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On the Existence of Cournot Equilibrium without Concave Profit Functions*

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This communication is concerned with the existence of equilibrium in Cournot's model of oligopoly [2, Chap. VII]. This question, of course, has been examined often (see, e.g., [1–3]). To our knowledge, however, all previous treatments of the problem have assumed (either directly or indirectly) that the reaction curves of the firms are single-valued continuous functions or convex-valued, upper hemicontinuous correspondences, so that the Brouwer–Kakutani fixed point theorem may be used. In the constant marginal cost case, this assumption amounts to a condition that marginal revenue always be decreasing. Given some regularity conditions, marginal revenue will be falling at any profit-maximizing output. However, to assume that this condition holds globally is extremely restrictive. For example, one can easily construct examples in which it is not met even though the demand arises from a single competitive consumer with homothetic preferences.[1]

We will consider here the case in which the price of the single homogeneous product is given by an upper hemicontinuous correspondence of the total production. Although this assumption is classical, it is nevertheless still restrictive, since it does not allow for general equilibrium effects. Moreover, we also assume costless production (although this

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[1] Note that obtaining a failure of concavity places restrictions on the marginal rates of substitution only along three rays.

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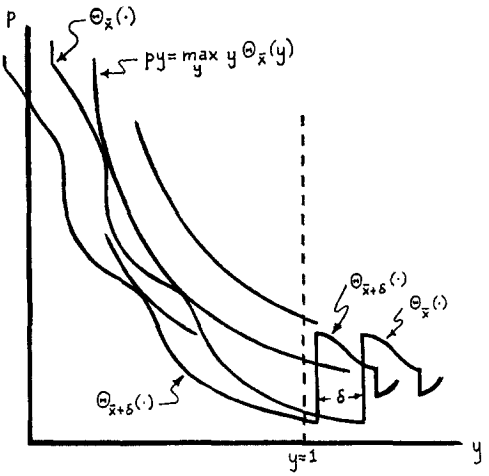

FIGURE 1

condition can be relaxed somewhat).[2] Yet even in this very simple model, the reaction curves of profit-maximizing firms need not be convex valued or upper hemicontinuous. However, we can still establish existence of Cournot equilibrium by using a simple fixed-point theorem for real-valued correspondences which does not assume that these properties hold.

Suppose there are m firms, each of which can costlessly supply the single homogeneous commodity in any nonnegative quantity less or equal to some bound B. Let y_i, $i = 1,..., m$ denote these quantities, and normalize $B = 1$. The demand for the commodity is given by the inverse demand correspondence Φ: $[0, m] \to R_+$. We assume that Φ is a closed correspondence with a compact range. This market structure is called a symmetric oligopoly. Given any aggregate output level $x = \sum_{j \neq i} y_j$ selected by the other firms, the objective of firm i is to maximize profit py by the choice of its output y, where $p = \varphi(x + y)$ and $\varphi(z) \equiv \max \Phi(z)$. A Cournot equilibrium is an $(n + 1)$-tuple $(\bar{p}, \bar{y}_1,..., \bar{y}_m)$ such that $p = \varphi(\sum_{i=1}^m \bar{y}_i)$, and for each i, $\bar{y}_i$ maximizes $y\varphi(y + \sum_{j \neq i} \bar{y}_j)$ on $[0, 1]$.

Graphically, the situation is depicted in Fig. 1, where the relationship between y and p for each $x = \sum_{j \neq i} y_j$ is given by the correspondence θ_x, where $\theta_x(y) \equiv \Phi(x + y)$. The level curves of profit are rectangular hyperbolae $py = c$. Profit is maximized for any x at those y where $\theta_x(y)$ just

[2] We can allow for costly production by all firms under a common constant returns to scale technology if input prices either are given competitively to the industry or depend only on total industry demand. This is done simply by reinterpreting the residual demand curve.

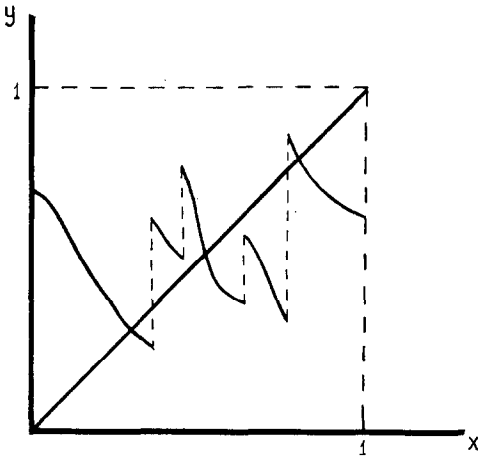

FIGURE 2

touches the highest level curve. Note that as x increases from $\bar{x}$ to $\bar{x} + \delta$, the relevant θ_x shifts left by the amount δ, since $\theta_{x+\delta}(y) = \Phi(x + y + \delta) = \theta_x(y + \delta)$.

It is clear that at $x = \bar{x}$ there are exactly two values of y which maximize profits. Thus, the reaction curve is not convex valued. Further, it need not have a closed graph: As $x_n \to x$, the profit maximizing choices y_n may converge to y, where y is not profit-maximizing given x. This occurs, for example, when the vertical section of the inverse demand enters the region [0, 1] from the right. Despite these problems, we are able to prove the following theorem.

THEOREM. *There exists a Cournot equilibrium* $(\bar{p}, \bar{y}_1, \ldots, \bar{y}_m)$, *where* $\bar{y}_1 = \cdots = \bar{y}_m = \bar{y}$, *in a symmetric oligopoly.*

To prove this theorem, we will first establish a result establishing the existence of fixed points for a class of real-valued correspondences. We will then show that the reaction curves of the firms belong to this class. To motivate the fixed-point results, consider the problem of proving existence of a fixed point for a real-valued function f from the unit interval [0, 1] to itself. If f is not continuous, there of course need not be such a point. However, as inspection of Fig. 2 suggests, a fixed point will exist if the only discontinuities in f take the form of "upward jumps," or, more formally, if f is continuous from the right [$x_n > x$, $x_n \to x$ implies $f(x_n) \to f(x)$][3] and upper semicontinuous from the left [$x_n < x$, $x_n \to x$

[3] In fact, it is sufficient that f be lower semicontinuous from the right.

implies $\limsup f(x_n) \leqslant f(x)$]. For a correspondence F, an appropriate generalization of these conditions is that F be closed from the right, [i.e., $x_n > x$, $x_n \to x$, $y_n \in F(x_n)$ and $y_n \to y$ implies $y \in F(x)$] and that the function given by taking $\inf F(x)$ for each x be upper semicontinuous from the left.

LEMMA. *For each $x \in [0, 1]$, let $F(x)$ be a nonempty subset of $[0, 1]$, and let f: $[0, 1] \to [0, 1]$ be defined by $f(x) = \inf\{y \mid y \in F(x)\}$. Suppose F is closed from the right and f is upper semicontinuous from the left. Then there exists $\bar{x} \in [0, 1]$ such that $\bar{x} \in F(\bar{x})$.*

Proof of the lemma. Let $\bar{x}$ be the smallest value of x such that there exists $y \in F(x)$ with $y \leqslant x$, and let $\bar{y} \leqslant \bar{x}$ belong to $F(\bar{x})$. Existence of such an $\bar{x}$ and $\bar{y}$ follows from F being closed from the right. Then $\bar{x}$ is a fixed point for F. To see this, note that if $\bar{x} = 0$, then $0 \leqslant \bar{y} \leqslant \bar{x} = 0$, while if $\bar{x} > 0$, we can approach $\bar{x}$ from the left via a sequence (x_n) for which $x_n \leqslant f(x_n)$. Then $\bar{x} \leqslant \limsup f(x_n)$, but, since f is upper semicontinuous from the left, $\limsup f(x_n) \leqslant f(\bar{x}) \leqslant \bar{y} \leqslant \bar{x}$. Thus $\bar{x} = \inf F(\bar{x})$. But $\bar{x} \geqslant \bar{y} \in F(\bar{x})$, so $\bar{x} \in F(\bar{x})$. Q.E.D.

Let R be the reaction correspondence of any firm, i.e., R: $[0, (m-1)] \to [0, 1]$ is defined for each x as the set of maximizers of $y\Phi(y + x)$. A value $\bar{y} \in [0, 1]$ such that $\bar{y} \in R((m-1)\,\bar{y})$ will define a Cournot equilibrium $(\varphi(m\bar{y}), \bar{y}, \ldots, \bar{y})$. Existence of such a $\bar{y}$ is equivalent to existence of a fixed point for the correspondence F: $[0, 1] \to [0, 1]$ defined by $F(x) = R((m-1)\,x)$. Since F will be nonempty valued and closed from the right if and only if R has these properties, while $f = \min F$ will be upper semicontinuous from the left if and only if $r = \min R$ is upper semicontinuous from the left, it is sufficient to show that R and r have these properties.

That R and thus F have closed, nonempty values follows from the profit function, py, being continuous and the set of all (p, y) such that $p \in \Phi(x + y)$ being compact for each x. To see that R is closed from the right, let $x_n \to x$, $x_n > x$, $y_n \in R(x_n)$, $y_n \to y$. If $y \notin R(x)$, then there exists $\bar{y} \in [0, 1]$ such that $\varphi(x + \bar{y})\,\bar{y} > \varphi(x + y)\,y$. Since $x_n > x$, unless $\bar{y} = 0$, the firm could have, for large enough n, chosen $\bar{y}_n \equiv \bar{y} + (x - x_n)$ when it selected y_n. For this choice, $\Phi(x_n + \bar{y}_n) = \Phi(x + \bar{y})$, and thus $\varphi(x_n + \bar{y}_n)\,\bar{y}_n = \varphi(x + \bar{y})\,\bar{y}_n \to \varphi(x + \bar{y})\,\bar{y} > \varphi(x + y)\,y$. But, since Φ has a closed graph, $\varphi(x + y)\,y \geqslant \limsup \varphi(x_n + y_n)\,y_n$. This yields a contradiction if $\bar{y} > 0$, while if $\bar{y} = 0$, then $0 = \varphi(x + \bar{y})\,\bar{y} > \varphi(x + y)\,y \geqslant 0$, another contradiction.

To show that the function r defined by $r(x) = \min R(x)$ is upper semi-

continuous from the right, we will show that if $x_n \to x$, $x_n < x$, then $r(x_n) \leqslant r(x) + (x - x_n)$. Then taking $\limsup r(x_n)$ yields the result.[4]

Let $\bar{y} = r(x)$. For any $y \geqslant \bar{y}$, $\varphi(x + \bar{y})\,\bar{y} \geqslant \varphi(x + y)\,y \geqslant \varphi(x + y)\,\bar{y}$, since $\varphi \geqslant 0$. Thus, for $y \geqslant \bar{y}$, $\delta \geqslant 0$, $(\bar{y} + \delta)\,\varphi(x + \bar{y}) \geqslant (y + \delta)\,\varphi(x + y)$. Rewrite this last inequality condition as "for any $u \geqslant (\bar{y} + \delta)$, $(\bar{y} + \delta)\,\varphi((x - \delta) + (\bar{y} + \delta)) \geqslant u\varphi(x - \delta + u)$." Note that, for any x, if u is such that for all $z \geqslant u$, $u\varphi(x + u) \geqslant z\varphi(x + z)$, then $r(x) \leqslant u$. This follows since if $u < r(x)$, then $u\varphi(x + u) \geqslant r(x)\,\varphi(x + r(x))$ implies $u \in R(x)$, a contradiction if $u \leqslant 1$, while the case $u > 1$ is trivial. Finally, this observation and the expression in quotation marks yield $r(x - \delta) \leqslant \bar{y} + \delta = r(x) + \delta$. Now, take $\delta = x - x_n$.

The model we have studied bears the following alternative interpretation (see [5]). Consider two commodities which are demanded in fixed proportions, say one-to-one, but which are produced by two separate firms. This market structure, called complementary monopoly, was also studied by Cournot [2, Chap. IX], who considered the example of copper and zinc used jointly to produce brass. It is natural to represent the demands for the two commodities, which must be identical at each pair of prices, by a correspondence Ψ: $[0, 2] \to R_+$ with a closed graph. Generically, we write $x \in \Psi(p + q)$, where x is the quantity demanded of either commodity and p and q represent the prices of the two commodities. Then, a Cournot equilibrium is a triple $(\bar{x}, \bar{p}, \bar{q})$ such that $\bar{p}\bar{x}$ is maximal given $\bar{q}$ and $\bar{q}\bar{x}$ is a maximal given $\bar{p}$, where $\bar{x} = \max \Psi(\bar{p} + \bar{q})$. Our theorem then provides conditions for the existence of Cournot equilibrium in situations of complementary monopoly.

It should be clear that the methods used here to prove existence depend crucially on the assumptions that the firms are identical and that price depends only on total output. Given the nonconvexities that arise even in this simple, symmetric case, one must conjecture that in more general models without this special structure, equilibrium might well fail to exist. In fact, we have recently constructed an example of a very simple economy with two firms, each of which costlessly produces a single commodity, in which no Cournot–Chamberlin equilibrium exists (see [4]).

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[4] We are indebted to Hervé Moulin for the following argument.

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