Fair Dynamic Routing in Large-Scale Heterogeneous-Server Systems: Technical Appendix

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In this technical appendix we provide proofs for the various results stated in the manuscript titled: "Fair Dynamic Routing in Large-Scale Heterogeneous-Server Systems". The technical appendix is divided into two subsections. Section A provides results that are useful to prove the Lemmas, Propositions, and Theorems stated in Sections 3 and 4 of the main paper. Section B shows the proofs of all the Lemmas, Propositions, and Theorems in the main paper.

To begin, we note that we require the following technicalities. All random variables are defined on a common probability space $(\Omega, \mathcal{F}, P)$. For each positive integer d, let $D([0,\infty), \Re^d)$ be the space of right continuous functions with left limits (RCLL) in $\Re^d$ having time domain $[0,\infty)$. We endow $D([0,\infty), \Re^d)$ with the usual Skorokhod J_1 topology, and let M^d denote the Borel σ-algebra associated with the J_1 topology. All stochastic processes are measurable functions from $(\Omega, \mathcal{F}, P)$ into $(D([0,\infty), \Re^d), M^d)$ for some appropriate dimension d. Suppose $\{\xi^n\}_{n=1}^{\infty}$ is a sequence of stochastic processes. The notation $\xi^n \Rightarrow \xi$ means that the probability measures induced by the ξ^n's on $(D([0,\infty), \Re^d), M^d)$ converge weakly to the probability measure on $(D([0,\infty), \Re^d), M^d)$ induced by the stochastic process ξ. Note that we suppress d from the notation unless necessary. We often reference the functional strong law of large numbers, the functional central limit theorem, and the continuous mapping theorem. A convenient reference for these theorems is [9] or [48] (numbered according to the reference list in the main body of the paper). Finally, we let "a.s." denote "almost surely" and "u.o.c." denote "uniformly on compact sets".

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A Auxiliary Results

Results Supporting the Proofs of the Results in Section 3

We will require two lemmas that provide (1) a stochastic upper bound between the process $\hat{X}$ in (3.1) under any admissible policy $\vec{u}$ and a particular reflected Brownian motion process, and (2) a stochastic lower bound between the process $\hat{X}$ and a particular Ornstein-Uhlenbeck process. Let $\leq_{st}$ represent stochastically less than.

Lemma A.1 *Suppose X^R is a reflected Brownian motion with infinitesimal drift $-\delta\sqrt{\mu}$, infinitesimal variance 2μ, and initial position $X^R(0)$. Suppose $\hat{X}$ satisfies equation (3.1) for some admissible control $u \in \mathcal{U}_P$. Then, assuming $\hat{X}(0) \leq X^R(0)$ a.s.,*

$$\hat{X}(t) \leq_{st} X^R(t) \text{ for every } t \geq 0.$$

Proof:
Let B be a standard Brownian motion and let

$$\begin{aligned} \hat{X}(t) &= \hat{X}(0) - \delta\sqrt{\mu}t + \int_0^t \left(\mu_1 u_1(s) + \mu_2 u_2(s)\right) \hat{X}(s)^- ds + \sqrt{2\mu}B(t) \\ X^R(t) &= X^R(0) - \delta\sqrt{\mu}t + \sqrt{2\mu}B(t) + L^R(t), \end{aligned}$$

where $L^R(0) = 0$, L^R is non-decreasing, and $\int_0^\infty X^R(t)dL^R(t) = 0$. Note that we have coupled $\hat{X}$ and X^R by using the same Brownian motion, B in the above equations. We establish that for these particular versions of the two processes $\hat{X}(t) \leq X^R(t)$ for every $t \geq 0$ with probability 1.

Our proof is by contradiction. Suppose there exists $t > 0$ for which $\hat{X}(t) > X^R(t)$. Since $\hat{X}(0) < X^R(0)$ a.s. and $\hat{X} - X^R$ has continuous sample paths, there exists $s \in [0, t)$ such that $\hat{X}(s) = X^R(s)$ with $\hat{X}(v) > X^R(v)$ for all $v \in (s, t]$. Because L^R is a non-decreasing process,

$$\begin{aligned} \hat{X}(t) - X^R(t) &= \hat{X}(s) - X^R(s) + \int_s^t \left(\mu_1 u_1(v)\hat{X}(v)^- + \mu_2 u_2(v)\hat{X}(v)^-\right) dv - \left(L^R(t) - L^R(s)\right) \\ &\leq \int_s^t \left(\mu_1 u_1(v)\hat{X}(v)^- + \mu_2 u_2(v)\hat{X}(v)^-\right) dv. \end{aligned}$$

X^R is a non-negative process and so $\hat{X}(v) > X^R(v)$ for all $v \in (s, t]$ implies $\hat{X}(v) > 0$ for all $V \in (s, t]$. Therefore, $\hat{X}(v)^- = 0$ for all $v \in (s, t]$, and so

$$\int_s^t \left(\mu_1 u_1(v)\hat{X}(v)^- + \mu_2 u_2(v)\hat{X}(v)^-\right) dv = 0.$$

We conclude that $\hat{X}(t) \leq X^R(t)$, which is a contradiction. ∎

Lemma A.2 *Suppose X^{O-U} is an Ornstein-Uhlenbeck process with infinitesimal drift $-\delta\sqrt{\mu} - \mu_1 x$, infinitesimal variance 2μ, and initial position $X^{O-U}(0)$. Suppose $\hat{X}$ satisfies equation (3.1) for some admissible control $u \in \mathcal{U}_P$. Then, assuming $X^{O-U}(0) \leq \hat{X}(0)$ a.s.,*

$$X^{O-U}(t) \leq_{st} \hat{X}(t) \text{ for every } t \geq 0.$$

Proof:
Let B be a standard Brownian motion. As in the proof of Lemma A.1, couple X^{O-U} and $\hat{X}$ as follows

$$\begin{aligned}\hat{X}(t) &= \hat{X}(0) - \delta\sqrt{\mu}t + \int_0^t (\mu_1 u_1(s) + \mu_2 u_2(s))\, \hat{X}(s)^- ds + \sqrt{2\mu}B(t)\\ X^{O-U}(t) &= \hat{X}(0) - \delta\sqrt{\mu}t - \mu_1 \int_0^t X^{O-U}(s)ds + \sqrt{2\mu}B(t).\end{aligned}$$

The proof is again by contradiction. Suppose there exists $t > 0$ for which $X^{O-U}(t) > \hat{X}(t)$. Since $X^{O-U}(0) \leq \hat{X}(0)$ a.s. and $X^{O-U}(t) - \hat{X}(t)$ has continuous sample paths, there exists $s \in [0, t)$ such that $\hat{X}(s) = X^{O-U}(s)$ and $\hat{X}(u) < X^{O-U}(u)$ for all $u \in (s, t]$. Therefore

$$X^{O-U}(t) - \hat{X}(t) = \int_s^t -\mu_1 X^{O-U}(s) - (\mu_1 u_1(v) + \mu_2 u_2(v))\, \hat{X}^-(v)dv.$$

There are three possible cases for any $v \in (s, t]$, enumerated below.

1. If $0 > X^{O-U}(v) > \hat{X}(v)$, then $\hat{X}(v)^- > X^{O-U}(v)^-$, and so

$$\begin{aligned}-\mu_1 X^{O-U}(v) - (\mu_1 u_1(v) + \mu_2 u_2(v))\, \hat{X}^-(v) &= \mu_1 X^{O-U}(v)^- - (\mu_1 u_1(v) + \mu_2 u_2(v))\, \hat{X}(v)^-\\ &\leq (\mu_1 u_1(v) + \mu_2 u_2(v)) \left(X^{O-U}(v)^- - \hat{X}(v)^-\right)\\ &\leq 0.\end{aligned}$$

2. If $X^{O-U}(v) \geq 0 > \hat{X}(v)$, then

$$-\mu_1 X^{O-U}(v) - (\mu_1 u_1(v) + \mu_2 u_2(v))\, \hat{X}^-(v) \leq 0.$$

3. If $X^{O-U}(v) > \hat{X}(v) \geq 0$, then

$$-\mu_1 X^{O-U}(v) - (\mu_1 u_1(v) + \mu_2 u_2(v))\, \hat{X}(v)^- = -\mu_1 X^{O-U}(v) \leq 0.$$

We conclude that

$$X^{O-U}(t) - \hat{X}(t) \leq 0,$$

which is a contradiction. ■

We will additionally use the following property of the standard normal probability density function ϕ and distribution function Φ.

Lemma A.3 *The function exp$(x^2/2)\Phi(x)$ is increasing in x.*

Proof:
We use relation 7.1.13 from [24]

$$e^{x^2}\int_x^\infty e^{-t^2}dt \le \frac{1}{x+\sqrt{x^2+\frac{4}{\pi}}},\ x \ge 0. \tag{A.1}$$

Observe that

$$\frac{d}{dx}\left(e^{x^2/2}\Phi(x)\right) = xe^{x^2/2}\Phi(x) + e^{x^2/2}\phi(x).$$

When $x \ge 0$, the right-hand side in the above expression is positive. Otherwise, when $x < 0$, by the inequality (A.1),

$$\begin{aligned} xe^{x^2/2}\Phi(x) + e^{x^2/2}\phi(x) &= \frac{x}{\sqrt{\pi}}\left(e^{x^2/2}\int_{-x/\sqrt{2}}^\infty e^{-y^2}dy\right) + \frac{1}{\sqrt{2\pi}} \\ &> \frac{x}{\sqrt{\pi}}\frac{1}{\frac{-x}{\sqrt{2}}+\sqrt{\frac{x^2}{2}+\frac{4}{\pi}}} + \frac{1}{\sqrt{2\pi}} \\ &= \frac{\sqrt{2}x + \frac{-x}{\sqrt{2}} + \sqrt{\frac{x^2}{2}+\frac{4}{\pi}}}{\sqrt{2\pi}\left(\frac{-x}{\sqrt{2}}+\sqrt{\frac{x^2}{2}+\frac{4}{\pi}}\right)} \\ &> 0. \end{aligned}$$

We conclude $\frac{d}{dx}\left(e^{x^2/2}\Phi(x)\right) > 0$ for all $x \in \Re$. ∎

Finally, it will be helpful to note that the second moments of the reflected Brownian motion X^R and Ornstein-Uhlenbeck X^{O-U} processes appearing in Lemmas A.1 and A.2 can be computed explicitly. Specifically, Theorem 1.1 in Whitt [1] shows that when $X^R(0)$ is a deterministic constant x^R,

$$\begin{aligned} &E\left[X^R(t)^2\right] \qquad\qquad (\text{A.2}) \\ &= \frac{2\mu}{\delta^2} + \frac{4\mu}{\delta^2}\left(\left(\frac{\delta}{2\sqrt{\mu}}x^R - 1\right)\left(\frac{\delta}{2}t\right)^{1/2} - \left(\frac{\delta}{2}t\right)^{3/2}\right)\phi\left(\sqrt{\frac{\delta}{2}t} - \frac{\sqrt{\delta}}{\sqrt{2}\sqrt{\mu}\sqrt{t}}x^R\right) \\ &\quad + \frac{4\mu}{\delta^2}\left(\left(\frac{\delta}{2}t - \frac{\delta}{2\sqrt{\mu}}x^R\right)^2 + \frac{\delta}{2}t - \frac{1}{2}\right)\left(1 - \Phi\left(\sqrt{\frac{\delta}{2}t} - \frac{\sqrt{\delta}}{\sqrt{2}\sqrt{\mu}\sqrt{t}}x^R\right)\right) \\ &\quad + \frac{4\mu}{\delta^2}\exp\left(\frac{\delta}{\sqrt{\mu}}x^R\right)\left(\frac{\delta}{2}t + \frac{\delta}{2\sqrt{\mu}}x^R - \frac{1}{2}\right)\left(1 - \Phi\left(\sqrt{\frac{\delta}{2}t} + \frac{\sqrt{\delta}}{\sqrt{2}\sqrt{\mu}\sqrt{t}}x^R\right)\right). \end{aligned}$$

Next, if B is a standard Brownian motion, then X^{O-U} solves the stochastic equation

$$X^{O-U}(t) = X^{O-U}(0) - \delta\sqrt{\mu}t - \mu_1\int_0^t X^{O-U}(s)ds + \sqrt{2\mu}B(t).$$

This integral equation can be solved explicitly by applying integration by parts to $\exp(\mu_1 t)X^{O-U}(t)$ to find

$$X^{O-U}(t) = X^{O-U}(0)\exp(-\mu_1 t) - \delta\frac{\sqrt{\mu}}{\mu_1}\left(1-\exp(-\mu_1 t)\right) + \sqrt{2\mu}\int_0^t \exp\left(-\mu_1(t-s)\right)dB(s). \tag{A.3}$$

Hence,

$$\begin{aligned} X^{O-U}(t)^2 &= \left(\hat{X}(0)\exp\left(-\mu_1 t\right) - \delta\frac{\sqrt{\mu}}{\mu_1}\left(1-\exp(-\mu_1 t)\right)\right)^2 + 2\mu\left(\int_0^t \exp\left(-\mu_1(t-s)dB(s)\right)\right)^2 \\ &\quad +2\left(\hat{X}(0)\exp(-\mu_1 t) - \delta\frac{\sqrt{\mu}}{\mu_1}(1-\exp(-\mu_1 t))\sqrt{2\mu}\int_0^t \exp(-\mu_1(t-s)dB(s)\right), \end{aligned}$$

and so the Ito isometry and the fact that the stochastic integral is a martingale imply

$$EX^{O-U}(t)^2 = \left(\hat{X}(0)\exp(-\mu_1 t) - \delta\frac{\sqrt{\mu}}{\mu_1}(1-\exp(-\mu_1 t)\right)^2 + 2\mu\int_0^t \exp(-2\mu_1(t-s))ds. \tag{A.4}$$

Results Supporting the Proofs of the Results in Section 4

We require knowledge concerning the behavior of the sequence of steady-state distributions under any admissible sequence of policies. The first result we discuss is the existence of a steady-state distribution for $\vec{X}^\lambda(t;\pi)$ for any fixed λ and for any stationary policy $\pi \in \Pi$ under a minor additional condition.

Proposition A.1 *Consider a fixed arrival rate λ and a given policy $\pi \in \Pi$ (omitted from the notation). Suppose that the process $\vec{X}(t) = (Z_1(t) + Q(t), Z_2(t))$ is a time-homogeneous Markovian and irreducible under the policy π. Then, $\vec{X}(t)$ has a steady-state distribution.*

Proof of Proposition A.1: The proof mimics the first part of the proof of Proposition 4.6 in [2], and is therefore omitted.

The second results we require is the tightness of the sequence of the steady-state distributions of the scaled processes $\vec{\hat{X}}^\lambda(\cdot)$ as $\lambda \to \infty$.

Proposition A.2 *Consider a sequence of systems indexed by the arrival rate λ with staffing levels $\vec{N}^\lambda$, working under a sequence of policies $\pi^\lambda = \pi(\lambda, \vec{N}^\lambda) \in \Pi$. Suppose that for each λ the steady-state distribution of $\vec{\hat{X}}^\lambda(\cdot;\pi^\lambda)$ exists. Then the sequence of these steady-state distributions is tight as $\lambda \to \infty$.*

Proof of Proposition A.2: The proof mimics the second part of the proof of Proposition 4.6 in [2], and is therefore omitted.

Finally, we establish that the sequences $\left[\hat{X}^{\lambda}(\infty)\right]^{+}$ and $\left[\hat{X}^{\lambda}(\infty)\right]^{-}$ are uniformly integrable (UI) as $\lambda\to\infty$, under any sequence of policies $\pi = \pi^{\lambda} \in \Pi$ for which those steady-state distributions exist.

Proposition A.3 *Consider a sequence of systems indexed by the arrival rate λ with staffing levels $\vec{N}^{\lambda}$, working under a sequence of policies $\pi^{\lambda} = \pi(\lambda, \vec{N}^{\lambda}) \in \Pi$. Suppose that for each λ the steady-state distribution of $\vec{\hat{X}}^{\lambda}(\cdot;\pi^{\lambda})$ exists. Then the sequences $\left[\hat{X}^{\lambda}(\infty)\right]^{+}$ and $\left[\hat{X}^{\lambda}(\infty)\right]^{-}$ are uniformly integrable (UI) as $\lambda\to\infty$.*

Proof of Proposition A.3 Consider a sequence of policies $\pi = \pi^{\lambda} \in \Pi$, under which the process $\vec{\hat{X}}^{\lambda}(\cdot, \pi)$ has a steady-state distribution for all values of λ. By Proposition 3.2 in [3], we have for all λ,

$$0 \leq Q^{\lambda}(\cdot) \overset{st}{\leq} Q_B^{\lambda}(\cdot),$$

and

$$Z_C^{\lambda}(\cdot) - N_C^{\lambda} \overset{st}{\leq} Z^{\lambda}(\cdot) - N^{\lambda} \leq 0,$$

where $Q_B^{\lambda}(\cdot)$ is the queue length process in an $M/M/N_B^{\lambda}$ system with arrival rate λ, service rates μ_2 and number of servers $N_B^{\lambda} = \left\lfloor \frac{N_1^{\lambda}\mu_1 + N_2^{\lambda}\mu_2}{\mu_2} \right\rfloor$. Similarly, $Z_C^{\lambda}(\cdot)$ is the total number of busy servers process in an $M/M/N_C^{\lambda}$ system with arrival rate λ, service rate μ_1 and number of servers $N_C^{\lambda} = \left\lceil \frac{N_1^{\lambda}\mu_1 + N_2^{\lambda}\mu_2}{\mu_1} \right\rceil$.

To complete the proof we need to establish that the sequences $\frac{Q_B^{\lambda}(\infty)}{\sqrt{N^{\lambda}}}$ and $\frac{Z_C^{\lambda}(\infty) - N_C^{\lambda}}{\sqrt{N^{\lambda}}}$ are uniformly integrable. But note that $\frac{Q_B^{\lambda}(\infty)}{\sqrt{N_B^{\lambda}}}$ and $\frac{Z_C^{\lambda}(\infty) - N_C^{\lambda}}{\sqrt{N_C^{\lambda}}}$ are uniformly integrable by Theorem 1 in [31] and by an argument paralleling the proof of Corollary 1 in [31]. The result then follows by noting that $\lim_{\lambda\to\infty} \frac{\sqrt{N_B^{\lambda}}}{\sqrt{N^{\lambda}}} = \sqrt{\frac{\mu}{\mu_2}}$ and $\lim_{\lambda\to\infty} \frac{\sqrt{N_C^{\lambda}}}{\sqrt{N^{\lambda}}} = \sqrt{\frac{\mu}{\mu_1}}$. ∎

B Proofs of Results in the Main Body of the Paper

Proof of Lemma 2.1:
Consider a fixed policy π (omitted from the notation). Then it follows from the system dynamics equations (2.6) and (2.7) and the identity $I_k(t) = N_k - Z_k(t)$ for all $t \geq 0$ that

$$X(t) = X(0) + A(t) - \sum_{k=1}^{2} \left(D_k(T_k(t)) - \mu_k T_k(t)\right) + \sum_{k=1}^{2} \mu_k \int_0^t I_k(s)ds - \left(\sum_{k=1}^{2} \mu_k N_k\right) t.$$

Proposition 2.1 shows that

$$\frac{T_k(t)}{t} = \frac{\int_0^t Z_k(s)ds}{t} \to q_k, \text{ a.s.},$$

as $t \to \infty$. Hence T_k is an increasing process and $T_k(t) \to \infty$ as $t \to \infty$. Recalling that D_k is a Poisson process with rate μ_k, we find

$$\frac{\sum_{k=1}^{2} D_k\left(T_k(t)\right) - \mu_k T_k(t)}{t} = \sum_{k=1}^{2} \frac{T_k(t)}{t}\left(\frac{D_k(T_k(t))}{T_k(t)} - \mu_k\right) \to 0, \text{ a.s.},$$

as $t \to \infty$. Therefore, dividing by t and letting $t \to \infty$ in (B.1) yields the equation

$$0 = \lambda + \mu_1 E\left[I_1(\infty)\right] + \mu_2 E\left[I_2(\infty)\right] - \mu_1 N_1 - \mu_2 N_2.$$

Combining the above equation with the equality $EI_1(\infty)/EI(\infty) = f_1$ yields the results of the lemma.

∎

Proof of Lemma 3.1:

(i): The existence of a strong solution to (3.1) follows by an argument similar to that in Proposition 1 in [32]. (Although the multidimensional diffusion appearing in their Proposition 1 has continuous drift, their arguments are not affected by the possible discontinuity in our drift. The key is that the infinitesimal drift does not grow superlinearly.) Note that the definition of a strong solution in 5.2.1 in Karatzas and Shreve [34] requires that $\hat{X}$ have continuous sample paths.

(ii): By Theorem (54.5) in Rogers and Williams [39], it is sufficient to observe that the speed density associated with $\hat{X}$ under an admissible control $u \in \mathcal{U}_P$

$$\begin{aligned} d(x) &:= \frac{1}{2\mu\text{exp}\left(\frac{-1}{\mu}\int_1^x m(y,u)dy\right)} \\ &= \frac{1}{2\mu}\text{exp}\left(\frac{\delta}{\sqrt{\mu}}(1-x) - \frac{1}{2}\left(u_1\mu_1 + u_2\mu_2\right)(x^2-1)\mathbf{1}\{x<0\}\right), \text{ for all } x \in \Re \end{aligned}$$

has finite mass; i.e., that

$$\int_{-\infty}^{\infty} d(x)dx < \infty.$$

(iii): The fact that $\hat{X}$ has a unique steady-state distribution implies that as $t \to \infty$,

$$\hat{X}(t) \Rightarrow \hat{X}(\infty). \qquad \text{(B.1)}$$

The continuous mapping theorem then implies that as $t \to \infty$

$$\hat{X}(t)^+ \Rightarrow \hat{X}(\infty)^+ \text{ and } \hat{X}(t)^- \Rightarrow \hat{X}(\infty)^-. \qquad \text{(B.2)}$$

The process $\hat{X}$ is Markovian, and so

$$u_i(t) = u_i\left(\hat{X}(t)\right),\ i \in \{1,2\},$$

which implies

$$u_i(t) \Rightarrow u_i(\infty) := u_i(\hat{X}(\infty)),\ i \in \{1,2\}.$$

Theorem 4.2 in [44] provides conditions under which multiplication is continuous in $D \times D$, and these are easily seen to be satisfied because $\hat{X}$ is continuous a.s.. Hence the continuous mapping theorem also implies that

$$u_i(t)\hat{X}(t) \Rightarrow u_i(\infty)\hat{X}(\infty),\ i \in \{1,2\}. \tag{B.3}$$

We now establish that the families $\{\hat{X}(t) : t \geq 0\}$, $\{\hat{X}(t)^+ : t \geq 0\}$, $\{\hat{X}(t)^- : t \geq 0\}$, and $\{u_i(t)\hat{X}(t)\} : t \geq 0\}, i \in \{1,2\}$ are all uniformly integrable. Let X^R and X^{O-U} be the reflected Brownian motion and Ornstein-Uhlenbeck processes appearing in Lemmas A.1 and A.2. It follows from these same Lemmas that

$$|\hat{X}(t)| \leq |X^R(t)| + |X^{O-U}(t)| \text{ for all } t \geq 0.$$

Furthermore, note that since $0 \leq u_i(t) \leq 1$ for all $t \geq 0$, $i \in \{1,2\}$,

$$0 \leq \hat{X}(t)^+ \vee \hat{X}(t)^- \vee |u_1(t)\hat{X}(t)| \vee |u_2(t)\hat{X}(t)| \leq |\hat{X}(t)|.$$

The explicit expressions for the transient second moments of X^R and X^{O-U} in (A.2) and (A.4) imply that there exists a M independent of t such that

$$EX^R(t)^2 < M \text{ and } EX^{O-U}(t)^2 < M \text{ for all } t \geq 0.$$

Hence $\{X^R(t) : t \geq 0\}$ and $\{X^{O-U}(t) : t \geq 0\}$ are uniformly integrable families. We conclude that the families $\{\hat{X}(t) : t \geq 0\}$, $\{\hat{X}(t)^+ : t \geq 0\}$, $\{\hat{X}(t)^- : t \geq 0\}$, and $\{u_i(t)\hat{X}(t) : t \geq 0\}, i \in \{1,2\}$ are all dominated by uniformly integrable families and so are themselves uniformly integrable.

The weak convergences in (B.1)-(B.3) and the uniform integrability established in the previous paragraph imply that as $t \to \infty$

$$\begin{aligned}
E\left[\hat{X}(t)\right] &\to E\left[\hat{X}(\infty)\right] \\
E\left[\hat{X}(t)^+\right] &\to E\left[\hat{X}(\infty)^+\right] \\
E\left[\hat{X}(t)^-\right] &\to E\left[\hat{X}(\infty)^-\right] \\
E\left[u_i(t)\hat{X}(t)\right] &\to E\left[u_i(\infty)\hat{X}(\infty)\right],\ i \in \{1,2\}.
\end{aligned}$$

The stated limits now follow. ∎

Proof of Lemma 3.2:
Let $u \in \mathcal{U}_P$ be an admissible control. Then (3.6) implies that

$$\mu V''(x) + m(x,u)V'(x) + x^+ + \triangle(x^- - d) \geq \kappa \tag{B.4}$$

for all $x \in \Re$. Define $T_n := \inf\{t \geq 0 : |\hat{X}(t)| > n\}$. By (3.1) and Ito's formula,

$$V\left(\hat{X}(t \wedge T_n)\right) = V(\hat{X}(0)) + \int_0^{t\wedge T_n} \mu V''(\hat{X}(s)) + m(\hat{X}(s), u(s))V'(\hat{X}(s))ds + \sqrt{2\mu}\int_0^{t\wedge T_n} V'(\hat{X}(s))dB(s).$$

Since V' is continuous and $|\hat{X}(t)| \leq n$ for all $t < T_n$, the stochastic integral is a martingale, and so

$$E\left[V(\hat{X}(t \wedge T_n))\right] = EV(\hat{X}(0)) + E\left[\int_0^{t\wedge T_n} \mu V''(\hat{X}(s)) + m(\hat{X}(s), u(s))V'(\hat{X}(s))ds\right].$$

We conclude from (B.4) that

$$E\left[\int_0^{t\wedge T_n} \hat{X}(s)^+ + \triangle\left(\hat{X}(s)^- - d\right)ds\right] + E\left[V(\hat{X}(t \wedge T_n))\right] \geq EV(\hat{X}(0)) + \kappa E[t \wedge T_n]. \tag{B.5}$$

We next argue that taking the limit as $n \to \infty$ on both sides of the inequality in (B.5) shows

$$E\left[\int_0^{t} \hat{X}(s)^+ + \triangle\left(\hat{X}(s)^- - d\right)ds\right] + E\left[V(\hat{X}(t))\right] \geq EV(\hat{X}(0)) + \kappa t. \tag{B.6}$$

The sample paths of $\hat{X}$ are continuous a.s., and so $t \wedge T_n \to t$ a.s. as $n \to \infty$. Futhermore, $t \wedge T_n \leq t$ for all n and so dominated convergence implies

$$\lim_{n\to\infty} E[t \wedge T_n] = t.$$

Monotone convergence implies

$$\lim_{n\to\infty} E\left[\int_0^{t\wedge T_n} \hat{X}(s)^+ + \triangle\hat{X}(s)^- ds\right] = E\left[\int_0^{t} \hat{X}(s)^+ + \triangle\hat{X}(s)^- ds\right].$$

Ito's formula applied to the function x^2 yields

$$\begin{aligned}\hat{X}(t \wedge T_n)^2 &= \hat{X}(0)^2 + 2\mu t - 2\delta\sqrt{\mu}\int_0^{t\wedge T_n} \hat{X}(s)ds + 2\sqrt{2\mu}\int_0^{t\wedge T_n} \hat{X}(s)dB(s) \\ &\quad +2\int_0^{t\wedge T_n} \hat{X}(s)\left(\mu_1 u_1(s)\hat{X}(s)^- + \mu_2 u_2(s)\hat{X}(s)^-\right)ds.\end{aligned}$$

By Lemma A.2, $X^{O-U}(t) \leq \hat{X}(t)$ a.s. for all $t \geq 0$. Therefore, also noting that $\hat{X}(t)\hat{X}(t)^- \leq 0$ for all $t \geq 0$,

$$\hat{X}(t \wedge T_n)^2 \leq \hat{X}(0)^2 + 2\mu t - 2\delta\sqrt{\mu}\int_0^{t\wedge T_n} X^{O-U}(s)ds + 2\sqrt{2\mu}\int_0^{t\wedge T_n} \hat{X}(s)dB(s).$$

The explicit expression for an Ornstein-Uhlenbeck process (A.3) then implies, noting that the stochastic integral is a martingale,

$$E\hat{X}(t \wedge T_n)^2 \leq \hat{X}(0)^2 + 2\mu t + 2\delta\sqrt{\mu}E\hat{X}(0)\int_0^t \exp(-\mu_1 s) + \delta\frac{\sqrt{\mu}}{\mu_1}\left(1 - \exp(-\mu_1 s)\right) ds.$$

By assumption, $V(\hat{X}(t \wedge T_n)) \leq b_1\hat{X}(t \wedge T_n)^2 + b_2$ for all $t \geq 0$, and so dominated convergence implies

$$EV(\hat{X}(t \wedge T_n)) \to EV(\hat{X}(t)),$$

as $n \to \infty$. We conclude the inequality (B.6) is valid.

Finally, we divide by t and take limits as $t \to \infty$ on both sides of the inequality (B.6) to establish

$$\lim_{t\to\infty} \frac{1}{t}E\int_0^t \hat{X}(s)^+ + \triangle(\hat{X}(s)^- - d)ds \geq \kappa.$$

(Note that the limit exists by Lemma 3.1 part (iii).) By assumption, $V(\hat{X}(0)) \leq b_1\hat{X}(0)^2 + b_2$ and $E\hat{X}(0)^2 < \infty$, and so

$$\frac{1}{t}EV(\hat{X}(0)) \to 0$$

as $t \to \infty$. Therefore, to complete the proof, it is sufficient to show

$$\lim_{t\to\infty} \frac{EV(\hat{X}(t))}{t} = 0. \tag{B.7}$$

Let X^R be a reflected Brownian motion, as in Lemma A.1. By assumption, Lemma A.1, and Lemma A.2, there exist versions of the respective processes such that

$$|V(\hat{X}(t))| \leq b_1\hat{X}(t)^2 + b_2 \leq b_1\left(X^R(t)^2 + X^{O-U}(t)^2\right) + b_2.$$

It follows from equation (A.2) and Corollary 1.1.1 in [1] that $EX^R(t)^2$ converges to a finite limit as $t \to \infty$. Hence

$$\frac{EX^R(t)^2}{t} \to 0,$$

as $t \to \infty$. Furthermore, equation (A.4) implies that

$$\frac{EX^{O-U}(t)^2}{t} \to 0,$$

as $t \to \infty$. We conclude that as $t \to \infty$

$$\frac{|EV(\hat{X}(t))|}{t} \leq \frac{E|V(\hat{X}(t))|}{t} \leq \frac{b_1\left(EX^R(t)^2 + EX^{O-U}(t)^2\right) + b_2}{t} \to 0,$$

and so (B.7) holds ∎

Proof of Lemma 3.3:
Suppose $\vec{u}$ is a threshold control at level L, meaning $\vec{u}$ satisfies (3.7) and the infinitesimal drift m satisfies (3.8). Exactly as in the proof of Lemma 3.2, for $T_n := \inf\{t \geq 0 : |\hat{X}(t)| > n\}$,

$$E\left[V(\hat{X}(t \wedge T_n))\right] = EV(\hat{X}(0)) + E\left[\int_0^{t\wedge T_n} \mu V''(\hat{X}(s)) + m(\hat{X}(s), u(s))V'(\hat{X}(s))ds\right].$$

It then follows from (3.9) that

$$E\left[\int_0^{t\wedge T_n} \hat{X}(s)^+ + \triangle\left(\hat{X}(s)^- - d\right) ds\right] + E\left[V(\hat{X}(t\wedge T_n))\right] = EV(\hat{X}(0)) + \kappa E[t\wedge T_n].$$

The exact same arguments as in the proof of Lemma 3.2 establish that

$$\begin{aligned}\lim_{n\to\infty} E[t\wedge T_n] &= t\\ \lim_{n\to\infty} E\left[\int_0^{t\wedge T_n} \hat{X}(s)^+ + \triangle\hat{X}(s)^- ds\right] &= E\left[\int_0^t \hat{X}(s)^+ + \triangle\hat{X}(s)^- ds\right]\\ \lim_{n\to\infty} EV(\hat{X}(t\wedge T_n)) &= EV(\hat{X}(t)),\end{aligned}$$

and so

$$E\left[\int_0^t \hat{X}(s)^+ + \triangle(\hat{X}(s)^- - d)ds\right] + EV(\hat{X}(t)) = EV(\hat{X}(0)) + \kappa t.$$

Again as in the proof of Lemma 3.2,

$$\lim_{t\to\infty} \frac{EV(\hat{X}(t))}{t} = 0.$$

The threshold policy is admissible, and so it follows from Lemma3.1 that

$$\lim_{t\to\infty} \frac{E\left[\int_0^t \hat{X}(s)^+ + \triangle\hat{X}(s)^- ds\right]}{t} = E\hat{X}(\infty)^+ + \triangle E\hat{X}(\infty)^-.$$

We conclude that

$$E\hat{X}(\infty)^+ + \triangle\left(E\hat{X}(\infty)^- - d\right) = \kappa.$$

∎

Proof of Lemma 3.4:
Note that $f(0,0) = 0$, and, therefore, the condition (3.13) implies

$$\frac{1}{\triangle\delta^2} - \frac{1}{\mu_2} > f(0,0) + \frac{1}{\delta\sqrt{\mu_2}} h\left(\frac{-\delta}{\sqrt{\mu_2}}\right).$$

It is well known that the hazard rate function associated with the standard normal distribution is increasing. Furthermore, $f(0,L)$ is increasing in L. To see this, note that

$$\begin{aligned}\frac{\partial}{\partial L} f(0,L) &= \sqrt{\frac{2\pi}{\mu_1}} \exp\left(\frac{1}{2}\frac{\delta^2}{\mu_1}\right) \left[\begin{array}{l} \frac{1}{\sqrt{\mu_2}}\left(\frac{d}{dL} h\left(L\sqrt{\frac{\mu_2}{\mu}} - \frac{\delta}{\sqrt{\mu_2}}\right)\right)\left(\Phi\left(\frac{\delta}{\sqrt{\mu_1}}\right) - \Phi\left(\frac{\delta}{\sqrt{\mu_1}} - L\sqrt{\frac{\mu_1}{\mu}}\right)\right) \\ + \frac{1}{\sqrt{\mu_2}}\sqrt{\frac{\mu_1}{\mu}}\phi\left(\frac{\delta}{\sqrt{\mu_1}} - L\sqrt{\frac{\mu_1}{\mu}}\right)\left(\left(\frac{\delta}{\sqrt{\mu_2}} - L\sqrt{\frac{\mu_2}{\mu}}\right) + h\left(L\sqrt{\frac{\mu_2}{\mu}} - \frac{\delta}{\sqrt{\mu_2}}\right)\right)\end{array}\right]\\ &> 0,\end{aligned}$$

because

1. The hazard rate function associated with a standard normal random variable is increasing;
2. Φ is an increasing function;
3. It is shown in the Internet supplement to [49] that $x + h(x) > 0$ for all $x \in \Re$.

To complete the proof, note that the right-hand side of (3.10) increases to ∞ as $L \to \infty$. ∎

Proof of Proposition 3.1:
We first show that the function V' defined in (3.11) has

$$V'(x) < 0 \quad \text{for all} \quad x < -L^\star_\triangle \tag{B.8}$$
$$V'(x) > 0 \quad \text{for all} \quad x \in (-L^\star_\triangle, 0]. \tag{B.9}$$

Since $V'(-L^\star_\triangle) = 0$, showing that the function $V'(x)$ is increasing for all $x \leq -L^\star_\triangle$ establishes (B.8). From Lemma A.3, the function

$$\exp\left(\frac{1}{2}\left(\frac{\delta}{\sqrt{\mu_2}} + \sqrt{\frac{\mu_2}{\mu}}x\right)^2\right)\Phi\left(\frac{\delta}{\sqrt{\mu_2}} + \sqrt{\frac{\mu_2}{\mu}}x\right)$$

is increasing in x, and so $V'(x)$ is increasing for all $x < -L^\star_\triangle$. To establish (B.9), first define the function

$$\begin{aligned} g(x) \quad := \quad & \frac{1}{\sqrt{\mu_1}}\left(\phi\left(\frac{\delta}{\sqrt{\mu_1}} - L^\star_\triangle\sqrt{\frac{\mu_1}{\mu}}\right) - \phi\left(\frac{\delta}{\sqrt{\mu_1}} + x\sqrt{\frac{\mu_1}{\mu}}\right)\right) \\ & + \left(\frac{1}{\sqrt{\mu_2}}\frac{\phi\left(\frac{\delta}{\sqrt{\mu_2}} - L^\star_\triangle\sqrt{\frac{\mu_2}{\mu}}\right)}{\Phi\left(\frac{\delta}{\sqrt{\mu_2}} - L^\star_\triangle\sqrt{\frac{\mu_2}{\mu}}\right)} + \frac{\delta}{\mu_2} - \frac{\delta}{\mu_1}\right)\left(\Phi\left(\frac{\delta}{\sqrt{\mu_2}} + \sqrt{\frac{\mu_1}{\mu}}x\right) - \Phi\left(\frac{\delta}{\sqrt{\mu_1}} - L^\star_\triangle\sqrt{\frac{\mu_1}{\mu}}\right)\right), \end{aligned}$$

and note that

$$V'(x) = \triangle\sqrt{\frac{2\pi}{\mu_1}}\exp\left(\frac{1}{2}\left(\frac{\delta}{\sqrt{\mu_1}} + \sqrt{\frac{\mu_1}{\mu}}x\right)^2\right) g(x) \text{ for } x \in \left[-L^\star_\triangle, 0\right].$$

Since

$$\exp\left(\frac{1}{2}\left(\frac{\delta}{\sqrt{\mu_1}} + \sqrt{\frac{\mu_1}{\mu}}x\right)^2\right) > 0 \text{ for all } x \in \Re,$$

and $V'(L^\star_\triangle) = 0$, it must be that $g(L^\star_\triangle) = 0$. Hence to establish (B.9), it is sufficient to show that the function g is increasing in x for all $x \in [-L^\star_\triangle, 0]$. For this, note that

$$\begin{aligned} g'(x) \quad &= \quad \frac{\sqrt{\mu_1}}{\sqrt{\mu}\sqrt{\mu_2}}\phi\left(\frac{\delta}{\sqrt{\mu_1}} + x\sqrt{\frac{\mu_1}{\mu}}\right)\left(\left(\frac{\delta}{\sqrt{\mu_2}} + x\sqrt{\frac{\mu_2}{\mu}}\right) + h\left(L^\star_\triangle\sqrt{\frac{\mu_2}{\mu}} - \frac{\delta}{\sqrt{\mu_2}}\right)\right) \\ &\geq \quad \frac{\sqrt{\mu_1}}{\sqrt{\mu}\sqrt{\mu_2}}\phi\left(\frac{\delta}{\sqrt{\mu_1}} + x\sqrt{\frac{\mu_1}{\mu}}\right)\left(\left(\frac{\delta}{\sqrt{\mu_2}} - L^\star_\triangle\sqrt{\frac{\mu_2}{\mu}}\right) + h\left(L^\star_\triangle\sqrt{\frac{\mu_2}{\mu}} - \frac{\delta}{\sqrt{\mu_2}}\right)\right) \\ &> \quad 0 \end{aligned}$$

as in item 3. in the proof of Lemma 3.5 above.

The inequalities (B.8) and (B.9) imply that the differential equation in (3.6) in the conditions of Lemma 3.2 is exactly the differential equation in (3.9) in the conditions of Lemma 3.3. Furthermore, as discussed directly following (3.12), there exist constants $b_1, b_2 \in \Re$ such that

$$|V(x)| \leq b_1 x^2 + b_2 \text{ for all } x \in \Re.$$

Therefore, by Lemmas 3.2 and 3.3, for κ as defined in (3.12),

$$\liminf_{t\to\infty} \frac{E\int_0^t \hat{X}(s)^+ + \triangle(\hat{X}(s)^- - d)ds}{t} \geq \kappa = E\hat{X}^\star(\infty)^+ + \triangle(E\hat{X}^\star(\infty)^- - d).$$

∎

Proof of Lemma 3.5:
We first show that there exists an $0 < L < \infty$ such that under a threshold control at level L, letting $\hat{X}_L$ to be the associated solution to the stochastic equation (3.1),

$$E\hat{X}_L(\infty)^- = d.$$

By the nature of the threshold policy with a threshold level L we have that

$$\begin{aligned} E\hat{X}_L(\infty)^- &= E[\hat{X}_L(\infty)^- | \hat{X}_L(\infty) < -L] P(\hat{X}_L(\infty) < -L) \\ &\quad + E[\hat{X}_L(\infty)^- | -L \leq \hat{X}_L(\infty) < 0] P(-L \leq \hat{X}_L(\infty) < 0), \end{aligned}$$

where the expression for $E[\hat{X}_L(\infty)^- | \hat{X}_L(\infty) < -L]$, $P(\hat{X}_L(\infty) < -L)$, $E[\hat{X}_L(\infty)^- | -L \leq \hat{X}_L(\infty) < 0]$ and $P(-L \leq \hat{X}_L(\infty) < 0)$ are given in (3.15)-(3.19).

It then follows that

$$\begin{aligned} \lim_{L\to 0} E\hat{X}_L(\infty)^- &= \frac{\frac{\delta\sqrt{\mu}}{\mu_2} + \sqrt{\frac{\mu}{\mu_2}} \frac{\phi\left(\frac{\delta}{\sqrt{\mu_2}}\right)}{\Phi\left(\frac{\delta}{\sqrt{\mu_2}}\right)}}{1 + \frac{\sqrt{\mu_2}}{\delta} \frac{\phi\left(\frac{\delta}{\sqrt{\mu_2}}\right)}{\Phi\left(\frac{\delta}{\sqrt{\mu_2}}\right)}} = \frac{\delta\sqrt{\mu}}{\mu_2} \\ \lim_{L\to\infty} E\hat{X}_L(\infty)^- &= \frac{\frac{\delta\sqrt{\mu}}{\mu_1} + \sqrt{\frac{\mu}{\mu_1}} \frac{\phi\left(\frac{\delta}{\sqrt{\mu_1}}\right)}{\Phi\left(\frac{\delta}{\sqrt{\mu_1}}\right)}}{1 + \frac{\sqrt{\mu_1}}{\delta} \frac{\phi\left(\frac{\delta}{\sqrt{\mu_1}}\right)}{\Phi\left(\frac{\delta}{\sqrt{\mu_1}}\right)}} = \frac{\delta\sqrt{\mu}}{\mu_1}. \end{aligned}$$

The assumption that $\mu_1 < \mu_2$ implies that

$$\frac{\delta\sqrt{\mu}}{\mu_2} < \frac{\delta\sqrt{\mu}}{f_1\mu_1 + (1-f_1)\mu_2} < \frac{\delta\sqrt{\mu}}{\mu_1}, \text{ for } 0 < f_1 < 1.$$

Since $E\hat{X}_L(\infty)^-$ is a continuous function of L and

$$d = \frac{\delta\sqrt{\mu}}{f_1\mu_1 + (1-f_1)\mu_2},$$

there must exist a finite $L > 0$ such that $E\hat{X}_L(\infty)^- = d$.

To finish the proof, it is enough to show that for any $L_0 > 0$ that satisfies $E\hat{X}_{L_0}(\infty)^- = d$, there exists a unique $\triangle^\star$ that satisfies condition (3.13) such that $L^\star(\triangle^\star)$ defined by equation (3.10) equals L_0. In particular, for L_0 such that $E\hat{X}_{L_0}(\infty)^- = d$, there must exist $\triangle^\star$ such that $L^\star(\triangle^\star) = L_0$.

Recall from the proof of Lemma 3.4 that the functions f and $\frac{d}{dL}h\left(L\sqrt{\frac{\mu_2}{\mu}} - \frac{\delta}{\sqrt{\mu_2}}\right)$ are increasing in L. Therefore, it follows from equation (3.10) that $L^\star_\triangle$ is decreasing in $\triangle$. Furthermore,

$$L^\star_\triangle \to \infty$$

as the penalty parameter $\triangle \to 0$, and

$$L^\star_\triangle \to 0,$$

as the penalty parameter $\triangle$ increases to the upper bound in condition (3.13); i.e., as

$$\triangle \to \left(\frac{\delta^2}{\mu_2} + \frac{\delta}{\sqrt{\mu_2}}h\left(\frac{-\delta}{\sqrt{\mu_2}}\right)\right)^{-1}.$$

We conclude that for any $L_0 > 0$, there exists $\triangle^\star$ such that $L^\star(\triangle^\star) = L_0$ and $\triangle^\star$ satisfies condition (3.13). ∎

Proof of Theorem 3.1:
The proof of Theorem 3.1 is immediate from Lemma 3.5 and the argument at the beginning of Section 3.3. ∎

Proof of Proposition 4.1 First note that $\hat{X}_{\eta,0} = \hat{X}^u_\eta$ and $\hat{X}_{\eta,1} = \hat{X}^l_\eta$. In particular, $E[\hat{X}_{\eta,0}(\infty)]^- = E[\hat{X}^u_\eta(\infty)]^-$, and $E[\hat{X}_{\eta,1}(\infty)]^- = E[\hat{X}^l_\eta(\infty)]^-$. Therefore, in view of (4.3) and (4.13) it is sufficient to show that $E[\hat{X}_{\eta,\gamma}(\infty)]^-$ is continuous in γ. We establish this continuity by examining the steady-state distribution of $\hat{X}_{\eta,\gamma}$ obtained from [11].

Fix $\eta > 0$. By (18.28) and (18.33) of [11] we have that the density $f_\gamma(\cdot)$ of $\hat{X}_{\eta,\gamma}$ in steady-state satisfies:

$$f_\gamma(x) = \begin{cases} C_{1,\gamma}f_1(x) & x \geq 0 \\ C_{2,\gamma}f_2(x) & -L+\eta \leq x < 0 \\ C_{3,\gamma}f_{3,\gamma}(x) & -L \leq x < -L+\eta \\ C_{4,\gamma}f_{4,\gamma}(x) & -L-\eta \leq x < -L \\ C_{5,\gamma}f_5(x) & x < -L-\eta, \end{cases} \tag{B.10}$$

where

$$f_1(x) = \frac{\delta}{\sqrt{\mu}}exp\{\delta x/\sqrt{\mu}\},$$

$$f_2(x) = \frac{\sqrt{\frac{\mu_1}{\mu}}\phi\left(\frac{x+\delta\sqrt{\mu}/\mu_1}{\sqrt{\mu/\mu_1}}\right)}{\Phi\left(\frac{\delta\sqrt{\mu}/\mu_1}{\sqrt{\mu/\mu_1}}\right) - \Phi\left(\frac{-L+\eta+\delta\sqrt{\mu}/\mu_1}{\sqrt{\mu/\mu_1}}\right)},$$

$$f_{3,\gamma}(x) = \frac{(b_{3,\gamma})^{-1}\phi\left(\frac{x-m_{3,\gamma}}{b_{3,\gamma}}\right)}{\Phi\left(\frac{-L+\eta-m_{3,\gamma}}{b_{3,\gamma}}\right) - \Phi\left(\frac{-L-m_{3,\gamma}}{b_{3,\gamma}}\right)},$$

with $m_{3,\gamma} = \frac{(1-\gamma)L(\mu_2-\mu_1)(\eta-L)/\eta-\delta\sqrt{\mu}}{\mu_1+(1-\gamma)L(\mu_2-\mu_1)/\eta}$ and $b_{3,\gamma} = \sqrt{\frac{\mu}{\mu_1+(1-\gamma)L(\mu_2-\mu_1)/\eta}}$.

$$f_{4,\gamma}(x) = \frac{(b_{4,\gamma})^{-1}\phi\left(\frac{x-m_{4,\gamma}}{b_{4,\gamma}}\right)}{\Phi\left(\frac{-L-m_{4,\gamma}}{b_{4,\gamma}}\right) - \Phi\left(\frac{-L-\eta-m_{4,\gamma}}{b_{4,\gamma}}\right)},$$

with $m_{4,\gamma} = -\frac{\gamma L(\mu_2-\mu_1)(\eta+L)/\eta+\delta\sqrt{\mu}}{\mu_2+\gamma L(\mu_2-\mu_1)/\eta}$ and $b_{4,\gamma} = \sqrt{\frac{\mu}{\mu_2+\gamma L(\mu_2-\mu_1)/\eta}}$, and

$$f_5(x) = \frac{\sqrt{\frac{\mu_2}{\mu}}\phi\left(\frac{x+\delta\sqrt{\mu}/\mu_2}{\sqrt{\mu/\mu_2}}\right)}{\Phi\left(\frac{-L-\eta+\delta\sqrt{\mu}/\mu_2}{\sqrt{\mu/\mu_2}}\right)}.$$

By continuity, $C_{1,\gamma}, ..., C_{5,\gamma}$ satisfy

$$C_{1,\gamma}f_1(0) = C_{2,\gamma}f_2(0),$$

$$C_{2,\gamma}f_2(-L+\eta) = C_{3,\gamma}f_{3,\gamma}(-L+\eta),$$
$$C_{3,\gamma}f_{3,\gamma}(-L) = C_{4,\gamma}f_{4,\gamma}(-L),$$
$$C_{4,\gamma}f_{4,\gamma}(-L-\eta) = C_{5,\gamma}f_{5,\gamma}(-L-\eta),$$

and

$$\sum_{i=1}^{5} C_{i,\gamma} = 1.$$

Note that all of the expressions for $f_{i,\gamma}(x)$ for $x = 0, -L+\eta, -L$ and $-L-\eta$ are continuous in γ and none are equal to 0 or to $\pm\infty$. Therefore, $C_{1,\gamma}, ..., C_{5,\gamma}$ are continuous in γ. Finally, by (18.29) of [11] we have that

$$E\left[\hat{X}_{\eta,\gamma}(\infty)\right]^{-} = -\sum_{i=2}^{5} C_{i,\gamma}\left(m_{i,\gamma} + b_{i,\gamma}\frac{\phi\left(\frac{s_{i+1}-m_{i,\gamma}}{b_{i,\gamma}}\right) - \phi\left(\frac{s_i-m_{i,\gamma}}{b_{i,\gamma}}\right)}{\Phi\left(\frac{s_i-m_{i,\gamma}}{b_{i,\gamma}}\right) - \Phi\left(\frac{s_{i+1}-m_{i,\gamma}}{b_{i,\gamma}}\right)}\right), \tag{B.11}$$

where $m_{2,\gamma} = -\delta\sqrt{\mu}/\mu_1$, $b_{2,\gamma} = \sqrt{\mu/\mu_1}$, $m_{5,\gamma} = -\delta\sqrt{\mu}/\mu_2$, $b_{2,\gamma} = \sqrt{\mu/\mu_2}$, and $s_2 = 0, s_3 = -L+\eta$, $s_4 = -L$, $s_5 = -L-\eta$, and $s_6 = -\infty$. Since all the components in (B.11) are continuous in γ and the denominators are all non-zero and finite, we have that $E[\hat{X}_{\eta,\gamma}(\infty)]^{-}$ is continuous in γ. ∎

Proof of Proposition 4.2 Proposition 4.1 establishes i. Therefore, to complete the proof of the proposition it is sufficient to show that $E[\hat{X}_{\eta,\gamma(\eta)}(\infty)]^{+}$ converges to $E[\hat{X}(\infty)]^{+}$ as $\eta \downarrow 0$. Notice that in light of (4.12) it is sufficient to show that $\lim_{\eta\to 0} E[\hat{X}^l_\eta(\infty)]^+ = E[\hat{X}(\infty)]^+$ and that $\lim_{\eta\to 0} E[\hat{X}^u_\eta(\infty)]^+ = E[\hat{X}(\infty)]^+$. We show the former. The latter follows similarly.

To show that $\lim_{\eta\to 0} E[\hat{X}^l_\eta(\infty)]^+ = E[\hat{X}(\infty)]^+$ we explicitly derive the expressions for the steady-state distributions of the processes involved, and then establish the desired convergence. We start with the process $\hat{X}$. By (4.6) and [11] we have that the density $g(\cdot)$ of $\hat{X}$ in steady-state satisfies:

$$g(x) = \begin{cases} b_1 g_1(x) & x \geq 0 \\ b_2 g_2(x) & -L \leq x < 0 \\ b_3 g_3(x) & x < -L. \end{cases}$$

Where

$$g_1(x) = \frac{\delta}{\sqrt{\mu}} \exp\{-\delta x/\sqrt{\mu}\},$$

$$g_2(x) = \frac{\sqrt{\frac{\mu_1}{\mu}}\phi\left(\frac{x+\delta\sqrt{\mu}/\mu_1}{\sqrt{\mu/\mu_1}}\right)}{\Phi\left(\frac{\delta\sqrt{\mu}/\mu_1}{\sqrt{\mu/\mu_1}}\right) - \Phi\left(\frac{-L+\delta\sqrt{\mu}/\mu_1}{\sqrt{\mu/\mu_1}}\right)},$$

and

$$g_3(x) = \frac{\sqrt{\frac{\mu_2}{\mu}}\phi\left(\frac{x+\delta\sqrt{\mu}/\mu_2}{\sqrt{\mu/\mu_2}}\right)}{\Phi\left(\frac{-L+\delta\sqrt{\mu}/\mu_2}{\sqrt{\mu/\mu_2}}\right)}.$$

By continuity of the function $g(x)$ the constants b_i satisfy:

$$b_1 g_1(0) = b_2 g_2(0),$$

and

$$b_2 g_2(-L) = b_3 g_3(-L).$$

Finally, because $g(x)$ is a proper density function we have

$$b_1 + b_2 + b_3 = 1.$$

Solving for b_1 we obtain

$$b_1 = \frac{g_2(0)g_3(-L)}{g_2(0)g_3(-L) + g_1(0)g_3(-L) + g_1(0)g_2(-L)}.$$

Finally, we have

$$E[\hat{X}(\infty)]^+ = b_1 \int_0^\infty x g_1(x)dx = b_1 \frac{\sqrt{\mu}}{\delta}.$$

Next we develop the expression for $E[\hat{X}^l_\eta(\infty)]^+$. By (4.8) and [11] we have that the density $f_\eta(x)$ of $\hat{X}^l_\eta$ in steady-state satisfies:

$$f_\eta(x) = \begin{cases} c_{1,\eta} f_1(x) & x \geq 0 \\ c_{2,\eta} f_2(x) & -L \leq x < 0 \\ c_{3,\eta} f_{3,\eta}(x) & -L-\eta \leq x < -L, \\ c_{4,\eta} f_{4,\eta}(x) & x < -L-\eta. \end{cases}$$

Here $f_1(x) = g_1(x)$ and $f_2(x) = g_2(x)$, and

$$f_{3,\eta}(x) = \frac{b_{3,\eta}^{-1}\phi\left(\frac{x-m_{3,\eta}}{b_{3,\eta}}\right)}{\Phi\left(\frac{-L-m_{3,\eta}}{b_{3,\eta}}\right) - \Phi\left(\frac{-L-\eta-m_{3,\eta}}{b_{3,\eta}}\right)},$$

with $b_{3,\eta} = \sqrt{\frac{\mu}{\mu_2+L(\mu_2-\mu_1)/\eta}}$ and $m_{3,\eta} = -\frac{\delta\sqrt{\mu}+L(\mu_2-\mu_1)(L+\eta)/\eta}{\mu_2+L(\mu_2-\mu_1)/\eta}$. Finally,

$$f_{4,\eta} = \frac{\sqrt{\frac{\mu_2}{\mu}}\phi\left(\frac{x+\delta\sqrt{\mu}/\mu_2}{\sqrt{\mu/\mu_2}}\right)}{\Phi\left(\frac{-L-\eta+\delta\sqrt{\mu}/\mu_2}{\sqrt{\mu/\mu_2}}\right)}.$$

By continuity of the function $f_\eta(x)$ the constants $c_{i,\eta}$ satisfy:

$$c_{1,\eta}f_1(0) = c_{2,\eta}f_2(0),$$

$$c_{2,\eta}f_2(-L) = c_{3,\eta}f_{3,\eta}(-L),$$

and

$$c_{3,\eta}f_{3,\eta}(-L-\eta) = c_{4,\eta}f_{4,\eta}(-L-\eta).$$

Finally, because $f_\eta(x)$ is a proper density function we have

$$c_{1,\eta} + c_{2,\eta} + c_{3,\eta} + c_{4,\eta} = 1.$$

Solving for $c_{1,\eta}$ we obtain

$$c_{1,\eta} = \frac{f_2(0)f_{4,\eta}(-L-\eta)}{f_2(0)f_{4,\eta}(-L-\eta) + f_1(0)f_{4,\eta}(-L-\eta) + f_1(0)f_2(-L)\left(\frac{f_{4,\eta}(-L-\eta)+f_{3,\eta}(-L-\eta)}{f_{3,\eta}(-L)}\right)}.$$

Finally, we have

$$E[\hat{X}_\eta^l(\infty)]^+ = c_{1,\eta}\int_0^\infty xf_1(x)dx = c_{1,\eta}\frac{\sqrt{\mu}}{\delta}.$$

In particular, we observe that $\lim_{\eta\to 0} E[\hat{X}_\eta^l(\infty)]^+ = E[\hat{X}(\infty)]^+$ if and only if $\lim_{\eta\to 0} c_{1,\eta} = b_1$. To establish the latter we make the following three key observations:

Observation 1: $\lim_{\eta\to 0} f_{4,\eta}(x-\eta) = g_3(x),\ \ \forall x \le -L.$
Observation 2: $\lim_{\eta\to 0} f_{3,\eta}(-L) = \infty.$
Observation 3: $\lim_{\eta\to 0}\frac{f_{3,\eta}(-L-\eta)}{f_{3,\eta}(-L)} = 1.$

In light of these observations, given the expressions for $c_{1,\eta}$ and b_1 the proposition immediately follows. Observation 1 is trivially true by the continuity of the functions ϕ and Φ. To prove Observation 2, note that

$$\frac{-L-m_{3,\eta}}{b_{3,\eta}} = \frac{\delta\sqrt{\mu}-\mu_1 L}{\sqrt{\mu(\mu_2+L(\mu_2-\mu_1)/\eta)}},$$

and that

$$\frac{-L-\eta-m_{3,\eta}}{b_{3,\eta}}=\frac{-L-m_{3,\eta}}{b_{3,\eta}}-\frac{\sqrt{\mu_2\eta^2+L\eta(\mu_2-\mu_1)}}{\sqrt{\mu}}.$$

In particular,

$$\lim_{\eta\to 0}\frac{-L-m_{3,\eta}}{b_{3,\eta}}=\lim_{\eta\to 0}\frac{-L-\eta-m_{3,\eta}}{b_{3,\eta}}=0.$$

Also note that $\lim_{\eta\to 0} b_{3,\eta}=0$. Therefore, Observation 2 follows from:

$$f_{3,\eta}(-L)=\frac{b_{3,\eta}^{-1}\phi\left(\frac{-L-m_{3,\eta}}{b_{3,\eta}}\right)}{\Phi\left(\frac{-L-m_{3,\eta}}{b_{3,\eta}}\right)-\Phi\left(\frac{-L-\eta-m_{3,\eta}}{b_{3,\eta}}\right)}\to\infty,\ \text{ as } \eta\to 0.$$

Finally, the above expressions also establish Observation 3, as

$$\frac{f_{3,\eta}(-L-\eta)}{f_{3,\eta}(-L)}=\frac{\phi\left(\frac{-L-\eta-m_{3,\eta}}{b_{3,\eta}}\right)}{\phi\left(\frac{-L-m_{3,\eta}}{b_{3,\eta}}\right)}\to\frac{1/\sqrt{2\pi}}{1/\sqrt{2\pi}}=1,\ \text{ as }\ \eta\to 0.$$

∎

Proof of Proposition 4.3

Recall that our policy is a special case of the QIR policy proposed in [26]. In particular, by Theorem 3.1 of [26], it is sufficient to establish that $v_i(\cdot)$, $i=1,2$ is locally Hölder continuous on the open interval $(0,\infty)$ for some exponent α_i, in order to establish state-space collapse. We show that $v_i(\cdot)$ is locally Hölder continuous with exponent $\alpha_i=1$.

Let K be a compact interval of $(0,\infty)$ such that K is a subset of one of the following intervals: $(0,L-\eta]$, $(L-\eta,L]$, $(L,L+\eta]$ or $(L+\eta,\infty)$. Then on K we have that $v_i(x)=c_{1,K}+\frac{C_{2,K}}{x}$ for some interval dependent constants $C_{1,K}$ and $C_{2,K}$. We wish to show that for all $x,y\in K$ we have $|v_i(x)-v_i(y)|\le C_K|x-y|$ for some constant C_k. Since the functions v_i are continuous, this will complete the proof. Let $C_k=\frac{C_{2,K}}{\underline{\mathbf{K}}^2}$, where $\underline{\mathbf{K}}=\inf_{x\in K}x$. Note, that by compactness of K we have $\underline{\mathbf{K}}>0$. Suppose that $x>y$, then

$$|v_i(x)-v_i(y)|=|C_{2,k}|\left(\frac{1}{y}-\frac{1}{x}\right)=|C_{2,k}|\frac{x-y}{xy}\le C_K|x-y|.$$

∎

Proof of Proposition 4.4 As in the proof of Proposition 4.2 in [2] (see (A.19) there), $\hat{X}$ satisfies the following decomposition:

$$\begin{aligned}\hat{X}^\lambda(t)\ &=\hat{X}^\lambda(0)+\tfrac{\sum_{k=1}^2\mu_kN_k^\lambda}{\sqrt{N^\lambda}}t-\delta\tfrac{\sqrt{\sum_{k=1}^2\mu_kN_k^\lambda}}{\sqrt{N^\lambda}}t\\ &\quad+\textstyle\sum_{k=1}^2\mu_k\int_0^t[X_k^\lambda(s)]^-ds-\tfrac{\sum_{k=1}^2\mu_kN_k^\lambda}{\sqrt{N^\lambda}}t+\tfrac{M^\lambda(t)}{\sqrt{N^\lambda}}+o(1)\\ &=\hat{X}^\lambda(0)-\delta\sqrt{\mu}t+\textstyle\sum_{k=1}^2\mu_k\int_0^t[X_k^\lambda(s)]^-ds+\tfrac{M^\lambda(t)}{\sqrt{N^\lambda}}+o(1)\\ &=\hat{X}^\lambda(0)-\delta\sqrt{\mu}t+\textstyle\sum_{k=1}^2\mu_k\int_0^t[X^\lambda(s)]^-\cdot v_k\left([X^\lambda(s)]^-\right)ds+\epsilon^\lambda(t)+\tfrac{M^\lambda(t)}{\sqrt{N^\lambda}}+o(1),\end{aligned}$$

where $M^\lambda/\sqrt{N^\lambda} \Rightarrow B$, where B is a Brownian motion with zero drift and infinitesimal variance 2μ. Also, by Proposition (4.3) we have that $sup_{t\leq T}|\epsilon^\lambda(t)| \to 0$ as $\lambda \to \infty$. Applying the continuous mapping theorem to the process $\vec{\hat{X}}^\lambda$ establishes that $\hat{X}^\lambda \Rightarrow \hat{X}$. It is left to show that weak convergence of the scaled queue length and number of idle servers processes holds. But this part easily follows from the continuous mapping theorem and Proposition 4.3. ∎

Proof of Theorem 4.1: The theorem follows from Propositions 4.4, A.1, A.2, and A.3. ∎

Proof of Theorem 4.2: For clarity of presentation, we drop the policy π' from the notation. It is useful to first write the process $\hat{X}$ in (2.29) in terms of the diffusion-scaled processes

$$\hat{A}^\lambda(t) := \frac{A^\lambda(t) - \lambda t}{\sqrt{\lambda}} \text{ and } \hat{D}_k^\lambda(t) := \frac{D_k^\lambda(N^\lambda t) - N^\lambda \mu_k t}{\sqrt{N^\lambda}}$$

as follows

$$\begin{aligned}
\hat{X}^\lambda(t) &= \hat{X}^\lambda(0) + \sqrt{\frac{\lambda}{N^\lambda}}\hat{A}^\lambda(t) - \sum_{k=1}^{2} \hat{D}_k^\lambda\left(\overline{T}_k^\lambda(t)\right) \\
&\quad + \left(\frac{\lambda - \sum_{k=1}^2 \mu_k N_k^\lambda}{\sqrt{N^\lambda}}\right) t - \frac{1}{\sqrt{N^\lambda}} \sum_{k=1}^{2} \mu_k \int_0^t I_k^\lambda(s) ds.
\end{aligned}$$

Next define

$$\hat{\mathcal{I}}^\lambda(t) := \int_0^t \mu_1 \hat{I}^\lambda(s) + \mu_2 \hat{I}^\lambda(s) ds,$$

and note that

$$\hat{X}^\lambda(t) = \hat{X}^\lambda(0) + \sqrt{\frac{\lambda}{N^\lambda}}\hat{A}^\lambda(t) - \sum_{k=1}^{2} \hat{D}_k^\lambda\left(\overline{T}_k^\lambda(t)\right) + \left(\frac{\lambda - \sum_{k=1}^2 \mu_k N_k^\lambda}{\sqrt{N^\lambda}}\right) t - \hat{\mathcal{I}}^\lambda(t). \quad \text{(B.12)}$$

It follows from assumptions (A1), (A2), and Proposition 2.1 that

$$\overline{T}_k^\lambda(t) = \int_0^t \overline{Z}_k^\lambda(s) ds \to t q_k, \text{ u.o.c., a.s.,}$$

as $\lambda \to \infty$. Let $e(t) = t$ for all $t \geq 0$ be the identity process and let B be a standard Brownian motion. The functional central limit theorem, random time change theorem, continuous mapping theorem, and assumptions (A3) and (A4) establish

$$\sqrt{\frac{\lambda}{N^\lambda}}\hat{A}^\lambda(t) - \sum_{k=1}^{2} \hat{D}_k^\lambda \circ \overline{T}_k^\lambda + \left(\frac{\lambda - \sum_{k=1}^2 \mu_k N_k^\lambda}{\sqrt{N^\lambda}}\right) e \Rightarrow -\delta\sqrt{\mu} + \sqrt{2\mu}B, \quad \text{(B.13)}$$

as $\lambda \to \infty$, also noting that $q_1 + q_2 = 1$ by (2.17).

Proposition 1 in [7] establishes that $(\hat{X}^\lambda, \hat{\mathcal{I}}^\lambda)$ is tight in D. Consider any subsequence on which

$$\left(\hat{X}^{\lambda_i}, \hat{\mathcal{I}}^{\lambda_i}\right) \Rightarrow \left(\hat{X}, \hat{\mathcal{I}}\right),$$

as $\lambda_i \to \infty$. The same arguments in Lemma 6 in [5] that establish the process $\hat{\mathcal{I}}^\lambda$ satisfies the assumptions of Lemma 5 in [6] (which is a very special case of a result in [35]) hold in our setting[3]. Hence by Lemma 5 in [6],

$$\int_0^\cdot d\hat{\mathcal{I}}^{\lambda_i}(s) \Rightarrow \int_0^\cdot d\hat{\mathcal{I}}(s), \tag{B.14}$$

as $\lambda_i \to \infty$. For each λ_i, it follows from the definitions of $\hat{\mathcal{I}}^\lambda$ and the fact that $\hat{I}_1^\lambda(t) + \hat{I}_2^\lambda(t) = \left[\hat{X}^\lambda(t)\right]^-$ that

$$\mu_1 \left[\hat{X}^{\lambda_i}(t)\right]^- \leq d\mathcal{I}^{\lambda_i}(t) \leq \mu_2 \left[\hat{X}^{\lambda_i}(t)\right]^- \text{ for all } t \geq 0,$$

and so also

$$\mu_1 \left[\hat{X}(t)\right]^- \leq d\hat{\mathcal{I}}(t) \leq \mu_2 \left[\hat{X}(t)\right]^- \text{ for all } t \geq 0.$$

Then, there exist unique processes u_1 and u_2 having $u_1(t) \in [0,1]$ and $u_2(t) \in [0,1]$ for all $t \geq 0$ such that

$$(\mu_1 u_1(t) + \mu_2 u_2(t)) \left[\hat{X}(t)\right]^- = d\hat{\mathcal{I}}(t). \tag{B.15}$$

Therefore, along the subsequence λ_i, from the expression for $\hat{X}^\lambda$ in (B.12), the weak convergences in (B.13) and (B.14), the equivalence in (B.15), and the continuous mapping theorem,

$$\hat{X}^{\lambda_i} \Rightarrow \hat{X},$$

where $\hat{X}$ satisfies the stochastic equation (3.1) under control $\vec{u} = (u_1, u_2)$.

The control $\vec{u}$ is admissible because Assumption (A5) guarantees condition (C1) is satisfied, and (B.15) shows condition (C2) holds. Furthermore, as in the proof of Lemma 2.1,

$$0 = \lambda_i + \mu_1 E\left[I_1^{\lambda_i}(\infty)\right] + \mu_2 E\left[I_2^{\lambda_i}(\infty)\right] - \mu_1 N_1^{\lambda_i} - \mu_2 N_2^{\lambda_i}.$$

Since $I_1^{\lambda_i}(\infty) + I_2^{\lambda_i}(\infty) = X^{\lambda_i}(\infty)^-$, it follows that

$$0 = \lambda_i + (\mu_1 - \mu_2) E\left[I_1^{\lambda_i}(\infty)\right] + \mu_2 E\left[X^{\lambda_i}(\infty)^-\right] - \mu_1 N_1^{\lambda_i} - \mu_2 N_2^{\lambda_i},$$

and so

$$\frac{\mu_1 N_1^{\lambda_i} + \mu_2 N_2^{\lambda_i} - \lambda_i}{\sqrt{N^{\lambda_i}} E\left[\hat{X}^{\lambda_i}(\infty)^-\right]} = (\mu_1 - \mu_2) \frac{E\left[\hat{I}_1^{\lambda_i}(\infty)\right]}{E\left[\hat{X}^{\lambda_i}(\infty)^-\right]} + \mu_2.$$

The definition of asymptotic feasibility and assumption (A3) then show that taking limits in the above equation yields

$$\frac{\delta\sqrt{\mu}}{E\left[\hat{X}(\infty)^-\right]} = (\mu_1 - \mu_2) f_1 + \mu_2,$$

[3] To see this, it is useful to realize that the process $\hat{\mathcal{I}}^\lambda$ is equivalent to the process Q_1^n appearing in Lemma 6 in [5], and that [5] refers to the arguments used to establish Lemma 6 in [6], in which the process Q_1^n appearing in Lemma 6 in [5] is equivalent to the process Y^n appearing in Lemma 6 in [6].

or, equivalently,

$$E\left[\hat{X}(\infty)^-\right] = \frac{\delta\sqrt{\mu}}{\mu_1 f_1 + (1-f_1)\mu_2}.$$

We conclude that the conditions of Theorem 3.1 are satisfied, and so

$$E\left[\hat{X}^\star_{\triangle^\star}(\infty)\right]^+ \leq E\left[\hat{X}(\infty)^+\right].$$

Since the subsequence λ_i was arbitrary,

$$\liminf_{\lambda\to\infty} E\left[\hat{X}^\lambda(\infty;\pi)^+\right].$$

∎