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HOTELLING'S "MAIN STREET" WITH MORE THAN TWO COMPETITORS*

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ABSTRACT. I analyze oligopolistic competition among three or more firms located on Hotelling's (1929) Main Street and show that in contrast with Hotelling's duopoly, the symmetric locational structure supports a noncooperative equilibrium in prices. However, in a two-stage game of location choice in the first stage, and price choice in the second stage, there exists no subgame-perfect equilibrium where the whole market is served. This is because, starting from any locational pattern, firms have incentives to move toward the central firm. This strong version of the *Principle of Minimum Differentiation* destroys the possibility of a locational equilibrium. The results are a direct consequence of the existence of boundaries in the space of location. The sharp difference between these results and those of the standard circular model (whose product space lacks boundaries) shows that the general use of the circular model as an approximation to the line interval model may be unwarranted.

1. INTRODUCTION

Hotelling's (1929) duopoly model of locationally differentiated products has been recently reexamined by D'Aspremont, Gabszewicz and Thisse (1979) and Economides (1984), among others. Similar models with a larger number of firms have been analyzed by Lancaster (1979), Salop (1979), Novshek (1980), and Economides (1983, 1989), among others. Problems of nonexistence of a noncooperative equilibrium arise in the context of a game of price competition with fixed differentiated products, as firms find it more profitable to undercut their opponents. In a game where firms choose product varieties, expecting to receive the equilibrium profits of the short-run price subgame played for the chosen locations, Hotelling claimed that firms will try to produce extremely similar products. This acclaimed "Principle of Minimum Differentiation" was shown to be incorrect by D'Aspremont et al. (1979) when consumers have a high reservation price and will buy a differentiated product at any cost. The opposite result, local monopolization

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of markets has been established when the reservation price is low (Economides, 1984).[1]

In duopoly, the problem of nonexistence of a noncooperative equilibrium in the short-run price game (where variety specifications are fixed) arises from the fact that some consumers (located in terms of their most preferred variety close to the edge of the market) are captured by their closest firm for a large range of prices. Then, at some low price, these consumers are lured by the distant firm. This creates incentives to undercut the opponent firm, and shatters the possibility of a price equilibrium.

For the oligopoly model of n firms located on a circumference, it has been shown (Salop, 1979) that a symmetric equilibrium (where successive firms are equidistant and firms charge the same price) exists in the short-run price game. Further, the symmetric configuration is a noncooperative equilibrium in the long-run varieties game (Economides, 1989).

The circumference model is a good paradigm for some characteristics, such as color. In the locational interpretation, the circumference model describes well the choices of consumers distributed along the coastline of a lake. For most goods, however, it is appropriate to use a line interval $[a, b]$ as the space of potential products. For a serious study of the spatial economy, the simplicity and symmetry of the circular model cannot sufficiently compensate for its lack of the appropriate structure. Thus, despite the mathematical difficulties. the analysis turns to the study of oligopolistic competition among $n \geq 3$ firms. each producing a product located *on the line interval* $[0, 1]$. This paper bridges the gap between the duopoly model on $[0, 1]$ of Hotelling (1929) and the oligopoly circumference model of Salop (1979).

The existence of endpoints in the space of characteristics produces results which are qualitatively different from the ones of the circumference model. We show that problems of nonexistence of equilibrium in the short-run price game are diminished. *Severe nonexistence problems arise, however, at the stage of variety choice* if firms anticipate the impact of their relocation on equilibrium prices in the following stage. As long as the whole market is served, irrespective of the locational pattern, all firms except the center firm have incentives to move towards the center firm. Therefore there exist no subgame-perfect equilibria in the two-stage game where locations are chosen in the first stage and prices in the second.[2]

Difficulties with the existence of equilibrium arise in this model at the variety choice stage and *not in the stage of price choice* as in the duopoly of Hotelling (1929)

[1]See Lerner and Singer (1937) and Eaton and Lipsey (1975) for an analysis of the Hotelling model with fixed prices.

[2]In a circular model, Schulz and Stahl (1985) demonstrate nonexistence of equilibrium in a simultaneous price-location game, as well as in a sequential location-price game with different costs. In a model of intersecting roadways, Braid (1989) also shows lack of existence of a noncooperative spatial equilibrium in the location stage as firms tend to agglomerate. Klein (1991) shows that a symmetric equilibrium exists for a two-dimensional disk market provided that there is sufficient elasticity in the demand generated by consumers "residing" at each point on the disk. This becomes possible essentially because of the existence of an added dimension of locational competition.

and D'Aspremont et al. (1979). The incentives to undercut in the price stage are diminished as the number of active firms increases from two to three or more. We analyze and establish the noncooperative equilibrium price structure of the symmetric locational configuration. The price equilibrium cannot be totally symmetric, as in the circle models of Salop (1979) and Novshek (1980), because the existence of endpoints in the market creates potential market power for the two firms near them. When reservation prices are high so that all consumers buy a differentiated product ("competitive" case), equilibrium prices are high near the edges of the market and decrease at a decreasing rate as we look at firms located closer to the center of the market. Thus, the equilibrium price structure is U-shaped as a function of distance from the left endpoint of the interval.

When the reservation prices are low, two more equilibrium structures can arise. The first is a "local monopolistic" price-location structure where firms seem not to be in direct competition, since at equilibrium there are some consumers between any two firms who prefer not to buy any differentiated product at the going prices. At the second type of equilibrium, all firms use prices which put them at the kink of the demand curve, so that the consumer who is indifferent between buying from firm j or $j + 1$ is also indifferent between buying and not buying any differentiated product.

We show that equal-profits equilibria do not exist when firms are in direct competition. It is possible to characterize and establish, however, the existence of free-entry equilibria.

The rest of the paper is organized as follows. The basic model is set up in Section 2, and in Section 3 we characterize the "competitive" equilibria of the price game. In Section 4 we analyze the choice of varieties and in Section 5 we show the existence of equispaced "competitive" equilibria in the price game and characterize them. In Section 6 we analyze "local monopolistic" and "kink" equilibria, and in Section 7 we discuss entry. In Section 8 we present concluding remarks.

2. THE MODEL

A consumer of type w is endowed with a utility function in money I (Hicksian composite good) and a unit of a differentiated product z of the following form:

$$U_w(I, z) = I + k - \lambda|w - z|$$

The utility function is separable, with a peak (in the space of characteristics) at w, and linear in the distance from w in the same space. Variable k represents the maximum amount of money that any consumer is willing to pay for a unit of the differentiated good, that is, k is his reservation price.

Given products $x_1, \ldots, x_n$ offered at prices $P_1, \ldots, P_n$, consumer w selects the one that maximizes $U_w(I - P_j, x_j)$. He has the option of buying no differentiated product and receiving utility $U_w^0(I) = I$. Consumers are distributed uniformly in [0, 1] according to the commodity type they like most (the peak of their utility functions in the space of commodities).

Production technology of every variety is through constant returns to scale with marginal cost m. Firm j sells product x_j at price $P_j = p_j + m$. Variable p_j is defined as the increment of price above the constant marginal cost m.

Competition is described with the help of two games. In the short-run game, all firms consider the products' specifications as given and use prices as strategic variables. Letting $\mathbf{x} = (x_1, \ldots, x_n)$, firm j's objective function is $\Pi_j \equiv \Pi_j(p_1, \ldots, p_n | \mathbf{x})$. Clearly, the game where firm j uses price P_j as its strategy is equivalent to a game where firm j uses the increment of price above marginal cost $p_j = P_j - m$ as its strategy. An n-tuple $(p_1^*, \ldots, p_n^*) \equiv \mathbf{p}^*(\mathbf{x})$ is a noncooperative equilibrium for the short-run game if no firm finds it profitable to depart unilaterally from its chosen strategy, that is, if

$$\Pi_j(p_1^*, \ldots, p_{j-1}^*, p_j, p_{j+1}^*, \ldots, p_n^* | \mathbf{x}) \leq \Pi_j(\mathbf{p}^* | \mathbf{x})$$

for all j and all p_j.

Under some circumstances, firms may change technology without cost in the long run. We then define a long-run game in which firms choose varieties. The long-run game in varieties is well defined for those n-tuples of varieties (locations) that result in a unique Nash equilibrium in the short-run game. In the long-run game, firms choose varieties (locations) expecting to receive the payoff that corresponds to the Nash equilibrium prices of the short-run game which is played for these locations. Let $\Pi_j^v(\mathbf{x}) \equiv \Pi_j(\mathbf{p}^*(\mathbf{x}), \mathbf{x})$ be the payoff function of player j in the long-run game, where $p_j^* \equiv p_j^*(\mathbf{x})$ is the Nash equilibrium increment of price above marginal cost in the short-run game played for locations $\mathbf{x} = (x_1, \ldots, x_n)$, all distinct. A subgame-perfect equilibrium of the two-stage game is a noncooperative equilibrium of the varieties stage where firms use $\Pi_j^v(\mathbf{x}), j = 1, \ldots, n$, as objective functions.

3. NONCOOPERATIVE EQUILIBRIA IN PRICES

In this section we derive the equilibrium price structure in the price game and distinguish those configurations of reservation price, spacing, and prices at which neighboring firms are in direct competition with each other. The "competitive" configuration, where firm j is in direct competition with its immediate neighbors, occurs under two conditions

$$p_j < \min\left[2k - 2m - \lambda(x_j - x_{j-1}) - p_{j-1}, 2k - 2m - \lambda(x_{j-1} - x_j) - p_{j+1}\right]$$

$$p_j > \max\left[p_{j-1} - \lambda(x_j - x_{j-1}), p_{j+1} - \lambda(x_{j+1} - x_j)\right]$$

The first condition guarantees that the worst-off consumers between firm j and its neighbors prefer to buy a differentiated product rather than not. The second condition guarantees that all firms have positive demand. Under these conditions the marginal consumers to its left and right of firm $j, j \neq 1, j \neq n$ are

$$\bar{z}_j = [x_j + x_{j-1} + (p_j - p_{j-1})/\lambda]/2 \qquad \bar{z}_{j+1} = [x_j + x_{j+1} + (p_{j+1} - p_j)/\lambda]/2$$

The profit function for firm j (for the "competitive" region) is

$$\Pi_j = p_j \int_{\bar{z}_j}^{\bar{z}_{j+1}} dz = p_j(\bar{z}_{j+1} - \bar{z}_j)$$

It is maximized with respect to p_j at[3]

$$p_j^m = [p_{j+1} + p_{j-1} + \lambda(x_{j+1} - x_{j-1})]/4$$

The first and last firms face competition from one side only. The profit function of the first firm is: $\Pi_1 = p_1\bar{z}_1 = p_1[x_1 + x_2 + (p_2 - p_1)/\lambda]/2$. If at the Nash equilibrium in prices, firm $j = 1$ faces competition from its closest (second) firm, i.e., under

$$p_2 - \lambda(x_2 - x_1) < p_1 < \min\,[k - m - \lambda x_1, 2k - 2m - p_2 - \lambda(x_2 - x_1)]$$

then its "candidate" Nash equilibrium increment of price above marginal cost is given by $p_1^m = [p_2 + \lambda(x_1 + x_2)]/2$. Similarly, $p_n^m = [p_{n-1} + \lambda(2 - (x_n + x_{n-1}))]/2$.

Let $\mathbf{p}^* = (p_1^*, \ldots, p_n^*)'$ be the vector of equilibrium increments of prices above marginal cost and let $\mathbf{A}$ be a matrix with all elements zero except $a_{i,i} = 1$ for all i; $a_{1,2} = -1/2$, $a_{i,i+1} = -1/4$, for all $i \neq 1$; and $a_{n,n-1} = -1/2$, $a_{i,i-1} = -1/4$ for all $i \neq 1, i \neq n$. Then the first-order conditions are summarized as

(1) $$\mathbf{A}\mathbf{p}^* = \mathbf{y}$$

where $y_1 = \lambda(x_1 + x_2)/2$, $y_n = \lambda[1 - (x_{n-1} + x_n)/2]$, and $y_j = \lambda(x_{j-1} - x_{j-1})/4$ for $j = 2, \ldots, n-1$.

The inverse of $\mathbf{A}$ exists and the first-order conditions can be solved for

(2) $$\mathbf{p}^* = \mathbf{A}^{-1}\mathbf{y}$$

The inverse of $\mathbf{A}$ is given by the following Lemma proved in Appendix A. Corollary 1, also proved in Appendix A, notes further properties of $\mathbf{A}^{-1}$.

LEMMA 1: *The inverse of A exists and is given by $A^{-1} \equiv B$ with representative element $b_{i,j}$, where* $b_{i,1} = 2[\rho_1^{n-i} + \rho_2^{n-i}]/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\}$, $b_{i,j} = 2[\rho_1^{n-j} + \rho_2^{n-j}][\rho_1^{i-1} + \rho_2^{i-1}]/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\}$ *for* $n - 1 \geq j \geq 2$ *and* $j \geq i$, $b_{i,j} = 2[\rho_1^{n-i} + \rho_2^{n-i}][\rho_1^{j-1} + \rho_2^{i-1}]/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\}$ *for* $n - 1 \geq j \geq 2$ *and* $i \geq j$, *and* $b_{i,n} = 2[\rho_1^{i-1} + \rho_2^{i-1}]/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\}$, *where* $\rho_1 = 2 + \sqrt{3}$ *and* $\rho_2 = 2 - \sqrt{3}$ *are the solutions of* $\rho^2 - 4\rho + 1 = 0$.

Note that matrix $\mathbf{A}$, as well as its inverse $\mathbf{A}^{-1}$, are independent of the locations of firms $\mathbf{x}$. This is due to the linearity of transportation costs. Matrix $\mathbf{A}$ is dependent on the locations of firms for nonlinear transportation costs as in Economides (1989).

COROLLARY 1: *All elements of A^{-1} are positive. The sum of the elements of each row of A^{-1} is equal to* 2.

PROPOSITION 1: *The first-order conditions for a short-run equilibrium have the solution $p^* = A^{-1} \cdot \mathbf{y}$, where A^{-1} is given by Lemma* 1. *At equilibrium, profits are*

[3]This formula holds only if all locations are distinct. If the locations of two firms coincide, then Bertrand competition drives the price differential over marginal cost to zero.

$\Pi_j(\mathbf{p}^*) = (p_j^*)^2/\lambda$ *for interior firms,* $j \neq 1, \neq n$ *and* $\Pi_j(\mathbf{p}^*) = (p_j^*)^2/(2\lambda)$ *for corner firms,* $j = 1$ *or* n.

One candidate equilibrium locational structure is the one where all firms have equal profits. This implies $p_j = p, j \neq 1$ or n and $p_1 = p_n = p\sqrt{2}$. Distances between consecutive firms starting from the second firm are all equal, $x_{j-1} - x_j = p/\lambda, j \in \{2, \ldots, n-2\}$, while distances closer to the edge are $x_1 = 1 - x_n = 3p(\sqrt{2} - 1)/(2\lambda)$, $x_2 - x_1 = x_n - x_{n-1} = p(2 - \sqrt{2})/\lambda$, and $p = \lambda/(\sqrt{2} + n - 2)$. These distances are smaller than the ones of interior firms, $x_2 - x_1 < x_1 < x_{j+1} - x_j$. However, this locational structure does not constitute a noncooperative equilibrium because the second [$(n-1)$st] firm has an incentive to undercut the first (nth) firm. The first firm has a high price and is relatively close to the second firm. It takes a relatively small decrease in the price of the second firm to undercut the price of the first firm, drive it out of business, and take over all customers in $[0, x_2]$. This is summarized in Proposition 2 and is formally proved in Appendix A.

PROPOSITION 2: *No (locational) market structure can support an equilibrium where all firms make equal profits.*

4. "COMPETITIVE" EQUILIBRIA WITH FLEXIBILITY IN RELOCATION

Before discussing in detail a locational structure which does constitute a short-run equilibrium (by fulfilling the appropriate *sufficient* conditions for existence of a price equilibrium), we show that in the long-run game with no relocation costs there is no equilibrium where the firms are in direct competition. Formally, we show that the two-stage game, of location (variety) choice in the first stage, and price choice in the second, does not have a perfect equilibrium.

From the first-order conditions of the price stage (which are necessary for equilibrium), we have deduced in Proposition 1 the equilibrium profits of the price stage parametrically for a varieties vector $\mathbf{x}$. Because firms anticipate the price equilibria when they choose varieties, the equilibrium profits of the price stage constitute the objective functions of the varieties' choice stage

$$\Pi_j^v(\mathbf{x}) \equiv \Pi_j(\mathbf{p}^*(\mathbf{x}), \mathbf{x}) = (p_j^*)^2/\lambda \qquad (j = 2, \ldots, n-1)$$

$$\Pi_j^v(\mathbf{x}) \equiv \Pi_j(\mathbf{p}^*(\mathbf{x}), \mathbf{x}) = (p_j^*)^2/(2\lambda) \qquad (j = 1 \text{ or } n)$$

Firm j has an incentive to relocate marginally from its position x_j towards the direction which makes $d\Pi_j^v/dx_j$ positive. For both internal and corner firms, $\text{sgn}(d\Pi_j^v/dx_j) = \text{sgn}(dp_j^*/dx_j)$. Differentiating $\mathbf{p}^* = \mathbf{A}^{-1}\mathbf{y}$ with respect to $x_j, j \neq 1, n$, we have $dp_j^*/dx_j = \mathbf{A}^{-1}\, d\mathbf{y}/dx_j$, where $dy_i/dx_j = 0$ for $i \neq j-1, j+1$ and $dy_{j-1}/dx_j = \lambda/4$, $dy_{j+1}/dx_j = -\lambda/4$. Thus, $dp_j^*/dx_j = \lambda(b_{j,j-1} - b_{j,j+1})/4$, which can be calculated directly from Lemma 1. For $j = 1$ we have $dy_1/dx_1 = dy_2/dx_2 = \lambda/2$, so that

$dp_1/dx_1 = \lambda(b_{j,j} + b_{j,j-1})/2$, and similarly for $j = n$. We show in Appendix A that

LEMMA 2: *$dp_j^*/dx_j > 0$ (<0) is equivalent to $(n + 1)/2 > j$ $(<j)$.*

Thus, *all firms, irrespective of the locational pattern, have incentives to relocate towards the middle firm in the market.* It immediately follows that there exists no perfect equilibrium in the game played for high reservation prices. There are two important features of this result. First, it is ordinal in nature. Side-firms try to move towards the *central firm* (or two central firms if n is even) and *not the central point* of the market. Only the *order* of the firms is important. The actual distances between firms play no role. Second, this result is immediately tied to the inevitable asymmetry of competition among more than two firms located in the interval [0, 1]. For a noncentral firm [$j \neq (n + 1)/2$ for n odd, or $j \neq n/2, j \neq 1 + n/2$ for n even], the world to its left is different than the world to its right. Thus the incentives of a movement dx_j on p_{j+1}^* and p_{j-1}^* are not of equal and opposite signs as in the circular model (Economides, 1983). From (1), $dp_j^*/dx_j = dp_{j-1}^*/dx_j + dp_{j+1}^*/dx_j$, but the right-hand side is not equal to zero for noncentral firms.[4] These results are summarized in Theorem 1.

THEOREM 1: *There is no equilibrium in a "competitive" configuration in the varieties (long-run) game which is subgame perfect in the price subgame because all firms want to relocate towards the central firm.*

A movement by firm j has a ripple effect through the industry. After firm j moves marginally to the right a new equilibrium is established where all firms to its right have lower prices (and profits) and the decrease is largest for firm $j + 1$. This reflects the increased intensity of competition after firm j came closer. At the new equilibrium, all firms to the left of j have higher prices and the increase is largest for firm $j - 1$. These effects are qualitatively independent of the location of the firm j in relation to the center of the market (and therefore qualitatively independent of the direction of change of its own price p_j^*). This is summarized in Proposition 3 which is proved in Appendix A.

[4]Even in the limit, as the number of firms goes to infinity, equilibrium price of off-center firm j changes significantly in its location

$$\lim_{n\to\infty} dp_j^*/dx_j = \{\lambda(\rho_1^2 - 1)/(2\sqrt{3})\}\{\lim_{n\to\infty} (\rho_1^{2n-4j+2} - 1)/[\rho_2^{(n-2j+2)}(\rho_1^{n-1} - \rho_2^{n-1})]\}$$

$$= \lambda(\rho_1^2 - 1)/(2\sqrt{3}) > 0 \text{ iff } n - 2j + 1 > 0, \qquad \text{i.e., } j < (n + 1)/2$$

This contrasts with the results of Economides (1983) and Braid (1989), where it is shown that marginal relocational effects on prices are zero in a symmetric locational pattern on an infinite line. However, the effect of relocation on equilibrium *profits* of an off-center firm is zero in the limit, because its equilibrium price is zero in the limit, $\lim_{n\to\infty} p_j^* = 0 \Rightarrow$

$$\lim_{n\to\infty} d\Pi_j(\mathbf{p}^*)/dx_j = (2/\lambda)(\lim_{n\to\infty} p_j^*)(\lim_{n\to\infty} dp_j^*/dx_j) = 0$$

Thus, the relocation tendencies in the limit conform to Economides (1983) and Braid (1989).

PROPOSITION 3: *A movement of firm j in the direction of firm i decreases the price and profits of firm i, and the absolute effect is larger the smaller the difference in the order of the firms,* $|i - j|$.

5. EQUISPACED "COMPETITIVE" EQUILIBRIUM

When the costs of changing product specification are high, the short-run equilibrium could be sustained in the long run. As pointed out in Section 3, not all locational structures allow for an equilibrium in the short-run game in prices, and in particular the equal-profits configuration does not constitute a noncooperative equilibrium. Here we propose and analyze the structure of n equispaced symmetrically located firms, $x_j - x_{j-1} = d$ and $x_1 = 1 - x_n = c$. We assume that the maximal number of firms are active in the market, so that $(n - 1)d < 1 < nd$.[5] Then $y_j = \lambda d/2$, for $j = 2, \ldots, n - 1$, and $y_1 = y_n = \lambda(c + d/2)$. This is called the *symmetric locational configuration.* In the expression for equilibrium increments of prices above marginal cost, $\mathbf{p}^* = \mathbf{A}^{-1}\mathbf{y}$, the sum of the first and the last elements of each row of $\mathbf{A}^{-1}$ plays a significant role since it adds to prices a proportion of c. Let e_j represent the sum of the first and the last elements of row j of $\mathbf{A}^{-1}$, $(e_1, \ldots, e_n)' \equiv \mathbf{A}^{-1}\ (1, 0, \ldots, 0, 1)'$. In our notation, $e_j = b_{j,1} + b_{j,n}$. Then, the candidate (first-order) Nash equilibrium price differentials over marginal cost fulfill

$$p_j^* = \lambda(d + ce_j) \tag{3a}$$

where

$$e_j = \frac{2[(2 - \sqrt{3})^{j-1}(1 + (2 + \sqrt{3})^{n-1}) + (2 + \sqrt{3})^{j-1}(1 + (2 - \sqrt{3})^{n-1})]}{\sqrt{3}[(2 + \sqrt{3})^{n-1} - (2 - \sqrt{3})^{n-1}]} \tag{3b}$$

PROPOSITION 4: *e_j (and therefore p_j^*) decreases in j for all $j \leq n/2$, increases in j for all $j \geq n/2$, and is strictly convex as a function of j.*[6] *Further, $p_j^*(n)$ is decreasing in n.*

Proof: See Appendix A.

By the above proposition the successive differences in prices increase as we go from the first to the last firm. This establishes a convex, symmetric, U-shaped price structure. The center firm charges the lowest price and the firms at the edge of the market charge the highest. See Figure 1.

It is easy to understand that the first and the last firms (which face competition from one side only) have monopoly power and that they exploit this power by charging high prices. The interesting element in this Nash equilibrium configuration is that some of the monopoly power of the first and last firms is transferred to their neighbors. The receivers transfer some of this monopoly power to their neighbors on the other side, and so on. All these transfers of monopoly power can be thought of as being done through prices. The corner firm quotes a

[5] That is, $(n - 1)d + 2c = 1$ and $c < d/2$.

[6] Except in the special case of $c = 0$ when prices of all firms are equal.

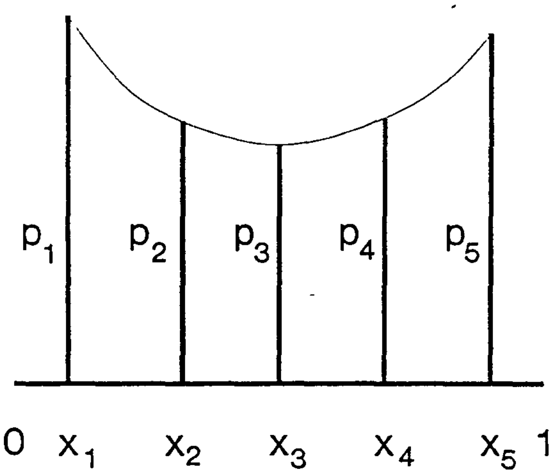

FIGURE 1: Equilibrium Prices.

high price and through this action it allows the second firm to charge a relatively high price and so on.[7]

From Proposition 1, profits of an interior firm in the variety stage are $\Pi_j^v = (p_j^*)^2/\lambda$ while the profits of an endpoint firm are $\Pi_j^v = (p_j^*)^2/(2\lambda)$. It is easily seen that $\Pi_1^v < \Pi_2^v$ so that the second firm has the highest profits in the industry.[8] The second firm is in a very privileged position as it is neighboring the firm with the highest price and at the same time it has significant potential demand. The first firm is in a privileged position too, since it faces competition from only one side. But its potential demand is lower than the potential demand of an interior firm since $c < d/2$. Its relative position in profitability depends crucially on its closeness to the edge, that is, on the length c. For $c = 0$, all firms have equal prices and the edge firms make the lowest profits. On the other extreme, for c nearly $d/2$ and $n \geq 5$ the edge firms are the second most profitable ones in the industry.

The prices derived from the first-order conditions [Equations (3a-b)] will constitute a Nash equilibrium if, for each firm j, p_j^* is the global maximizer of Π_j when opponents play $\mathbf{p}_{-j}^* \equiv (p_1^*, \ldots, p_{j-1}^*, p_{j+1}^*, \ldots, p_n^*)$. As long as both neighbors of firm j have positive demand and compete directly with firm j, Π_j is concave in p_j. However, for a low enough price, firm j can drive one or both of its neighbors out of business. Say firm $j + 1$ is pushed out of business by firm j. At the price where this happens there is an upward jump in the demand and profits of firm j as all the customers to the right of x_{j+1} previously served by firm $j + 1$ are won by firm j. For lower prices, firm j competes with firms $j - 1$ and $j + 2$ and its profit function is concave in p_j until it drives another neighbor out of business. Clearly, then, it has to be checked that the profits of firm j at the first-order prices are higher than the

[7]This result is interesting in view of ten Raa (1984, p. 113) who found (in a nonstrategic setting) that a unimodal density of housing was necessary to result in a spatial equilibrium.

[8]$\Pi_1^* < \Pi_2^* \Leftrightarrow p_1^* < p_2^*\sqrt{2} \Leftrightarrow ce_1 + d < \sqrt{2}(ce_2 + d) \Leftrightarrow (e_1 - e_2\sqrt{2})/(\sqrt{2} - 1) < d/c$, which is true because $\min(d/c) = 2$ while the left-hand side varies from 0.94 for $n = 3$ to 1.73 for large n.

profits at the prices which drive one or more of its neighbors out of business. We call such prices "undercutting prices."

Three factors are critical in establishing the profitability or lack of profitability of undercutting. First, the decrease in the price which is necessary to achieve undercutting. Calling p_j^u the highest price of firm j which undercuts firm $j-1$, this decrease has to be at least $p_j^* - p_j^u = p_j^* - p_{j-1}^* + \lambda d$, which depends on the difference of equilibrium prices of neighbors and on the distance between them. The second factor is the extent of equilibrium demand which will have to be sold at the lower undercutting prices. The third factor is the extent of gain in the demand achieved by undercutting. For p_j^* to be the global maximizer in a "competitive" configuration (where all consumers are served and they are strictly better off buying a differentiated product) it is also necessary that firm j does not find it profitable to increase its price so high that it becomes a local monopolist (and not all consumers are served in $[x_j, x_{j+1}]$). If the reservation price is sufficiently low, firms are forced to be local monopolists, while, if the reservation price is sufficiently high, a Nash equilibrium of a "competitive" configuration exists. It can be shown that,

THEOREM 2: *For sufficiently high reservation prices, i.e., either* $\{(k-m)/\lambda \geq 3d/2 + c(2+3e_2)/4$ *and* $0 \leq c \leq 2d/(6-e_2)\}$ *or* $\{(k-m)/\lambda \geq d + c(2+e_2/2)$ *and* $2d/(6-e_2) \leq c \leq d/2\}$, *a "competitive" price equilibrium exists in the symmetric locational configuration for* $n \geq 4$. *For* $n = 3$ *it is further required that* $c/d < 0.435$.

The proof of Theorem 2 involves a detailed examination of all the possibilities of undercutting and the determination that undercutting is unprofitable. It uses the intuition developed above on the merits of undercutting. The proof, based on four lemmata, is available from the author upon request. It is also contained in Economides (1992).

Theorem 2 extends the results of D'Aspremont, Gabszewicz and Thisse (1979) that corrected Hotelling (1929). They showed that in the original model of Hotelling a price equilibrium exists for locations $x_1 \in [0, 1/4]$ and $x_2 \in [3/4, 1]$ (in a symmetric setting). This is similar to the condition used in Theorem 2 that the length of the interval to the left of the first and to the right of the last firm is smaller than half the distance between consecutive firms, $c < d/2$.

Note that in this paper it is not the finite reservation price—Hotelling's reservation price was infinite—which helps establish the existence of equilibrium as in the duopoly model of Economides (1984). Theorem 2 says that a "competitive" Nash equilibrium exists for large and even infinite reservation prices k.

6. "LOCAL MONOPOLISTIC" AND "KINK" EQUILIBRIA

When the reservation price k is low, the possibility arises for the existence of equilibria where all firms are local monopolists, so that between any two firms there are some consumers who prefer not to buy any differentiated product at the going prices. In the local monopolistic price region, demand is $D_j = 2(k - m - p_j)/\lambda$, and the internal maximizing increment of price above marginal cost is $p_j^* = (k-m)/2$. A local monopolistic equilibrium will exist if the distances between the firms are large relative to the reservation price, that is, if for all j

(4) $$x_j - x_{j-1} \geq (k - m)/\lambda$$

For any reservation price k it is the equispaced configuration which accommodates the largest number of local monopolistic firms, $n = \lambda/(k - m)$. If there are $n < \lambda/(k - m)$ firms in the market then there is an infinity of locational equilibria, all fulfilling Equation (4).

In the long run, firms in local monopolistic configurations have no incentives to relocate. Marginal relocations which do not violate (4) leave profits unchanged. If a relocation of firm j violates (4), then the demand faced by firm j will be lower after the relocation, and will result in a lower equilibrium price and profits. Thus, it has just been shown that

THEOREM 3: *There exist subgame perfect n-firm locational equilibria if* $n \leq \lambda/(k - m)$, *that is, for low reservation prices. Such equilibria fulfill* $x_j - x_{j-1} > (k - m)/\lambda$, *so that at equilibrium there are some consumers between any two firms who do not buy any differentiated product.*

At the price where the gap of nonserved consumers located between the firms closes, the demand function exhibits a kink which results in a decrease of the derivative of the profit function at that point. This creates the possibility that the global maximizer of Π_j is at the kink. We call this location-price configuration, where consumers who are indifferent between buying from the two firms are also indifferent between buying and not buying a differentiated product, a *kink configuration.*[9] Calling z the marginal consumer between x_j and x_{j+1}, at the kink configuration, $p_j + \lambda(z - x_j) = (k - m) = p_{j+1} + \lambda(x_{j+1} - z)$, so that

(5) $$p_j + p_{j+1} + \lambda(x_{j+1} - x_j) = 2(k - m)$$

For this configuration to constitute a noncooperative equilibrium in prices it is required that for all j

(6a) $$\lim_{\epsilon \to 0} \partial\Pi_j(2k - 2m - p_{j+1} - \lambda(x_{j+1} - x_j) - \epsilon)/\partial p_j > 0$$

(6b) $$\lim_{\epsilon \to 0} \partial\Pi_j(2k - 2m - p_{j+1} - \lambda(x_{j+1} - x_j) + \epsilon)/\partial p_j < 0$$

Conditions (6a–b) are equivalent[10] to $\lambda d \leq (k - m) \leq 3\lambda d/2$. For the symmetric locational configuration introduced in Section 5, Condition (5) implies immediately that $p_{j-1} = p_{j+1}$. There are $n - 2$ conditions on n prices. Clearly, there is no unique kink equilibrium. The symmetric configuration suggests that the price differential over marginal cost at the kink configuration is equal for all firms, $p = k - m - \lambda d/2$. Undercutting is unprofitable as it involves the use of a negative price differential over marginal cost, $p^u \leq p - \lambda d = k - m - 3\lambda d/2 < 0$. Thus, kink

[9]Beckmann (1972) and Salop (1979) observed the importance of kink equilibria.

[10]Evaluating $\partial\Pi_j/\partial p_j$ at $p_j = k - m - \lambda d/2 + \epsilon$ we have that $\lim_{\epsilon \to 0} \partial\Pi_j(k - m - \lambda d/2 + \epsilon)/\partial p_j = 2(\lambda d + m - k)/\lambda < 0$ implies $\lambda d < k - m$. Similarly, $\lim_{\epsilon \to 0} \partial\Pi_j(k - m - \lambda d/2 - \epsilon)/\partial p_j > 0$ implies $3\lambda d < 2(k - m)$.

equilibria exist in the short run for an intermediate range of prices.[11] At the symmetric equilibrium selection, all firms have equal prices and the first firm has the lowest profits since it has the lowest potential demand.

THEOREM 4: *Price equilibria at the kink of the demand function exist for the symmetric locational configuration for intermediate reservation prices* k, $\lambda d \leq k - m \leq 3\lambda d/2$.

For the symmetric locational configuration introduced in Section 5, we note that equilibrium prices do not depend on c for the local monopolistic configuration and the symmetric selection of the kink configuration. In the "competitive" case equilibrium prices do not depend on c only when $c = 0$.

For the symmetric locational configuration introduced in Section 5, an asymmetric kink equilibrium price structure has alternating prices $(p^1, p^2, p^1, p^2, \ldots)$. It can be shown that $4k/3 - \lambda d < p^1, p^2 < 3k/2 - \lambda d$, with $p^1 + p^2 = 2k - \lambda d$. Equilibrium existence requires, again, $\lambda d \leq k - m \leq 3\lambda d/2$. Note that these equilibria are independent of c.

COROLLARY 2: *A continuum of asymmetric price equilibria at the kink of the demand function with alternating prices* $(p^1, p^2, p^1, p^2, \ldots)$ *exists for the symmetric locational configuration for intermediate reservation prices* k, $\lambda d \leq k - m \leq 3\lambda d/2$ *with* $p^1 + p^2 = 2k - \lambda d$.

In general, for every varieties vector $\mathbf{x} = (x_1, \ldots, x_n)$, there is a continuum of kink configurations where Equations (5) and (6a–b) hold for all j. Further, these equilibria result in differing prices and profits. A marginal relocation of a firm from a kink configuration will again result in a kink configuration. However, there is no accepted way to select among these noncooperative equilibria. Since it is unclear which short-run equilibrium will result, we cannot evaluate the long-run profit function. Unfortunately, not much else can be said about the long-run pattern in this intermediate case.

7. ENTRY

In the very long run we allow entry and exit. Potential entrants have the same production technology as the already active firms and entry entails an extra set-up cost F. We have shown (in Proposition 2) that no noncooperative equilibrium exists where all firms make equal profits. Thus, we have to specifiy the relative profits position of the entrant. Let us assume that the entrant has pessimistic expectations. When he enters he expects to receive the lowest profits among the active firms in the market, $\Pi^E(n-1) = \Pi^m(n) \equiv \min_i \Pi_i^*(n) - F$, where $\Pi^E(n-1)$ are the expected profits of the entrant given $n - 1$ firms already active in the market.

[11]It is also checked that $\lim_{\epsilon \to 0} \partial\Pi_1(k - m - \lambda d/2 - \epsilon)/\partial p_1 = (m - k + \lambda(2c + 3d/2))/(2\lambda) > 0$, and $\lim_{\epsilon \to 0} \partial\Pi_1(k - m - \lambda d/2 + \epsilon)/\partial p_1 = (\lambda c - k + m + \lambda d)/\lambda < 0$.

The free-entry[12] noncooperative equilibrium number of firms n^* with pessimistic entry expectations (of potential entrants) has to fulfill $\Pi^m(n^*) \geq 0$ and $\Pi^m(n^* + 1) < 0$. Any expectations of the entrant other than pessimistic can only lead to perpetual entry disequilibrium. If the nth entrant expects to get more than $\Pi^m(n)$ he will enter even when $\Pi^m(n)$ is negative, which should result in the exiting of the firm making the lowest profits. However, note that, whatever the expectations of the potential entrant, the number of active firms in the industry will be the same n^* defined above. The industry can only hold a maximum of n^* profitable firms.

I assume that the locational structure remains invariant when the additional firm enters. There is no locational structure implied by an equal profits equilibrium, since by Proposition 2 such an equilibrium does not exist. Further, by Theorem 1 there exists no location-price structure which constitutes a perfect equilibrium in the variety-price game. Thus, any locational structure which results in a short-run equilibrium can be considered.

I consider in detail the symmetric structure studied in Section 5. For every number of firms n, there exist many combinations of distances d and c which fulfill the original requirement $(n - 1)d + 2c = 1 < nd$. I pin down the locational structure by defining $c = \theta d$, $0 \leq \theta \leq 1/2$, and considering θ as a parameter.[13] Thus, the ratios of the distances between firms are kept independent of the number of firms. Define this as the *fixed-distance-ratios symmetric configuration*. At the competitive equilibrium the worst-off firm is the middle one, $j = (n + 1)/2$ for odd n (alternatively $j = n/2$ or $j = 1 + n/2$ for even n). Quantities $p^*_{n/2}(n)$ and $\Pi^*_{n/2}(n)$ decrease in n (and asymptotically tend to zero). Thus, there exists a unique solution $\bar{n}$ to $\Pi^*_{n/2}(\bar{n}) = F$ which is equivalent to $\Pi^m(\bar{n}) = 0$. Then the free-entry equilibrium number of firms is $n^* = I(\bar{n})$ where $I(x)$ represents the integer part of x.

THEOREM 5: *In the symmetric fixed-distance-ratios competitive locational structure there exists a unique free-entry equilibrium number of firms n^*.*

COROLLARY 3: *The equilibrium number of firms n^* decreases in the setup cost F.*

Similar analysis can be applied to the equilibria at the kink. For the symmetric selection, the edge firm makes the lowest profits,[14] which increase in d and thus decrease in n. Thus, Theorem 5 and Corollary 3 also hold for kink equilibria configurations. In the local monopolistic configuration (minimum) equilibrium profits $\Pi^m = (k - m)^2/(2\lambda) - F$ are independent of the number of firms n. Thus, there can be three cases: either $\Pi^m(n) < 0$ for all n and there are no firms in the

[12]This is a free-entry equilibrium in the sense that all firms have access to the same technology and can enter without a cost impediment compared to other firms which are already active in the market.

[13]When there are n firms in the market the distance between interior firms is $d = 1(n - 1 + 2\theta)$.

[14]The profits of the edge firm are $(k - \lambda d/2)(c + d/2) = d(\mu + 1/2)(k - \lambda d/2)$, which has derivative $(\mu + 1/2)(k - \lambda d) > 0$.

industry; or $\Pi^m(n) > 0$ for all n which result in local monopolistic configurations and then entry drives the distance down to the kink configuration; or $\Pi^m(n) = 0$ for all n, a knife-edge case with an indeterminate number of firms. Therefore Theorem 5 holds for the local monopolistic case.[15]

8. CONCLUDING REMARKS

We have analyzed oligopolistic competition of more than two firms with products defined by their characteristics, using [0, 1] as the space of characteristics. It has been shown that standard results of the traditional model (which used the circumference as the product space) do not hold for the [0, 1] model. In particular, *there exists no subgame-perfect equilibrium in the two-stage varieties-prices game in which the whole market is served.* Equilibrium existence breaks down in the stage of variety choice as firms, starting from any locational pattern, would like to relocate closer to the central firm. This result can be thought of as a strong form of the "Principle of Minimum Differentiation." Indeed, it is so strong that no locational equilibrium exists as a direct consequence of it, provided that firms can relocate costlessly.

For a symmetric locational structure, I have shown the existence of a noncooperative equilibrium price structure. Such an equilibrium exists even for high reservation prices—although for such prices it fails in duopoly. Firms near the endpoints of the space of characteristics quote the highest prices, and prices decrease as one moves to interior firms.

Whatever the locational pattern, it has been shown that there is no short-run noncooperative price equilibrium where all firms make equal profits. Despite the profits inequality, we showed the existence and characterization of free-entry equilibria.

Traditionally the symmetric circumference model of differentiated products has been used as an approximation to models that use an interval $[a, b]$ as the product space. This paper plainly shows that the results of the two models are quite different, and therefore *we can no longer use the simple circular symmetric model to approximate most real-world differentiated markets where products are naturally ordered by a characteristic which has a natural real interval domain.*

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[15]Of course, since local monopolistic profits are independent of the number of firms, Corollary 2 fails for local monopolistic equilibria.

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APPENDIX A:

Proof of Lemma 1

I first derive the first column of $\mathbf{B}$. By the definition of $\mathbf{B}$ as $\mathbf{B} = \mathbf{A}^{-1}$

$$-b_{i,1}/4 + b_{i+1,1} - b_{i+2,1}/4 = 0 \tag{A1}$$

$$b_{1,1} - b_{2,1}/2 = 1 \tag{A2}$$

$$-b_{n-1,1}/2 + b_{n,1} = 0 \tag{A3}$$

From (A1) it follows that $b_{i,1} = k_{1,1}\rho_1^{i-1} + k_{2,1}\rho_2^{i-1}$, where $\rho_1 = 2 + \sqrt{3}$ and $\rho_2 = 2 - \sqrt{3}$ are the solutions of $\rho^2 - 4\rho + 1 = 0$. Imposing (A2) and (A3) it follows that $k_{1,1} = 2\rho_2^{n-1}/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\}$ and $k_{2,1} = 2\rho_1^{n-1}/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\}$ so that

$$b_{i,1} = 2[\rho_1^{n-i} + \rho_2^{n-i}]/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\} \tag{A4}$$

Similarly, the elements of the last column of $\mathbf{B}$ are

$$b_{i,n} = 2[\rho_1^{i-1} + \rho_2^{i-1}]/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\} \tag{A5}$$

I now derive the general interior column of $\mathbf{B}, j \neq 1$ or n. By the definition of $\mathbf{B}$

$$b_{1,j} - b_{2,j}/2 = 0 \tag{A6}$$

$$-b_{i,j}/4 + b_{i+1,j} - b_{i+2,j}/4 = 0 \qquad (i \le j - 2 \text{ or } i \ge j) \tag{A7}$$

(A8) $$-b_{j-1,j}/4 + b_{j,j} - b_{j+1,j}/4 = 1 \qquad (i = j-1)$$

(A9) $$-b_{n-1,j}/2 + b_{n,j} = 0$$

From (A7) it follows that $b_{i,j} = k_j[\rho_1^{i-1} + \rho_2^{i-1}]$ for $i \leq j$ and $b_{i,j} = k_j'[\rho_1^{n-i} + \rho_2^{n-i}]$ for $i \geq j$. For $i = j$ these two expressions gave the same value. Thus, $k_j'/k_j = [\rho_1^{j-1} + \rho_2^{j-1}]/[\rho_1^{n-j} + \rho_2^{n-j}]$ and $k_j = 2[\rho_1^{n-j} + \rho_2^{n-j}]/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\}$. Therefore

(A10) $$b_{i,j} = 2[\rho_1^{n-j} + \rho_2^{n-j}][\rho_1^{i-1} + \rho_2^{i-1}]/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\} \qquad (i \leq j)$$

(A11) $$b_{i,j} = 2[\rho_1^{j-1} + \rho_2^{j-1}][\rho_1^{n-i} + \rho_2^{n-i}]/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\} \qquad (i \geq j) \quad \blacksquare$$

Proof of Corollary 1

Clearly all elements $b_{i,j}$ of $\mathbf{B} \equiv \mathbf{A}^{-1}$ are positive. Let $\iota = (1, \ldots, 1)'$ be a vector of 1s, let $v_i = \sum_{j=1}^n b_{i,j}$ be the sum of row i and let $\mathbf{d}$ be the vector of v_is. $\mathbf{A} \cdot \mathbf{B} = \mathbf{I}$ implies $\iota = \mathbf{A} \cdot \mathbf{v}$. This has a unique solution, $\mathbf{v} = \mathbf{A}^{-1} \cdot \iota$. The solution is $\mathbf{v} = 2\iota$, since $1 = 2\sum_{j-1}^n a_{i,j}$, for all i. Thus, $\sum_{j-1}^n b_{i,j} = 2$. $\blacksquare$

Note that the proof only used the invertability of **A** and the fact that the rows of **A** and **I** summed to a constant. Thus it applies to similar matrices, like $\mathbf{A}^*$ which appears in the first-order conditions of a circular market which is identical to **A** except for $a_{1,n}^* = a_{n,1}^* = -1/4$ and $a_{1,2}^* = a_{n,n-1}^* = -1/4$.

Proof of Proposition 2

I derive the equal profits locational structure for odd $n = 2q - 1$. The proof for $n = 2q$ is similar. The equilibrium prices and locations have to fulfill

[A12(1)] $$p(\sqrt{2} - 1/2) = \lambda(x_1 + x_2)/2$$

[A12(2)] $$p(3 - \sqrt{2})/4 = \lambda(x_3 - x_1)/4$$

[A12(j)] $$p/2 = \lambda(x_{j+1} - x_{j-1})/4$$

Add relations [A12(1)] and (q) to relations [A12(2)] through $(q-1)$ multiplied by 2. The left-hand side is LHS $= p[\sqrt{2} - 1/2 + 3/2 - \sqrt{2}/2 + 2(q-3)/2 + 1/2] = p[\sqrt{2} + 2q - 3]/2 = p(\sqrt{2} + n - 2)/2$. The right-hand side, after some cancellations is RHS $= \lambda[x_{q+1}/2 + x_{q-1}/2 + x_q]/2 = \lambda x_q = \lambda/2$, so that $p(n) = 1/(\sqrt{2} + n - 2)$. $x_2 - x_1 = p(2 - \sqrt{2})/\lambda$, $x_1 = 3p(\sqrt{2} - 1)/(2\lambda)$. The undercutting price for the second firm is $p^u = p\sqrt{2} - \lambda(x_2 - x_1) = 2p(\sqrt{2} - 1)$. Demand at this price is $D^u > x_2 = p(\sqrt{2} + 1)/(2\lambda)$, so that undercutting profits minus candidate equilibrium profits are $\Pi^u - \Pi^* > p^2[2(\sqrt{2} - 1)(\sqrt{2} + 1)/2 - 1]/\lambda = 0$. Thus, undercutting by the second firm is profitable. $\blacksquare$

Proof of Lemma 2

For $j \neq 1, n$, we have $dp_j/dx_j = \lambda(b_{j,j-1} - b_{j,j+1})/4$. From Lemma 1, $b_{j,j-1} - b_{jj+1} = 2\{[\rho_1^{n-j+1} + \rho_2^{n-j+1}][\rho_1^{j-1} + \rho_2^{j-1}] - [\rho_1^{n-j} + \rho_2^{n-j}][\rho_1^j + \rho_2^j]\}/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\}$. The term in the first outer brackets is equal to $\rho_1^{n-2j+2} + \rho_2^{n-2j+2} - \rho_1^{n-2j} - \rho_2^{n-2j} =$

$(\rho_1^2 - 1)(\rho_1^{n-2j} - \rho_2^{n-2j+2}) = (\rho_1^2 - 1)(\rho_1^{2n-4j+2} - 1)\rho_1^{-(n-2j+2)}$. Thus, $dp_j/dx_j > 0 \Leftrightarrow 2(n - 2j + 1) > 0 \Leftrightarrow j < (n + 1)/2$. Similarly, $dp_j/dx_j < 0 \Leftrightarrow j > (n + 1)/2$. For $j = 1$, we have $dp_1/dx_1 = \lambda(b_{j,j} + b_{j,j-1})/2 > 0$, which is positive since all bs are positive. A similar proof shows that $dp_n/dx_n < 0$. ■

Proof of Proposition 3

The mathematical statement of Proposition 3 is: $dp_i^*/dx_j > 0$ for $i < j$ and increases with i, while $dp_i^*/dx_j < 0$ for $i > j$ and decreases in absolute value with i.

Now, $d\mathbf{p}/dx_j = \mathbf{A}^{-1} d\mathbf{y}/dx_j$ implies $dp_i^*/dx_j = \lambda(b_{i,j-1} - b_{i,j-1})/4$, which for $i < j$ is proportional and of the same sign as $\rho_1^{n-j+1} + \rho_2^{n-j+1} - (\rho_1^{n-j-1} + \rho_2^{n-j-1}) = (\rho_1^2 - 1)(\rho_1^{n-j-1} - \rho_2^{n-j+1}) > 0$, and the proportionality coefficient, $2(\rho_1^{i-1} + \rho_2^{i-1})/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\}$, is increasing in i. For $i > j$, dp_i^*/dx_j is proportional and of the same sign as $\rho_1^{j-2} + \rho_2^{j-2} - (\rho_1^j + \rho_2^j) = (1 - \rho_1^2)(\rho_1^{j-2} - \rho_2^j) < 0$ and the proportionality coefficient, $2(\rho_1^{n-1} + \rho_2^{n-i})/\{\sqrt{3}[\rho_1^{n-1} - \rho_2^{n-1}]\}$, is decreasing in i. ■

Proof of Proposition 4

The difference of successive differences of the e_js is $\delta = (e_{j+1} - e_j) - (e_j - e_{j-1}) = e_{j+1} + e_{j-1} - 2e_j = e_{j+1} + e_{j-1} - 4e_j + 2e_j = 0 + 2e_j > 0$, which establishes convexity.

Define the difference of successive e_js as $f(j) = e_{j+1} - e_j$. $f(j)$ is proportional to (and of the opposite sign as) $(1 + \sqrt{3})[\rho_1^{n-j-1} - \rho_1^{j-1}] + (\sqrt{3} - 1)[\rho_2^{j-1} - \rho_2^{n-j-1}]$. For even $n = 2q$, $f(n/2) = 0$. By the convexity argument, $f(j)$ is increasing in j. Therefore $f(j) < 0$ for $j < n/2$ and $f(j) > 0$ for $j > n/2$. For odd $n = 2q + 1$, observe that $n > 2j$ implies that both the first and second brackets of the above expression are positive, and thus $f(j) < 0$ for $j < n/2$. Similarly, $n < 2j$ implies that both the first and the second brackets are negative and therefore $f(j) < 0$.

To show that $e_j(n)$ decreases in n, first note that e_j can be written as $e_j = 2(\rho_2^{j-1} + \rho_2^{n-j})/[\sqrt{3}(1 - \rho_2^{n-1})]$. Now, since $\rho_2 < 1$, the denominator is increasing in n while the numerator is decreasing in n, and therefore e_j decreases in n. ■