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The Demand for Life Insurance: An Application of the Economics of Uncertainty: A Comment

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IN HIS THEORETICAL STUDY on the demand for life insurance R.A. Campbell [1], has derived simple demand-for-insurance equations in an attempt to relate households' optimal responses to human capital uncertainty. One major conclusion of his paper is that for households characterized by risk aversion, the optimal amount of human capital insurance is a decreasing function of the "load factor," λ, which is defined as a percentage markup from the actuarially fair value of insurance.

Campbell's conclusions follow directly from a Taylor Series expansion of his objective function. In this comment, the demand-for-insurance equations are derived once again, but this time the derivation will be direct, without use of the Taylor approximation. It is demonstrated that Campbell's approximate functions do not converge in the limit to the exact solutions for logarithmic utility functions. For this class of utility functions, Campbell's approximate solution dictates nonoptimal holdings of life insurance for virtually all policies whose load factors are greater than zero. The reason for this is that Campbell's approximations contain utility derivatives only up to the second order. As will be shown, for the logarithmic utility function, third and higher derivatives are significant.

The organization of this note is as follows. First, the optimal amount of insurance coverage, I^*, will be derived and compared with Campbell's alternative, *INS*. Second, it is demonstrated that *INS* does not approximate I^* as the time horizon, Δt, becomes infinitely small; i.e., as $\Delta t \rightarrow 0$, $\lim I^* \neq INS$. Third,[1] it is shown that at one point, where the bequest and utility functions are identical and where insurance is sold with no load ($\lambda = 0$), *INS* and I^* coincide. Fourth, the optimal (I^*) and approximate (*INS*) life insurance coverage (and their difference) are calculated for some realistic parameter values. Finally, a general result on approximations is given.

At the outset, it is useful to reintroduce the problem, notation, and assumptions of Campbell. A wage earner of age x and retirement age R has present value of wage income from x to R ${}_RH_x \equiv H$ and current marketable assets of $W_x \equiv W$. At instant x he faces a lottery. With probability $1 - q\Delta t$ he survives at time $x + \Delta t$ and has wealth $W + H$ and utility function $V(.)$ and with probability $q\Delta t$ he dies, has wealth W, and bequest utility function $B(.)$. The assumption is made that both the utility and the bequest functions are increasing and concave, i.e. $V' > 0$,

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[1] This is also stated by Campbell, footnote 14.

$V'' < 0$, $B' > 0$, $B'' < 0$, and that these functions are proportional to each other: $k = \frac{B(.)}{V(.)}$ where k is the household's intensity for bequests. The wage earner is offered life insurance for the period Δt, i.e. a contract to receive I if he dies in $(x, x + \Delta t)$ and to pay PREM before the outcome of the lottery is disclosed. Given a "loading factor" λ of the insurance firm the premium is PREM $= I(1 + \lambda)q\Delta t$.[2]

The general problem of the consumer is to choose the optimal insurance coverage $I^* \geq 0$ that maximizes expected utility (Campbell's Equation 5):

$$\operatorname*{Max}_I E[U] = \operatorname*{Max}_I (1 - q\Delta t)V(W + H - Iq\Delta t(1 + \lambda)) + q\Delta t B(W + I(1 - q\Delta t(1 + \lambda)))$$

Rather than expand this state-dependent utility function by a Taylor Series, the more direct approach is taken here of differentiating the objective function with respect to I.

$$\frac{dE[U]}{dI} = -(1 - q\Delta t)(1 + \lambda)q\Delta t V'(W + H - Iq(1 + \lambda)\Delta t) + q\Delta t(1 - q\Delta t(1 + \lambda)B'(W + I(1 - q(1 + \lambda)\Delta t))$$

At the optimal level of insurance, I^*,

$$\frac{dE[U(I^*)]}{dI} = 0$$

implying for $\Delta t \neq 0$ that

$$(1 - q\Delta t)(1 + \lambda)V'(W + H - I^*(1 + \lambda)q\Delta t) = (1 - q\Delta t(1 + \lambda))B'(W + I^*(1 - q\Delta t(1 + \lambda))) \qquad (1)^3$$

The second order condition,

$$\frac{d^2E[U(I^*)]}{dI^2} = (1 - q\Delta t)(1 + \lambda)^2 q^2(\Delta t)^2 V''(W + H - Iq(1 + \lambda)\Delta t) + q\Delta t B''(W + I(1 - q(1 + \lambda)\Delta t))(1 - q(1 + \lambda)\Delta t)^2 < 0 \qquad (2)$$

is met if $V''(.) < 0$ and $B''(.) < 0$.

(Note that the above condition (2) is sufficient but not necessary.) By Campbell's assumption, these conditions are met. Therefore, the expected utility maximizer solves (1) for I^*. Assuming, as Campbell does, that the bequest function is a positive linear transformation of the utility function,

$$B'(.) = kV'(.) \qquad k > 0 \qquad (3)$$

then (1) may be rewritten as

$$(1 - q\Delta t)(1 + \lambda)V'(W + H - I^*(1 + \lambda)q\Delta t) = (1 - q\Delta t(1 + \lambda))kV'(W + I^*(1 - q\Delta t(1 + \lambda))) \qquad (4)$$

[2] The notation is slightly different. Here $q \equiv {}_R q_x$.

[3] Note that (1) holds only if $\Delta t \neq 0$, but this creates no restrictions because, although it may be small, Δt is always positive.

At this point it is helpful to consider specific forms of the utility function before solving for the optimal amount of insurance coverage and comparing this amount with that derived by Campbell. The class of utility functions considered explicitly by Campbell is that exhibiting constant relative risk aversion, amongst which the logarithmic function is found. This particular form will serve as the counter-example offered here, although it is not difficult to prove that there is divergence between I^* and Campbell's approximate coverage INS for other members of this class. Letting $V(.) = Ln(.)$ and noting that $V'(.) = 1/(.)$, the following may be derived:

$$\frac{(1 - q\Delta t)(1 + \lambda)}{W + H - I^*(1 + \lambda)\Delta tq} = \frac{k(1 - q\Delta t(1 + \lambda))}{W + I^*(1 - q\Delta t(1 + \lambda))}$$

This can be solved for I^*:

$$I^* = \frac{k(1 - q\Delta t(1 + \lambda))(W + H) - (1 - q\Delta t)(1 + \lambda)W}{(1 + \lambda)[(1 - q\Delta t)(1 - q\Delta t(1 + \lambda)) + k(1 - q\Delta t(1 + \lambda))q\Delta t]} \tag{5}$$

Under the logarithmic case, $c = 1$ in Campbell's Equation 10 and the approximately optimal (where terms involving higher than second order derivatives of the utility function have been omitted) amount of insurance is given as:

$$INS = \frac{H}{k}(2k - 1 - \lambda) - \frac{W}{k}(1 + \lambda - k)$$

Therefore, in general, $I^* \neq INS$. Also, I^* is not linear as INS is. Moreover, as the time interval approaches zero,

$$\lim_{\Delta t \to 0} I^*(q, \lambda, k) = \frac{k(W + H) - (1 + \lambda)W}{(1 + \lambda)}$$
$$\neq \frac{H}{k}(2k - 1 - \lambda) - \frac{W}{k}(1 + \lambda - k) = INS \tag{6}$$

Therefore, the INS function does not approximate the optimal coverage function $I^*(q, \lambda, k)$ for small Δt.

At one point, the two insurance demand functions will be equal. Consider the case where the bequest function is identical to the utility function ($k = 1$):

$$INS(k = 1) = H(1 - \lambda) - W\lambda = H - \lambda(W + H) \tag{7}$$

$$I^*(k = 1) = \frac{H(1 - q\Delta t(1 + \lambda)) - \lambda W}{(1 + \lambda)(1 - q\Delta t(1 + \lambda))} \tag{8}$$

In the special case where insurance is sold at its actuarially fair value (profitless sale where $\lambda = 0$), the two functions are equal and optimal insurance coverage is H.

$$INS(k = 1, \lambda = 0) = H, \qquad I^*(k = 1, \lambda = 0) = \frac{H(1 - q\Delta t)}{1 - q\Delta t} = H \qquad (9)^4$$

[4] But if $k \neq 1$ these functions do not coincide at $\lambda = 0$: $INS(\lambda = 0) - I^*(\lambda = 0) = \frac{H(2k - 1) - W(1 - k)}{k} - \frac{kH - W(1 - k)}{1 + (k - 1)q\Delta t}$. Further, $\lim_{\Delta t \to 0} (INS(\lambda = 0) - I^*(\lambda = 0)) = -\frac{(H + W)(k - 1)^2}{k}$ which is zero only at $k = 1$.

Table I

Table of Divergence between I* and INS

HUMAN WEALTH = 200,000		NON-HUMAN WEALTH = 20,000			
$\lambda = 0.1$ Ages:		20	35	50	60
I*	:	179996.3	179996.1	179983.9	179958.5
INS	:	178000	178000	178000	178000
I* - INS	:	1996.346	1996.076	1983.879	1958.473
(I*-INS)/I*	:	0.01109103	0.01108956	0.01102254	0.0188291
$\lambda = 0.2$ Ages:		20	35	50	60
I*	:	163326	163325.5	163301.1	163250.1
INS	:	156000	156000	156000	156000
I* - INS	:	7326.021	7325.482	7301.063	7250.105
(I*-INS)/I*	:	0.0448552	0.04485205	0.04470921	0.04441103
$\lambda = 0.3$ Ages:		20	35	50	60
I*	:	149219.8	149219	149182.3	149105.7
INS	:	134000	134000	134000	134000
I* - INS	:	15219.8	15218.99	15182.33	15105.67
(I*-INS)/I*	:	0.1019958	0.101991	0.1017703	0.1013085
$\lambda = 0.4$ Ages:		20	35	50	60
I*	:	137128.2	137127.2	137078.2	136975.5
INS	:	112000	112000	112000	112000
I* - INS	:	25128.23	25127.15	25078.21	24975.71
(I*-INS)/I*	:	0.1832462	0.1832398	0.1829482	0.1823368
$\lambda = 0.5$ Ages:		20	35	50	60
I*	:	126648.4	126647	126585.8	126457.3
INS	:	90000	90000	90000	90000
I* - INS	:	36648.38	36647.03	36585.8	36457.29
(I*-INS)/I*	:	0.2893711	0.2893635	0.2890198	0.2882973

The differences between functions $I^*(\lambda, k = 1)$ and $INS(\lambda, k = 1)$ are now investigated. Let $f(\lambda)$ denote the difference between I^* and INS:

$$f(\lambda) = I^*(\lambda, k = 1) - INS(\lambda, k = 1)$$
$$= \frac{\lambda}{(1+\lambda)(1 - q\Delta t(1+\lambda))}[\lambda H(1 - q\Delta t(1+\lambda)) + W(\lambda - q\Delta t(1+\lambda))^2] \quad (10)$$

Clearly $f(0) = 0$, as seen above. We are interested in the case when the interval Δt is small. As Δt tends to zero,

$$\lim_{\Delta t \to 0} f(\lambda) = \frac{\lambda^2(H + W)}{1 + \lambda} > 0$$

For small $q\Delta t$, INS underestimates I^*, the optimal coverage. It is easy to see that f is always increasing in H, and increasing in W for any λ that is not extremely small.[5]

Table I shows the amounts of optimal coverage I^* and approximate coverage INS, as well as their difference for some values of H, W and λ, for the one-year

[5] A sufficient condition for $\frac{\partial f}{\partial W} > 0$ is $q\Delta t < \frac{\lambda}{(1+\lambda)^2}$

probabilities of death of white males of ages 20–60.[6] As seen in Table I for $H =$ 200,000 and $W = 20{,}000$, for small λ of the order of 10% the difference between I^* and *INS* is of the order of 1%. However the divergence between I^* and *INS* rises rapidly with the loading factor λ to reach the level of 29% at $\lambda = .5$. This table was calculated for the same levels of H and W at all ages. It is expected that W is significantly higher at higher ages, and this will increase the difference between I^* and *INS*.

Finally, I provide a justification for the divergence between I^* and *INS*, its approximation for small Δt proposed by Campbell. $V'(.)$ and $B'(.)$ can be expanded around $W + H$ up to first order with remainder and then substituted in Equation (1) that determines the optimal coverage I^*. Then limits can be taken as $\Delta t \to 0$. The resulting expression is:

$$V'(W+H)(1+\lambda-k) - V''(W+H)k(I-H) = \frac{k}{2}V''(W+H+\Theta(I-H))(I-H)^2, \qquad 0 \le \Theta \le 1 \quad (11)^7$$

This expression involves the third derivative of the bequest function. In R.A. Campbell's paper the right-hand side of the above equation is zero (Equation (7)). It is clear then that for any bequest function (like the logarithm function) that has a nonzero third derivative the approximation *INS* will diverge for the optimal coverage I^*.

Conclusion

The classical term insurance problem was examined and its optimal solution was derived and contrasted with the approximate solution of R.A. Campbell. The case of the logarithmic utility function was examined in detail. It was found that the optimal and the approximate life insurance coverage coincide if the "load charge" is zero, i.e. if the insurance firm charges the actuarially fair premiums. If there is some positive "load charge" for reasonably low probabilities of death, Campbell's approximation underestimates the optimal coverage.

Finally, a general approximation formula was given for the optimal life insurance coverage. In this approximation, the third derivative of the bequest function was essential.

These results point to the fact that the optimal life insurance cannot be adequately described through approximations which use up to the second derivative of the utility and bequest functions, even when these functions exhibit constant relative risk aversion.

REFERENCES

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3. U.S. Department of Health, Education, and Welfare. *Vital Statistics of the United States* 1975, Vol. II.

[6] Source: *Vital Statistics of the United States*, 1975, Vol. II, U.S. Department of Health, Education, and Welfare. The one-year ($\Delta t = 1$) probabilities of death for white males of ages 20, 35, 50, and 60 are: $q = .001824, .001958, .007990$, and $.02030$ respectively.

[7] A full proof is available from the author upon request.