Dynamic Oligopoly with Network Effects*

By Nicholas Economides**, Matt Mitchell*** and Andrzej Skrzypacz****

July 2005

Abstract

We analyze oligopolistic competition in a multi-period dynamic setting for goods with network effects. Two or more infinitely-lived firms produce incompatible products differentiated in their inherent quality. Consumers live for a single period and receive the network effect of the previous period's sales. We show existence and characterize Markov perfect equilibria that are unique given market shares at the beginning of time and fast convergence to the long run equilibrium. We find that, generally, small network effects help the higher quality firm realize higher prices, sales, and profits. Intermediate network effects lead eventually to monopoly of the firm that provides the higher inherent quality, irrespective of original market shares. Strong network effects lead to a stable monopoly equilibrium in the long run which is achieved by the firm of sufficiently high starting market share. Although the case of monopoly resulting under strong network effects and determined by original market shares has been understood in the academic literature and drives the traditional theory of "tilting" of networks to monopoly, our finding that, for intermediate network effects, the resulting monopoly is only determined by inherent quality is new and qualitatively different than traditional theories of titling to monopoly. We also find that, in the case of not-too-high network effects, the dominance of the high quality firm is accentuated as consumers become more patient. Finally, we analyze the impact of the intensity of network effects on the number of firms that survive at the long run equilibrium.

Key words: Dynamic oligopoly, network effects, foreclosure, incompatibility, monopoly

JEL Classification: L13, D43

*We thank Roy Rander, Tom Sargent, and participants at presentation at the Society of Economic Dynamics 2004 Meeting, the Stern School of Business Economics/IS workshop, the Stern School of Business IO Seminar, the Telecommunications Policy Research Conference, and U.C. Santa Cruz for their comments and suggestions.

** Stern School of Business, New York University, 44 West 4th Street, New York, NY 10012, (212) 998-0864, fax (212) 995-4218, http://www.stern.nyu.edu/networks/, e-mail: neconomi@stern.nyu.edu, and Executive Director, NET Institute, http://www.NETinst.org .

*** University of Iowa, Email: matthew-mitchell@uiowa.edu, tel. (319) 335-0849.

**** Graduate School of Business, Stanford University, Stanford, CA 94305-5015. Email: skrzypacz_andrzej@gsb.stanford.edu, tel. (650) 736-0987.

Dynamic Oligopoly with Network Effects

1. Introduction

Many markets exhibit network effects, that is, the good traded is more valuable when its expected sales are higher, everything else kept constant.[1] Key markets with network effects include telecommunications, markets for computer hardware and software or other equipment conforming to the same technical standard, railroads, financial and commodity exchanges, financial credit card and bank networks, electricity, and many others.

Firms often have the possibility of choosing whether their products will be compatible and interoperable with products of competitors.[2] If products are fully compatible, then each firm is able to reap the full network effects of sales of all competitors. However, in many network industries, incompatibility prevails. In high technology industries, firms are often able to choose incompatibility if they so desire because of the intellectual property protection afforded to their products. Thus, we often have incompatibility by design, as in the case of computer operating systems (Windows vs. Mac), computer platforms (e.g. Adobe, Mathematica) or in household hardware (e.g. video recorders, Beta vs. VHS). In other industries, a network platform can be made incompatible with others by contract. For example, traditionally Visa and MasterCard imposed incompatibility between their shared network and American Express by not allowing the American Express Bank (Centurion Bank) to issue a Visa or MasterCard.[3] In an extreme example of imposition of incompatibility by contract, AT&T up to the 1930s refused to interconnect its long distance and local telephone networks with the telephone networks of competing carriers.[4]

In network industries, where the existence of compatible products in important to consumers, past sales as summarized by the installed base can be important for present decisions of consumers (and therefore of present decisions of the firms). If incompatibility prevails, the presence of network effects can intensify competition as a firm can use a low price resulting in high market share as an aggressive competitive strategy. This issue becomes more complex in multi-period competition, as firms

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[1] Network effects arise because the value of the traded good is influenced positively from the availability of complementary goods, and more complementary goods are sold given higher sales of the traded good.

[2] Sometimes, compatibility is imposed by regulation, such as in the case of voice telecommunications in the United States since the 1930s, or because the technical compatibility standards have been negotiated and agreed by all competitors, as in the case of FAX, or because the network was set up to run on public and unalterable technical standards as in the case of the Internet.

[3] In the credit card industry, the MasterCard and Visa networks did not allow the American Express Bank to issue cards in these networks. Following a legal challenge by the US DOJ, Visa and MasterCard agreed to allow the American Express Bank to issue their cards.

[4] In telecommunications, before the imposition of regulation, AT&T refused to interconnect with networks of independent competitors resulting in parallel incompatible local telecommunications networks in many parts of the United States. AT&T, a monopolist in long distance at the time, allowed interconnection of local networks only if they were merged in the Bell System. This resulted in the absorption of the large majority of the independents in AT&T by the 1930s.

may have incentives use low prices in the current period, forego current profits, and reap higher profits in the future. In some cases, low enough prices in early periods can guarantee that a firm achieves a monopoly position in the long run. The exact way in which multi-period dynamic competition works is determined by a number of factors which we will examine in this paper.

We will focus on oligopolistic competition among long-lived firms and relatively short-lived consumers. Many markets with network effects have these features. An example for the more applied reader is competition between the Windows and the Apple (Mac) operating systems. We assume that the two firms potentially live for ever and are maximizing the present value of their long run profits. In every period, sales of the two operating systems induce the creation of computer software applications compatible with each of these operating systems by independent software developers. The existence of these two sets of applications, with each set compatible with only one operating system, provides benefits to an operating system that can be summarized as the "network effect" of the respective operating system. Consumers in our model live one period and receive the benefit of network effects of sales of applications in existence at the time of their purchases, that is, applications sales of the immediately preceding period.

In network industries, often, there are two crucial determinants of the value of a good to a consumer: first the inherent quality of the good; and second the value of the existence of complementary goods as summarized by the network effect. Taking this into account, our model assumes that firms sell products of different inherent qualities. Thus, we embed a standard model of network effects and vertical differentiation into a dynamic model of price competition. We are interested to determine the existence of the equilibrium path and characterize it. We also determine the impact of network effects on market shares and prices in relation to inherent quality differences and in relation to starting market shares. Finally, we examine the role of the intensity of network effects on the number of viable firms.

We restrict pricing strategies to be Markov-perfect where each period's price choice is based only on relevant variables of the immediately previous period. We characterize dynamic pricing in monopoly and duopoly. Under duopoly, we show that small enough network effects lead to the existence of a Markov-perfect equilibrium where firms choose linear strategies. For myopic firms and small network effects, in duopoly, stable dynamics converge to long run equilibrium where the high quality firm has more than 2/3 of the market. For intermediate network effects, in the long run, the high quality firm monopolizes the market. The long run winner is determined by inherent quality differences, and not by initial market share conditions. For strong network effects, there are two stable monopoly equilibria. There is also a third interior unstable steady state, so that even when there are interior solutions, the evolution is unstable. Which long run equilibrium is reached is determined only by the initial market share conditions. Although the case of monopoly resulting under strong network effects and determined by original market shares has been understood in the academic literature and drives the traditional theory of "tilting" of networks to monopoly, our finding that, for intermediate network effects, the resulting monopoly is only determined by inherent quality is new and qualitatively different than traditional theories of titling to monopoly.

For patient firms and not-too-strong network effects, we establish a unique Markov-perfect equilibrium where the equilibrium path market share difference is linear in the price differences between the firms in the preceding period. Higher network effects increase the inequality of the market structure. We show that the convergence to the long run equilibrium is fast with a half-life of one period or less. For large network effects, we can show that this type of equilibrium does not exist.

We also find conditions for survival of more than two firms depending on the intensity of the network effects.

This paper builds on two strands of literatures, the literature on network effects and the literature on dynamic games. Both of these literatures are large and it would be counterproductive to summarize them here. However, we note that the intersection of these literatures, that is, dynamic games with network effects is thin. We note Farrell and Saloner (1985, 1986) and Katz and Shapiro (1985). More recently, Mitchell and Skrzypacz (2005) study a dynamic duopoly model with network effects similar to the one studied here. Whereas the present paper studies a model where people differ in their taste for quality, firms provide goods of different levels of inherent quality, and these quality levels plays a crucial role in the equilibrium market structure and in comparative statics, Mitchell and Skrzypacz (2005) focus on long run market shares in a model where there is horizontal differentiation but taste for quality is identical across consumers. Our model is based on infinite horizon recursive models as described in Ljungqvist and Sargent (2000), and shares some computational methodologies with Judd (2003).

2. Dynamic Monopoly

Let firm i = 1 sell a product of inherent quality v_i. Consumers also receive a "network effect" proportional to sales of the previous period.[5] The network effect summarizes the benefit to a consumer from the existence of complementary goods. Thus, the utility of consumer θ in (current) period t who buys good i is

$$U_{\theta,t,i} = k + \theta v_i + Ax_{i,t-1} - p_{i,t} \tag{1}$$

where $k + \theta v_i$ is the inherent quality of the good i, A is the intensity of the network effect, $x_{i,t-1}$ are sales of good i in period t-1, and $p_{i,t}$ is the price of good i in period t. We assume that types θ are distributed uniformly on [0, 1].

2.1 Myopic Single Product Monopolist

Profits for a myopic monopolist are:

$$\Pi_{1,t}(p_{1,t}, x_{1,t-1}) = p_{1,t}\, x_{1,t}(p_{1,t}, x_{1,t-1}) = p_{1,t}(1 - (p_{1,t} - Ax_{1,t-1} - k)/v_1),$$

They are maximized at

$$p^*_{1,t} = (k + v_1 + Ax_{1,t-1})/2$$

[5] Results are similar if consumers receive network effects from current period sales as well.

provided that

$$v_1 > k + Ax_{1,t-1}.$$

Under this condition, that is, for small inherent product value k, the monopolist does not provide full coverage. Otherwise, that is, for large k, the monopolist provides full coverage and charges

$$p^*_{1,t} = k + Ax_{1,t-1}.$$

2.2 Myopic Two Product Monopolist

Let the monopolist produce two goods i = 1, 2 with respective qualities v_1 , v_2 and prices p_1, p_2. Without loss of generality, we assume $v = v_1$, $v_2 > 0$. A myopic monopolist maximizes $p_1x_1 + p_2x_2$. Direct maximization yields

$$p_{1,t} = k + (A + v)/2,\ x^*_{1,t} = (1 + A\Delta_{t-1}/v)/2,\ x^*_{2,t} = (1 - A\Delta_{t-1}/v)/2$$

and therefore the new market share difference is

$$\Delta_t = A\Delta_{t-1}/v.$$

Therefore, for small network effects, A/v < 1, the monopolist eventually offers equal market shares. For large network effects, A/v > 1, eventually the whole market is served by the product that has the largest initial market share.

3. The Dynamic Duopoly Model

Let two firms sell quality-differentiated products. Consumers also receive a "network effect" proportional to sales of the previous period.[6] The network effect summarizes the benefit to a consumer from the existence of complementary goods. Thus, the utility of consumer θ in (current) period t who buys good i is, as defined earlier

$$U_{\theta,t,i} = k + \theta v_i + Ax_{i,t-1} - p_{i,t} \qquad (1)$$

where $k + \theta v_i$ is the inherent quality of the good i, A is the intensity of the network effect, $x_{i,t-1}$ are sales of good i in period t-1, and $p_{i,t}$ is the price of good i in period t. We assume that types θ are distributed uniformly on [0, 1]. Without loss of generality, we assume that $v_1 - v_2 = v > 0$ so that firm 1 has higher quality.

Consumers with high willingness to pay for quality prefer to buy from firm 1, *ceteris paribus.* We also assume that k, the value of the good to the lowest type when there are zero prior sales, is sufficiently high so that all consumers buy from one or the other firm. This implies that the market is fully covered and, since the market is of size 1, sales of firm i are identical to its market share.[7]

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[6] Results are similar if consumers receive network effects from current period sales as well.

[7] We have assumed that consumers receive network effects from sales of the same good in the immediately previous period. Thus, sales in period t-1 act as the "installed base" that generates a network effect for a consumer of period t. Thus, sales of periods t-2 and earlier do not matter for the

We define the market share difference as

$$\Delta_{t-1} = x_{1,t-1} - x_{2,t-1}. \tag{2}$$

Given the definition of $x_{i,t}$ as market share, in general, $x_{i,t}$ œ [0, 1] and Δ_t œ [-1, 1]. The marginal consumer in period t is θ_t^*:

$$\theta_t^*(p_{1,t}, p_{2,t}, \Delta_{t-1}) = (p_{1,t} - p_{2,t} - A\Delta_{t-1})/v \tag{3}$$

and sales for the two firms in period t are

$$x_{1,t}(p_{1,t}, p_{2,t}, \Delta_{t-1}) = 1 - \theta_t^*, \quad x_{2,t}(p_{1,t}, p_{2,t}, \Delta_{t-1}) = \theta_t^*. \tag{4}$$

We assume that consumers live one period while firms live forever. Firms are choosing current period prices anticipating correctly the continuation of the game. We seek Markov-perfect equilibria. Assuming zero costs, the profits of firm i from current period sales are

$$\pi_{i,t}(p_{1,t}, p_{2,t}, \Delta_{t-1}) = p_{i,t}\, x_{i,t}(p_{1,t}, p_{2,t}, \Delta_{t-1}). \tag{5}$$

Profits of firm i starting from period t and continuing indefinitely are

$$\Pi_{i,t} = \pi_{i,t} + \beta V_{i,t}, \tag{6}$$

where $V_{1,t}$ is the value from the continuation of the game beyond period t, and β in [0, 1] is the discount rate. At the Markov-perfect equilibria we will be describing, the continuation value will depend only on the current state of the industry, so that $V_{1,t} = V_{1,t}(\Delta_t)$ and therefore

$$\Pi_{i,t}(p_{1,t}, p_{2,t}, \Delta_{t-1}) = \pi_{i,t}(p_{1,t}, p_{2,t}, \Delta_{t-1}) + \beta V_{1,t}(\Delta_t). \tag{7}$$

4. Duopoly Equilibrium

4.1 Myopic Firms

We first consider the benchmark case when firms maximize just current period profits either because they are myopic, or, future profits had zero present value because the discount rate is infinite, *i.e.,* $\beta = 0$. For myopic firms,

$$\Pi_{i,t}(p_{1,t}, p_{2,t}, \Delta_{t-1}) = \pi_{i,t}(p_{1,t}, p_{2,t}, \Delta_{t-1}) = p_{1,t}\, x_{1,t}(p_{1,t}, p_{2,t}, \Delta_{t-1}). \tag{8}$$

First, we consider the case of small network effects, $A/v < 1$. In this case, solutions are always interior, that is, no firm ever goes out of business. Substitution from (3) and (4) and profit maximization yields equilibrium prices, sales, and profits

$$p_{1,t} = (2v + A\Delta_{t-1})/3, \quad p_{2,t} = (v - A\Delta_{t-1})/3, \tag{9a}$$

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network effect of consumer of period t. We have also assumed that a consumer does not receive network effects from present period sales. We believe that these assumptions could be relaxed without significant qualitative change in the results.

$$x_{1,t} = (2 + A\Delta_{t-1}/v)/3, \quad x_{2,t} = (1 - A\Delta_{t-1}/v)/3, \tag{9b}$$

$$\pi_{1,t}(\Delta_{t-1}) = (2v + A\Delta_{t-1})^2/(9v), \quad \pi_{2,t}(\Delta_{t-1}) = (v - A\Delta_{t-1})^2/(9v). \tag{9c}$$

This equilibrium implies that the difference in market shares in period t is

$$\Delta_t = x_{1,t} - x_{2,t} = 1/3 + [2A/(3v)]\Delta_{t-1}. \tag{10}$$

Thus the difference in market shares Δ_t converges to

$$\Delta^* = v/(3v - 2A) \text{ œ } (1/3, 1), \tag{11a}$$

and prices, sales, and profits converge to

$$p_1^* = v(2v - A)/(3v - 2A), \quad p_2^* = v(v - A)/(3v - 2A), \tag{11b}$$

$$x_1^* = (2v - A)/(3v - 2A) > 2/3, \quad x_2^* = (v - A)/(3v - 2A) < 1/3, \tag{11c}$$

$$\Pi_1^* = v(2v - A)^2/(3v - 2A)^2, \quad \Pi_2^* = v(v - A)^2/(3v - 2A)^2. \tag{11d}$$

Prices, sales, and profits increase (respectively decrease) for firm 1 (respectively firm 2) with the strength of the network effect. An increase in the inherent quality of firm 1 (or equivalently a decrease in the inherent quality of firm 2) results in higher prices for both firms, lower sales for firm 1, higher sales for firm 2, and higher profits for both firms.

Second, we consider the case of intermediate strength network effects, $2 > A/v > 1$. Market share difference evolves according to

$$\Delta_t = x_{1,t} - x_{2,t} = \min\{1/3 + [2A/(3v)]\Delta_{t-1}, 1\}. \tag{12}$$

For $\Delta_{t-1} > v/A$, the high quality firm captures the whole market in period t, $\Delta_t = 1$. However, for no Δ_{t-1} does firm two capture the whole market at time t. In the long run, $\Delta^* = 1$.

Finally, suppose network effects are strong, $A/v > 2$. In that case, even when there are interior solutions, the evolution is unstable. When either firm reaches a monopoly position ($\Delta_{t-1} = 1$ or $\Delta_{t-1} = -1$) it maintains that position. Besides the two stable monopoly equilibria, there is always a third, unstable steady state at $\Delta^* = v/(3v - 2A)$. For initial Δ_0 greater than $v/(3v - 2A)$, firm one eventually dominates. For Δ_0 less than $v/(3v - 2A)$, firm 2 eventually controls the entire market.

Notice that the existence of network externalities has a fundamentally different effect from simple quality differences, because it makes past market shares matter. Initial conditions have persistent effects in the model; while the initial market share difference Δ_0 does not always affect the long run market share, its effects typically disappear only asymptotically.[8]

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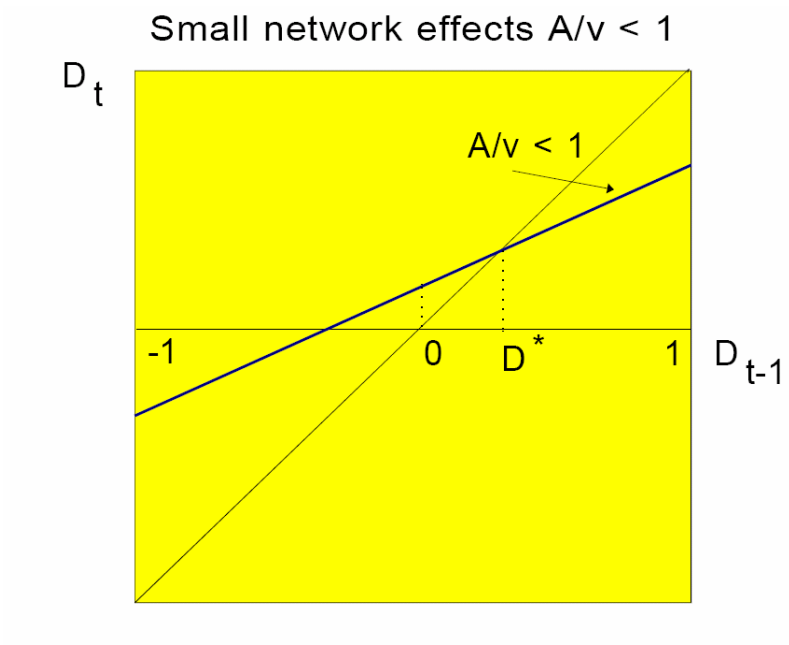
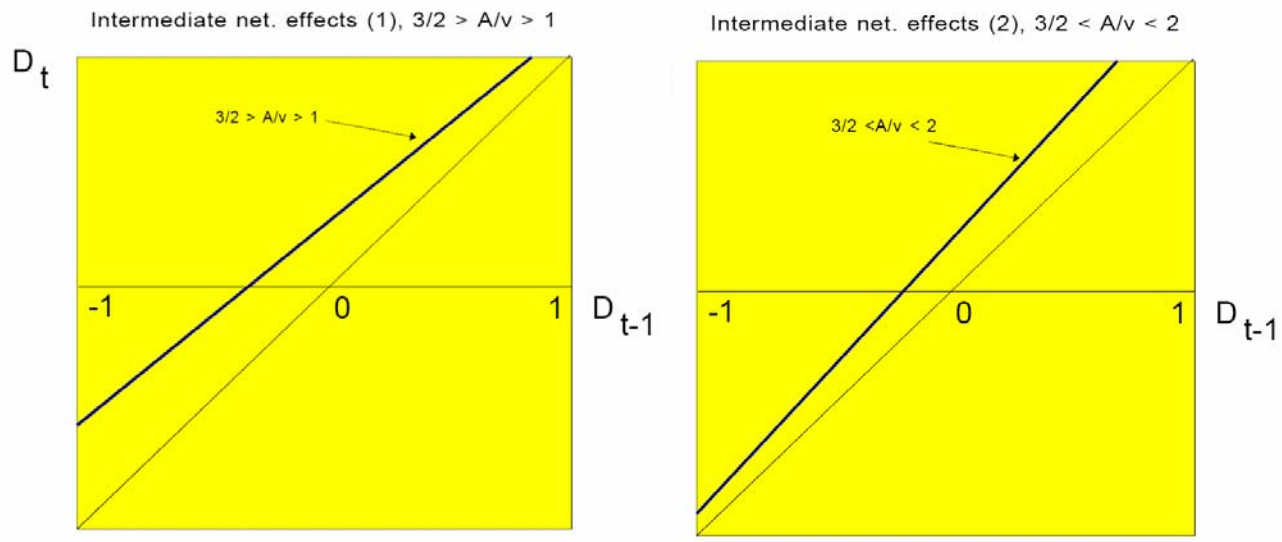
[8] To see the importance of this distinction, suppose $A/v > 1$. In that case, for almost all initial conditions, the long run outcome is for one firm to dominate the market. Suppose one correctly

Figure 1

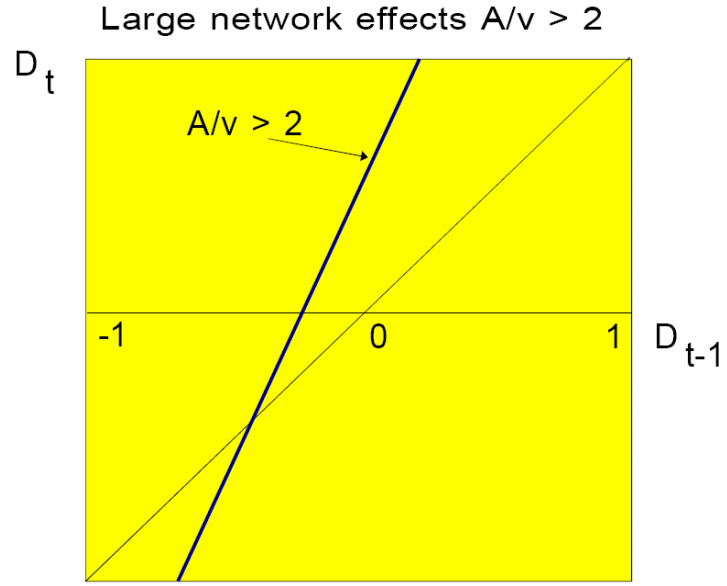


Figures 2 and 3



identifies that there are some quality elements for which consumers' tastes are heterogeneous (v), and some characteristics (the network effect) that everyone values identically. However, suppose that the identically-valued characteristic is assumed to be differences in k rather than a network effect. The market share of firm one is then $(2 + (k_1 - k_2)/v)/3$. The estimate of "$k_1 - k_2$" is actually $A\Delta_0$. If Δ_0 is small, using the model without network effects one might conclude that the market share difference will hover around 1/3 when in fact it will diverge to 1 or -1.

Figure 4



4.2 Dynamic Duopoly Equilibrium in a Two-period Game

We now discuss a two-period dynamic game where the firms discount the future with discount rate β. Since there is no continuation after period 2, the second period profits are directly seen from (9c) setting t = 2 as

$$\pi_{1,2}(\Delta_1) = (2v + A\Delta_1)^2/(9v), \quad \pi_{2,2}(\Delta_1) = (v - A\Delta_1)^2)/(9v). \tag{13}$$

Given period 1 prices, the difference in quantities entering period 2 will be:

$$\Delta_1 = x_{1,1} - x_{2,1} = 1 + 2(A\Delta_0 - p_{1,1} + p_{2,1})/v. \tag{14}$$

At the first period, firms maximize

$$\Pi_i(p_{i,1}, p_{j,1}, \Delta_0) = \pi_{i,1}(p_{i,1}, p_{j,1}, \Delta_0) + \beta\pi^*_{i,2}(\Delta_1), \tag{15}$$

where $\pi^*_{i,2}(\Delta_1)$ are the equilibrium profits in the last (second) period of the game. Solving for the optimal prices in the first period we get:[9]

$$\begin{aligned} p_{1,1} &= A\frac{9v^2 - 8\beta A^2}{27v^2 - 16\beta A^2}\Delta_0 + \frac{2}{3}\frac{27v^3 + 2\beta A\left(8\beta A^2 - 9Av - 15v^2\right)}{27v^2 - 16\beta A^2}, \\ p_{2,1} &= -A\frac{9v^2 - 8\beta A^2}{27v^2 - 16\beta A^2}\Delta_0 + \frac{1}{3}\frac{27v^3 + 4\beta A\left(8\beta A^2 - 3Av - 12v^2\right)}{27v^2 - 16\beta A^2}. \end{aligned} \tag{16}$$

[9] Second order conditions are met for $A < (3v)/(2\sqrt{\beta})$.

We point out two regularities in the equilibrium price strategies: first, they are linear in the state variable, Δ_0; second, the coefficient on the state variable in the strategies of the two firms differs only in sign. As we will see, these two features survive in the infinite-horizon game.

The equilibrium difference in market shares after the first period is:

$$\Delta_1 = v\frac{9v + 18A\Delta_0 + 8\beta A}{27v^2 - 16\beta A^2}. \tag{17}$$

The simple comparative statics are that Δ_1 is increasing in β, A and Δ_0, and decreasing in v. Similar comparative statics turn out to be true for the long-run market equilibrium in the infinite horizon game that we study next.

4.3 Dynamic Duopoly Equilibrium in an Infinite Horizon Game for Patient Firms

4.3.1 Existence and Uniqueness

We now consider the game with patient firms, *i.e.*, with a positive discount factor. We construct and show the existence of Markov-perfect equilibria, in which firms choose prices conditional on the current state of the industry, Δ_{t-1}. In the two previous models (myopic firms and a two-period game) the equilibrium pricing strategies were linear in the state variable. Therefore we seek equilibrium price strategies that are linear in the state variable. We will show that for sufficiently small A > 0, such an equilibrium exists. The strategy of the proof is to show that profit maximization in each time period by firms using strategies that are linear in the state variable (market share difference) defines an operator that maps quadratic value functions to themselves and satisfies Blackwell's sufficient conditions for a contraction mapping.

Proposition 1: For not-too-large network effects, A in [0, v/[2(1 + β)], and β in [0, 1) there exists a unique Markov-perfect equilibrium in which the firms follow linear pricing strategies, $p_{i,t} = r_i + q_i\Delta_{t-1}$.

Proof: See Appendix.

Corollary 1: For not-too-large network effects, A in [0, v/[2(1 + β)], and β in [0, 1), at the equilibrium path, the difference in market shares difference converges over time to $\Delta^* \in$ [1/3, 1].

Proof: The evolution of market share, $\Delta_t = \alpha\Delta_{t-1} + \Delta^+$, has $\alpha \in (0, 1)$ and Δ^+ increasing in A. When A = 0, $\Delta^* = 1/3$, so Δ^* must be at least 1/3 for any A > 0.

Note that when A = 0, the market share difference is always 1/3. Therefore the presence of network externalities, as in the earlier cases we considered, increase the disparity between firms.

4.3.2 Comparative Statics

We now turn to characterizing the equilibrium constructed above. As we have shown, the firms follow price strategies $p_{i,t} = q_i\Delta_{t-1} + r_i$. We know already that $q_1 = -q_2 \in (0, A/3)$. Also, from equation (A14) (see the Appendix) we can verify that $r_1 > r_2$. Therefore in equilibrium, if $\Delta_{t-1} \geq 0$, the price of firm 1 will be higher than the price charged by firm 2. We now consider the comparative statics of the strategies at the equilibrium path.

Proposition 2: For not-too-large network effects, A in [0, v/[2(1 + β)], and β in [0, 1), at the equilibrium path, q_1, the influence of the market share difference on price, is decreasing in β (as the future becomes more important) and increasing in the strength of network effects A and quality difference v.

Proof: Follows from equation (A13) and implicit function theorem.

Now, let's consider the long-run difference of market shares, Δ^*. It can be written as:

$$\Delta^* = \frac{1+4\beta y}{1+2\sqrt{1-4\beta y^2}} \tag{18}$$

where

$$y = (A - q_1)/v. \tag{19}$$

That allows us to prove the following comparative statics:

Proposition 3: For not-too-large network effects, A in [0, v/[2(1 + β)], and β in [0, 1), the long-run difference in market shares, Δ^*, is increasing in β and A, and decreasing in v.

Proof: First note that Δ^* is increasing in y. Second, using equation (A13) and the implicit function theorem we can show that y is increasing in A. Therefore Δ^* is increasing in A. Third, from the previous proposition we know that q_1 is decreasing in β, so y is increasing in β. Summing up, Δ^* is increasing in β because of its direct and indirect effect (through y). Finally, from the previous proposition, q_1 is increasing in v. Therefore y is decreasing in v and so does Δ^*. QED

It may look odd that an increase in the quality difference v results in the high quality firm selling less (and the low quality firm selling more). But consider the equilibrium in a world with no network effects, *i.e.,* A = 0. Then the market shares of the two firms would be 2/3 and 1/3 resulting in a market share difference of $\Delta^* = 1/3$, which we may call the vertical differentiation effect. Now with some network effect A > 0, the high quality firm captures a little more market share, so that $\Delta^* > 1/3$. We may call this the network effect. The bigger is v, the more important is the vertical differentiation effect, and therefore the closer the equilibrium to $\Delta = 1/3$. Put another way, the greater is v, the more the high quality firm wants to substitute high prices for volume, because differentiation in tastes is bigger.

Notice that the long run market share difference is not influenced by the initial market share difference Δ_0; all that matters is the relative values of v, A, and β.

However, for any finite time T, the initial Δ_0 plays a role and its role disappears only asymptotically.

In terms of the speed of convergence to the long run equilibrium, from equations (A15-A16) we can write the market share difference at a weighted sum of the initial market share difference and the long run equilibrium:

$$\Delta_t = \alpha^t \Delta_0 + \frac{1-\alpha^t}{1-\alpha}\Delta^+ = \alpha^t \Delta_0 + \left(1-\alpha^t\right)\Delta^* \tag{20}$$

To calculate the time t by which the distance between the starting market share and the long-run steady state shrinks to a half (*i.e.*, the half-life of the process), we need to solve:

$$\Delta_t - \Delta^* = \frac{1}{2}\left(\Delta_0 - \Delta^*\right) \tag{21}$$

which is solved at

$$t = -\frac{\ln 2}{\ln \alpha}. \tag{22}$$

Therefore the half-life is increasing in α and hence the speed of convergence is decreasing in α. When α is less than ½ the half-life (t) is less than one period. When α is above 0.7 the half-life is two periods; when α is 0.9 the half-time is more than 6.5 periods.

What can we say about size of α at the equilibrium path? From (A16), $\alpha = 2(A - 2q_1)/v$. From proposition 2 we have that q_1 is decreasing in β, so α and half-life are increasing (and the speed of convergence is decreasing) in β. Since we know that q_1 is between 0 and A/3, it follows that α is between (2/3)A/v and 2A/v. In the equilibrium existence range (see Proposition 2), A in [0, v/2(1+ β)], the upper bound on α is 1/(1+ β). Since α is increasing in β, a uniform upper bound on α is ½ in that range! It follows that the half-life is at most one period, that is, the convergence is quite fast!

We can also show that q_1 is increasing in v so the half-life is decreasing in v, and the speed of convergence is increasing in v, and that α is increasing in A, so the half-life is increasing in A (and the speed of convergence is decreasing in A).[10]

<u>Proposition 4</u>: For not-too-large network effects, A in [0, v/[2(1 + β)], and β in [0, 1), the process converges fast to the long run equilibrium with the distance between the starting market share and the long-run steady state shrinks to a half (*i.e.*, the half-life of the process) being at most one period. The half-life is also decreasing in the quality difference v and increasing the intensity of the network effect A.

[10] This follows from direct calculation. The proof is available from the authors upon request.

5. The Planner's Problem

A myopic planner maximizes single-period welfare

$$\max_{x_{2,1}} \int_0^{x_{2,1}} (k + v_2\theta + Ax_{2,0})d\theta + \int_{x_{2,1}}^{1} (k + v_1\theta + A(1 - x_{2,0}))d\theta \tag{23}$$

which represents the sum of the valuations of consumers that buy each of the goods (including the value of network effects) and $x_{2,t}$ is sales of the second firm in period t. It is easy to show that first order condition implies

$$\Delta_1 = 1 + 2A\Delta_0/v \tag{24}$$

and second order conditions are satisfied.[11] Thus, if the myopic planner is faced with an initial condition of $\Delta_0 \geq 0$, he chooses immediately $\Delta_t = 1$ for all $t \geq 1$, *i.e.,* he assigns a monopoly to the high quality firm. If the initial market share is very much in the low quality firm's favor, $\Delta_0 < -v/A$, then the planner chooses immediately $\Delta_1 = -1$, and gives a monopoly to the low quality firm. For

$$\Delta_0 \in (-v/A, 0), \tag{25}$$

the planner allows both firms to stay in business and chooses next period's market share difference as

$$\Delta_t = 1 - 2x_{2,t} = 1 + 2A\Delta_{t-1}/v. \tag{26}$$

The steady state $\Delta_t = 1$ for all $t \geq 1$ exists for all A and v if $\Delta_0 \geq 0$, and for small A, $A/v < 1$, it is the unique steady state. For large A, $A/v \geq 1$, there exists another stable steady state at $\Delta_1 = -1$ (monopoly of firm 2). For large A, the myopic planner will always choose monopoly by one of the firms, and he will give the monopoly to firm 2 for sufficiently high starting market share of firm 2. For large A, if at the starting point the lower quality product has sufficiently large advantage $\Delta < v/(v - 2A)$, then the social planner will diverge over time to $\Delta = -1$. In that case, there is also an unstable steady state

$$\Delta = v/(v - 2A). \tag{27}$$

Compared to the market outcomes, we note that, for small network effects, $A/v < 1$, the planner tends to accentuate inequality by giving the monopoly to the higher quality firm. For $2 > A/v \geq 1$, $\Delta = -1$ is a stable steady-state for the planner, but it isn't for the market outcome. Thus, for intermediate network effects, the market over-provides quality. For $2 < A/v$, the market and the planner choose similarly with small differences in the cutoff points of the parameters.

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[11] The first order condition is $k + x_{2,1}v_2 + Ax_{2,0} - k - x_{2,1}v_1 - A(1 - x_{2,0}) = 0$, which simplifies to $x_{2,1}v - A(1 - 2x_{2,0}) = 0$. As $v > 0$, the SOC is satisfied. Solving the FOC we obtain: $x_{2,1} = A(1 - 2x_{2,0})/v = -A\Delta_0/v$, and therefore $\Delta_1 = 1 - 2x_{2,1} = 1 + 2A\Delta_0/v$.

6. Competition With More Than Two Firms

Our analysis of the two firm case shows that network externalities can lead, eventually, to one firm dominating the market; we know that without network externalities, the degree of consumer heterogeneity assumed here is sufficient for many firms with different levels of quality to make positive profits. However, the existence of network externalities may disadvantage small firms significantly and may lead to their exit from the market. We explore this possibility further by considering competition with more than two active firms. For analytic tractability, we focus on the myopic case.

The question we are interested in is whether N firms can survive in a stable steady state for a given level of network externalities. To simplify matters, let quality differences among consecutive firms be equal, $v_i - v_{i+1} = v$, for all i from 1 to N - 1. This specification is similar to the usual sort of "quality ladder" model, for instance as in Gabzsewicz and Thisse (1980), where quality differences across firms are constant "rungs" in the ladder.

From the analysis of section 5.1, we know that when $A/v > 1$ there is only one firm in the long run. This result is not limited to the case where N = 2. Specifically, consider the case when N = 3. Let $\mathbf{\Delta}_{t-1}$ denote the (column) vector of market share differences across consecutive firms, *i.e.,*

$$\Delta_{i,t-1} = x_{i,t-1} - x_{i+1,t-1} \tag{28}$$

Similarly, let **p** denote the vector of prices. Then the marginal consumers between the firms are defined by

$$\theta_{t,1}^*(\mathbf{p}_t, \mathbf{\Delta}_{t-1}) = (p_{1,t} - p_{2,t} - A\Delta_{1,t-1})/v, \quad \theta_{t,2}^*(\mathbf{p}_t, \mathbf{\Delta}_{t-1}) = (p_{2,t} - p_{3,t} - A\Delta_{2,t-1})/v, \tag{29}$$

and sales for the three firms in period t are

$$x_{1,t}(\mathbf{p}_t, \mathbf{\Delta}_{t-1}) = 1 - \theta_{t,1}^*, \; x_{2,t}(\mathbf{p}_t, \mathbf{\Delta}_{t-1}) = \theta_{t,1}^* - \theta_{t,2}^*, \; x_{3,t}(\mathbf{p}_t, \mathbf{\Delta}_{t-1}) = \theta_{t,2}^*. \tag{30}$$

The analysis follows a similar path as in section 5.1. First suppose $A/v \leq 1/4$. In that case, maximizing $p_i x_i$ for each firm always leads to an interior solution; the evolution of market share differences follows

$$\begin{aligned} \Delta_{1,t} &= \frac{1}{4} + \frac{A}{4v}(3\Delta_{1,t-1} - \Delta_{2,t-1}) \\ \Delta_{2,t} &= \frac{1}{4} + \frac{A}{4v}(3\Delta_{2,t-1} - \Delta_{1,t-1}) \end{aligned} \tag{31}$$

This is a stable, monotone system, so for any initial value, this system converges monotonically to

$$\overline{\Delta}_1 = \overline{\Delta}_2 = \frac{v}{4v - 2A} \in (1/4, 2/7)\,. \tag{32}$$

This corresponds to market shares of

$$x_1 = \frac{1}{6}\frac{7v-2A}{2v-A} \in (7/12, 13/21), x_2 = \frac{1}{3}, x_3 = \frac{1}{6}\frac{v-2A}{2v-A} \in (1/12, 1/21)\,. \tag{33}$$

Clearly, the high quality firm gains market share at the expense of the low quality firm as A increases.[12]

When 1/4 < A/v < 1/2, it is possible that $x_3 = 0$ for some $\mathbf{\Delta}_{t-1}$. In that case, the dynamics revert to those in the two firm case; however, these dynamics always lead to a situation where firms choose interior market shares, at which point the evolution follows equation (31), and the long run market shares are as described for A/v < 1/4.

When 1/2 < A/v < 1, then there is no steady state with positive values for all three firms, since in that case the only candidate is the one described for A/v < 1/4, which is not interior. Since the dynamics for interior market shares is monotone and there is no steady state on the interior, the system must eventually reach a boundary, with firm 3 producing nothing. The question is, once firms 1 and 2 are in competition, are their market shares such that firm 3 remains out.

When $x_{3,t-1} = 0$, firm 3 produces nothing as long as $(A/v)(1 + 3x_{2,t-1}) \geq 1$. Competition between firms 1 and 2 leads to monotonic convergence to $x_2 = (v - A)/(3v - 2A)$. At this level of x_2, firm 3 remains producing nothing if

$$(A/v)(1 + 3(v - A)/(3v - 2A) \geq 1, \tag{34}$$

or equivalently, A/v ≥ 3/5. Thus, for 1/2 < A/v < 1, we have two possible cases. For 3/5 < A/v < 1, firm 3 eventually has zero market share, and remains there forever. However, for 1/2 < A/v < 3/5, we have cycles: whenever all three firms are producing a positive amount, firm 3 is being driven to zero market share; however, firm 3 eventually returns when the competition between firms 1 and 2 goes on for long enough.[13]

When A/v > 1, the evolution is unstable if all three firms have positive market share, so that there are no stable long run outcomes with three firms. We know from the earlier analysis for N = 2 that there is no more than one firm when A/v > 1; which firm is in business in the long run depends on the magnitude of A relative to v and the initial conditions.

We can generalize our study of the number of firms in the long run to the case of N firms. We look for a steady state where N firms all have positive market share, to see under what conditions the model predicts that N firms might survive.[14] If the N

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[12] However, note that profits need not be decreasing in A for firm 3.

[13] These cycles are reminiscent of the ones found in two-sector models of capital accumulation, where one type of capital is used in two different production functions for consumption and investment. We conjecture that the cycles we find here are, like in the two-sector growth model, not robust to environments with continuous time. Instead of cycles, the dynamics would be damped oscillations, perhaps including multiple steady states.

[14] Unlike the case of N = 2 and N = 3, the global dynamics will not be studied, although it could be very interesting. Moving from N = 2 to N = 3 already generates the possibility of cycles.

firms all have positive market share, we can write the first order conditions as a linear function of $\mathbf{\Delta}$ and $\mathbf{p}$

$$B\mathbf{p}_t + C\mathbf{\Delta}_{t-1} + c = 0. \tag{35}$$

The market share differences evolve according to the linear relationship

$$\mathbf{\Delta}_t = E\mathbf{p}_t + F\mathbf{\Delta}_{t-1} + f. \tag{36}$$

Since $\mathbf{p}_t = B^{-1}(C\mathbf{\Delta}_{t-1} + c)$, the evolution of $\mathbf{\Delta}$ can be rewritten as $\mathbf{\Delta}_t = M\mathbf{\Delta}_{t-1} + m$, where $M = EB^{-1}C + F$ and $m = EB^{-1}c + f$. A steady state with N firms exists if and only if there is a solution to $\mathbf{\Delta} = M\mathbf{\Delta} + m$ where all of the market shares arising from $\mathbf{\Delta}$ are positive. That steady state is locally stable if all of the eigenvalues of M are between zero and one.[15]

The stability issue turns out to be simpler than the question of interiority. All of the eigenvalues are of the form

$$\lambda_i = \rho_i \frac{A}{v} \tag{37}$$

where ρ_i is a positive, real number. The following table lists the largest value of ρ_i as a function of N:

N	Largest Value of ρ_i
3	1.000
4	1.178
5	1.250
6	1.282
7	1.299
8	1.308
9	1.314
10	1.319

As N gets large, this largest value converges to 4/3.[16] In other words, if A/v < 3/4, there is the possibility of convergence to a steady state with many firms.

For N > 3, it is still possible to follow the same steps to compute the long run market share at an interior solution for all firms. For instance, for the case where N = 7,

[15] Note that stability is global if the firms always choose interior market shares; otherwise the dynamics from M only describe a portion of the state space.

[16] This result has been shown numerically for up to 271 firms. The convergence is already becoming apparent from the table. When N = 31 the largest eigenvalue is within 1/1000 of 4/3.

$$x_4 = \frac{(2\text{A-v})^2}{(28\text{A}^2\text{-}80\text{Av}+45\text{v}^2)}$$

$$x_5 = \frac{(2\text{A-v})^3 (A-v)(2A-7v)}{(28\text{A}^2\text{-}80\text{Av}+45\text{v}^2)(\text{-}26\text{v}^3+55\text{v}^2\text{A-}32\text{vA}^2+4\text{A}^3)}$$

$$x_6 = \frac{(2\text{A-v})^4 (A-2v)}{(28\text{A}^2\text{-}80\text{Av}+45\text{v}^2)(\text{-}26\text{v}^3+55\text{v}^2\text{A-}32\text{vA}^2+4\text{A}^3)}$$

$$x_7 = \frac{1}{2}\frac{(2\text{A-v})^5}{(28\text{A}^2\text{-}80\text{Av}+45\text{v}^2)(\text{-}26\text{v}^3+55\text{v}^2\text{A-}32\text{vA}^2+4\text{A}^3)}$$

which are all positive if A/v < 1/2. Surprisingly, once A/v is below 1/2, it is not simply that three firms can survive; in fact, many firms can. We conjecture that the number of firms that can exist in this case is infinite. The calculations show that at least 7 firms can survive for any such A, which itself is an interesting feature of the model. The model is able to distinguish, using the relative strength of network effects, whether there will be one, two, or many firms in the long run.

7. Concluding Remarks

This paper analyzes dynamic duopolistic and oligopolistic competition for incompatible goods with network effects that also differ in their inherent quality levels. We show that, for myopic firms, small network effects increase the dominance of the high quality firm. Large network effects lead to "tipping" to a long run monopoly of the firm of the highest initial market share, as expected. Intermediate strength network effects also lead to long run monopoly. However, the monopolist in this case is the firm with the highest quality level, irrespective of initial market shares. This is a new and interesting result showing that inherent quality combined with intermediate strength network effects can lead to long run monopoly, and the crucial variable determining who is the monopolist is the inherent quality of the products and not their initial market shares.

For patient forward-looking firms and not-too-large network effects, we show existence and uniqueness of a Markov-perfect equilibrium where prices are linear in last period's market share difference. Higher network effects increase the inequality of the market structure. We show that the convergence to the long run equilibrium is fast with a half-life of one period or less. For large network effects, we can show that this type of equilibrium does not exist.

For three myopic firms we can show that when the network effect is small, all three firms coexist. As the network effect increases, we observe cycles where the third (lowest quality) firm stops producing, and later on restarts production, and the cycle is repeated. For stronger network effects, eventually, firm 3 goes out of business and never restarts. For very strong network effects, only one firm is active in the long run. On the other extreme, we also show that for small network effects there can be a number active firms at equilibrium.

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Appendix

Proof of Proposition 1

Suppose that firm 2 follows a linear strategy $p_{2,t} = r_2 + q_2 \Delta_{t-1}$. Then the best-response problem of firm 1 is:

$$\max_{p_1} \ \pi_{1,t}(p_1, r_2 + q_2\Delta_{t-1}) + \beta V_1(\Delta_t), \tag{A1}$$

where $V_1(\Delta_t)$ is the continuation profit for firm 1. Since Δ_t is linear and monotonic in $p_{1,t}$, instead of writing the best response of firm 1 in terms of its selection of its current price, we can equivalently write it in terms of choosing the current difference in market shares Δ_t.[17] Using the demand functions and the price strategy of firm 2, the price and current sales of firm 1 can be written as:

$$p_{1,t} = v(1 - \Delta_t)/2 + (A + q_2)\Delta_{t-1} + r_2, \quad x_{1,t} = (\Delta_t + 1)/2. \tag{A2}$$

Substituting in (A1) we get that the total profits from optimal strategy of firm 1 are:

$$V_1(\Delta_{t-1}) = \max_{\Delta_t \in [-1,1]} \frac{(\Delta_t + 1)}{2}\left(\frac{v(1-\Delta_t)}{2} + (A + q_2)\Delta_{t-1} + r_2\right) + \beta V_1(\Delta_t). \tag{A3}$$

This maximization problem maps bounded functions into bounded functions on the interval [-1, 1], it satisfies monotonicity and discounting. Thus, it satisfies the sufficient conditions for Blackwell's Contraction Theorem and therefore it has a unique solution for the value function $V_1(\Delta)$.[18]

To find the equilibrium, we first assume that the optimal Δ_t is interior for all Δ_{t-1}, and we will verify that this is true after we determine the candidate equilibrium. Given that the current profits are quadratic in Δ_t, we guess that the value function is quadratic as well:

$$V_1(\Delta_t) = a_1 + b_1\Delta_t + c_1\Delta_t^2. \tag{A4}$$

If the Δ_t resulting from the optimal choice of p_t is interior, it has to satisfy the FOC.[19] Solving it we obtain:

$$\Delta_t^* = (2\beta b_1 + (A + q_2)\Delta_{t-1} + r_2)/(v - 4\beta c_1). \tag{A5}$$

We note that this best response is linear in Δ_{t-1}. Substituting it into (A3) we see that, if the solution is interior, the operator maps quadratic functions into quadratic functions. Therefore the value function is indeed quadratic. Finally, given that the $p_{1,t}$

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[17] Substitution of (3) and (4) in (2) yields $\Delta_t = 1 - 2(p_{1,t} - p_{2,t} - A\Delta_{t-1})/v$. Solving this for $p_{1,t}$ and substituting $p_{2,t} = r_2 + q_2 \Delta_{t-1}$ yields (A2).

[18] See Stokey and Lucas (1989), Theorem 3.3.

[19] The sufficient second order condition, $2\beta c_1 < v/2$, is also satisfied.

is linear in Δ_t, we have that the best response of firm 1 to a linear pricing strategy is a linear pricing strategy.

The next step is to find the coefficients c_1 and b_1 to describe the strategy.[20] We obtain:

$$c_1 = (v - z')/8\beta, \quad b_1 = \frac{1}{2}\left(A + q_2\right)\left(\frac{v + z' + 2r_2}{v + z' - 2\beta\left(A + q_2\right)}\right), \tag{A6}$$

where

$$z' = \sqrt{v^2 - 4\beta\left(A + q_2\right)^2}\ . \tag{A7}$$

The best response of firm 1 is $p_{1,t} = r_1 + q_1\Delta_{t-1}$ where

$$\begin{aligned} q_1 &= z'\frac{q_2 + A}{v + z'}, \\ r_1 &= \frac{1}{2}\left(\frac{v\left(v + z' - 4\beta b_1\right) + 2r_2 z'}{v + z'}\right). \end{aligned} \tag{A8}$$

Now we turn to the best response problem of firm 2. Assuming that firm 1 follows a linear pricing strategy, we can show in the same way as above, that if the solution of choosing optimal price yields an interior Δ_t, then the value function of firm 2 is also quadratic in Δ_t and the best response pricing strategy is linear in Δ_t. In particular we obtain:

$$V_2(\Delta_t) = a_2 + b_2\Delta_t + c_2\Delta_t^2, \tag{A9}$$

where

$$c_2 = (v - z)/8\beta, \quad b_2 = \frac{1}{2}\left(A - q_1\right)\left(\frac{v - z - 2r_1}{v + z - 2\beta\left(A - q_1\right)}\right), \quad z = \sqrt{v^2 - 4\beta\left(A - q_1\right)^2}\ . \tag{A10}$$

The best response of firm 2 is $p_{2,t} = r_2 + q_2\,\Delta_{t-1}$, where

$$\begin{aligned} q_2 &= z\frac{A - q_1}{v + z}, \\ r_2 &= \frac{1}{2}\left(\frac{v\left(v - z + 4\beta b_2\right) + 2r_1 z}{v + z}\right). \end{aligned} \tag{A11}$$

Now we can find the equilibrium. Combining (A8) and (A4) we get that $q_1 = -q_2$. That simplifies matters considerably. In particular $z = z'$ and the equilibrium q_1 is a solution to:

$$A/q_1 - (2 + v/z) = 0. \tag{A12}$$

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[20] This is done using standard dynamic programming method: by substituting (A4) and (A5) into (A2) and matching the coefficients on Δ. There are two solutions for c_1 and we pick the one consistent with the myopic model.

That implies that q_1 is positive and at most A/3. We can check that the LHS of that expression changes sign when we evaluate it at $q_1 = 0$ and $q_1 = A/3$. Finally the above equation can be rewritten as:

$$-1 + z\,(A - 2q_1)/(vq_1) = 0 \tag{A13}$$

and it is easy to verify that the LHS is decreasing in q_1, so there is a unique solution in q_1 in the range (0, A/3).

Exact expressions for r_1 and r_2 can be also obtained, but more importantly:

$$r_1 - r_2 = R = v(z - 2\,\beta(A - q_1))/(v + 2z). \tag{A14}$$

We have to still verify that the constructed strategies lead to interior equilibrium demands for all Δ_{t-1}. We can write the evolution of the difference in market shares as:

$$\Delta_t = 1 - 2(p_1 - p_2 - A\Delta_{t-1})/v = \alpha\Delta_{t-1} + \Delta^+ \tag{A15}$$

where

$$\alpha = 2(A - 2q_1)/v > 0,\ \ \Delta^+ = 1 - 2R/\mathrm{v} > 0. \tag{A16}$$

If there is convergence in the long run, the long run market share difference will be $\Delta^* = \Delta^+/(1 - \alpha)$. It is required that

$$\Delta^* = \Delta^+/(1 - \alpha) \leq 1. \tag{A17}$$

This simplifies to $R \geq A - 2q_1$, and for this to hold it is sufficient to have $v \geq 2A(1 + \beta)$.[21] Therefore, under this condition we have that indeed the best responses are interior, so the best responses and value functions assumed are correct and an equilibrium exists. QED.

[21] Using $q_1 = Az/(2z + v)$ and $R = v(z - 2\beta(A - q_1))/(v + 2z)$, $R \geq A - 2q_1$ simplifies to $Av/(2z + v) \leq R$ *i.e.*, $z \geq A + 2\beta(A - q_1)$ or equivalently $v^2 \geq (4\beta + 4\beta^2)q_1^2 - (12\beta A + 8\beta^2 A)q_1 + 8\beta A^2 + A^2 + 4\beta^2 A^2$. The RHS is decreasing in q_1, so it is sufficient to evaluate it at $q_1 = 0$: $v^2 \geq A^2(1 + 8\beta + 4\beta^2)$ so for $\Delta^* = \Delta^+/(1 - \alpha) \leq 1$ it is sufficient to have $v \geq 2A(1 + \beta)$.