

ON HOTELLING'S "STABILITY IN COMPETITION"

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THE PURPOSE OF THIS NOTE is to show that the so-called *Principle of Minimum Differentiation*, as based on Hotelling's 1929 celebrated paper (Hotelling [3]), is invalid. Firstly, we assert that, contrary to the statement formulated by Hotelling in his model, nothing can be said about the tendency of both sellers to agglomerate at the center of the market. The reason is that no equilibrium price solution will exist when both sellers are not far enough from each other. Secondly, we consider a slightly modified version of Hotelling's example, for which there exists a price equilibrium solution everywhere. We show however that, for this version, there is a tendency for both sellers to maximize their differentiation. This example thus constitutes a counterexample to Hotelling's conclusions.

We shall first recall Hotelling's model and notations. On a line of length l , two sellers A and B of a homogeneous product, with zero production cost, are located at respective distances a and b from the ends of this line ($a + b \leq l$; $a \geq 0, b \geq 0$). Customers are evenly distributed along the line, and each customer consumes exactly a single unit of this commodity per unit of time, irrespective of its price. Since the product is homogeneous, a customer will buy from the seller who quotes the least delivered price, namely the mill price plus transportation cost, which is assumed linear with respect to the distance. Let p_1 and p_2 denote, respectively, the mill price of A and B and let c denote the transportation rate.

The situation described above gives rise to a two-person game with *players* A and B , *strategies* $p_1 \in S_1 = [0, \infty[$, and $p_2 \in S_2 = S_1$, and *payoff functions* given by the profit functions:

$$\begin{aligned} \pi_1(p_1, p_2) &= ap_1 + \frac{1}{2}(l - a - b)p_1 + \frac{1}{2c}p_1p_2 - \frac{1}{2c}p_1^2, && \text{if } |p_1 - p_2| \leq c(l - a - b); \\ &= lp_1, && \text{if } p_1 < p_2 - c(l - a - b); \\ &= 0, && \text{if } p_1 > p_2 + c(l - a - b); \\ \pi_2(p_1, p_2) &= bp_2 + \frac{1}{2}(l - a - b)p_2 + \frac{1}{2c}p_1p_2 - \frac{1}{2c}p_2^2, && \text{if } |p_1 - p_2| \leq c(l - a - b); \\ &= lp_2, && \text{if } p_2 < p_1 - c(l - a - b); \\ &= 0, && \text{if } p_2 > p_1 + c(l - a - b). \end{aligned}$$

The profit function of seller A is illustrated in Figure 1 for a fixed value \bar{p}_2 .

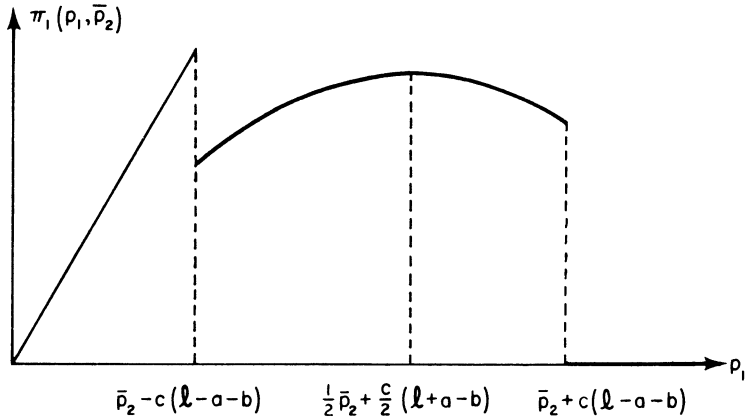


FIGURE 1

Clearly a particular feature of these profit functions is the presence of two discontinuities which appear at the price where a whole group of buyers is indifferent between the two sellers.

A strategy p_1 of player A is a *best reply* against a strategy p_2 of player B when it maximizes $\pi_1(\cdot, p_2)$ on the whole S_1 for the given p_2 . Similarly for player B . A *Nash-Cournot equilibrium point* is a pair (p_1^*, p_2^*) such that p_1^* is a best reply against p_2^* and vice-versa.

In the following proposition we shall treat the problem of existence of such an equilibrium for every location a and b . More specifically, we shall derive necessary and sufficient conditions on a and b for such an equilibrium to exist, and compute all equilibrium points.

PROPOSITION: *For $a + b = l$, the unique equilibrium point is given by $p_1^* = p_2^* = 0$. For $a + b < l$, there is an equilibrium point if, and only if*

$$(1) \quad \left(l + \frac{a-b}{3} \right)^2 \geq \frac{4}{3}l(a+2b),$$

$$(2) \quad \left(l + \frac{b-a}{3} \right)^2 \geq \frac{4}{3}l(b+2a),$$

and, whenever it exists, an equilibrium point is uniquely determined by

$$(3) \quad p_1^* = c \left(l + \frac{a-b}{3} \right),$$

$$(4) \quad p_2^* = c \left(l - \frac{a-b}{3} \right).$$

PROOF: The case $a + b = l$ is immediate. Then both sellers are located at the same place and, as in Bertrand [1], there always exists an equilibrium uniquely

determined by $p_1^* = p_2^* = 0$. So let $a + b < l$. We shall begin by showing that any equilibrium must satisfy the condition $|p_1^* - p_2^*| < c(l - a - b)$.

Suppose first on the contrary that (p_1^*, p_2^*) is an equilibrium but $|p_1^* - p_2^*| > c(l - a - b)$. Then, one of the two sellers—the one who charges the strictly larger (and hence positive) price—gets a null profit and so may gain by charging a positive price equal to the delivered price of the other. But this contradicts the fact that (p_1^*, p_2^*) is an equilibrium. Suppose now that $|p_1^* - p_2^*| = c(l - a - b)$, say, for instance, $p_2^* - p_1^* = c(l - a - b)$. If $p_1^* = 0$, then the profit of A is zero and so he would profit by charging a positive price less than $p_2^* + c(l - a - b)$. If $p_1^* > 0$, two cases may arise. Either A gets the whole market and so B , who charges a positive price, can increase his profit by decreasing his price. Or A gets only a fraction of the market, i.e., $q_1 < l$, and it is then sufficient for A to charge a slightly lower price to capture the whole market and make a larger profit: indeed for $0 < \varepsilon < (l - q_1)p_1^*/l$ we have $\pi_1(p_1^* - \varepsilon, p_2^*) = l(p_1^* - \varepsilon) > q_1 p_1^* = \pi_1(p_1^*, p_2^*)$. In any case we always get a contradiction. Accordingly any equilibrium (p_1^*, p_2^*) must satisfy the condition $|p_1^* - p_2^*| < c(l - a - b)$.

A consequence of this condition is that, for any equilibrium (p_1^*, p_2^*) , p_1^* must maximize $ap_1 + \frac{1}{2}(l - a - b)p_1 + (1/2c)p_2^*p_1 - (1/2c)p_1^2$ in the open interval $]p_2^* - c(l - a - b), p_2^* + c(l - a - b)[$, and similarly for p_2^* . Taking first order conditions we get (3) and (4). Hence, we shall now verify that the pair of prices given by (3) and (4) is indeed an equilibrium. Recall that to be an equilibrium strategy p_1^* must maximize $\pi_1(p_1, p_2^*)$ not only in the above interval but on the whole domain S_1 , and similarly for p_2^* . Let us see that this is true only on a restricted set of possible locations. Indeed, given a and b , for p_2^* to be an equilibrium strategy against p_1^* , we must have in particular that, for any $\varepsilon > 0$,

$$(*) \quad \pi_1(p_1^*, p_2^*) = \frac{c}{2} \left[l + \frac{a-b}{3} \right]^2 \geq l[p_2^* - c(l - a - b) - \varepsilon].$$

The right hand side of the inequality is the profit of A , should he quote a delivered price slightly smaller than p_2^* . But condition (*) can be rewritten as (1). By symmetry we get condition (2).

To show that conditions (1) and (2) are also sufficient for (p_1^*, p_2^*) to be an equilibrium it remains only to check that they imply $|p_1^* - p_2^*| < c(l - a - b)$. This completes the proof of our proposition.

Note in passing that if we consider only symmetric locations around the center ($a = b$), then the necessary and sufficient conditions (1) and (2) reduce to $a = b \leq l/4$. In other words, both the duopolists must be located outside the quartiles to get a Cournot equilibrium in prices.

If conditions (1) and (2) are strictly verified, then, as noted by Hotelling, both $\partial\pi_1(p_1^*, p_2^*)/\partial a$ and $\partial\pi_2(p_1^*, p_2^*)/\partial b$ are strictly positive, which implies a tendency of both sellers towards the center. But a major consequence of the preceding proposition is that, as far as the Cournot equilibrium is taken as the market solution, nothing can be said on this solution when conditions (1) and (2) are violated. Hotelling seems to be unaware of this difficulty while deriving the

implications of his model, and in particular the tendency of both sellers to agglomerate at the center of the market.¹ Indeed should conditions (1) and (2) be violated, i.e., should the firms be located relatively close to each other, the Cournot equilibrium could no longer serve as a reference point since it no longer exists!²

Having reached this negative outcome, it seems natural to work out an example, which is as close as possible to Hotelling's one, but avoiding the difficulty exhibited above.³ If, for this alternative example, the principle of minimal differentiation could be retrieved, the defect in Hotelling's argumentation would be immaterial. Unfortunately, this principle is invalidated by the following reexamination.

A slightly modified version of Hotelling's example for which there exists a price equilibrium solution for *any* pair of locations (a, b) obtains if, in place of considering linear transportation costs we assume that these costs are quadratic with respect to the distance, i.e., for any distance x , transportation costs are given by cx^2 . Under this assumption, an easy computation leads to the following expressions for the demand and profit functions:

$$\begin{aligned}
 q_1(p_1, p_2) &= a + \frac{p_2 - p_1}{2c(l - a - b)} + \frac{l - a - b}{2}, \\
 &\quad \text{if } 0 \leq a + \frac{p_2 - p_1}{2c(l - a - b)} + \frac{l - a - b}{2} \leq l; \\
 &= l, \quad \text{if } a + \frac{p_2 - p_1}{2c(l - a - b)} + \frac{l - a - b}{2} > l; \\
 &= 0, \quad \text{if } a + \frac{p_2 - p_1}{2c(l - a - b)} + \frac{l - a - b}{2} < 0; \\
 q_2(p_1, p_2) &= b + \frac{p_1 - p_2}{2c(l - a - b)} + \frac{l - a - b}{2}, \\
 &\quad \text{if } 0 \leq b + \frac{p_1 - p_2}{2c(l - a - b)} + \frac{l - a - b}{2} \leq l; \\
 &= l, \quad \text{if } b + \frac{p_1 - p_2}{2c(l - a - b)} + \frac{l - a - b}{2} > l; \\
 &= 0, \quad \text{if } b + \frac{p_1 - p_2}{2c(l - a - b)} + \frac{l - a - b}{2} < 0;
 \end{aligned}$$

¹ In footnote (8) of his paper, Hotelling remarks however that, for some values of a and b , the pair of prices defined by (3) and (4) cannot be an equilibrium, but proposes then another pair of prices as an equilibrium. By our proposition, we know that they are not. It seems that Hotelling has neglected to consider strategies through which a merchant undercuts the delivered price of the other, and attracts to him the whole market. These strategies are particularly advantageous when both merchants are close to each other.

² Here we only consider equilibrium with price strategies. However, it is easily verified that if each seller's strategy is a price location pair, which has to be chosen simultaneously, then again no Nash equilibrium exists.

³ This example is particularly illustrative in regard to footnote (9) of Hotelling's paper.

$\pi_1(p_1, p_2) = p_1 \cdot q_1(p_1, p_2)$ and $\pi_2(p_1, p_2) = p_2 \cdot q_2(p_1, p_2)$. These profit functions ensure the existence of a price equilibrium, *whatever the locations a and b may be*. It is indeed easily checked that the pair of prices (p_1^*, p_2^*) defined by

$$(5) \quad p_1^* = c(l - a - b) \left(l + \frac{a - b}{3} \right),$$

$$(6) \quad p_2^* = c(l - a - b) \left(l + \frac{b - a}{3} \right),$$

is the unique Nash–Cournot equilibrium point for fixed a and b , and that this is true without any condition on these location parameters. We verify however that, if we substitute these equilibrium prices in the profit functions of both players, both $\partial\pi_1(p_1^*, p_2^*)/\partial a$ and $\partial\pi_2(p_1^*, p_2^*)/\partial b$ are negative! Consequently, at any given pair of locations, each merchant gains an advantage from moving away as far as possible from the other.⁴

The preceding example, far from confirming the minimal differentiation principle, suggests that this principle cannot be based on spatial competition. Certainly many comments derived from Hotelling's contribution should be carefully reexamined before taking them as granted. The outcome of this note should not however be considered as too negative. Indeed, although Hotelling's example suggested the contrary, one should expect intuitively that product differentiation must be an important component of oligopolistic competition. It seems to be clear that oligopolists should gain an advantage by dividing the market into submarkets in each of which some degree of monopoly would reappear.⁵ But this important subject would need more imagination.

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⁴ In other terms, for the game where the strategies are the locations and the payoff functions the profits $\pi_1(p_1^*(a, b), p_2^*(a, b))$ and $\pi_2(p_1^*(a, b), p_2^*(a, b))$ —which can be viewed as a sequential game where first locations, and then prices, are chosen—the equilibrium locations are the two extremes. As a referee pointed out to us, Hay [2] and Prescott and Visscher[5] use a similar sequential approach. In particular, Prescott and Visscher analyze the existence problem by numerical methods in a revised Hotelling problem and find equilibrium locations "far apart." We should stress however that the existence is not restored simply because the discontinuities of the demand functions are eliminated as, for example, by introducing the assumption of strictly convex transportation costs. We have indeed worked out an example which verifies the latter assumption and does not possess any equilibrium prices.

⁵ An example of this advantage is studied in Jaskold Gabszewicz and Thisse [4] and Salop [6].

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