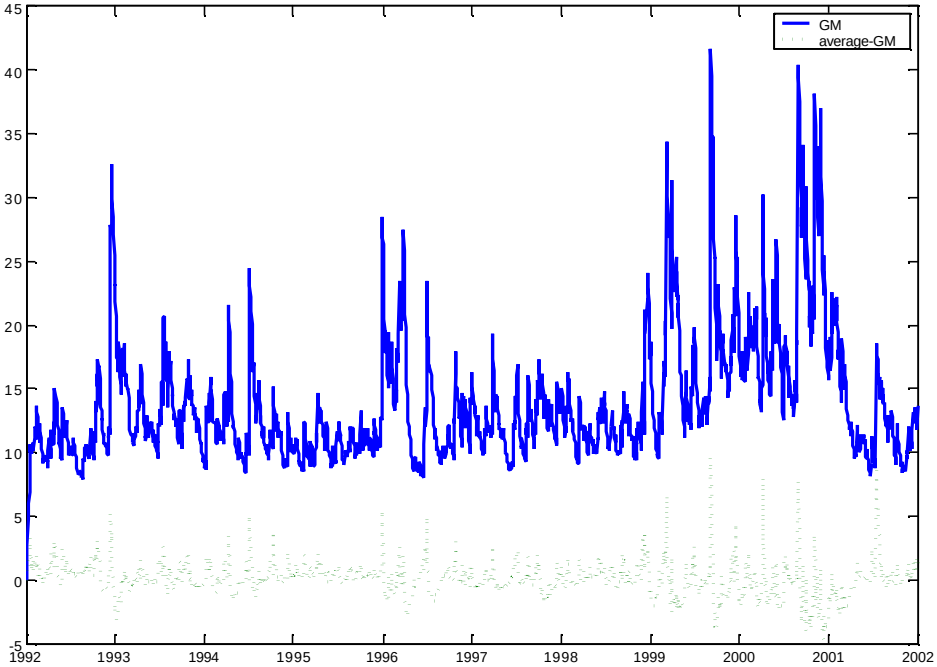

GM
IBM
30
20
10
0
-10
-20
-30
-40
-50
-60
1992
1993
1994
1995
1996
1997
1998
1999
2000
2001
2002

Figure 3


GM
average-GM
45
40
35
30
25
20
15
10
5
0
-5
1992
1993
1994
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1996
1997
1998
1999
2000
2001
2002

Table 1 – Comparison between the Variance Sensitivity Analysis (VSA) methodology and the most popular multivariate GARCH models, as of 11 March 2002: 1) Dynamic Conditional Correlation (DCC), 2) Orthogonal GARCH (OGARCH), 3) Exponentially Weighted Moving Average (EWMA). For each portfolio we report: a) the univariate GARCH annualised volatility associated to the minimum variance weights implied by the estimated variance-covariance matrix, b) the percentage difference w.r.t. VSA, and c) the computation time to estimate the model.

	PORTFOLIO WITH 2 ASSETS			**PORTFOLIO WITH 5 ASSETS**			**PORTFOLIO WITH 10 ASSETS**			**PORTFOLIO WITH 20 ASSETS**			**PORTFOLIO WITH 30 ASSETS**		
	Min Vol	*% VSA*	*Seconds*	*Min Vol*	*% VSA*	*Seconds*	*Min Vol*	*% VSA*	*Seconds*	*Min Vol*	*% VSA*	*Seconds*	*Min Vol*	*% VSA*	*Seconds*
DCC	30.64%	0.01	9	28.26%	7.20	16	18.77%	10.19	121	16.35%	23.06	185	12.92%	25.99	351
OGARCH	30.68%	0.15	5	28.06%	6.45	10	20.15%	18.34	18	17.55%	32.06	35	14.23%	38.84	53
EWMA	30.66%	0.09	2	27.53%	4.42	2	17.69%	3.86	2	15.17%	14.20	2	11.00%	7.34	2
VSA	30.64%	–	71	26.36%	–	214	17.03%	–	105	13.29%	–	169	10.25%	–	465

* The idea of this paper came about through several discussions I had with Vladimiro Ceci and Walter Vecchiato. In fact, this paper builds on an earlier draft (ECB WP No. 194) co-authored with Ceci and Vecchiato, who in the meantime have gone on to other pursuits. In addition to Vladimiro Ceci and Walter Vecchiato, I would like to thank Eric Renault (the Editor), an anonymous Associate Editor, two anonymous referees, Lutz Kilian, Luca Lotti, Neil Shephard, Kevin Sheppard and the Editorial Board of the ECB Working Paper Series, for useful comments and suggestions. The codes to replicate the results in the paper can be downloaded from http://www.simonemanganelli.org. The views expressed in this paper are those of the author and do not necessarily reflect those of the ECB.

[1] The zero-mean assumption is made only for the sake of simplicity and implies no loss of generality.

[2] The (n+1)-th weight is given by 1 minus the sum of the other weights. The corresponding (n+1)-th asset is the pivotal asset against which the sensitivity analysis is performed. By changing the pivotal asset, one obtains different sensitivity measures. Computing these sensitivity measures for each single asset of the portfolio, it is possible to compute a matrix of sensitivities analogous to the variance-covariance matrix.

[3] We left out the constant for the sake of simplicity.

[4] As usual, we assume that the weights sum to 1: $a_{n+1} = 1 - \sum_{i=1}^{n} a_i$.

[5] The average first derivative for IBM is –7.86 and for GM is 6.26. That is, the variance sensitivity of the portfolio containing only IBM was about 25% higher than that of the GM portfolio.

[6] Brandt and Diebold (2004) and Ledoit, Santa-Clara and Wolf (2003) provide two interesting alternative strategies to variance-covariance matrix estimation that will not be discussed here.

[7] Annualised volatility is given by the square root of (252 * daily variance).

[8] Assets are progressively aggregated in the order reported in Appendix A.

[9] Convergence for a 30 asset portfolio occurs in less than 15 iterations, for randomly chosen initial conditions.

[10] Computation time of VSA actually decreases when moving from portfolios with 5 assets to portfolios with 10 assets. This result, however, is probably due to the quality of the initial conditions fed to the optimisation algorithm.