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Vol 47, No. 1, February 1989
Printed in Belgium

Notes, Comments, and Letters to the Editor

Symmetric Equilibrium Existence and Optimality in Differentiated Product Markets*

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Received December 20, 1985; revised January 7, 1988

Equilibrium existence and optimality are analysed in a market for products differentiated by their variety. A game of three stages is analysed. Firms enter in the first stage, choose varieties in the second stage, and choose prices in the third stage. The existence of subgame-perfect equilibria is established. At equilibrium products are symmetrically located in the space of characteristics and are offered at equal prices. The surplus maximizing solution is characterized, and it is shown that surplus maximizing product diversity is lower than the equilibrium one. *Journal of Economic Literature* Classification Numbers: 022, 611, 615, 933.

1. INTRODUCTION

Ever since Hotelling's acclaimed "Principle of Minimum Differentiation" [13] attention has been focused on the issue of equilibrium patterns of differentiated products in duopoly and oligopoly. Hotelling modelled duopoly competition as a two stage game. In the last stage firms played a noncooperative game in prices taking as given the choices of varieties made in the first stage. In the first stage firms chose varieties non-cooperatively expecting to receive the Nash equilibrium profits of the price game played for the chosen varieties. Hotelling looked for equilibria which (in modern terminology) were subgame-perfect. He claimed that the emerging perfect equilibrium pattern was one of firms producing nearly identical varieties. Hence the Principle of Minimum Differentiation. D'Aspremont *et al.* [1] corrected an error in the original paper of Hotelling by showing that there exists no perfect equilibrium in his formulation. Nonquasiconcave profit

* I thank Phoebus Dhrymes for helpful discussions and two anonymous referees and an associate editor of this journal for helpful comments.

0022-0531/89 $3.00

functions result is discontinuous best reply functions in many price subgames. Equilibrium existence fails in many subgames where firms are "located" (in variety space) close to each other. Thus no perfect equilibrium exists. D'Aspremont *et al.* [1] also modified the utility functions[1] of the representative consumer and thus established existence of a perfect equilibrium where firms produce the most different (feasible) varieties.[2]

Here we discuss the perfect equilibrium pattern in oligopolistic competition of three or more firms. Each firm decides whether or not to enter the market, which variety to produce, and at what price to sell it. These choices are made in a game of three stages. In the first stage firms decide simultaneously whether or not to enter the market. In the second stage each firm decides on the variety it will produce. In the third stage, given the choices of varieties of all other firms, each firm chooses its price. We seek non-cooperative equilibria that are subgame-perfect in any subgame that starts at the second or third stage.[3]

This structure of the game is natural as firms may decide on price in the short run, on variety in the long run, and on entry in the very long run. Note also that no interesting equilibrium exists in a game of simultaneous choice of prices and varieties. As shown in Novshek [20] and Economides [9], in such a game the only Nash equilibria are "local monopolistic" where firms do not compete directly for the marginal consumer, and there are consumers between any neighboring firms who weakly prefer not to buy any differentiated product at equilibrium. Recently, Schulz and Stahl [23] have shown that equilibrium fails in the game of simultaneous choice of varieties and prices, even when the strategy space is restricted to preclude the use of the global relocation strategies employed in [20, 9].[4]

Our setting resembles that of Salop [22], the significant difference being that here we include a full-fledged stage of choice of varieties, while Salop [22] allowed only symmetric varieties. The difference is important because in Salop's model an equilibrium does not exist in *any* price

[1] In Hotelling [13] consumers had linear disutility of distance in the product space. D'Aspremont *et al.* [1] specified quadratic disutility of distance.

[2] The problem of existence of equilibrium and the perfect equilibrium patterns has also been discussed in modifications of Hotelling's [13] problem by Novshek [20], Economides [8, 9], and Eaton and Wooders [6].

[3] See Selten [24] for a detailed discussion of equilibrium perfectness. Briefly, a Nash equilibrium of game G is "subgame-perfect" if any subgame G_r results in a Nash equilibrium which is the truncation to the subgame of the Nash equilibrium of game G.

[4] Other sequential games have also been considered. Eaton and Wooders [6] consider entry equilibria in a game where active firms are fixed in locations but have price flexibility against entrants, while entrants are flexible in both prices and locations. Presscott and Visscher [21] discuss a sequential game in locations with subsequent simultaneous price choice.

subgame, although an equilibrium exists in the price subgame defined by symmetric choices of varieties. To establish a perfect equilibrium, as we do in this paper, it is necessary to have an equilibrium for every subgame that may arise.

We show that, under symmetry assumptions on the distribution of consumers' preferences, at the resulting perfect equilibrium products are equidistant in the space of characteristics. Thus we observe neither "minimal product differentiation" as Hotelling [13] claimed, nor "maximal product differentiation" as d'Aspremont *et al.* [1] showed in duopoly.[5]

In Section 2 the basic model is presented and analysed. In Section 3 we solve backwards for the perfect equilibrium starting with the third stage subgames where firms choose prices. Section 4 analyses the choice of varieties in the second stage. Section 5 discusses the entry decisions of the first stage. In Section 6 we compare market product diversity with surplus maximizing product diversity. In Section 7 we conclude.

2. The Model

Consider an economy with a homogeneous good m (Hicksian composite good) and n differentiated products $x_1, ..., x_n$ offered at prices $p_1, ..., p_n$. Consumers are allowed to buy one unit of a differentiated good. They have single-peaked preferences over differentiated products. A typical consumer has a utility function separable in the homogeneous good,

$$U_w(m, x_j, p_j) = m - p_j + V_w(x_j), \qquad V_w(x_j) = k - (x_j - w)^2,$$

where x_j represents one unit of differentiated product x_j. $V_w(x)$ has a single peak at $x = w$. Thus consumer "w" likes variety "w" most. The term $(x_j - w)^2$ measures the loss of utility that consumer "w" incurs when he consumes product "x_j" rather than "w." Here the "disutility of distance" function is assumed to be quadratic. In general "w" will not be available and he will buy variety $\tilde{x}$ which maximizes utility $U_w(\cdot)$ over the set of available choices $\{(x_j, p_j), j = 1, ..., n\}$. The maximum a consumer is willing to pay for a differentiated product is k, his reservation price. The product space is assumed to be a circumference of radius $1/2\pi$. Consumers are distributed uniformly on the circumference with density μ, according to the peaks of the utility functions $V_w(\cdot)$. Each firm produces one differentiated product. The technology of production of any variety is summarized by the

[5] The symmetric location equilibrium pattern on a circumference can be seen as a result of firms maximizing the minimal distance between them or of firms minimizing the maximal distance between them. Thus it could be interpreted as a maximal product differentiation pattern.

cost function $C_j(q_j) = F + c_j(q_j)$ for $q_j > 0$ and $C_j(0) = 0$, with $c_j'(\cdot) \geq 0$, $c_j''(\cdot) \geq 0$ and $F > 0$.

We differ from Salop [22] and Hotelling [13] and we follow D'Aspremont *et al.* [1] in assuming quadratic disutility of distance rather than linear. Our specification guarantees that the equilibrium prices of any third stage subgame are very responsive to changes in the specification of varieties, and that competition intensifies (and prices go to marginal cost) as product specifications become very similar. It is exactly the lack of responsiveness of the first-order-condition prices to changes in specifications which lead to undercutting and non-existence of equilibrium in [13]. Intensification of competition as product specifications become very similar is a desirable feature of a model of differentiated products. In this class of models it is also indispensable for the existence of equilibrium.

3. Equilibria in the Last Stage Subgame

We begin by analysing the last stage of the game. When they reach this stage, firms have already chosen varieties $x_1, ..., x_n$. In the present stage they choose prices $p_1, ..., p_n$ non-cooperatively. It is easy to show that firm j will supply a set of consumers which are represented on the circumference by an interval (which can be of zero length). Consider the maximum demand which a firm can face in this market. Let this be called the (local) monopoly demand, since it is realized when firm j faces no direct competition by other firms. Monopoly demand is $D_j^M = 2\mu(k - p_j)^{1/2}$. It can be broken into demand from consumers located to the right and demand from consumers located to the left: $D_j^M/2 = D_j^{MR} = D_j^{ML}$. These demand functions are concave in p_j. Now consider the demand for firm j when a neighboring firm, through its choice of variety and price, competes directly with it. The demand faced by firm j is continuous with a linear part in the region of compatition with the neighbor. For concreteness consider demand for j when it competes with $j+1$, i.e., at prices p_j in $[\bar{p}_j, \bar{\bar{p}}_j]$, where

$$\bar{p}_j = k - (x_{j-1} - x_j + (k - p_{j+1})^{1/2})^2,$$
$$\bar{\bar{p}}_j = k - (x_{j+1} - x_j - (k - p_{j+1})^{1/2})^2.$$

Demand from consumers to the right of x_j is

$$D_j^R = \mu\{(x_{j+1} - x_j)/2 - (p_{j+1} - p_j)/[2(x_{j+1} - x_j)]\},$$

linear in p_j, where $p_j = \lambda\bar{p}_j + (1-\lambda)\bar{\bar{p}}_j$ and $0 \leq \lambda \leq 1$.

In Fig. 1 the (local) monopolistic demand is replaced by the line segment (AB) for p_j in $[\bar{p}_j, \bar{\bar{p}}_j]$. If firm $j+1$ is in direct competition with firm $j+2$

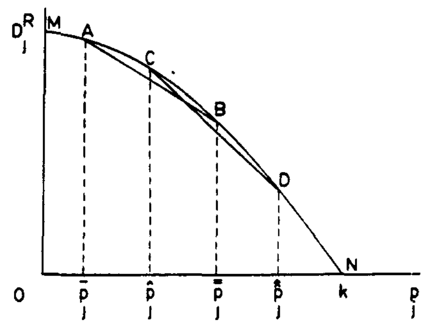

FIG. 1. The demand function of firm j generated from consumers located to its right.

then firm j is not able to realize the lower part (including B) of the linear segment of the demand. Let $\hat{p}_j$ ($\hat{\hat{p}}_j$) be the minimal (maximal) price of j such that firm j would be in direct competition with firm $j+2$, if firm $j+1$ were not active. These prices correspond to points C, D on the demand function; if firm $j+1$ were not active, firm j would face the monopoly demand except for the linear section (CD). When firm $j+1$ is active, firm j faces the lower envelope of the local monopolistic demand (MN) and segments (AB) and (CD). Points A, C, B, D lie in that order on (MN) and the lower envelope D_j^R, is (weakly) concave. Similar arguments can be applied to show that the demand from the left, D_j^L, is concave, and thus the demand of firm j, $D_j \equiv D_j^L + D_j^R$, is concave.

PROPOSITION 1. *The demand function for firm j, D_j, is concave.*

Non-decreasing marginal cost is sufficient to ensure the existence of a noncooperative equilibrium by a direct application of Kakutani's fixed point theorem. The proof is straightforward. See Friedman [10].

THEOREM 1. *Given varieties $x_1, ..., x_n$, each produced exclusively by the corresponding firm, the profit function Π_j of firm j is concave in p_j for all j. Therefore there exists a non-cooperative equilibrium in any price subgame played for these varieties.*

In a more general framework more than one firm is allowed to produce the same product. Say product x' is produced by firms $j = 1, ..., k$. Then, whatever the residual demand for product x' (determined by the price strategies of firms that produce products different than x') a Nash equilibrium in prices will entail all firms that produce x' pricing at marginal cost. (This is just the standard Bertrand result applied to this game.) Therefore, any differentiated product produced by more than one firm will be priced at marginal cost. In such a setting, a firm which produces exclusively a differentiated product (say x_i, $i \neq j$) can consider the price for x' as given and equal to marginal cost. Π_j for $i \neq j$ is concave. Theorem 2 follows.

THEOREM 2. *Given varieties $x_1, \ldots, x_n$ and assuming that no firm produces more than one product (but one product can be produced by more than one firm) a non-cooperative equilibrium exists in the price subgame they imply. Equilibrium prices for any firms which produce the same variety are equal to marginal cost.*

We can establish uniqueness of equilibrium using the following lemma.

LEMMA 1. $\partial^2\Pi_j/\partial p_j^2 + \sum_{j \neq i} |\partial^2\Pi_j/\partial p_j\, \partial p_i| < 0$.

Proof. See Appendix.

Lemma 1 assures us that the best reply mapping are contractions. It is then immediate that the fixed point is unique.

THEOREM 3. *The price subgame played for varieties $(x_1, \ldots, x_n)$ has a unique equilibrium.*

There are three types of non-cooperative equilibria. For low reservation prices, $k \leqslant \bar{k}$, firms will be "local monopolists" in no direct competition with neighboring firms. Between any two firms there will be consumers who prefer not to buy any differentiated product at equilibrium. For high reservation prices firms are in direct competition and all consumers in the market buy a differentiated product. We call these equilibria "competitive." For intermediate reservation prices the possibility arises that all consumers are served but the marginal consumer between consecutive firms is indifferent between buying a differentiated product and not buying any differentiated product. Such equilibria we name "kink" equilibria because they correspond to a kink in the demand curve.

These types of equilibria are better illustrated for the symmetric locational configuration, $x_j - x_{j-1} = d$, all j. Let all other firms charge p. For high reservation prices the first order condition of firm j can be written as[6]

$$p_j = c'(D_j(p_j, p)) + D_j d/\mu,$$

[6] In general $\partial\Pi_j/\partial p_j = D_j + (p_j - C'(D_j))\, \partial D_j/\partial p_j$. For the linear part of the demand, $D_j = \mu[(p_{j+1} - p_j)/(x_{j+1} - x_j) - (p_j - p_{j-1})/(x_j - x_{j-1}) + x_{j+1} - x_{j-1}]/2$. Under symmetry $\partial D_j/\partial p_j = -\mu/d$ and $\partial\Pi_j/\partial p_j = 0 \Rightarrow p_j - C'(D_j) = D_j d/\mu$.

which is solved by $p_j = p^C \equiv d^2 + c'(\mu d)$. This will be the equilibrium price if p^C falls on the linear part of the demand, which is equivalent to[7]

$$k - c'(\mu d) > 5d^2/4.$$

For low reservation prices firm j is "local monopolist." Its first order condition can be written as

$$3p^{LM} = 2k + c'(2\mu(k - p^{LM})^{1/2}).$$

The solution of this equation will be the equilibrium price if, at $p_j = p^{LM}$, firm j does not face direct competition from other firms, i.e., $p^{LM} + (d/2)^2 > k$. This implies that, at a "local monopolistic" equilibrium, maximum marginal cost is $c'(\mu d)$ and maximal price is $[2k + c'(\mu d)]/3$. This, together with the previous condition, defines the region of existence of "local monopolistic" equilibria as

$$k - c'(\mu d) < 3d^2/4.$$

For intermediate reservation prices it is possible to have an equilibrium at the kink of the demand curve at price

$$p^T = k - (d/2)^2.$$

The necessary and sufficient conditions for existence of "kink" equilibria are

$$\lim_{\varepsilon \to 0} \partial \Pi_j(k - (d - (k - p^T)^{1/2})^2 - \varepsilon)/\partial p_j > 0,$$

$$\lim_{\varepsilon \to 0} \partial \Pi_j(k - (d - (k - p^T)^{1/2})^2 + \varepsilon)/\partial p_j < 0.$$

These conditions are equivalent to[8]

$$3d^2/4 \leqslant k - c'(\mu d) \leqslant 5d^2/4.$$

[7] $p_j - p^C$ falls in the linear part of the demand when $p^C < \bar{p}$ where $\bar{p}$ is the maximal price which places firm j on the linear part of the demand. $\bar{p}$ is defined by $(k - p^C)^{1/2} + (k - \bar{p})^{1/2} = d$, i.e., $\bar{p} = p^C - d^2 + 2d(k - p^C)^{1/2}$. Then $p^C < \bar{p} \Leftrightarrow p^C < k - (d/2)^2 \Leftrightarrow k - C'(\mu d) > 5d^2/4$.

[8] $\lim_{\varepsilon \to 0} \partial \Pi_j(k - (d - (k - p^T)^{1/2})^2 - \varepsilon)\partial p_j = \lim_{\varepsilon \to 0} \partial \Pi_j(k - (d/2)^2 - \varepsilon)\,\partial p_j = \mu(d^2 + p^T - 2p_j + C'(\mu d))/d = \mu(5d^2/4 - k + C'(\mu d))/d > 0$ iff $k - C'(\mu d) < 5d^2/4$. $\lim_{\varepsilon \to 0} \partial \Pi_j(k - (d - (k - p^T)^{1/2})^2 + \varepsilon)\partial p_j = \lim_{\varepsilon \to 0} \partial \Pi_j(k - (d/2)^2 + \varepsilon)\partial p_j = \mu(2k - 3p^T + C'(\mu d))/(k - p^T)^{1/2} = 2\mu(3d^2/4 - k + C'(\mu d))/d < 0$ iff $k - C'(\mu d) > 3d^2/4$.

We have established:

THEOREM 4. *Depending on the reservation price and marginal cost functions, there are three types of non-cooperative equilibria in a price subgame. The equilibrium prices for the symmetric configuration are*

(I) $p^C = d^2 + c'(\mu d)$ *for* $5d^2/4 < k - c'(\mu d)$*—"competitive" equilibria,*

(II) $p^T = k - (d/2)^2$ *for* $3d^2/4 \leqslant k - c'(\mu d) \leqslant 5d^2/4$*—"kink" equilibria,*

(III) *the solution of* $3p^{LM} = 2k + c'(2\mu(k - p^{LM})^{1/2})$ *for* $0 < k - c'(\mu d) < 3d^2/4$*—"local monopolistic" equilibria.*

Equilibrium prices increase in the distance between firms in the "competitive" region, decrease in the "kink" region, and remain constant in the "local monopolistic" region. The "kink" configuration, corresponding to the kink of the demand curve, acts as a focal point of competition for a range of values of the parameters. As the distance between firms increases, this focal point is achieved at lower prices. Thus, "kink" equilibrium prices decrease in the distance d.

4. Choice of Varieties

We now examine the second stage of the game where the strategic variables are the varieties to be produced. Firms evaluate choices of varieties at the equilibrium of the price subgame that they imply. First we note that, at all perfect equilibria of a subgame that starts at the second stage, each product is produced by only one firm. If a product, say x', was produced by more than one firm, by Theorem 2, all firms producing it would price it at marginal cost and incur losses equal to their fixed costs. By producing a slightly different product, any of these firms can charge above marginal cost and do better than at x'. Hence at equilibrium no product is produced by more than one firm.

In contemplating any change of variety x_j, firm j takes into account not only the resulting change in its own price, p_j^*, but also the resulting changes in the prices of all other firms. The objective function of firm j is $\Pi_j^\vee(\mathbf{x}) \equiv \Pi_j(\mathbf{x}, \mathbf{p}^*(\mathbf{x}))$, where $\mathbf{p}^*(\mathbf{x})$ is the vector of equilibrium prices in the price subgame. Existence of equilibrium depends on the quasiconcavity of $\Pi_j^\vee$ in x_j. Quasiconcavity of Π_j in x_j and p_j would not suffice. Below we prove that this game has at least one perfect equilibrium—the symmetric one. The existence of other locational (varietal) structures as perfect equilibria, although unlikely, cannot be ruled out.

We now restrict our attention to the constant marginal cost technology and examine existence of subgame perfect "competitive" equilibria. When both firms $j-1$ and $j+1$ have positive demands, Π_j is maximized at

$$p_j^m = \frac{(x_{j+1}-x_j)(x_j-x_{j-1})}{2(x_{j+1}-x_{j-1})}\left[\frac{p_{j+1}}{x_{j+1}-x_j}+\frac{p_{j-1}}{x_j-x_{j-1}} + x_{j+1}-x_{j-1}+\frac{c(x_{j+1}-x_{j-1})}{(x_{j+1}-x_j)(x_j-x_{j-1})}\right]. \tag{1}$$

As a corollary of Theorems 3 and 4 we have:

COROLLARY 1. *For symmetric locations* ($x_{j+1}-x_j=d$ *for all* j) *there exists a unique symmetric equilibrium in the price subgame, at prices* $\bar{p}_j=d^2+c$ *for all* j.

When each product is produced exclusively by one firm, the equilibrium profits of the last stage game are

$$\Pi_j^{\vee}(\mathbf{x}) = \frac{\mu(x_{j+1}-x_j)(x_j-x_{j-1})}{8(x_{j+1}-x_{j-1})}\left[\frac{p_{j+1}^*}{x_{j+1}-x_j}+\frac{p_{j-1}^*}{x_j-x_{j-1}} + x_{j+1}-x_{j-1}-\frac{c(x_{j+1}-x_{j-1})}{(x_{j+1}-x_j)(x_j-x_{j-1})}\right]^2 - F. \tag{2}$$

We will now show that the symmetric configuration, $x_{j+1}-x_j=d$, $p_j=\bar{p}$, for all j, constitutes a subgame-parfect equilibrium. Let all other firms except j be at symmetric positions (i.e., $x_{i+1}-x_i=d$ for all $i\neq j$, $i\neq j+1$, and $x_{j+1}-x_{j-1}=2d$). Necessary conditions for the existence of a symmetric perfect equilibrium are that $\Pi_j^{\vee}(x_j)$ is maximized at the symmetric position $x_j=(x_{j+1}+x_{j-1})/2$ and that $\Pi_j^{\vee}(x_j)$ is quasiconcave for x_j in (x_{j-1}, x_{j+1}). These results are established in Lemma 2. Further, taking a position at the location of another firm or between any other existing firms gives lower profits than $\Pi_j^{\vee}$ at symmetry. Therefore the symmetric configuration is a perfect equilibrium.

LEMMA 2. $\Pi_j^{\vee}(\mathbf{x})$ *is quasiconcave in* x_j *and maximized at* $x_j=(x_{j-1}+x_{j+1})/2$ *when* $x_{i+1}-x_i=d$ *for all* $i\neq j$, $i\neq j+1$, *and* $x_{j+1}-x_{j-1}=2d$.

The full proof is given in the Appendix. An outline of the proof follows. $\Pi_j^{\vee}$ can be expressed as a function of the relative deviation of the location of firm j from symmetry $z=(x_j-x_{j-1})/d-1$ as

$$\Pi_j^{\vee}(\mathbf{x}) = \Pi_j^*(d,z) = \mu[p_j^*(z)-c]^2/[d(1-z^2)] - F,$$

where the equilibrium price $p_j^*(z)$ is determined by the first order conditions (1) that can be summarized as

$$\mathbf{A}(z)[\mathbf{p}^*(z)-\mathbf{c}]=\mathbf{b}(z).$$

The locational structure of competition in product specification, and in particular the fact that each firm competes directly only with its two immediate neighbors, implies that z affects only few elements of the matrix $\mathbf{A}(z)$ that are positioned near the element a_{jj}. We take advantage of this fact by writing

$$\mathbf{A}(z)=\mathbf{U}+\mathbf{V}(z),$$

where $\mathbf{U}$ is independent of z and has a well-known factorization $\mathbf{U}=\mathbf{MM}'/s$, where s is a scalar. Then $\mathbf{A}^{-1}(z)$ is approximated as

$$\mathbf{A}^{-1}(z)\simeq\mathbf{U}^{-1}-\mathbf{U}^{-1}\mathbf{V}(z)\,\mathbf{U}^{-1}+\mathbf{U}^{-1}\mathbf{V}(z)\,\mathbf{U}^{-1}\mathbf{V}(z)\,\mathbf{U}^{-1}.$$

We then derive $\mathbf{p}^*(z)=\mathbf{c}+\mathbf{A}^{-1}(z)\,\mathbf{b}(z)$. Quasiconcavity of Π_j^* in z (and therefore of $\Pi_j^\vee$ in x_j) follows by direct computation.

THEOREM 5. *Given that n firms have entered in the market at stage* 1, *there exists a perfect symmetric equilibrium in the subgame that begins with the choice of varieties.*

Proof. Given Lemma 2, the result of Theorem 5 follows directly from an application of Brouwer's fixed point theorem, as in Nash [19]. Q.E.D.

5. ENTRY

Let $\Pi^*(n)$ denote the profits of a firm at a symmetric equilibrium of n firms in the subgame that starts with the choice of varieties. From (2) it follows that $\Pi^*(n)=(\mu/n^3)-F$. Clearly, profits decrease with the number of firms. The *free entry Nash equilibrium* number of firms n^* is defined by $\Pi^*(n^*+1)<0$, $\Pi^*(n^*)\geqslant 0$, the condition that an entrant will make negative profits at a post-entry equilibrium while before entry all firms were making nonnegative profits. Define $I[x]$ as the integer part of x, i.e., the largest integer smaller or equal to x.

THEOREM 6. *The three stage (entry, variety choice, and price choice) game has a perfect symmetric equilibrium with* $n^*=I[(\mu/F)^{1/3}]$ *firms, symmetric varieties at distances* $x_j^*-x_{j-1}^*=1/n^*$, *and equal prices* $p_j^*=c+(1/n^*)^2$.

The number of active firms is increasing with population density and decreasing with the fixed cost. Prices decrease in population density and increase with marginal and fixed cost.

6. Optimal Versus Market Product Diversity

The relation between optimal and market product diversity in models of differentiated products is an issue of debate. In models of single-peaked preferences (in the space of characteristics), where bilateral relations between firms differ (because of the existence of neighbors), as in Lancaster [15] and Salop [22], it is generally true that the number of products at the market equilibrium is higher than the optimal number of products. On the other hand, in the model of Dixit and Stiglitz [3] (where consumers have a taste for variety and all firms see other firms symmetrically) the equilibrium number of products may be lower than the optimal.[9] Here we shall compute the optimal (surplus maximizing) product diversity and compare it with the perfect equilibrium product diversity. The result is in line with those of the single-peaked preference models.

Theorem 7. *At the perfect symmetric equilibrium the number of varieties produced exceeds the optimal (surplus maximizing) number of varieties.*

Proof. Under symmetry the total consumers' surplus is $CS = (\mu/d)[(k-p)d - 2\int_0^{d/2} z^2\,dz] = \mu[k-p-d^2/12]$. Total producers' surplus is $PS = \mu(p-c) - F/d$, so that total surplus is $TS(d) = \mu(k-c-d^2/12) - F/d$ which is concave in d. The internal maximizer is $\tilde{d} = (6F/\mu)^{1/3}$. This will be the optimal distance between firms (and the optimal number of firms will be $n_0 = 1/\tilde{d}$) if it is not bigger than 1 and total maximized surplus is non-negative, i.e., if $F \leqslant 4\mu(k-c)^{3/2}/3$. If $F > 4\mu(k-c)^{3/2}/3$ no differentiated product should be produced and $n_0 = 0$. If $F < 4\mu(k-c)^{3/2}/3$ and $\tilde{d} > 1$ the internal maximizing number of products cannot fit in the market; exactly one product should be produced ($n_0 = 1$) if $TS(1) \geqslant 0$ ($\Leftrightarrow F/\mu \leqslant k-c-1/12$) and no products should be produced otherwise.

At equilibrium the distance between neighboring firms in the "competitive" region is $d_e = (F/\mu)^{1/3}$. In the "kink" region equilibrium profits are $\Pi^T = \mu d[k-(d/2)^2 - c] - F < \mu d^2/2 - F$ (using the boundary conditions of the "kink" region) so that $d_e < (2F/\mu)^{1/3} < (6F/\mu)^{1/3} = \tilde{d}$. The "local monopolistic" equilibrium is a perfect equilibrium when $\Pi^{LM} = 0$, or equivalently when $F = 4\mu[(k-c)/3]^{3/2} \equiv \mu d^3/2$, so that $d_e = (2F/\mu)^{1/3} < (6F/\mu)^{1/3} = \tilde{d}$.

[9] Dixit and Stiglitz [3] claim that equilibrium diversity is lower then optimal. This is disputed by Lancaster [15].

Thus, for all internal maxima, $d_e < \bar{d}$ and equivalently $n_e > n_0$; i.e., the equilibrium product diversity is higher than optimal. This can also be checked at the corner maxima. Q.E.D.

It is important to realize that here profits do not give the right signals to firms that consider entry. Optimal product diversity is attained at positive and, in general, non-negligible profit level. $n_e/n_0 \geq 6^{1/3} \simeq 1.82$, so that there are at least 82% more products at equilibrium than is optimal. Equilibrium product diversity significantly exceeds optimal diversity even when fixed costs tend to zero, although then prices tend to marginal cost and the difference between equilibrium and optimal total surplus tends to zero.

Up to this point, optimality was discussed in terms of total surplus and implicitly utility transfers have been assumed to be feasible. However, the addition of a new product does not necessarily make all consumers better off when the new equilibrium configuration involves relocation of some products. Consider symmetric equilibrium configurations. Although prices of all products are reduced when an extra product is added, a consumer "located" at the position of a product produced at the old equilibrium but not available at the new equilibrium may find his welfare reduced. The movement from the symmetric equilibrium with n products to the one with $n+1$ products will be welfare-improving for all consumers (without utility transfers) if $p^*(n) > p^*(n+1) + (1/[2(n+1)])^2$, since the distance between equilibrium locations in consecutive symmetric equilibria cannot exceed $1/[2(n+1)]$. The above relation is satisfied only for $n \leq 8$. Thus, increases in the number of products are welcomed by all consumers only when the original product diversity was small.

7. Conclusion

We analysed equilibrium existence and optimality in a market for products differentiated by their variety. Firms choose entry into the market, variety specifications, and prices in stages. At each stage a non-cooperative equilibrium is established which is subgame-perfect. At the resulting perfect equilibrium, products are equidistant in the space of characteristics, and each product is produced exclusively by one firm. Thus we observe neither "minimal product differentiation" as Hotelling claimed [13], nor "maximal product differentiation" as d'Aspremont *et al.* [1] showed in duopoly. Our results have the flavor of both extremes. The equidistant spacing is the one which minimizes the maximal distance between any two firms. It is also the one which maximizes the minimal distance between any two firms.

We show that the perfect equilibrium product diversity is significantly higher than the total surplus maximizing product diversity, even when fixed costs are small and the differentiated sector is nearly competitive.

APPENDIX

Proof of Lemma 1. When firm j faces the monopoly demand, all terms in the summation are zero and the result is immediate from the concavity of the profit function. When a firm faces competition by neighbors, demand is $D_j = (x_{j+1} - x_{j-1})/2 + (p_{j+1} - p_j)/[2(x_{j+1} - x_j)] + (p_{j-1} - p_j)/[2(x_j - x_{j-1})]$ so that $\partial \Pi_j/\partial p_j = D_j - (p_j - C'(D_j))(1/(x_{j+1} - x_j) + 1/(x_j - x_{j-1}))/2$, $\partial^2 \Pi_j/\partial p_j^2 = -[1/(x_{j+1} - x_j) + 1/(x_j - x_{j-1})][2 + C''(D_j)[1/(x_{j+1} - x_j) + 1/(x_j - x_{j-1})]/2]/2$, $\partial^2 \Pi_j/\partial p_j p_{j+1} = 1/[2(x_{j+1} - x_j)] + C''(D_j)[1/(x_{j+1} - x_j) + 1/(x_j - x_{j-1})]/[4(x_{j+1} - x_j)]$, $\partial^2 \Pi_j/\partial p_j p_{j-1} = 1/[2(x_j - x_{j-1})] + C''(D_j)[1/(x_{j+1} - x_j) + 1/(x_j - x_{j-1})]/[4(x_j - x_{j-1})]$, and $\partial^2 \Pi_j/\partial p_j \partial p_i = 0$ for $i \neq 1, j-1, j+1$. Therefore $\partial^2 \Pi_j/\partial p_j^2 + \sum_{j \neq i} |\partial^2 \Pi_j/\partial p_j \partial p_i| = -[1/(x_{j+1} - x_j) + (x_j - x_{j-1})]/2 < 0$. When the equilibrium is at the kink of the demand curve, equilibrium prices follow $(k - p_j)^{1/2} + (k - p_{j-1})^{1/2} = x_j - x_{j-1}$ which is equivalent to $p_j = k - (x_j - x_{j-1} - (k - p_{j-1})^{1/2})^2 = p_{j-1} + 2(x_j - x_{j-1})(k - p_{j-1})^{1/2} - (x_j - x_{j-1})^2$. Thus, $|dp_j/dp_{j-1}| = |1 - (x_j - x_{j-1})/(k - p_{j-1})^{1/2}| < 1$, and the resulting mapping is a contraction. Q.E.D.

Proof of Lemma 2. Let all firms be at symmetric positions except for firm j. Assume that $x_j = x_{j-1} + d + e$, $x_{j+1} = x_j + d - e$, and $x_i = x_{i-1} + d$ for $i \neq j, j+1$. Let $d_i = x_{i+1} - x_i$, all i. Then $d_i = d$ for all i except for $i = j$ when $d_j = d - e$ and $i = j - 1$ when $d_{j-1} = d + e$. We will show that under these assumptions profits for firm j are maximized at $e = 0$.

The first order condition for firm i is (dropping the *'s from the p's)

$$2(d_i + d_{i-1})(p_i - c) - d_{i-1}(p_{i+1} - c) - d_i(p_{i-1} - c) = (d_i + d_{i-1})(d_i d_{i-1}).$$

For any firm other than firm $j-1$, j, $j+1$, this reduces to

$$4(p_i - c) - (p_{i+1} - c) - (p_{i-1} - c) = 2d^2.$$

Letting $z = e/d$, z in $[0, 1)$, the first order conditions for firms $j-1$, j, $j+1$ are $-(1 + z)(p_{j-2} - c) + 2(2 + z)(p_{j-1} - c) - (p_j - c) = (2 + z)(1 + z)d^2$, $(z - 1)(p_{j-1} - c) + 4(p_j - c) - (1 + z)(p_{j+1} - c) = 2(1 - z^2)d^2$, $-(p_j - c) + 2(2 - z)(p_{j+1} - c) + (z - 1)(p_{j+2} - c) = (2 - z)(1 - z)d^2$.

All the first order conditions can be summarized in matrix form as $\mathbf{A}(\mathbf{p}-\mathbf{c})=\mathbf{b}$ where

$$\mathbf{A}=\begin{bmatrix} 4 & -1 & & & & & & & & & -1 \\ -1 & 4 & -1 & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & -1 & 4 & -1 & & & & & & \\ & & & -1-z & 4+2z & -1 & & & & & \\ & & & & -1+z & 4 & -1-z & & & & \\ & & & & & -1 & 4-2z & -1+z & & & \\ & & & & & & -1 & 4 & -1 & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & -1 & 4 & -1 \\ -1 & & & & & & & & & -1 & 4 \end{bmatrix}$$

and $\mathbf{b}^T=(1,\ldots,1,(1+z/2)(1+z),\ 1-z^2,\ (1-z/2)(1-z),1,\ldots,1)$. $\mathbf{A}$ can be written as $\mathbf{A}=\mathbf{U}+\mathbf{V}$ where

$$\mathbf{U}=\frac{1}{s}\begin{pmatrix} 1 & -s & & & \\ -s & 1+s^2 & -s & & \\ & \cdot & \cdot & \cdot & \\ & & -s & 1+s^2 & -s \\ & & & -s & 1 \end{pmatrix} \quad \text{with } s=2-\sqrt{3} \text{ so that}$$

$$\mathbf{U}=\begin{pmatrix} 2+\sqrt{3} & -1 & & & & \\ -1 & 4 & -1 & & & \\ & & & \ddots & & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 2+\sqrt{3} \end{pmatrix}$$

and

$$\mathbf{V}=\begin{bmatrix} s & 0 & & & & & & 0 & -1 \\ 0 & 0 & & & & & & & 0 \\ & & \ddots & & & & & & \\ & & & -z & 2z & 0 & & & \\ & & & & z & 0 & -z & & \\ & & & & & 0 & -2z & z & \\ & & & & & & & \ddots & \\ 0 & & & & & & & 0 & 0 \\ -1 & 0 & & & & & & 0 & s \end{bmatrix}.$$

The matrix $s\mathbf{U}$ has a well known factorization $s\mathbf{U}=\mathbf{M}\mathbf{M}'$ where

$$\mathbf{M}'=\begin{pmatrix} \sqrt{1-s^2} & -s & & & \\ & 1 & -s & & \\ & & \ddots & & \\ & & & 1 & -s \\ & & & & 1 \end{pmatrix}.$$

Therefore I can write $\mathbf{A}=\mathbf{U}+\mathbf{V}=(1/s)\,\mathbf{M}'\mathbf{M}+\mathbf{V}=(1/s)(\mathbf{M}'\mathbf{M}+s\mathbf{V})=(1/s)\,\mathbf{M}'(\mathbf{I}+\mathbf{M}'^{-1}s\mathbf{V}\mathbf{M}^{-1})\mathbf{M}$ so that $\mathbf{A}^{-1}=s\mathbf{M}^{-1}(\mathbf{I}+\mathbf{M}'^{-1}s\mathbf{V}\mathbf{M}^{-1})^{-1}\mathbf{M}'^{-1}$. Since the eigenvalues of $s\mathbf{M}'^{-1}\mathbf{V}\mathbf{M}^{-1}$ are less than unity, I can expand: $(\mathbf{I}+\mathbf{M}'^{-1}s\mathbf{V}\mathbf{M}^{-1})^{-1}=\sum_{k=0}^{\infty}(-1)^k(\mathbf{M}'^{-1}s\mathbf{V}\mathbf{M}^{-1})^k$, so that $\mathbf{A}^{-1}\simeq\mathbf{U}^{-1}-\mathbf{U}^{-1}\mathbf{V}\mathbf{U}^{-1}\mathbf{V}\mathbf{U}^{-1}\mathbf{V}\mathbf{U}^{-1}-\cdots$, where $\mathbf{U}^{-1}=s\mathbf{M}^{-1}\mathbf{M}'^{-1}$, i.e.,

$$\mathbf{U}^{-1}=\frac{s}{1-s^2}\begin{bmatrix} 1 & s & s^2 & \cdot & s^{n-2} & s^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s^{j-1} & s^{j-2} & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s^{n-2} & & & \cdot & 1 & s \\ s^{n-1} & & & \cdot & s & 1 \end{bmatrix}.$$

Then $\mathbf{p}-\mathbf{c}=\mathbf{A}^{-1}\mathbf{b}\simeq\mathbf{U}^{-1}\mathbf{b}-\mathbf{U}^{-1}\mathbf{V}\mathbf{U}^{-1}\mathbf{b}$.

We are interested in the jth element of the $\mathbf{p}-\mathbf{c}$ vector. I can arbitrarily select the non-symmetric firm as $j=n/2$. The jth element of $\mathbf{U}^{-1}\mathbf{b}$ is (after some computation)

$$\frac{2d^2s}{1-s^2}\left(\frac{1+s}{1-s}(1-s^{n/2})+z^2(s-1)\right).$$

Further, the elements of row j of $\mathbf{U}^{-1}\mathbf{V}\mathbf{U}^{-1}(1-s^2)^2/s^2$ are

for $1\leqslant i\leqslant j-2$,

$$y_{ji}=(s^{n-j+1}-s^{j-1})\,s^{n-i}+2zs^{j-i}-zs^{j-i+1}-2zs^{j-i+2}+zs^{j-i+3},$$

for $j+2\leqslant i\leqslant n$,

$$y_{ji}=(s^{n-j+1}-s^{j-1})\,s^{n-i}-zs^{i-j+3}+2zs^{i-j+2}+zs^{i-j+1}-2zs^{i-j}.$$

$$y_{j,j-1}=(s^{n-j+1}-s^{j-1})\,s^{n-j+1}+zs^4-2zs^3-2zs^2+2zs+z.$$

$$y_{j,j}=(s^{n-j+1}-s^{j-1})\,s^{n-j}.$$

$$y_{j,j+1}=(s^{n-j+1}-s^{j-1})\,s^{n-j-1}-zs^4+2zs^3+2zs^2-2zs-z.$$

Thus, the jth element of $\mathbf{U}^{-1}\mathbf{V}\mathbf{U}^{-1}\mathbf{b}(1-s^2)^2/s^2$ is

$$2d^2\left(\sum_{i=1}^{n} y_{ji} + y_{j,j-1}(b_{j-1}-1) + y_{j,j}(b_j-1) + y_{j,j+1}(b_{j+1}-1)\right),$$

where $b_{j-1}-1=(3z+z^2)/2$, $b_j-1=-z^2$, $b_{j+1}-1=(-3z+z^2)/2$. Now

$$\sum_{i=1} y_{ji} = s^{n/2}\left[\left(s-\frac{1}{s}\right)\left(\frac{1-s^n}{1-s}\right)+2z(-s^2+s-1)\right],$$

and

$$\begin{aligned} &y_{j,j-1}(b_{j-1}-1)+y_{j,j}(b_j-1)+y_{j,j+1}(b_{j,j+1}-1) \\ &\quad = 2d^2\left[s^n\left(\frac{3z}{2}\left(s-\frac{1}{s}\right)^2+z^2\left(\frac{1}{2}\left(s-\frac{1}{s}\right)^2-1\right)\right)+3z^2(5s^2+1)\right]. \end{aligned}$$

Utilizing the relation $s^2-4s+1=0$, we deduce

$$\begin{aligned} (p_j-c)/2d^2 &\simeq \frac{1}{2}-\frac{s}{1+s}z^2-\frac{1}{4}z^2(5s^2-1) \\ &\quad -\frac{1}{12}\left[s^{n/2}\left(\frac{s+1}{s}(s^n-1)-6sz+6\right)+s^n(18z+5z^2)\right]. \end{aligned}$$

Disregarding high powers of s we have $(p_j-c)/2d^2 \simeq \frac{1}{2}(1-\lambda z^2)$, where $\lambda = 2s/(1+s)+\frac{1}{2}(5s^2+1)$.

Equilibrium profits of firm j located at $x_j = x_{j-1}+d(1+z)$ are

$$\Pi_j^*(d,z)=\frac{\mu(p_j-c)^2}{d(1-z^2)}-F=\mu d(1-\lambda z^2)^2/(1-z^2)-F,$$

and

$$\frac{d\Pi_j^*}{dz}=\frac{2z(1-\lambda z^2)}{(1-z^2)^2}[1-2\lambda+\lambda z^2]\,\mu d^3.$$

Clearly $d\Pi_j^*/dz=0$ at $z=0$ and $d\Pi_j^*/dz<0$ if $z<\min(1/\lambda^{1/2}, ((2\lambda-1)/\lambda)^{1/2})$. Now $\lambda=(31s-5)/3 \simeq 1.1$. $\lambda>1$ implies $((2\lambda-1)/\lambda)^{1/2}>1>1/\lambda^{1/2}$ so that equilibrium profits are falling in z for $z<1/\lambda^{1/2}\simeq .99$.

Q.E.D.

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium