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# THE PRINCIPLE OF MINIMUM DIFFERENTIATION REVISITED*

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Hotelling's model of differentiated products is examined and modified so that consumers have relatively low reservation prices for the differentiated products. The problem of existence of a Nash equilibrium in prices in a duopoly with given products is analysed. When the reservation price is relatively low, a Nash equilibrium in prices exists for a larger range of products than when the reservation price is infinite. Next I examine product competition with instantaneous adjustment to the Nash equilibrium prices of the previous game. Firms competing in a Nash fashion have tendencies to move away from each other and try to achieve 'local monopolistic' positions. This is in sharp contrast with the acclaimed 'Principle of Minimum Differentiation' of the original model of Hotelling (of high reservation price) where firms clustered in the product space.

## 1. Introduction

The first important contribution to the study of oligopoly with commodities differentiated by their characteristics was made by H. Hotelling (1929). He studied duopolistic competition in a market for commodities differentiated in variety by one of their characteristics. Competition happened in two stages. In the last stage firms played a non-cooperative game in prices given the choices of varieties made in the first stage. In the first stage firms chose varieties non-cooperatively expecting to receive the Nash equilibrium profits of the last stage game.

Hotelling claimed existence of a Nash equilibrium in prices for any varieties. Next Hotelling sought equilibria in varieties which were subgame perfect (in the price subgame), and claimed that such an equilibrium must involve firms producing nearly identical varieties. Hence the 'Principle of Minimum Differentiation', and the associated suboptimal product diversity.

Recently, the model was re-examined by C d'Aspremont, J. Jaskold Gabszewicz and J.F. Thisse (1979). They point to a mistake in the original

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paper of Hotelling to prove that a Nash equilibrium in prices (given the 'locations') does not always exist. It does not exist when 'locations' (products) are relatively close, but not identical. The reason for non-existence is that for relatively close locations it pays for each competitor to undercut the opponents and capture the whole market. This is caused by the non-quasiconcavity (and not by the discontinuities) of the profit functions. We shall discuss this in detail further on. Once undercutting is optimal, no Nash equilibrium exists. Now, suppose that the market is such that firms offer quite different varieties. Nash equilibrium in prices exists and the firms tend to relocate (in the product space) closer to each other. As the varieties produced become very similar, no Nash equilibrium in prices exists and therefore nothing can be said about relocation tendencies.

The non-existence of equilibrium discovered by d'Aspremont et al. (1979) prompts a closer look at Hotelling's model. By relaxing the assumption of an infinite reservation price for the differentiated product, I derive drastically different results from the ones mentioned above. In the price game with varieties (locations) fixed, Nash equilibria exist for not-too-high reservation prices relative to the distance between products.[1] In the varieties (locations) game with instantaneous adjustment to the Nash equilibrium prices, firms tend to produce very different products. This is contrary to the acclaimed 'Principle of Minimum Differentiation'. At equilibrium both firms are local monopolists.[2] Further, the welfare of the society as measured by the total surplus is maximized at the 'local monopolistic' positions.

Section 2 describes the model. In section 3 we establish the equilibria of the price game. In section 4 we examine the equilibria of the varieties game. In section 5 we conclude.

## 2. The model

There are two kinds of commodities. A homogeneous good, $m$, and a set of differentiated commodities $C=[0,1]$. Each differentiated commodity is defined by a point in $C$. Without possible confusion point $x\in[0,1]$ will be identified with one unit of commodity $x$. Each point $x$ in $C$ can be thought of as representing the amount of the characteristic embodied in one unit of commodity $x$.

Commodity bundles can have one or none units of a differentiated commodity. A commodity bundle that has a unit of a differentiated good $x$

[1] The idea of introducing a finite reservation price was also used by Lerner and Singer (1937) and Smithies (1941).

[2] D'Aspremont et al. (1979) also demonstrate an equilibrium of the varieties game where firms produce the most diverse products possible. This occurs when the disutility of distance is quadratic in distance and the reservation price is very high. In the present paper the disutility of distance is kept linear [as in Hotelling (1929)] and a relatively not-too-high reservation price is introduced.

cannot have any units of any other differentiated good $y \neq x$. Thus consumers are restricted to buy one or none units of differentiated goods.[3]

The consumption set is $(C \cup \{v\}) \times [0, \infty)$. $(z, m) \in (C \cup \{v\}) \times [0, \infty)$. $m$ represents the amount of the homogeneous good (Hicksian composite good). $z$ represents one unit of the differentiated commodity when it lies in $C$, and no units of any differentiated commodity when $z = v$.

Consumption agent $w$ is endowed with a utility function $U_w(z, m)$.

(1) $U$ is separable: $U_w(z, m) = m + V_w(z)$.
(2) $V_w(z = v) = 0$: No utility is gained or lost if no differentiated product is consumed.
(3) $V_w(z \neq v)$ is single peaked at $z = w$. It can be written as $V_w(z) = V_w(w) - f(d(z, w))$, where $f(d)$ is increasing in $d$ and $d(z, w)$ is the distance between $z$ and $w$. Further, $f(0) = 0$. [$f(.)$ is sometimes called the 'transportation cost function' since it represents the transportation cost in the formally equivalent location model.][4]
(4) $V_w(w) = k$, for all $w$, so that the reservation price of all consumers is the same.
(5) Consumers are price takers. They maximize their utility given the commodities and prices offered in the market.

Consumers are distributed uniformly on $C$, the product space.[5] $w$ denotes the peak of the utility of consumer '$w$' in the space of characteristics.

There are two production agents (firms). Firm 1 produces differentiated commodity $x$, and firm 2 produces differentiated product $y$. The same firm is not allowed to produce more than one differentiated product so that the model is kept simple. There are no costs of production.[6] Firms are active players that select strategies to maximize an objective (profit) function. The formal games are described next.

In the price game the strategic variable of each firm is the price of its product. The products $x, y$ are given and no variation is allowed. Firm 1's (respectively 2's) objective (profit) function $\Pi_x$ (respectively $\Pi_y$) is a function of its own price $P_x$ (respectively $P_y$) with parameters the price of the opponent firm and both varieties (locations) $x, y$, i.e.,

$$\Pi_x = \Pi_x(P_x \mid P_y, x, y) \quad \text{and} \quad \Pi_y = \Pi_y(P_y \mid P_x, x, y).$$

[3]This assumption follows the tradition of Hotelling and is made in an effort to keep the results tractable.

[4]In Hotelling (1929) $f(.)$ was linear.

[5]Uniformity of the distribution of consumers is assumed for simplicity of calculations, following the tradition of Hotelling, and in our pursuit of the creation of a paradigm. The results may be sensitive to changes in the distribution of the characteristics of the consumers.

[6]Zero costs are assumed so that the emphasis of the analysis is on demand and competition rather than on production. A technology of a fixed cost plus a constant marginal cost gives very similar results.

The solution concept is the Nash equilibrium. A pair of prices $(P_x^*, P_y^*)$ is a Nash equilibrium for this game if $P_x^*$ maximizes the objective function $\Pi_x$ at the price $P_y = P_y^*$ of the opponent firm and $P_y^*$ maximizes $\Pi_y$ at price $P_x = P_x^*$, i.e., if

$$\Pi_x(P_x^* | P_y^*, x, y) \geqq \Pi_x(P_x | P_y^*, x, y), \quad \text{all } P_x,$$

$$\Pi_y(P_y^* | P_x^*, x, y) \geqq \Pi_y(P_y | P_x^*, x, y), \quad \text{all } P_y.$$

Then each firm has no incentive to change the strategy it plays at a Nash equilibrium if the opponent firm plays a Nash equilibrium strategy. Therefore, both firms have no incentive to depart alone from the Nash equilibrium.

Let $Q$ be the set of all product locations $(x, y)$ such that an equilibrium exists in the price game.

In the varieties game the strategic variable of each firm is its product (location). Let $\Pi_x^*(x, y) \equiv \Pi_x(P_x^*(x, y), P_y^*(x, y), x, y)$, $\Pi_y^*(x, y) \equiv \Pi_y(P_x^*(x, y), P_y^*(x, y), x, y)$, $(x, y) \in Q$, be the Nash equilibrium profits of the price game. In the varieties game, the objective (payoff) functions are $\Pi_x^*$, $\Pi_y^*$ defined above. That is, the payoff function of the varieties game is the Nash equilibrium payoff of the price game. The varieties game is played within the set of the Nash equilibria of the price game. The definition of the varieties game requires a unique Nash equilibrium of the price game. When there are more than one Nash equilibrium in the price game for some locations $(x, y)$ we will *select* one Nash equilibrium to be the Nash equilibrium of the game. As will be seen later *the* equilibrium selected will be the symmetric one.

The solution concept used is again the Nash equilibrium. A pair $(x^*, y^*)$ of products is the Nash equilibrium if

$$\Pi_x^*(x^*, y^*) \geqq \Pi_x^*(x, y^*), \qquad \text{all } (x, y^*) \in Q,$$

$$\Pi_y^*(x^*, y^*) \geqq \Pi_y^*(x^*, y), \qquad \text{all } (x^*, y) \in Q.$$

The two-stage game structure can be interpreted as follows: the price game is a short-run game, so that there is not time to change the product and therefore competition can occur only in prices. Also, there may be costs associated with a change in the product produced (or a change in location). The varieties game is a long-run game. Firms have observed in the short run the non-cooperative equilibrium prices when products were fixed (in the price game), and are in position to calculate parametrically the Nash equilibrium prices and profits for any product they may offer. Therefore it is reasonable that the firms will be restricted to payoffs that are Nash equilibrium payoffs of the price game. Formally, the price game can be thought of as a truncation of the varieties game. In that sense Nash equilibria of the varieties game are subgame perfect in the price subgame.

In assuming price competition and subsequent product (location) competition, we diverge from a significant collection of papers, primarily in spatial competition and voting, which assume prices fixed in some sense while there is competition in locations (products). The two-stage structure assumes an advanced level of sophistication by competitors. When both price and variety are simultaneous strategic variables, equilibrium fails to exist when there are no gaps between consumers served by competing firms. In that game there are only 'local monopolistic' Nash equilibria. [See Economides (1982b).]

## 3. Existence of Nash equilibria in the price game

Hotelling (1929) as corrected by d'Aspremont et al. (1979) established the equilibria of the price game, assuming that the reservation price was so high that all consumers bought the differentiated product at equilibrium. On the other hand intuition suggests that there will be equilibria for relatively low reservation prices such that not all consumers buy the differentiated product. For very low reservation prices each firm must be a local monopolist selling only in its neighborhood. It also seems reasonable that there are equilibria of differing types covering the two extreme cases, Hotelling's case and the 'local monopolistic' one. Here I establish all the equilibria of the price game when there are consumers at the edges of the market not served.

I start with an examination of the general demand functions for single peaked utilities in section 3.1. Starting with section 3.2 I assume linear disutility in distance in the space of characteristics. Section 3.2 establishes the result of Hotelling (and d'Aspremont et al.) in the framework of a finite reservation price. Section 3.3 establishes the existence and characterizes equilibria such that there are consumers at the edges of the market not buying any differentiated product. Sections 3.4 and 3.5 establish comparative statics results.

### 3.1. *The demand functions for general, single-peaked, symmetric utility functions*

Consider a market with two firms, each producing only one variety. Given varieties $x$ and $y$ and their respective prices $P_x$, $P_y$, each consumer $w$ decides to buy one unit from $x$ or from $y$ or not to buy at all. For simplicity we shall assume that in the choice over the above decision problem, the consumer is in all three cases inside the budget set, i.e., can afford both products.

The utility of consumer $w$ who has money $m$ when he buys one unit of commodity $x$ at price $P_x$ is: $U_w(x, m) = m + V_w(w) - f(d(x, w)) - P_x$. If the same consumer buys one unit of commodity $y$ at price $P_y$, his utility is: $U_w(y,m)=m+V_w(w)-f(d(y,w))-P_y$. On the other hand, if the consumer does not buy any of the two differentiated commodities, his utility is: $U_w^0(m)=m$. Therefore he is picking the largest of $m$, $m+V_w(w)-f(d(x,w))-P_x$, and $m+V_w(w)-f(d(y,w))-P_y$. The choice is formally equivalent to minimizing $V_w(w)$, $P_x+f(d(x,w))$, and $P_y+f(d(y,w))$. $V_w(w)$ is interpreted as the reservation price of consumer $w$. $P_x+f(d(x,w))\equiv g_x(w)$ is the utility cost to consumer $w$ of buying one unit of the differentiated commodity $x$. $P_y+f(d(y,w))\equiv g_y(w)$ is the utility cost to consumer $w$ of buying one unit of the differentiated commodity $y$. It is clear then that the optimization problem of the consumer is to pick up the commodity that minimizes his utility cost, i.e., minimizes among $V_w(w)$, $g_x(w)$, $g_y(w)$. Fig. 1 is a diagrammatic representation of the problem in a space of characteristics of dimension 1.

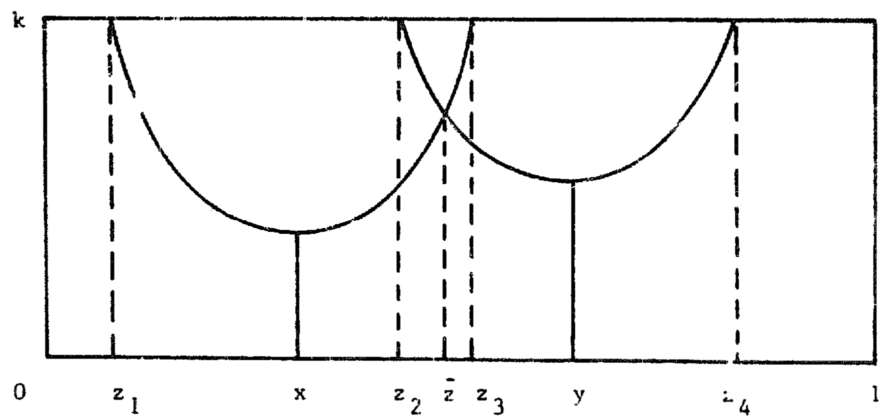


Fig. 1. Consumers' choice with convex disutility of distance.

$\bar{z}$ represents the most disadvantaged consumer between $x$ and $y$. He faces the highest utility cost among all consumers in $[x,y]$ and is indifferent between buying product $x$ or product $y$. Consumer $z_1$ is indifferent between buying product $x$ or not buying any differentiated product. Consumer $z_4$ is indifferent between buying product $y$ or not buying any differentiated product. In the particular case represented in fig. 1, all consumers with $w$ in $[z_1,\bar{z})$ will buy product $x$, all consumers with $w$ in $[\bar{z},z_4]$ will buy product $y$ and the rest, those in $[0,z_1]$ and $[z_4,1]$ will not buy any differentiated product. Points $z_1$, $z_3$ are solutions of $P_x+f(d(x,w))=k$; points $z_2$, $z_4$ are solutions of $P_y+f(d(y,w))=k$; and $\bar{z}$ satisfies $P_x+f(d(x,\bar{z}))=P_y+f(d(y,\bar{z}))$.

One of the first questions to ask concerns the nature of the similarities of the consumers that buy the same product. Is it true, for example, that closely located consumers (in terms of locations or of preferences) will tend to buy the same differentiated commodity? The answer comes out of a simple examination of the intersections of the three functions $V_w(w)=k$, $g_x(w)$, and $g_y(w)$.

First, it is clear that $g_x(w) = P_x + f(d(x, w))$ intersects $V_w(w) = k$ twice, once, or not at all, depending on whether $P_x$ is less, equal, or greater than $V_w(w)$. Correspondingly, the set $\Omega_x = \{w/V_w(w) > g_x(w)\}$ of consumers $w$ that prefer to buy one unit of $x$ at price $P_x$ to not buying the differentiated product at all is an interval, a single point, or empty.

On the other hand, the number of intersections between $g_x(w) = P_x + f(d(x, w))$ and $g_y(w) = P_y + f(d(y, w))$ can be more than one if $f$ is concave. In fig. 2 firm $x$ does not only have a demand from consumers to the left $\bar{z}_1$, but also from consumers to the right of $\bar{z}_2$. It is easy to see that concavity is required for the above phenomenon.

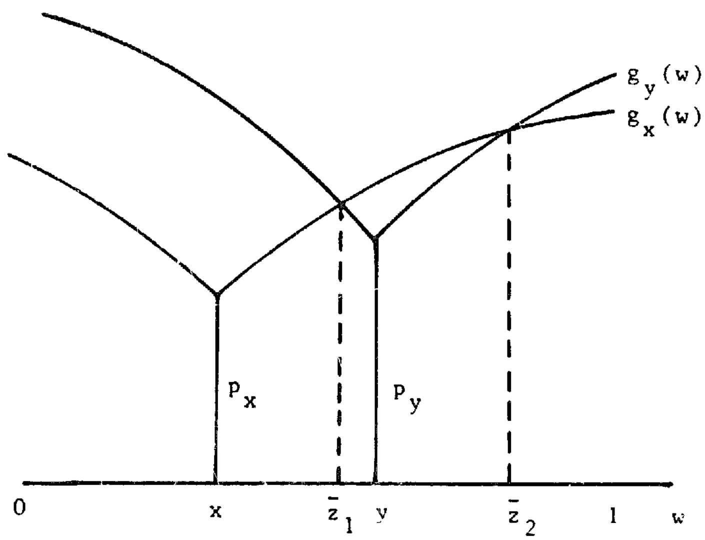


Fig. 2. Consumers' choice with concave disutility of distance.

The following proposition characterizes the nature of the similarities of the consumers that buy the same good.

*Proposition 1.* *Let there be two firms, each producing a different variety in the model of section 2. Let the transportation cost function $f(d)$ be strictly convex in distance $d$, so that the utility $V_w(z)$ is strictly concave in $|z-w|$. Then:*

(*1*) *Curves $g_x(w) \equiv P_x + f(|x-w|)$ and $g_y(w) \equiv P_y + f(|y-w|)$ intersect no more than once.*

(*2*) *The set $H_x = \{w | g_x(w) < g_y(w)\}$ of consumers $w$ that prefer to buy $x$ rather than $y$ at the going prices $P_x, P_y$, is connected. Similarly, $H_y = \{w | g_y(w) < g_x(w)\}$ is connected.*

(*3*) *Function $g_x(w)$ [respectively $g_y(w)$] intersects $V_w(w) = k$ twice, once, or not at all depending on whether $P_x$ (respectively $P_y$) is less, equal or greater than $k$.*

(4) *The set* $\Omega_x = \{w \mid V_w(w) \geq g_x(w)\}$, *of consumers that prefer to buy product* $x$ *than not to buy at all, is empty, a single point, or an interval* $[w_1, w_2]$. *Similarly for* $\Omega_y = \{w \mid V_w(w) \geq g(w)\}$.

(5) *The set of consumers* $w$ *who by product* $x \in C_x = \Omega_x \cap H_x \cap [0, 1]$ *is empty, a single point or an interval* $(z_1, z_2)$. *Similarly, the set* $C_y = \Omega_y \cap H_y \cap [0, 1]$, *of consumers* $w$ *who buy* $y$, *is empty, a single point or an interval* $(z_1, z_2)$.

When the transportation cost functions are weakly convex results (2)–(5) are also immediately established. Therefore, when the 'transportation cost function' is convex, consumers with similar preferences will buy the same product. If the 'transportation cost function' is concave, then the set of consumers that buy the same differentiated product $x$ is disconnected, a union of disjoint intervals. See Economides (1982a) for a detailed proof.

### 3.2. Existence of a Nash equilibrium in the price game when the reservation price is high

In the rest of this paper the utility will be assumed to be decreasing linearly in the space of characteristics (equivalently the transportation cost function will be linear in distance). Formally, $f(d) = d$. $V_w(z) = k - \|z - w\|$. It is clear that without loss of generality I can take the slope of $f(.)$ to be equal to one. The first result to be presented is the result of d'Aspremont et al. (1979) on existence of a Nash equilibrium in prices when the reservation price of all consumers is high and the transportation cost is linear in distance.

*Proposition 2. Assume linear transportation cost function. Let the reservation price* $k$ *be sufficiently high to induce all consumers in* $[0, 1]$ *to buy a unit of the differentiated product. Assume* $x \neq y$, *i.e., different products. A Nash equilibrium in the price game exists if and only if:* $x^2 + y^2 + 2xy - 8x + 28y - 20 > 0$, $x^2 + y^2 + 2xy - 32x + 4y + 4 > 0$, *and is described by prices:* $P_x^* = \frac{1}{3}(2 + x + y)$, $P_y^* = \frac{1}{3}(4 - x - y)$.

These conditions are fulfilled for $x$ far from $y$ and fail for close locations. However, if $x = y$, a Nash equilibrium in prices exists at prices $P_x^* = P_y^* = 0$.

Now, if each firm relocates in the commodity space according to profit incentives, assuming instantaneous adjustment of prices to Nash equilibrium levels and no movement by its competitors, the firm that produces $x$ will move towards the right and the firm that produces $y$ will move toward the left, so they will tend to move toward each other. The firms will tend to approach each other as much as possible. This is the famous 'clustering' result of Hotelling or, as it is sometimes called, the 'Principle of Minimum Differentiation'.

However, as the firms come close enough they violate the conditions for existence of a Nash equilibrium in prices. If, in the region where the price Nash equilibrium doesn't exist, the firms follow the best reply (zero conjectural variation) rules and they start at $(P_x^*, P_y^*)$ (the maximizers of the quadratic portions of their respective profit functions) then they will undercut each other.

The initial movement will be from $(P_x^*, P_y^*)$ to $(P_y^* - (y-x),\ P_x^* - (y-x))$, i.e., to lower than the candidate Nash equilibrium prices. Eventually the best reply rule will lead the firms to higher prices than the candidate Nash equilibrium prices. The game will oscillate above and below the candidate Nash equilibrium price pair.

The essential reason for non-existence of a price Nash equilibrium in the model of Hotelling is that the reservation price of all consumers is taken sufficiently high to induce all of them to buy one unit of the differentiated commodity. (This assumption does not correspond to reality, as usually differentiated commodities may be non-necessities.) It is the existence of a 'guaranteed' demand (even at reasonably high prices) up to the edges of the commodity space which creates the incentives for undercutting which shatters the existence of a Nash equilibrium in the original model of Hotelling.

### *3.3. Existence and characterization of Nash equilibria in the price game when the reservation price is not-too-high*

When the reservation price is not-too-high there will be some consumers at the edges of the market who will face a utility cost higher than their reservation price and will therefore not buy any of the differentiated commodities. This, and subsequent sections assume a relatively not-too-high reservation price so that there are consumers (at the edges of the market) who do not buy the differentiated product.

#### *3.3.1. Derivation of the demand and profit functions in the price game*

Let $z_1(P_x)$, $z_3(P_x)$, $z_1(P_x) \leqq z_3(P_x)$ be the solutions in $z$ of $k = |x-z| + P_x$. Similarly let $z_2(P_y)$, $z_4(P_y)$, $z_2(P_y) \leqq z_4(P_y)$ be the solutions in $z$ of $k = |y-z| + P_y$. See fig. 3. Then $z_1(P_x) = x + P_x - k$, $z_2(P_y) = y + P_y - k$, $z_3(P_x) = x + k - P_x$, and $z_4(P_y) = y + k - P_y$.

The condition that there are consumers at the edges of the market who are not served by either firm when prices are $(P_x, P_y)$ is translated to the following four conditions: $z_1(P_x) > 0 \Leftrightarrow k < x + P_x$, $z_3(P_x) < 1 \Leftrightarrow k < (1-x) + P_x$, $z_2(P_y) > 0 \Leftrightarrow k < y + P_y$, or $z_4(P_y) > 1 \Leftrightarrow k < (1-y) + P_y$. These four conditions are equivalent to:

$$k < \min(x + P_x,\ 1 - x + P_x,\ y + P_y,\ 1 - y + P_y).$$

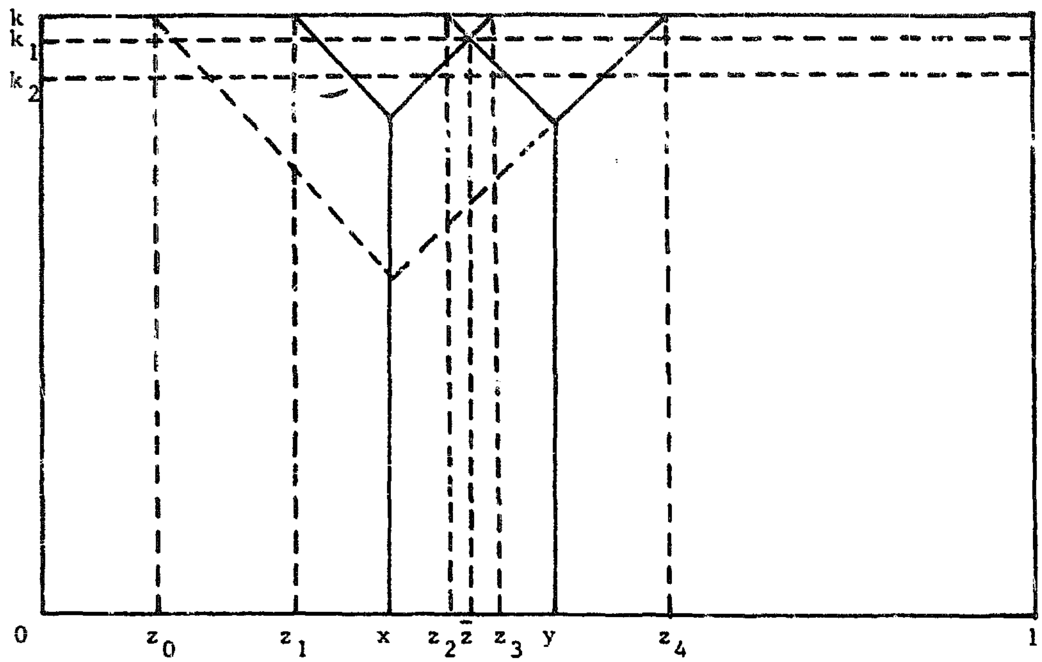


Fig. 3. Alternative equilibrium configurations: 'competitive' equilibrium for high reservation price ($k$), 'touching' equilibrium for intermediate reservation price ($k_1$), and 'local monopolistic' equilibrium for low reservation price ($k_2$).

Under the above condition and $P_y+(y-x)>k$ the demand is depicted in fig. 4.[7] The line that connects $(0,2k)$ and $(k,0)$ is the monopoly demand line. At $P_x=P_y-(y-x)$, firm $y$ enters the competition and, as discussed earlier, there is a discontinuity of the demand and profit functions at that point, since at that price all consumers to the right of $y$ are indifferent between buying $x$ or $y$. However, since there are consumers at the edges of the market who decide not to buy the commodity, the absolute jump will be smaller than if all consumers had to buy the commodity. Observe that in fig. 3, the jump is $z_4-y$ in contrast to $1-y$. Further, it is clear that the absolute jump is smaller the smaller the reservation price $k$. Also, we expect Nash equilibrium prices for the low reservation price case to be lower than the Nash equilibrium prices for the infinite reservation price case. Since the undercutting price has a fixed difference from the Nash equilibrium price, the undercutting demand will be sold at higher prices when the reservation price is high. Therefore, the incentive to undercut will be smaller when the reservation price is low.

*Definition 1*

$$\Pi_x(P_x,P_y)=2P_x(k-P_x) \quad \text{if} \quad 0\leqq P_x<P_y-(y-x),$$

[7]When $p_y+(y-x)<k<2k-p_y-(y-x)$ the second line segment of the demand function ends at $p_x=p_y+(y-x)$. For higher values of $p_x$ the demand is zero.

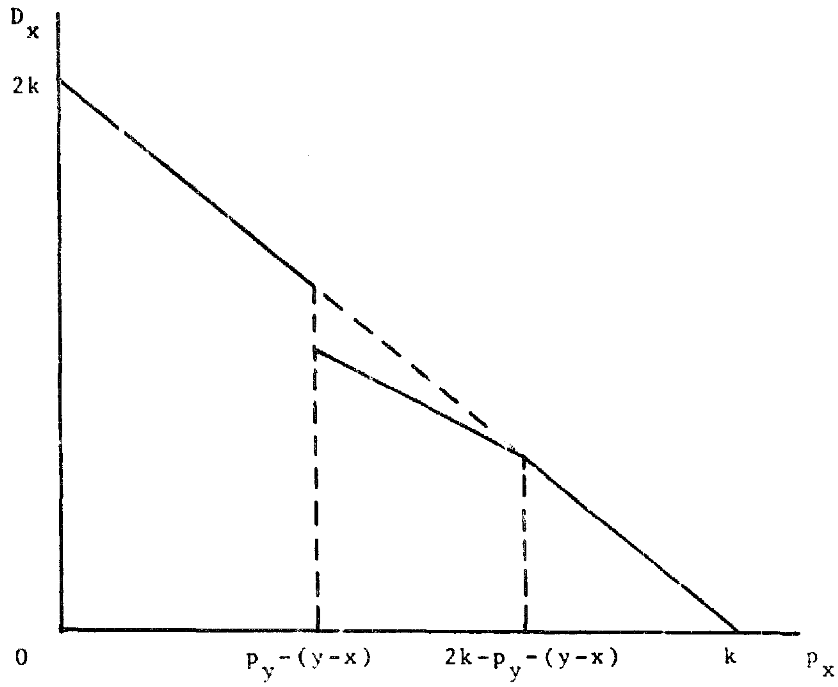


Fig. 4. The demand function.

$$\Pi_x(P_x,P_y)=P_x\left(\frac{y-x}{2}+\frac{2k-3P_x+P_y}{2}\right) \text{ if } \quad P_y-(y-x)<P_x<2k-P_y-(y-x),$$

$$=2P_x(k-P_x) \qquad \text{if} \quad 2k-P_y-(y-x)<P_x\leqq k.$$

*Definition 2*

$$\Pi_y(P_x,P_y)=2P_y(k-P_y) \quad \text{if} \quad 0\leqq P_y<P_x-(y-x),$$

$$=P_y\left(1-\frac{y-x}{2}+\frac{2k-3P_y+P_x}{2}\right) \text{ if } \quad P_x-(y-x)<P_y<2k-P_x-(y-x),$$

$$=2P_y(k-P_y) \qquad \text{if} \quad 2k-P_x-(y-x)<P_y\leqq k.$$

*3.3.2. Existence of 'local monopolistic' Nash equilibria*

The profit function is piece-wise concave and can take distinct shapes depending on the relative positions of $k/2$, $2k-P_y-(y-x)$, as explained below.

For $P_x\in(2k-P_y-(y-x),k]$ the profit function is $\Pi_x(P_x,P_y)=2P_x(k-P_x)$. In this region of prices firm $x$ does not compete with firm $y$ for the marginal consumer. $P_x^*=k/2$ is a local maximum of $\Pi_x(P_x,P_y)=2P_x(k-P_x)$ if it falls in the region $(2k-P_y-(y-x),k)$. Obviously, $k>k/2$. It is required that $2k-P_y-(y-x)<k/2$, i.e., that, $P_y>3k/2-(y-x)$. When the symmetric opponent plays $P_y^*=k/2$, this relation is equivalent to $k/2>3k/2-(y-x)\Leftrightarrow k<(y-x)$. Observe that the equilibrium point $(P_x^*=k/2,\ P_y^*=k/2)$ is unique for given $k$, $x$, $y$. I call this equilibrium 'local monopolistic' because the firms do not compete for the marginal consumer. See in fig. 3 the configuration corresponding to reservation price $k_2$.

Therefore, under the condition $k<(y-x)$, for the 'local monopolistic' case, the prices for the unique Nash equilibrium are $P_x^*=P_y^*=k/2$ and the Nash equilibrium profits are $\Pi_x^*=\Pi_y^*=k^2/2$. It is intuitively clear that such an equilibrium will exist for low enough reservation prices.

### 3.3.3. Existence of 'competitive' Nash equilibria

If $k/2$ does not fall in the region $(2k-P_y-(y-x),k]$ the 'local monopolistic' equilibrium does not exist. Then the maximizer $P_x^*$ of $\Pi_x(P_x,P_y)$ has to be lower than $2k-P_y-(y-x)$. It is clear that no Nash equilibrium can exist when $0\leqq P_x<P_y-(y-x)$ or $0\leqq P_y<P_x-(y-x)$, because then one firm has the whole market while the opponent has zero profits which it can always improve. Therefore the Nash equilibrium price $P_x^*$ has to be in $[P_y-(y-x),2k-P_y-(y-x)]\equiv[A,B]$.

We first investigate conditions under which $P_x^*$ is in $(P_y-(y-x),2k-P_y-(y-x))$. Such an equilibrium will be called a 'competitive' Nash equilibrium, since firms will be competing for the marginal consumer. See in fig. 3 the configuration corresponding to reservation price $k$.

The conditions that both firms charge prices in the region $(A,B)$ are

$$P_y-(y-x)<P_x<2k-P_y-(y-x), \tag{1}$$

$$P_x-(y-x)<P_y<2k-P_x-(y-x). \tag{2}$$

These conditions are equivalent to the following two conditions:

$$|P_x-P_y|<(y-x), \tag{3}$$

$$P_x+P_y<2k-(y-x). \tag{4}$$

The first condition implies that there will be no undercutting to drive the opponent out of business. The second condition says that the most disadvantaged consumer between $x$ and $y$, the one that is indifferent to buying from $x$ or from $y$, still prefers to buy a unit of the differentiated commodity rather than not buy at all. This establishes this case as the 'competitive' one, with competition geared towards the consumers that lie between $x$ and $y$.

Given the piecewise-concave shape of the profit function, for $P_x^*$, the local maximizer in $(A, B)$, to be the global one it is necessary to have the profits at $P_x^*$ higher than the profits at the left peak at $A$. Further, $\Pi_x = 2P_x(k-P_x)$ for $P_x$ in $(0, P_y-(y-x))$, since its maximum is at $P_x = k/2$, and $k/2 > P_y-(y-x)$ by the assumptions of this section. Finally, it can be checked [see Economides (1982a)] that $\Pi_x$ is continuous at $P_x = 2k-P_y-(y-x)$ and $\Pi_x$ is decreasing in $(2k-P_y-(y-x), k)$. Therefore, the condition on the profit peaks and (4) are necessary and sufficient for the existence of the 'competitive' Nash equilibrium.

The first-order condition prices are

$$P_x^* = P_y^* = ((y-x)+2k)/5 \tag{5}$$

and the condition on the profit peaks becomes (after some calculations)

$$k/(y-x) < (7+5\sqrt{10})/6. \tag{6}$$

At the proposed Nash equilibrium prices $(P_x^*, P_y^*)$, condition (4) becomes:

$$7/6 < k/(y-x). \tag{7}$$

Condition (3) is obviously fulfilled, since $P_x^* = P_y^*$. Therefore, under the condition

$$7/6 < k/(y-x) < (7+5\sqrt{10})/6 \tag{8}$$

we have 'competitive' Nash equilibria at prices

$$P_x^* = P_y^* = ((y-x)+2k)/5 \tag{9}$$

resulting in profits $\Pi_x^* = \Pi_y^* = (3/2)(P_x^*)^2$. Note that the 'competitive' Nash equilibrium is unique for given varieties $x, y$, and reservation price $k$.

### 3.3.4. Existence of 'touching' Nash equilibria

The derivative of the profit function at $P_x = P_x^1 \equiv 2k - P_y - (y-x)$ is discontinuous and drops as $P_x$ moves from $P_x^1 - \varepsilon$ to $P_x^1 + \varepsilon$. This creates the possibility that the point $P_x = P_x^1$ is a global maximizer of the profit function, for some values of the parameters.

At $P_x = 2k - P_y - (y-x)$ two seemingly contradictory events take place: (1) The most disadvantaged consumer between $x$ and $y$ (at $\bar{z}$) is faced with a utility cost $k$ if he buys from either one of the producers. (2) The firms are in direct competition. See in fig. 3 the configuration corresponding to reservation price $k_1$.

We call this case 'touching' since at this configuration all three alternatives yield the same utility to the consumer at $\bar{z}$. This configuration is based on the assumption that the profit function is decreasing at $B+\varepsilon$, increasing at $B-\varepsilon$, and that the peak at $B$ is higher than the peak at $A$. These assumptions are equivalent to the following conditions:

$$10k/7-(y-x)<p_x<3k/2-(y-x), \tag{10}$$

$$10k/7-(y-x)<p_y<3k/2-(y-x). \tag{11}$$

Note that the 'touching' Nash equilibrium is neither an isolated occurrence, nor a borderline case. It exists for a range of values of the parameters. For example, if $k$ is held constant there is a region of values of $x$ and $y$ such that 'touching' equilibria exist. Using the condition

$$P_x+P_y=2k-(y-x) \tag{12}$$

the above conditions imply

$$1<k/(y-x)<7/6. \tag{13}$$

An essential property of a 'touching' Nash equilibrium is that it is not uniquely determined by the reservation prices, the locations and the transportation costs. For any 3-tuple $k$, $x$, $y$ there is a continuum of equilibria that follow conditions (10)–(12).

The question arises, which 'touching' Nash equilibrium is going to be picked up by the competitors. Symmetry and Harsanyi's (1975) 'tracing procedure' suggest equal Nash equilibrium prices for both firms, that is $P_x^*=P_y^*=k-(y-x)/2$. We will assume that the symmetric Nash equilibrium $(P_x^*, P_y^*)$, with $P_x^*=P_y^*=k-(y-x)/2$ is the Nash equilibrium picked and the profits are: $\Pi_x^*=\Pi_y^*=(y-x)[k-(y-x)]/2$. Formally, given the existence of multiple Nash equilibria under (10)–(13) we define a selection that picks up *the* Nash equilibrium as the symmetric one with $P_x^*=P_y^*=k-(y-x)/2$.

### *3.3.5. Summary of results on the existence of 'competitive' 'touching' and 'local monopolistic' Nash equilibria in the price game*

It is now clear that under the condition

$$k<\min(x, 1-y) \tag{14}$$

which guarantees that the consumers at the edges of the market do not buy the commodity at zero prices, a Nash equilibrium exists for $k<(y-x)((7+5\sqrt{10})/6)$, i.e., for $k$ relatively not-too-high compared to the distance between the firms. The region $C=[0,(7+5\sqrt{10})/6]$ can be partitioned into three regions according to which kind of Nash equilibrium exists when $k/(y-x)$ falls in each of the regions.

Let $C_1=[0,1]$, $C_2=(1,7/6]$, $C_3=(7/6,(7+5\sqrt{10})/6]$. Note that $C_1$, $C_2$, $C_3$ form a partition of $C$; i.e., they are disjoint and their union is $C$.

Under (14) the following three types of equilibria exist depending on the value of $\theta=k/(y-x)$:

(I) For $\theta\in C_1$ a unique 'local monopolistic' Nash equilibrium exists at prices $P_x^*=P_y^*=k/2$.

(II) For $\theta\in C_2$ a continuum of 'touching' Nash equilibria exist. The symmetric selection is at prices $P_x^*=P_y^*=k-(y-x)/2$.

(III) For $\theta\in C_3$ a unique 'competitive' Nash equilibrium exists at prices $P_x^*=P_y^*=(2k+(y-x))/5$.

Below, given $k$, we define the set $G$ of the locations where we have existence of a Nash equilibrium. This set incorporates condition (14) and the upper bound on $\theta=k/(y-x)$.

*Definition 3.* $G(k)=\{x,y\,|\,k<\min(x,1-y,(y-x)(7+5\sqrt{10})/6)\}$.

Now the set $G$ is partitioned into the sets $G_1$, $G_2$, $G_3$. These sets are disjoint and their union is $G$. Set $G_1$ contains all locations $(x,y)$ such that a 'local monopolistic' Nash equilibrium exists. Set $G_2$ contains all locations $(x,y)$ such that 'touching' Nash equilibria exist. Set $G_3$ contains all locations $(x,y)$ such that a 'competitive' equilibrium exists.

*Definition 4.* $G_1(k)=\{x,y\,|\,k<\min(x,1-y,y-x)\}$.

*Definition 5.* $G_2(k)=\{x,y\,|\,(y-x)<k<\min(x,1-y,7(y-x)/6)\}$.

*Definition 6.* $G_3(k)=\{x,y\,|\,7(y-x)/6<k<\min(x,1-y,(7+5\sqrt{10})/6)\}$.

Theorem 1 summarizes the results on the equilibria of the price game. It essentially comes through because more emphasis is given to consumers located between the commodities offered rather than to consumers at the edges of the market.

*Theorem 1. In the duopoly game in prices, for all products $(x,y)\in G(k)$ (Definition 3) a Nash equilibrium (in prices) exists.*[8]

[8]Conditions $k<x$, $k<(1-y)$ that define the set $G$ guarantee that at zero prices there will be consumers at the edges of the market that prefer not to buy any differentiated product. These conditions are the strongest possible to guarantee that there are consumers at the edges that don't buy the differentiated product. It is possible to relax these conditions to some extent and still get existence.

(I) *For all points in* $G_1$ *(Definition 4), the Nash equilibrium is called 'local monopolistic' and occurs at prices* $P_x^* = P_y^* = k/2$.

(II) *For all points in* $G_2$ *(Definition 5), the Nash equilibrium is called 'touching' and occurs at prices* $P_x^* = P_y^* = k-(y-x)/2$.

(III) *For all points in* $G_3$ *(Definition 6), the Nash equilibrium is called 'competitive' and occurs at prices* $P_x^* = P_y^* = (2k+(y-x))/5$.

We see that if $y-x$ is large (provided that there are consumers at the edges that are not served) there exists a 'local monopolistic' Nash equilibrium. As $y-x$ is shortened, a continuum of 'touching' Nash equilibria exist. For smaller $y-x$ there exist 'competitive' Nash equilibria. For even smaller $y-x$ existence fails as it is more profitable to undercut.

The proof of Theorem 1 is intuitively clear from the discussion of existence in sections 3.3.2–3.3.4. Still some technical details have to be checked. For a detailed exposition see Economides (1982a).

The lack of existence of price equilibria for a large number of locations (products) is a significant problem. It is not clear what the appropriate price behavior of firms is when they are given locations (products) which imply non-existence of a price equilibrium. One could allow use of mixed strategies (in prices), and recent work by Dasgupta and Maskin (1982a) guarantees the existence of an equilibrium despite the discontinuities of the objective functions. However, there is a difficulty in the interpretation of mixed strategies. Further the uniqueness of a mixed strategy equilibrium (which is required for the definition of the payoff functions of the game in varieties) is far from guaranteed. Thus we limit our analysis to pure strategies.

### 3.4. *Comparative statics I: Effect of change in products on the Nash equilibrium prices in the short run*

Simple calculations on the equilibrium prices of the price game establish the following proposition:

*Proposition 3. The Nash equilibrium prices of the price game are continuous in* $(y-x)/k$ *in the region of existence of a Nash equilibrium, i.e. for* $k < \min(1-y, x, (y-x)(7+5\sqrt{10})/6)$. *Further, the Nash equilibrium prices increase linearly for* $6/(7+5\sqrt{10}) < (y-x)/k < 6/7$, *decrease linearly for* $6/7 < (y-x)/k < 1$, *and are constant for* $1 < (y-x)/k$.

This result is surprising at first glance. There are Nash equilibrium that are 'competitive' and are realized at higher prices than the 'local monopolistic' ones. Further, all 'touching' Nash equilibria, where there is essentially competition for the marginal consumer, have higher Nash equilibrium prices than the 'local monopolistic' ones. A closer look at the situation reveals that the result is not so surprising. In the 'touching' region competition is more fierce than in the 'competitive' region. This is because in the 'touching' and 'local monopolistic' regions the firms cannot increase their prices expecting all consumers located between $x$ and $y$ to continue buying. The marginal consumer who receives the same (zero) utility if he buys or does not buy a differentiated commodity (from $x$ or $y$) is reached as soon as the 'touching' case is reached. On the other hand in the 'competitive' case, all consumers between $x$ and $y$ can be expected to continue buying at slightly higher prices. While the firms remain in the 'touching' case and increase their distances they have to lower the prices to hold the marginal consumer indifferent between buying and not buying and at the same time sell to all consumers in $(x, y)$. This decrease in the price continues until the 'local monopolistic' case is reached. For locations further apart there is a gap between consumers served by $x$ and consumers served by $y$.

### *3.5. Comparative statics II: Effects of change in products on the consumers' and producers' surplus and total welfare at the Nash equilibria of the short-run (price) game*

*Definition 7.* $P(z) \equiv \min(P_x + \|z-x\|, P_y + \|z-w\|)$, the minimum (among all alternatives) utility cost to consumer $z$ of buying one unit of the differentiated commodity.

*Definition 8.* The consumers' surplus $CS$ in the market for differentiated products is the sum of the surpluses that all consumers appropriate in the market by buying the differentiated product. Formally,

$$CS(P_x, P_y, x, y) \equiv \int_0^1 \max(k - P(z), 0)\,dz.$$

*Definition 9.* The producers' surplus $PS$ in the market for differentiated products is the sum of the profits of the firms:

$$PS(P_x, P_y, x, y) \equiv \Pi_x(P_x, P_y) + \Pi_y(P_x, P_y).$$

*Definition 10.* The total surplus $TS$ is the sum of the consumers and the producers surplus: $TS(P_x, P_y, x, y) \equiv CS(P_x, P_y, x, y) + PS(P_x, P_y, x, y)$.

*Definition 11.* $CS^* \equiv CS(P_x^*, P_y^*)$, $PS^* \equiv PS(P_x^*, P_y^*)$, $TS^* \equiv TS(P_x^*, P_y^*)$, where $P_x^*$, $P_y^*$ are the Nash equilibrium prices, where the Nash equilibrium exists and is unique. In the 'touching' case where there are multiple Nash equilibria, the symmetric Nash equilibrium is selected and total surplus, $TS^*$, is defined for that equilibrium.

Detailed analysis of function $TS^*$ reveals that it is maximized at a 'local monopolistic' Nash equilibrium. Within the set of 'competitive' Nash equilibria $TS^*$ is maximized at the marginal point between the 'competitive' and the 'touching' cases. Further, $TS^*$ increases with the distance between the firms within the set of 'touching' equilibria, and is maximal at the marginal point between the 'touching' and the 'local monopolistic' cases. All 'local monopolistic' equilibria have the same surplus. Therefore $TS^*$ is maximized at the 'local monopolistic' configuration. It is evident that this result comes through because the producers' surplus is considerably larger than the consumers' surplus, and dominates their sum. See Economides (1982a) for a detailed proof.

*Theorem 2. Total surplus at the Nash equilibria $TS^*(x, y)$ is maximized at a 'local monopolistic' Nash equilibrium as products $x, y$ vary.*

## 4. Nash equilibrium existence in the varieties game

### *4.1. Marginal relocation tendencies at the Nash equilibria of the price game*

Hotelling's clustering intuition was based on three effects that the high reservation price, forcing all consumers to buy the differentiated product, had on the market:

(1) It created enclaves of consumers at the edges of the market where the local firm had a monopoly for a large range of prices.
(2) Sometimes (when the firms were relatively close) it was optimal for each firm to undercut its opponent, therefore capturing the local enclave of the opponent.
(3) Provided that undercutting was not the best reply, i.e., assuming that a Nash equilibrium existed in prices, since the competition was essentially for the consumers located in-between firms, it was more profitable to locate closer to the opponent because this not only increased the local monopoly enclave of the firm but also increased the demand by consumers located in-between the firms. (Of course, the side effect of this process was that, as the local enclaves became large, the incentive to undercut increased, and eventually there was no Nash equilibrium.)

On the other hand, if the reservation price is not-too-high:

(1) There are consumers at the edges of the market who prefer not to buy the differentiated commodity.
(2) The monopoly power of the local firm over consumers that are in the local 'enclave' is lowered, since the firm has to compete in price to gain their business.
(3) Since the local enclaves are smaller, the incentives to undercut are

lowered, and therefore there are better chances for existence of Nash equilibrium.

(4) Competition is easier in the enclaves than at the region between the firms. Hence firms will make higher profits (at the Nash equilibrium prices) if they are located closer to the edges of the market. As long as there are consumers at the edges of the market that do not buy the product, firms have some incentive to move towards the edges.

*Definition 12.* For $(x,y)\in G(k)$, let $\Pi_x^*(x,y)\equiv \Pi_x(P_x^*,P_y^*)$, $\Pi_y^*(x,y)\equiv \Pi_y(P_x^*,P_y^*)$ be the payoff functions of the varieties game, where $P_x^*=P_x^*(x,y)$, $P_y^*=P_y^*(x,y)$ denote the Nash equilibrium prices of the price game, which are:

$$P_x^*=P_y^*=k/2 \qquad \text{if} \quad k/(y-x)<1,$$

$$P_x^*=P_y^*=k-(y-x)/2 \qquad \text{if} \quad 7/6>k/(y-x)>1,$$

$$P_x^*=P_y^*=(2k+(y-x))/5 \qquad \text{if} \quad (7+5\sqrt{10})/6>k/(y-x)>7/6.$$

We consider marginal changes in the position of each firm assuming no reaction by the opponent. We call the direction of increase of $\partial\Pi_x^*/\partial x$ $(\partial\Pi_y^*/\partial y)$ 'the marginal relocation tendency of firm 1 (2)'.

In the 'local monopolistic' region, i.e., for $2k-P_y-(y-x)<P_x\leqq k$, profits are $\Pi_x(P_x^*,P_y^*)=\Pi_y(P_x^*,P_y^*)=k^2/2$. Then $\partial\Pi_x^*/\partial x=\partial\Pi_y^*/\partial y=0$, and the marginal relocation tendency is zero, except if the relocation causes the firms to fall outside the region of 'local monopoly'.

In the 'competitive' region, i.e., for $|P_x-P_y|<y-x$ and $P_x+P_y<2k-(y-x)$, we have $\Pi_x(P_x^*,P_y^*)=3(P_x^*)^2/2$, $\Pi_y(P_x^*,P_y^*)=3(P_y^*)^2/2$. Then, $\partial\Pi_x^*/\partial x=3P_x^*(\partial P_x^*/\partial x)=-\frac{3}{5}P_x^*<0$, and $\partial\Pi_y^*/\partial y=3P_y^*(\partial P_y^*/\partial y)=\frac{3}{5}P_y^*>0$. Therefore, firms have incentives to relocate marginally away from each other and reach the 'touching' region.

For the selection of the symmetric Nash equilibria of the 'touching' type, i.e., for $1<k/(y-x)<7/6$, profits are $\Pi_x(P_x^*,P_y^*)=\Pi_y(P_x^*,P_y^*)=(y-x)[k-(y-x)/2]$. Then $\partial\Pi_x^*/\partial x=-k+(y-x)<0$, $\partial\Pi_y^*/\partial y=k-(y-x)>0$. Therefore firms have incentives to relocate away from each other and reach the 'local monopolistic' region. Proposition 4 clearly follows:

*Proposition 4. Let $\Pi_x^*(x,y)\equiv\Pi_x(P_x^*,P_y^*)$, $\Pi_y^*(x,y)\equiv\Pi_y(P_x^*,P_y^*)$ where $P_x^*,P_y^*$ denote the Nash equilibrium profits of the price game. Then $\partial\Pi_x^*/\partial x<0$, $\partial\Pi_y^*/\partial y>0$, for $y-x<k<\min(x,1-y,(7+5\sqrt{10})/6(y-x))$ (i.e., in the 'competitive' and 'touching' cases), and $\partial\Pi_x^*/\partial x=\partial\Pi_y^*/\partial y=0$, for $0\leqq k<\min(x,1-y,y-x)$ (i.e., in the 'local monopolistic' case).*

Inequality $k<\min(x,1-y)$ guarantees that consumers at the edges of the market prefer not to buy the differentiated commodity. We see that the marginal relocation tendencies are opposite to the ones proved by Hotelling in the model where the reservation price was infinite. Note that this result does not contradict the fact that the Nash equilibrium prices decrease in the distance $y-x$ in the 'touching' case. As $y-x$ increases in the 'touching' case and prices simultaneously decrease, the expansion of the demand that accompanies the moves overcompensates the price drop and the end result is that Nash equilibrium profits increase in $y-x$.

### 4.2. Nash equilibrium existence in the varieties game

As was discussed in section 4.1 firms have incentives to relocate marginally away from each other if they expect their opponents not to react in terms of locations and the Nash equilibrium in prices to be established at the new locations. This is true for all Nash equilibria of the price game of the 'competitive' and 'touching' type.

One can envisage this process as taking place over time. Say, originally the firms are at a 'competitive' Nash equilibrium in prices. Then they are allowed to relocate marginally and are given the Nash equilibrium profits of the new locations. By Proposition 4 they will relocate away from each other. This will increase the distance between them, and will eventually lead them outside the region of existence of a 'touching' Nash equilibrium. The relocation tendencies are the same for a 'touching' Nash equilibrium. Over time, firms can be thought to follow relocation tendencies and move away from each other. This will increase their distance even more, and will lead them away from the region of existence of a 'touching' equilibrium into the region of existence of a 'local monopolistic' equilibrium. Once at a 'local monopolistic' equilibrium there are no incentives to move. Therefore one would expect over time this firm to end at an equilibrium at the boundary between the 'touching' and the 'local monopolistic' cases, which essentially is a 'local monopolistic' equilibrium. A firm that will start from a 'local monopolistic' position will have no incentive to move. Therefore, one expects in the long run all firms to be (or try to be) at 'local monopolistic' positions.

The varieties game formalizes long-run competition among firms. Firms compete in location where payoffs are the Nash equilibrium profits of a price game if this Nash equilibrium exists, and zero otherwise. This guarantees that the payoff function is well-defined everywhere, and that firms will not choose to produce varieties which result in non-existence of equilibrium in the price game.

*Definition 13*

$$\Pi_x^s(.) = \Pi_x^*(.) \quad \text{if} \quad (x, y) \in G(k),$$

$$\Pi^s_x(.) = 0 \quad \text{if} \quad (x, y) \notin G(k)$$

and similarly for $\Pi^s_y(.)$.

The following theorem establishes the existence of an equilibrium in the varieties game. At the equilibrium of the varieties game, the short-run price equilibrium is a 'local monopolistic' one.

*Theorem 3. In the varieties (long-run) game, a (perfect) Nash equilibrium exists if there exist $x, y, 0 < x < y < 1$, such that $k < \min(x, 1-y, y-x)$.*

*Proof.* Consider an outcome of the 'competitive' type $(x, y, P^*_x, P^*_y)$ of the product game with correct anticipation of the Nash equilibrium in prices. By Proposition 4, it is superior for firm 1 to move to point $x'$, where $x' < x$, provided that firm 2 remains at $y$. Clearly then any 'competitive' type outcome cannot be a Nash equilibrium in the product game.

Also consider an outcome of the 'touching' type in the same product game. By Proposition 4 it is superior for firm $x$ to move to $x'$, where $x' < x$, provided that firm 2 remains at $y$. Clearly then any 'touching' type outcome cannot be a Nash equilibrium in the product game.

All outcomes in the product game that are price Nash equilibria of the 'local monopolistic' type give the same profits irrespective of the locations.[9] Therefore these profits cannot be increased. Then it is clear that as long as locations fulfill the condition $k < \min(x, 1-y, y-x)$ which guarantees the 'local monopolistic' configuration, there is no incentive for any firm to change location.

Thus the long-run equilibrium will be a local monopolistic one. If there is a fixed cost in the production technology, the long-run equilibrium may involve zero profits and invite no entry. To tackle the problem of entry in the general case one may add another stage of competition, preceding the varieties stage, where firms choose whether or not to enter in the industry. Then the varieties game is played given the number of firms that have entered. An analysis of Hotelling's problem when there are more than two firms in the market can be found in Economides (1982e). There it is shown that long-run equilibrium existence is not guaranteed when there are three or more firms in the industry. Nevertheless, this seems to be a result of the limitation of the choice of firms imposed by the one-dimensional location space rather than a general result of price-location models.

## 5. Summary and concluding remarks

This research was done in the framework of Hotelling's (1929) model for

[9]These profits (and the prices at which they are attained) are the same as the ones that would result if there was only one firm in the market for differentiated products.

differentiated products defined by their characteristics. Consumers were endowed with single-peaked utility functions in the space of characteristics and were distributed uniformly in the product space according to the position of the peak of their utility functions. Given the utility functions, the problem of utility maximization for the consumers is equivalent to cost minimization in a base-pricing model where the transportation cost is born by the buyers.

When the reservation price is very high, d'Aspremont et al. (1979) have proved that the duopolistic price game (of competition in prices given fixed locations of the firms) there are locations (products) such that no Nash equilibrium exists. When the reservation price is not-too-high so that there are consumers at the edges of the market that prefer not to buy any differentiated product at low prices, I proved that the region of existence of equilibria in the price game is extended. There are three types of equilibria. For relatively high reservation prices only 'competitive' Nash equilibria exist, where all consumers between the locations of the firms are served and all are strictly better off by buying a differentiated product. For lower reservation prices in addition to 'competitive' equilibria there are also (for different locations) 'touching' equilibria where all consumers between the firms are served and the most disadvantaged consumer between the locations of the firms is indifferent between buying a differentiated product and not buying at all. For even lower reservation prices in addition to the above equilibria there are also (for different locations) 'local monopolistic' Nash equilibria, such that there are consumers between the firms who are not served.

In the varieties game (of variety competition within the set of Nash equilibria of the price game) Hotelling predicted that, for an infinite reservation price, marginal relocations (following the direction of increasing profits) will bring the firms closer together. This was the acclaimed 'Principle of Minimum Differentiation'. For a not-too-high reservation price (so that the Nash equilibria of the price game exist) I prove that the varieties game's marginal relocations (following the direction of increasing profits) tend to drive firms far apart and towards a 'local monopolistic' configuration. A Nash equilibrium of the varieties game exists if the reservation price is not-too-high. It is at a 'local monopolistic' configuration with the firms choosing to produce very different products.[10] The essential reason for this result is that, for relatively not-too-high reservation prices, firms are not guaranteed the purchases of the consumers who are located near the edges of the market.

[10] This is not the only Nash equilibrium of the second stage game, although it is the only one where all the market is not covered by the firms. There are other equilibria of the second stage game where all the market is covered. These equilibria are characterized by the fact that consumers at the edges of the market buy the differentiated product but differ from Hotelling's equilibria in competition in the middle of the market.

Social welfare as measured by total surplus is maximized at the non-cooperative equilibrium of the varieties game where firms are local monopolists. This is a direct result of the domination of total surplus by profits.

In conclusion we note that the general idea of 'clustering' that has been pretty generally accepted since proposed by Hotelling (1929) is contradicted by the results of this research. The tendency to move far apart is discovered in the varieties game. The problems of non-existence of an equilibrium in the one-characteristic game in prices noted by d'Aspremont et al. (1979) diminish when the reservation price is not-too-high.

This paper points in the direction of non-robustness of the non-existence result of equilibrium in the price game. Indeed, the author has found that in almost all variations of Hotelling's model price equilibria exist. In Economides (1982d) the existence of equilibrium (in the price game) when there are three or more firms in the market is investigated. Two-sided competition by most firms results in lower prices, limited gains by undercutting and guaranteed existence of equilibrium. In a two-dimensional analogue of Hotelling's model, an equilibrium in prices exists at least for all symmetric locations [Economides (1982c)]. Variation of utility functions so that they are more convex in the distance in the space of characteristics also results in a wider range of existence of equilibrium in the price game [d'Aspremont et al. (1979) and Economides (1982e)]. Therefore the non-existence problem does not seem to be as essential as it previously seemed. On the other hand, none of the results points towards 'Minimum Differentiation' in the long run. On the contrary, the present paper points toward maximal (profitable) differentiation.

## References


D'Aspremont, C., J. Jaskold Gabszewicz and J.F. Thisse, 1979, On Hotelling's 'Stability in Competition', Econometrica 47, 1145–1150.

Cournot, A., 1960, Researches into the mathematical principles of the theory of wealth, translated by N.T. Bacon (Kelley, New York).

Dasgupta, P. and E. Maskin, 1982a, The existence of equilibrium in discontinuous economic games, 1: Theory, Discussion paper 82/54 (ICERD, London School of Economics, London).

Dasgupta, P. and E. Maskin, 1982b, The existence of equilibrium in discontinuous economic games, 2: Applications, Discussion paper 82/55 (ICERD, London School of Economics, London).

Dixit, A. and J. Stiglitz, 1977, Monopolistic competition and optimum product diversity, American Economic Review 67, 297–308.

Eaton, B.C. and R. Lipsey, 1975, The principle of minimum differentiation reconsidered: Some new developments in the theory of spatial competition, Review of Economic Studies 42, 27–50.

Economides, N.S., 1981, Oligopoly in markets for products differentiated by their characteristics, unpublished Ph.D. dissertation (University of California, Berkeley, CA).

Economides, N.S., 1982a, The principle of minimum differentiation revisited, Discussion paper no. 134 (Columbia University, New York).

Economides, N.S., 198 b, Nash equilibrium in oligopolistic competition in prices and varieties, Discussion paper no. 150 (Columbia University, New York).

Economides, N.S., 1982c, Nash equilibrium in duopoly with products defined by two characteristics, Discussion paper no. 152 (Columbia University, New York).
Economides, N.S., 1982d, Oligopoly in differentiated products with three or more competitors, Discussion paper no. 153 (Columbia University, New York).
Economides, N.S., 1982e, Minimal and maximal product differentiation, manuscript.
Economides, N.S., 1983a, Symmetric equilibrium existence and optimality in differentiated products markets, Discussion paper no. 197 (Columbia University, New York).
Economides, N.S., 1983b, Existence of equilibrium in quality competition, Discussion paper no. 199 (Columbia University, New York).
Fetter, F., 1924, Economic law of market areas, Quarterly Journal of Economics 38, 520–529.
Friedman, J., 1977, Oligopoly and the theory of games (North-Holland, Amsterdam).
Harsanyi, 1975, The tracing procedure: A Bayesian approach to defining a solution for n-person non-cooperative games, International Journal of Game Theory 4, no. 2, 61–94.
Hotelling, H., 1929, Stability in competition, The Economic Journal 39, 41–57.
Lancaster, K., 1979, Variety, equity and efficiency (Columbia University Press, New York).
Lerner, A. and H.W. Singer, 1937, Some notes on duopoly and spatial competition, Journal of Political Economy 45, 145–186.
Novshek, W., Equilibrium in simple spatial (or differentiated product) models, Journal of Economic Theory 22, no. 2, 313–326.
Salop, S., 1979, Monopolistic competition with outside goods, Bell Journal of Economics 10, no. 1, 141–156.
Selten, R., 1975, A reexamination of the perfectness concept in extensive game forms, International Journal of Game Theory 4, 25–55.
Shaked, A. and J. Sutton, 1982, Relaxing price competition through product differentiation, Review of Economic Studies 49, 3–14.
Smithies, A., 1941, Optimal location in spatial competition, Journal of Political Economy, 423–439.
Spence, M., 1976, Product selection, fixed cost and monopolistic competition, Review of Economic Studies 43, 217–236.
Zeuthen, F., 1933, Theoretical remarks on price theory: Hotelling's case with variations, Quarterly Journal of Economics 49, 231–253.