A General Framework for the Study of Decentralized Distribution Systems

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We develop a general framework for the analysis of decentralized distribution systems. We carry the analysis in terms of a simplified model which entails $N$ retailers who face stochastic demands and hold stocks locally and/or at one or more central locations. An exogenously specified fraction of any unsatisfied demand (demand greater than locally available stock) at a retailer could be satisfied using excess stocks at other retailers and/or stocks held at a central location. We consider inventory ordering and allocation decisions. The operational decisions of inventory and allocation of stocks and the financial decision of allocation of revenues/costs must be made in a way consistent with the individual incentives of the various independent retailers. We develop a “coopetitive” framework for the sequential decisions of inventory and allocation. We introduce the notion of claims that allows us to separate the ownership (with decision rights) and the location of inventories in the system. For the cooperative shipping and allocation decision, we use the concept of core and develop sufficient conditions for the existence of the core. For the inventory decision, we develop conditions for the existence of a pure strategy Nash Equilibrium. For this decentralized system, we show that there exists an allocation mechanism that achieves the first-best solution for inventory deployment and allocation. We develop conditions under which the first-best equilibrium will be unique. Our model can be easily generalized to include complicated ownership structures such as “super dealers,” partnerships, “inventory speculators,” and situations in retail e-commerce settings such as “click-through arrangements,” separation of “demand generators,” and “fulfillment houses,” etc. It can also be applied to situations involving capacity allocations and product substitutions.

(Decentralization; Distribution; Game Theory; Inventory; e-Business)

1. Introduction

Information technology is changing the way companies structure and operate their distribution network to serve customer needs. Consider the following typical scenario faced by a customer seeking to buy a specific new car at a dealership. If the right car, or an acceptable substitute is not available on her lot, the dealer may make the following suggestion, “Do not worry. I will get the car for you.” Where does the dealer get the car from? Naturally, she must have access to an information system that provides her with timely data on the availability of unsold cars on the lots of participating dealers and in some centralized distribution centers. Based on this information, the

1 This is the first-hand experience of the first author purchasing a Honda.
dealer attempts to locate a specific car that is acceptable to the customer and could be delivered within acceptable time and cost. If such a car can be identified, both the selling dealer, who “owns the sale,” and the supplying dealer, who “owns the unit,” must also agree on the appropriate compensation to each party. If all of this can be accomplished, the dealers involved in this transaction can enjoy the benefits of ex post, cooperative, pooling of stocks.

The concept of sharing (pooling) of stock is, of course, not new. It is well known that when demands are uncertain, centralization of stocks helps a firm reduce inventory investment and/or improve customer service. While traditionally firms have exploited this “risk pooling” benefit of “centralization” by physically consolidating their stocks, the example above makes use of information technology to extract the same benefits from “virtual pooling” under which the “pooled” stocks are disbursed among the various locations, but are shipped from location to location to satisfy demand.

Examples of such arrangements are quite common in practice, and are often mentioned in the literature. For example, Narus and Anderson (1996) cite the case of the machine tool builder Okuma America Corporation, a subsidiary of Japan’s Okuma Corporation. Each of Okuma’s 46 distributors in North and South America carry machine tools and selected repair parts in its inventory. A shared information technology system called Okumalink keeps distributors informed about the location and availability of machine tools and parts in Okuma warehouses in Charlotte and Japan. When a customer orders a machine tool or a part that a distributor does not have, the distributor checks Okumalink to determine the item’s availability in the distribution network and arrange for intrachannel exchanges of machine tools and parts electronically. Another example, involving an alliance of independent distributors who cooperate to serve their customers is the Grainger Industrial Supply Operations, a division of W. W. Grainger. Lee and Whang (2001) discuss an example of a Korean Cement company that engages product exchange with its competitor.

There is an extensive and well developed literature on the analysis of the classical distribution systems, which include inventory, allocation, and transportation decisions (see § 2). These analyses, however, assume a single (central) decision maker. In such systems, often demand at each location is satisfied using local stocks. Additional profits, over and above the standard “newsvendor profits,” can be generated by transshipping excess inventory to locations with excess demand. However, the distribution system in the examples cited earlier consist of multiple decision makers who make decisions regarding inventory, allocation and transportation. To operationalize the sharing of stocks (via transshipments) in the decentralized system, two components are essential. First, we need an information system that provides accurate data on (excess) demand and (excess) stock. Without this there can be no exchange. Second, we need to secure the willingness of the various parties to cooperate with each other. This willingness, in turn, depends on the existence of an associated process for allocating the relevant costs and revenues in a way that is consistent with the self-interests of the relevant parties. It is this last component that is the focus of this paper.

To highlight the essence of the results, and to economize on notational complexity, we carry the analysis in the paper of a “bare bones” model that is introduced below, and is described in more detail in § 3. The model allows for N retailers who could hold inventory locally as well as at some given central location. The costs included in the model, namely purchase, salvage, and transportation, as well as the revenues, are assumed linear and the parameters could vary across the retailers. Each retailer is faced with her own stochastic demand, which may be correlated with the demand of other retailers. Some portion of the unsatisfied demand at a given dealer can be satisfied by shipping excess units from other locations. However, as discussed in § 6, the model can be extended easily to accommodate a large number of supply-chain structures, some of which are quite novel and appropriate for the e-tailing environment.

There are two broad classes of decisions to be made within this general framework that are made at two different stages: (a) inventory decisions for local and...
We analyze the cooperative shipping decision using the notion of a core. Specifically, for a given pattern of stocking positions by the various agents, and a given demand pattern, we study the cooperative game defined by the excess profits that could be generated by cooperation. In general, not every game has a nonempty core. However, games whose cores are nonempty are much more conducive to cooperation. We demonstrate (Theorem 4.1) that the core of this game is not empty, and suggest a set of market prices that reflect the economic true value of each unit of excess inventory or excess demand. Thus, our result indicates that cooperation at this stage is possible and offers a price mechanism to support it.

As we mentioned earlier, the inventory decision is taken unilaterally by the various agents. In making her decision, a given agent takes account of her expectation of demand at the various locations, as well as of inventory stocking decisions by the other agents. We analyze this decision using the concept of Nash equilibrium. We develop conditions for the existence of a pure strategy Nash equilibrium. We show that there exists an allocation mechanism that achieves the "first-best" solution for inventory deployment and allocation (Theorems 5.1–5.2). Thus, we demonstrate that there exists an allocation mechanism that maximizes the value of information in a decentralized system.

Several straightforward extensions of the model are feasible. The model can be extended to include product substitution. One can also replace "inventory" with "capacity," and the results still hold with the appropriate modification in interpretations. The ex post part of our "bare bones" model applies directly to more complex ex ante decisions outlined in § 6. All that is needed is that by the time the cooperative decision of Stage 2 takes place, each unit of demand, and each unit of inventory, is owned individually by a single decision maker.

The rest of the paper is organized as follows. In § 2 we give a brief literature review. In §§ 3 and 4 we develop the general framework. In § 3 we present a basic model of the \( N \) retailer system with local and centralized inventories and discuss the solution from a single decision-maker perspective. This solution is called the first-best. In § 4 we develop the general
framework for analysis of a distribution network when each retailer is an independent decision maker. In § 5 we demonstrate the existence of an allocation mechanism for the decentralized system to achieve the first-best solution. In § 6, we conclude with extensions and generalizations of the basic framework presented here and directions for future research.

2. Literature Review

Our basic problem framework is similar to the single warehouse-multiple retailer stochastic inventory literature, which has been extensively analyzed. Of particular relevance is the literature on distribution systems with transshipments. Karmarkar (1977, 1979, 1981a,b) develops a very general framework for multilevel inventory problems with transshipments. All these models assume that transshipment is anticipatory, i.e., the transshipment quantities are determined before demands are realized. Multilation models with nonanticipatory transshipments (Das 1975, Hoadley and Heyman 1977, Lee 1987, Klein 1990, Robinson 1990) are closer to our model because transshipments occur after realization of demands. The notion of a central stock in our framework is similar to the concept of physical centralization of stocks. The benefits of centralization in a news vendor environment with multiple retailers, fixed lead times, and backlogging is shown by Eppen (1979) and Eppen and Schrage (1981). All of these studies, however, assume (i) a single ownership of the entire system and (ii) that demands are independent across retailers.

Work on competitive multilocation distribution systems with stochastic demands and transshipments is limited. There has been some work in decentralized approaches for multiechelon systems; see for example, Lee and Whang (1999), Cachon and Zipkin (1999), or Federgruen et al. (2001). However, these pieces of work either assume one agent at each echelon or model deterministic demand. In contrast, our interest is in echelon(s) with multiple agents and stochastic demand. The most relevant literature in this area can be classified using three attributes, namely “search,” “system,” and “interaction.” “Search” refers to the activity of looking for the goods at other retailers when it is unavailable at the local retailer. It could be performed either by the customer or the retailer, depending on who has the information regarding stock availability. In addition, the allocation of profits/revenues from satisfying excess demand (at one retailer) using excess stock (at another retailer) is impacted by whether the search is customer- or retailer driven. In a customer-driven search, naturally, all proceeds from the sale go to the retailer who satisfies excess demand. On the other hand, in a retailer-driven search the allocation of excess profit generated can be negotiated between the two retailers involved in the exchange process. We note that strategic interaction between retailers in decentralized distribution systems with stochastic demand may occur even without search; see, for example, Cachon (2001). “System” refers to the number of players modeled in the analysis and is further categorized as duopoly or oligopoly. Finally, “interaction” is further categorized into horizontal or vertical. Horizontal interaction refers to the competitive decision making by the retailers; search clearly implies such interaction. Vertical interaction refers to the competitive dynamics between retailers and their upstream supplier. Based on this categorization, we briefly review six papers (Parlar 1988, Lippman and McCardle 1997, Anupindi and Bassok 1999, Rudi et al. 1999, Mahajan and van Ryzin 1999, van Ryzin and Mahajan 1999). Of these papers Anupindi and Bassok (1999) and van Ryzin and Mahajan (1999) consider both horizontal and vertical interaction.

Parlar (1988) analyzes the substitutable product inventory problem with random demands. While the model is presented in the context of substitutable products, the framework carries over to two retailers carrying the same or substitute goods. Because any spillover (excess) demand is satisfied by the retailer who has stock (and who also keeps the revenues from the sale), search can be considered to be performed by the customer. Lippman and McCardle (1997) analyze a competitive newsboy (oligopoly) model. They start with aggregate industry demand and specify rules to split these demands to the various firms. Different splitting rules impose different correlation structures between firm demands. Once the local demands are satisfied, excess stock at one location is
used to satisfy excess demand at another in a manner very similar to the one considered by Parlar and, hence, the search can be considered to be performed by the customer. They then examine the relationship between equilibrium inventory levels and the demand splitting rules and provide conditions under which there is a unique equilibrium. Mahajan and van Ryzin (1999) also consider inventory competition under substitutable goods in an oligopoly setting. Unlike others, they posit a dynamic model of consumer choice and show that competition leads to overstocking in equilibrium that becomes severe competition increases.

Anupindi and Bassok (1999) study the issue of centralization of stocks in a system with a single manufacturer and two independently owned retailers. Retailers face stochastic demand with lost sales and search is performed by the customer. They study the equilibrium behavior parameterized by customer search and show that centralization of stocks need not always benefit the manufacturer; whether it does or not depends on the extent of search. This work thus considers a duopoly system but studies both horizontal and vertical interactions. van Ryzin and Mahajan (1999) analyze a similar two-echelon setting with key differences. First, demand is modeled as derived from dynamic consumer choice. Second, they analyze two distinct models of the channel. In one they have several manufacturers competing to stock a single good with one retailer and in the other they have several retailers competing for a single good supplied by one manufacturer. They then investigate whether trading mechanisms like Vendor Managed Inventory (VMI) and Retailer Managed Inventory (RMI) lead to channel coordination.

While this paper was being written, we came across the work of Rudi et al. (1999) who discuss a duopoly model for inventory decisions with retailer-driven search and horizontal interaction. They explore conditions under which decentralized inventory decision making leads to the first-best solution. The paper can be considered as a special case of the general framework presented here though the concept of ex post cooperation in a duopoly system is almost trivial. Furthermore, they differ in the incentive mechanisms employed.

3. Inventory Management in a Centralized System

Before we develop a general framework for the analysis of decentralized distribution systems in the next section, we first present a basic system that allows us to discuss management of inventory from the perspective of a central decision maker. We call this the first-best solution and it will ultimately form a benchmark case.

Consider a set \( N = \{1, \ldots, N\} \) of retailers of a common product, each of which faces its own demand and manages his own inventory and sales. We use the well known framework of the single period, single item uncapacitated newsvendor problem. Specifically, for each \( n \in N \), let \( c_n \) be the unit cost, \( v_n \) the unit revenue, and \( h_n \) the unit salvage value, respectively. We let these parameters depend on the retailer \( n \), reflecting any possible differences among the retailers in terms of location, size, purchase power, salvage opportunities etc. In addition, each retailer \( n \in N \) is faced with its own demand distribution, \( D_n \), with \( \bar{D} = (D_1, \ldots, D_N) \). We assume that the demand profile \( \bar{D} \) is distributed according to a continuous joint cumulative distribution function (CDF) \( F(\bar{D}) \), with a corresponding set of marginal (not necessarily independent) CDF’s denoted \( F_n(D_n) \). Finally, each retailer \( n \in N \) orders (and pays for) a certain quantity of stock \( X_n \) before the actual demand \( D_n \) is realized. We denote the profile of stock ordered by \( \bar{X} = (X_1, \ldots, X_N) \). After the demand \( D_n \) at retailer \( n \) is realized the (direct) sales is \( S_n = \min \{X_n, D_n\} \) and the leftover (local) inventories for salvage is \( H_n = \max \{X_n - D_n, 0\} \). The optimal level of inventory in this model is obtained by each retailer solving his own, independent, newsvendor problem. This describes the inventory and sales decisions when no inventory is shared between the various retailers.

Pooling refers to the sharing of inventory among several retailers so that shortages at one retailer can be satisfied from surpluses at another. It is well known (Eppen 1979, Eppen & Schrage 1981) that pooling allows the system to increase its overall profit relative to the basic (no-pooling) system. Pooling can be achieved by physical pooling, virtual pooling or product substitution; see Anupindi et al. (1999). Physical pooling means holding some common inventory in a
centralized location so that it is available on short notice to each of the retailers. Virtual pooling means allowing demand in one location (retailer) to be met from inventory located at another through transshipments.\(^2\)

To add pooling (both physical and virtual), we make the following additions to our model described thus far. First, we include a set of \(\text{centralized warehouses } W = \{1, \ldots, W\}\). For each warehouse \(w \in W\), we let \(c_w\) be the purchasing cost, \(v_w\) the salvage value, and \(Y_w\) stock position at warehouse \(w\) with \(Y = (Y_1, \ldots, Y_W)\). Next, for each \(i \in W \cup \mathcal{N}, n \in \mathcal{N}\), let \(t_{i,n}\) be the per-unit additional shipping, handling, or discounting costs associated with meeting demand at retailer \(n\) from location (retailer or warehouse) \(i\). Finally, let \(\beta_{i,n}\) be the (deterministic) fraction of customers at retailer \(n\) who will accept service out of location \(i\). Clearly, \(0 \leq \beta_{i,n} \leq 1\). The remaining \(1 - \beta_{i,n}\) consumers prefer to balk rather than accept service from an alternate location.

Note that in some situations it may be advantageous to ship units from a given location for sale at another, even if there is demand for the unit at the original location. Such shipment is profitable if the differential in selling prices is high enough to cover the shipping costs plus the differential in salvage costs. We call this arrangement pooling of stocks. In most cases, however, it is natural to assume that each agent covers his own demand first, and only the residual inventory is shipped. We call this latter, more common arrangement, pooling of residuals. Note that the two arrangements coincide if the selling price differentials are low relative to the shipping costs. For simplicity of notation, we restrict ourselves here to pooling of residuals for the local inventory and pooling of stocks for the inventory in the central warehouses. Our approach, however, can be easily modified to accommodate any variation of pooling of residuals and stocks.

We now consider the decision making process for a general system in which both physical and virtual pooling are feasible, but in which all inventory decisions are done centrally, by a single decision maker. Managing this system requires implementing two sets of decisions at two different points in time:

1. **Inventory Deployment Decisions**: Before the demand profile \(\hat{D}\) is realized, the decision maker needs to decide on the quantity of inventory to be stocked at each retailer and at each warehouse, namely \(\hat{X}\) and \(\hat{Y}\).
2. **Shipping Decisions**: After the demand profile \(\hat{D}\) is realized it is first satisfied, to the extent possible, from local inventory \(\hat{X}\). Then a decision is made on the disposition of the residual (local) and common inventory towards the various retailers to meet the residual demands.\(^3\) Denote by \(q_{i,n}\) the number of units shipped from location \(i \in W \cup \mathcal{N}\) (warehouse or retailer) to retailer \(n \in \mathcal{N}\). Let \(\hat{Q} = \{q_{i,n}, i \in W \cup \mathcal{N}, n \in \mathcal{N}\}\) be the shipping pattern.

Mathematically, the decision maker’s problem is a two-stage stochastic program. In the first stage the inventory decisions are made and in the second stage, given inventory levels and realized demands, the optimal shipping pattern is determined. In a centralized system with pooling, one seeks to implement both the inventory deployment decision and the shipping decision such that the performance (expected profits) of the entire system is optimized. This can be achieved by the following backward induction process. Recall that \(S_n = \min(X_n, D_n)\) is the sales from local inventory at \(n\) and \(H_n = \max(X_n - D_n, 0)\) is the residual inventory at \(n\) after local demand is satisfied from local inventory. Then,

\[
\text{(1)} \quad P_X(\hat{X}, \hat{Y}, \hat{D}, \hat{Q}) = \sum_{n \in \mathcal{N}} \left[ r_n S_n + v_n H_n - c_n X_n \right] - \sum_{w \in W} (c_w - v_w) Y_w 
+ \sum_{i \in W \cup \mathcal{N}, n \in \mathcal{N}} (r_n - v_n - t_{i,n}) q_{i,n},
\]

where the superscript \(c\) indicates a centralized sys-

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\(^2\) In a multiproduct setting, pooling of inventory could also occur through product substitution by allowing demand for a particular product to be satisfied by another, related product. For the sake of notational simplicity, we do not include the case of product substitution in the formal model presented below. However, as detailed in § 6, all our results can be easily extended to include this form of pooling.

\(^3\) We disallow for inventory shipments between locations purely to save on inventory holding costs or exploit any arbitrage opportunity in salvage markets.
tem. Note that the first term represents the sum of unpooled (generated from local transactions only) profits of each player, the second term represents the net cost of common inventory, and the third term the excess profits because of shipping of residual and common inventories to satisfy residual demands.

(2) Shipping Decision: For each potential state \((\tilde{X}, \tilde{Y}, \tilde{D})\), find the shipping pattern \(\tilde{Q}\) that maximizes profit. Specifically,

\[
P_{\tilde{X}}(\tilde{X}, \tilde{Y}, \tilde{D}) = \max_{\tilde{Q}} P_{\tilde{X}}(\tilde{X}, \tilde{Y}, \tilde{D}, \tilde{Q})
\]

subject to

\[
\sum_{i \in \mathcal{N}} q_{i \alpha} \leq H_{\alpha}, \quad \forall \ i \in \mathcal{N},
\]

\[
\sum_{i \in \mathcal{N}} q_{i \alpha} = Y_{\alpha}, \quad \forall \ \alpha \in \mathcal{W},
\]

\[
\sum_{i \in \mathcal{N} \cup \mathcal{W}} q_{i \alpha} / \beta_{i \alpha} \leq E_{\alpha}, \quad \forall \ \alpha \in \mathcal{N},
\]

\[
q_{i \alpha} \geq 0, \quad \forall \ i \in \mathcal{N} \cup \mathcal{W}, \quad n \in \mathcal{N},
\]

where \(E_{\alpha} = \max (D_{\alpha} - X_{\alpha}, 0)\) is the residual demand at \(n\) which cannot be satisfied from local inventory. Constraint (2b) ensures that the total quantity shipped from retailer \(i\) is no more than the residual inventory available to that retailer; similarly constraint (2c) ensures the total shipments from a central location \(w\) do not exceed the total stock at this location. Finally, constraint (2d) ensures that total quantity shipped to a retailer \(n\) does not exceed the residual demand faced by it.

(3) For each inventory profile \((\tilde{X}, \tilde{Y})\) compute the expected value of the profits

\[
f_{\tilde{X}}(\tilde{X}, \tilde{Y}) = P_{\tilde{D}} P_{\tilde{X}}(\tilde{X}, \tilde{Y}, \tilde{D}).
\]

(4) Inventory Decision: Find the optimal inventory profile \((\tilde{X}, \tilde{Y})\) that maximizes the expected profit

\[
\Pi_{\tilde{X}} = \max_{\tilde{X}, \tilde{Y}} f_{\tilde{X}}(\tilde{X}, \tilde{Y}).
\]

We say that \(\Pi_{\tilde{X}}\) represents the first-best profits and \((\tilde{X}^*, \tilde{Y}^*) = \arg \max_{\tilde{X}, \tilde{Y}} f_{\tilde{X}}(\tilde{X}, \tilde{Y})\) is the first-best solution.

4. Inventory Management in a Decentralized System with Pooling

We now examine an inventory system in which the various retail outlets belong to independent agents, who are interested in maximizing their own performance rather than that of the entire system. Obviously, cooperation via pooling (physical or virtual) allows the agents to increase the overall profit ("increase the size of the pie"), as indicated in the previous section. At the same time, each agent must also protect his own profit ("piece of the pie"), and thus may be put in a position of conflict with the other agents. This tension between cooperation and competition introduces an additional level of complexity to the problem. Below we analyze this interaction.

4.1. Solution Concepts

We now discuss several approaches for analyzing situations that include a combination of unilateral and negotiated actions. Recall the decision-making framework that was necessary to support the global optimum in the case of a centralized system with pooling. How are these steps to be implemented in a decentralized system with pooling? The following set of decisions need to be made:

(1) Inventory Decisions: Before the demand profile \(\tilde{D}\) is realized:

(a) Each agent, \(n \in \mathcal{N}\) must determine his own inventory position, \(X_{\alpha}\).

(b) An agreement among the agents must be achieved on the level of common inventories at the various centralized warehouses \(Y_{\alpha}\), \(w \in \mathcal{W}\).

(c) An agreement must be reached among the retailers as to how the costs of the pooled inventory \(\tilde{Y}\) are to be allocated to the various agents.

(2) Shipping Decisions: In addition, the following decisions need to be made after the demand profile is realized:

(a) deployment of the residual and common available inventory towards the various residual demands at the respective locations after demand profile is realized, and
(b) allocation of the excess profits generated by sales at the various retailers because of the shipping decisions for a given realized demand profile.

Observe that while the allocation of profits is done after the demand profile is realized, the players need to reach an agreement regarding the allocation mechanism before the demand profile is realized. Hence, the implementation has to proceed in three steps. In the first step, retailers agree on the allocation mechanism to be used to allocate the surplus profits from the ex post shipping decision. The second and third steps are then, respectively, the inventory and shipping decisions described above. How can we analyze the implementation of these decisions in a decentralized system? Before we present the details, a few basic definitions are useful.

Coalition is a set of agents who cooperate to collectively achieve the maximal profits for themselves.

Grand Coalition is a coalition of all agents (represented by the set $\mathcal{N}$) in the game.

Allocations are the shares of profit allocated to various members of a coalition. We represent it by $\bar{\alpha} = (\alpha_n; n \in \mathcal{N})$, where $\alpha_n$ is the share of profits allocated to agent $n$.

4.1.1. Complete Cooperation. Several authors (Gerchak and Gupta 1991, Robinson 1993, Hartman and Dror 1996) have studied models of complete cooperation among the agents for special cases of the general distribution system. By complete cooperation we mean here that the agents somehow reach a cooperative decision on both inventory shipments and allocations (possibly with the help of a third party). That is, they collectively achieve the centralized optimum, while preserving the interests of each agent. This possibility can be investigated using the concept of the core (Luce and Raiffa 1985) applied to the combined problem of inventories and allocations. To see how this is done let $\bar{\alpha} = (\alpha_n; n \in \mathcal{N})$ be an allocation of $\Pi_{\mathcal{N}}$. Define for each coalition $S \subseteq \mathcal{N}$ the value $V^*_S$ as the maximal total profits for the members of $S$, if they were to act as a single decision maker, ignoring the members of $\mathcal{N} \setminus S$. Clearly, $V^*_S = \Pi^*_S$. Similarly, $V^*_S$ is determined by solving the centralized system in $\S$ 3 for the coalition $S$. We then say that an allocation $\bar{\alpha}$ is a core allocation if it satisfies the following set of equations:

$$\sum_{j \in S} \alpha_j \geq V^*_S \quad \forall S \subseteq \mathcal{N} \quad (5a)$$

$$\sum_{j \in \mathcal{N}} \alpha_j = V^*_\mathcal{N} \quad (5b)$$

The utility of this concept derives from the fact that if inequality (5a) is violated for a given coalition $S$, then members of this coalition would consider breaking away from the other participants and attempt to do better on their own. Thus, situations in which the core is empty are typically not conducive to supporting full cooperation.

4.1.2. Cooperation: A Competitive-Cooperative System. In several situations, complete cooperation as defined earlier may not be possible. Indeed, we conjecture that the core as defined by Equation (5) is empty for the situation we examine. We note, however, that the analyses outlined in the previous section ignore crucial factors in the situation under study here. Specifically, we note that our agents are faced with two fundamentally different types of decisions with respect to the interaction with the other agents. The first type, such as the decision of how much inventory a given agent buys on his own, can be made unilaterally, without the need to reach an agreement with the other agents. In contrast, the second type of decision, such as on the level of the common pool of inventory, or on the shipping and allocation decision, the agents must somehow reach a certain level of consensus, that is, must act in a nonunilateral fashion. Clearly, these two different types of decision prob-

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4 These models effectively assume identical cost structures and no transshipment costs. Thus, there is a central pool of jointly owned inventory that is used to satisfy demands of the various retailers.

5 Even for simple systems like those considered by Gerchak and Gupta (1991), Robinson (1993), and Hartman and Dror (1996), it is as yet unknown if the core, defined by Equation (6), exists.

6 Obviously, when an agent makes this decision he takes into account his predictions on the future actions of the other agents, as well as the way profits will be allocated to each agent. In that sense, his decision is tied into the actions of the other agents. However, once the agent has reached a decision, he can implement his action unilaterally.
lems require different modes of analysis. Note also that the inventory and the shipping/allocation decisions are separated in time, while the former are done prior to realization of demand (ex ante), the latter decisions are done after demand realization (ex post). As we see below, the introduction of claims render all the ex ante decisions as unilateral, while all the ex post decisions remain nonunilateral. This suggests a solution concept in which agents act cooperatively ex post, even though their ex ante actions are determined in a competitive fashion. Such a hybrid approach could be termed as a "cooperation" borrowing from Brandenburger and Nalebuff (1996).

4.1.3. Claims. We say that the common inventory level \( \bar{Y} \) is claimed if each unit of inventory that comprises \( \bar{Y} \) is separately owned (regardless of its location) by some retailer. Let \( Y_{wn} \) denote the inventory at centralized location \( w \in W \) claimed by retailer \( n \). We require that \( \sum_{n=1}^{N} Y_{wn} = Y_{w} \) for all \( w \in W \). Claims establish ownership of each unit of inventory in the system. They are established by purchasing inventory before demand is realized and entitle the owner, in turn, to determine its use ex post. The notion of claims allows us to separate the ownership and location of inventories. Observe that even if all inventory is claimed, it might be kept at centralized warehouses to avail of the transshipment cost advantages. Effectively, this implies that we have converted the cooperative decision making on common inventory (\( \bar{Y} \)) into a unilateral (noncooperative) decision making. That is, the three-step task of inventory decision outlined in \( \S \) 4.1 reduces to a single step, namely:

1') Inventory Decisions: Before the demand profile \( \bar{D} \) is realized:

(a) Each agent, \( n \in N \) must determine his inventory position, \( (X_{wn}, Y_{wn}, \ldots, Y_{wn}) = \bar{Z}_{n} \).

The shipping decisions remain unchanged. The decision-making framework with claims thus reduces to a set of noncooperative decisions for the level of local and pooled, but claimed, inventories and a cooperative-shipping decision.

4.2. Analysis
We now examine how the competitive-decision-making situation just outlined can be analyzed. We present analyses in the reverse order (in the spirit of backward induction in dynamic programming), starting with the cooperative-shipping problem.

4.2.1. The Cooperative-Shipping Decision. We first consider the cooperative decision for a given level of inventories \( \bar{X} \) and \( \bar{Y} \), and a given realization of demand. Recall that by the time we reach this point, each retailer has satisfied his local demand from local stock to the extent possible. Also, each retailer has collected the revenues resulting from this transaction and, in addition, paid for the entire stock of his local and claimed pooled inventory. What remains to be done is to agree to a pattern of shipping for the residual and pooled inventory to meet residual demands, and to divide the excess profits. Naturally, each agent is interested in promoting shipping and allocation patterns that would maximize his own share of the allocation.

While it is possible to postulate and model the explicit (and very complex) process of negotiations and bargaining, which ultimately leads to the determination of shipping pattern and the allocation of excess profits, the analysis is often quite intractable and the results are very sensitive to the assumptions about the minute details of the postulated process. An alternative approach utilized by game theorists, which we take here, is to abstract away from such detailed analysis, and to concentrate instead on some reasonable properties of the final outcome. The concept of core, introduced in \( \S \) 4.1.1 for the entire problem, is of this type.

Specifically, let \( [Z] = (Z_{1}, \ldots, Z_{n}) \) denote the inventory position of all players. For each coalition \( S \subseteq N \), let \( W_{S}^{*}([Z], \bar{D}) \) be the maximal excess profit available to members of \( S \) (in addition to what could be achieved without pooling) for a specific demand realization. Then \( W_{S}^{*}([Z], \bar{D}) \) is expressed as follows:

\[
W_{S}^{*}([Z], \bar{D}) = \max_{Q} \sum_{i \in S, w \in W} (r_{i} \cdot v_{i} - t_{i,n})q_{i,n},
\]

subject to

\[
\sum_{n \in S} q_{i,n} \leq H_{i,S}, \quad \forall i \in S,
\]

\[
\sum_{w \in W} q_{i,n} \leq \sum_{w \in W} Y_{i,w}, \quad \forall i \in W,
\]
\[
\sum_{i \in S \cup W} q_{i,n} / \beta_{i,n} \leq E_{n,S} \quad \forall \ n \in S,
\]
\[
q_{i,n} \geq 0 \quad \forall \ i \in S \cup W, \quad n \in S,
\]
which are functions of \((\tilde{X}, \tilde{D})\). Let \(\alpha_j([Z], \tilde{D})\) be the allocation of residual profits to player \(j\). As in § 4.1.1, we define the core of the game induced by Equation (6) as a solution to the set of inequalities
\[
\sum_{j \in S} \alpha_j([Z], \tilde{D}) \geq W^*_j([Z], \tilde{D}) \quad \forall \ S \subseteq \mathcal{N},
\]
\[
\sum_{j \notin S} \alpha_j([Z], \tilde{D}) = W^*_S([Z], \tilde{D}),
\]
Inequality (7a) insists that the members of \(S\) receive at least as much as they could have generated by acting on their own. Inequality (7b) requires that the retailers will implement the centrally optimal shipment pattern for the entire system. If a solution to Equation (7) exists, it means that competition and cooperation, in this case, for the shipping decision do not conflict: The retailers can agree on a shipment pattern and an allocation of profits that at the same time maximizes the “total pie” and gives each retailer (and subsets of retailers) as much as can be generated by acting independently. We call such games Snapshot Allocation Games or SAG([Z], \tilde{D}). In SAG([Z], \tilde{D}), we seek allocation mechanisms in a core, where the value of the game is derived by optimizing the shipping decisions for a given realization of the random demands.

The existence of the core for SAG([Z], \tilde{D}) is not obvious. In fact several intuitive allocations (e.g., those based on predetermined transfer prices) are not in the core of SAG([Z], \tilde{D}).

**Example 1.** Consider a symmetric system of four agents with all costs identical across agents. For instance, let \(r_n = 10\) and \(v_n = 5\) \(\forall n\) and \(t_{i,n} = 1\) \(\forall i, n\).

Then any shipment of excess supply to excess demand generates equal excess profit of \(10 - 5 - 1 = 4\) per unit. Consider an allocation system where the entire excess profit is allocated to the retailer(s) whose residual demands are satisfied. Thus, these retailers receive the revenue from sales but pay for transportation costs plus a transfer price equal to the salvage value. Consider the case when \(H_1 = 3, H_2 = 1, E_3 = 5,\) and \(E_4 = 2\). Total excess profits generated is equal to 16. Based on the allocation mechanism just described, \(\tilde{\alpha} = (0, 0, 16, 0)\). However, the coalition \([1, 4]\) can generate a surplus of 8 by themselves, thus contradicting the core constraints (7a). We note that if we were to allocate all excess profits to retailers 1 and 2, the core constraints of Equation (7) would be satisfied. This rule, however, would break down if excess supply were larger than excess demand.

The above example demonstrates that predetermined transfer prices, as commonly utilized, are problematic because they do not take into account “market conditions” and thus may allocate surplus revenues to the “wrong” agents. Our main theorem shows that this can be avoided.

**Theorem 4.1.** If all inventory is claimed, the core of SAG([Z], \tilde{D}) is nonempty. In particular, the allocation based on dual prices of the shipping decision in Equation (6) for the grand coalition \(\mathcal{N}\) is an allocation in the core of SAG([Z], \tilde{D}). That is,
\[
\alpha_n([Z], \tilde{D}) = v_n H_n + \sum_{w \in W} \gamma_w Y_{n,w} + \delta_n E_n,
\]
\(\forall \ n \in \mathcal{N},\)

where \(v_n, \gamma_w,\) and \(\delta_n\) are the dual prices associated with the constraints (6b), (6c), and (6d), respectively, of the optimal shipping problem (6) for the grand coalition \(\mathcal{N}\).

Note how the pricing allocation of Theorem 4.1 handles the situation in Example 1. Specifically, the dual price is zero for a retailer \(i\) with \(H_i > 0\) if total excess supply is too large and vice versa. Thus, these market prices represent the realistic value of excess demand and excess inventory.

In a different context of marketing-manufacturing incentives, Porteus and Whang (1991) illustrate the use of “market-based” mechanisms to provide incentives for coordination. In their setting, a firm consists of...
of the owner (the principal), a manufacturing manager, and several product managers (the agents). The product managers (marketing function) set inventory levels that are impacted by the (shared) capacity level affected by manufacturing manager’s efforts. All agents exert effort unobservable to the owner. They show that an incentive system in which the product managers pay the “shadow price” of the constrained capacity induces the first-best effort. This result is similar in spirit to Theorem 4.1, which induces the first-best shipping decision.

Note that the set of dual prices referred to earlier is not necessarily unique. Further, the core may contain allocations that are not related to dual prices such as illustrated in the following example.

Example 2. Consider the distribution system of Example 1 and let $H_1 = H_2 = H_3 = 2,$ and $E_4 = 10.$ Then the total surplus from pooling is $6 \times 4 = 24.$ The dual allocation of Theorem 4.1 is $\bar{\alpha} = (8, 8, 8, 0).$ But, the allocation $\bar{\alpha} = (0, 0, 0, 24)$ is also in the core. In fact any allocation of the form $\alpha_n \leq 8$ for $n = 1, 2, 3$ and $\alpha_4 = 24 - (\alpha_1 + \alpha_2 + \alpha_3)$ is in the core.

Example 2 sheds some more light on the economic meaning of the core in general and of the dual allocations of Theorem 4.1 in particular. Note that the dual allocation assigns surplus to actual units of excess demand and excess inventory that reflects the relative scarcity of supply and demand and the relative economic advantage of the various locations (in the case of Examples 1 and 2, all the locations are equally attractive). Thus, for the case of Example 2, the dual allocation assigns all surplus profits to holders of the (scarce) inventory and none to the holders of demand, which is abundant. If retailers 1 and 2 were owned jointly by a single agent, or if the inventory held by retailer 1 were to be split between two identical, but independent, retailers, the allocation of surplus to each unit of inventory or demand would not be affected. In contrast, the allocation $\bar{\alpha} = (0, 0, 0, 24)$ reflects the fact that retailer 4 holds a lot of power (in fact a “veto” power) over shipments because she is the only agent with excess demand. Note that this allocation depends crucially on the ownership structure: If the 10 units held by retailer 4 were to be split between two identical, but independent, owners (giving each 5 units of excess demand), then these retailers would have lost their veto power and would be allocated zero surplus. In general, a core allocation reflects a balance between the economic “value” of the units and the structural power of the owners. For more on the relation between the core and the dual allocation the reader is referred to Samet and Zemel (1984).

The discussion above highlights a particularly attractive property of the dual allocation of Theorem 4.1. Because this allocation is based on transfer prices, which reflect only the “economic value” of the various supply and demand units and not on the ownership of these units, Theorem 4.1 is applicable to any situation in which each unit of sales and inventory are owned by an individual decision maker; i.e., are “claimed.” Thus, the result holds for much more complex situations as outlined in § 6. If these conditions hold, the results suggest an allocation mechanism that will induce cooperation on the best shipping solution.

4.2.2. The Inventory Decision. How does a retailer set the inventory levels to maximize his own profits? Obviously, his actual profits will depend on local and claimed inventory positions of all players represented by $[Z] = (\hat{Z}_1, \ldots, \hat{Z}_N)$, on the realization of demand $\bar{D}$, on the shipping pattern $\bar{Q}$ to satisfy excess demand, and on the allocation of surplus profits $\alpha_n([Z], \bar{D})$. Thus, the retailer needs to determine his optimal level of $\hat{Z}_n$ given his expectations about demand, shipment, and allocations.

In the previous section we discussed allocation rules based on dual prices. More generally, we define an Allocation Rule $m$ (AR-$m$) as a mapping that specifies surplus allocations ($\bar{\alpha}^m$) for each possible $[Z]$ and $\bar{D}$. Clearly, the total surplus generated is a function of the shipping decision ($\bar{Q}$) determined by Equation (6). We restrict our attention below to allocation rules that are based on the optimal shipping decision in Equation (6). In particular, we denote by AR-$d$ the allocation rule based on dual prices of Theorem 4.1 (with an appropriate way of breaking ties in case of degeneracy).

Given an AR-$m$, we denote the total profits of retailer $n$ as $P^n_m([Z], \bar{D})$. Specifically,
\[ P^m_n([Z], \bar{D}) = r_n S_n + v_n H_n - c_n X_n - \sum_{w=1}^\omega (c_w - v_w) Y_{w,n} + \alpha^m_n([Z], \bar{D}). \] (9)

Let \( I^m_n([Z]) = E_0 P^m_n([Z], \bar{D}). \) Then a Nash Equilibrium (NE) of the inventory problem for a given allocation rule AR-m satisfies
\[ I^m_n([Z^m]) \ni I^m_n([Z^m]_w \cup \bar{Z}_n) \quad \forall \ n \in \mathcal{N}_0, \quad \forall \ n \in \mathcal{N}_0. \] (10)

That is, \( \bar{Z}^m_n \) is the optimal policy for retailer \( n \) for a given AR-m and actions of all other players.

**Theorem 4.2.** Consider the payoff function \( I^m_n([Z]) \) for player \( n \). Suppose:

1. \( I^m_n([Z]) \) is simultaneously continuous in \([Z], \) and
2. \( I^m_n([Z]) \) is unimodal in \( \bar{Z}_n \) for every \([Z]_w. \)

Then there exists a NE in pure strategies for the inventory decision game in a decentralized system with claims.

Sometimes, it is easier to check the above conditions on \( P^m_n([Z], \bar{D}) \) rather than the expected payoff function \( I^m_n([Z]) \). We can refine the conditions in Theorem 4.2 further if we make some assumptions on the demand distribution functions. Specifically, suppose that the demand distribution function belongs to the class of Polya Frequency Functions (P.F.F.), see Karlin (1968). Requiring that demand distribution be a P.F.F. is not very restrictive. In fact, a gamma distribution, uniform distribution on the interval \((0,1)\), and a truncated normal distribution are all P.F.F.s. Then we have the following result from Karlin (1968).

**Proposition 4.1.** If \( P^m_n([Z], \bar{D}) \) is unimodal in \( \bar{Z}_n \) for every \([Z]_w. \) and the demand distribution functions are P.F.F.s, then \( I^m_n([Z]) \) is unimodal in \( \bar{Z}_n \) for every \([Z]_w. \)

Thus, Proposition 4.1 along with the conditions (1) and (2) of Theorem 4.2 applied to \( P^m_n([Z], \bar{D}) \) ensure existence of a pure strategy NE.

5. Achieving First-Best

Section 4.2 gives a solution mechanism for determining inventory deployment levels and allocation decisions for the decentralized system. We derived the first-best solution in § 3. A fundamental question remains: Can the players in the decentralized system achieve the first-best? In this section we demonstrate the existence of an allocation mechanism that achieves first-best, show when this equilibrium will be unique, and illustrate with numerical examples.

5.1. Existence

Observe that first-best is achieved whenever the players in the decentralized system choose the same inventory vector as in the centralized system and make the same shipping decisions. Furthermore, notice for the same inventory vector (not necessarily first-best) both the centralized system and the decentralized system will make the same shipment decisions as long as the allocation of surplus resulting from the shipping decisions is in the core of \( \text{SAG}([Z], \bar{D}) \).

Therefore, first-best is achieved whenever there exists an allocation rule that is in the core of \( \text{SAG}([Z], \bar{D}) \) such that the corresponding Nash Equilibrium inventory decision is identical to the inventory decision in the centralized system. The purpose of this section is to show existence of such an allocation rule.

In the previous section we demonstrated (Theorem 4.1) that for any given inventory vector, allocations based on the dual were in \( \text{SAG}([Z], \bar{D}) \). While this ensures that the “first-best” is achieved for the second stage shipping decision, we do not know if it will also lead to first-best for the first-stage inventory decision. The next proposition illustrates that the answer is negative.

**Proposition 5.1.** Consider a duopoly. Allocations based on AR-D do not give the first-best inventory decision.

This is rather unfortunate because the dual allocation was a constructive way of identifying a member of the core. We now show that there does exist an allocation rule that achieves first-best and is in the core. To do so, we proceed in two steps. First we show (Theorem 5.2) the existence of an allocation rule that gives the players incentive to choose the first-best inventory solution if they cooperate in the shipping decision. Allocations based on this rule, however, need not be in the core. In the second step, we show (Corollary 5.1) that there always exists a set of side payments that force the allocations in the first step to be a member of the core. This second step is based on a general result (Theorem 5.1), which states that
there always exists a set of side payments that allows allocations based on any allocation rule to be made a member of the core while leaving the NE unchanged. We state and prove this general result first and then detail the two-step procedure to achieve the first-best.

**Theorem 5.1.** Consider an allocation policy AR-m with allocations \( \alpha_m^w([Z], \bar{D}) \) to player \( n \in \mathcal{N} \). Let \( f_n^w(\cdot) \) be the profit function of player \( n \) under AR-m. We assume that \( f_n^w(\cdot) \) are continuous and unimodular, so that the existence of a Nash Equilibrium in pure strategies is guaranteed. Let \( [Z]^w \) be a Nash Equilibrium in pure strategies. Then there exist constants \( w_n([Z], \bar{D}) \), which by using we can construct a new allocation policy, denoted by AR-m, with allocations

\[
\alpha_m^w([Z], \bar{D}) = \alpha_n^w([Z], \bar{D}) + w_n([Z]^w), \bar{D}),
\]

such that (i) \( [Z]^w = [Z]^m \) and (ii) AR-m are in core of \( SAG([Z], \bar{D}) \) whenever \( [Z] = [Z]^m \).

Now we detail the two-step process. For the first step observe that if the payoff functions of the retailers are set such that each will get some constant fraction (independent of inventory level) of the total expected system profit, then each retailer will make the system-optimal decision. We formalize this in Theorem 5.2. Recall from § 3 that \( P_n^w(\tilde{X}, \tilde{Y}, \bar{D}) \) represents the maximal profits of the centralized system for each potential state \( \bar{D} \) and \( f_n^w(\tilde{X}, \tilde{Y}) \) its expectation. Because all pooled inventory is claimed, we rewrite these as \( P_n^w([Z], \bar{D}) \) and \( f_n^w([Z]) \), respectively.

**Theorem 5.2.** Let \( \gamma_n \in (0, 1) \) be a constant such that \( \sum_{n=1}^{N} \gamma_n = 1 \). Consider a fractional allocation rule (denoted by AR-f) with the following allocation policy:

\[
\alpha_f^w([Z], \bar{D}) = \gamma_n P_n^w([Z], \bar{D})
\]

\[
= \gamma_n \left[ r_n S_n - v_n H_n - c_n X_n - \sum_{w=1}^{W} (c_w - v_w) Y_{w,n} \right] \tag{11}
\]

Then \( \alpha_f^w([Z], \bar{D}) \) induces the same equilibrium inventory levels as the first-best. That is, \( \tilde{X}^w = \tilde{X}^c \) and \( \sum_{w=1}^{W} Y_{w,n}^{w,n} = Y_{w,n}^c, \forall w \in W \).

If allocations based on AR-f above were in the core, then it induces the first-best stocking levels. This, however, may not be true in general. If so, then we can use Theorem 5.1 to construct a new allocation mechanism AR-c, which is in the core and the resulting NE is first-best. Basically, allocations under AR-c differ from those outlined in the first step by (demand-dependent) constants. This second step is summarized in the following corollary.

**Corollary 5.1.** Consider a modified fractional allocation rule (denoted by AR-c) that allocates the residual profits to player \( n \in \mathcal{N} \) as follows:

\[
\alpha_c^w([Z], \bar{D}) = \alpha_f^w([Z], \bar{D}) + w_n([Z]^c), \bar{D})
\]

where \( \alpha_f^w([Z], \bar{D}) \) is given by Equation (11) and \( [Z]^c \) is the first-best solution. Then the NE using \( \alpha_c^w([Z], \bar{D}) \) is first-best and allocations at the first-best solution are in the core of \( SAG([Z], \bar{D}) \).

Notice that \( w_n([Z]^c), \bar{D}) \) is a constant that does not depend on \( [Z] \) but on \( [Z]^c \) and the realization of demands. The final profits of each player is given by the newsvendor profits obtained from the sale of local stocks to satisfy one's own demands plus the expected value of allocations based on the dual.

One interpretation of this mechanism to achieve first-best is as follows. The key problem with the allocation mechanism in Theorem 5.2 is that while it achieves first-best, it may not be in the core. For a two-agent system, this is an "individual rationality" problem and often a side-payment scheme is used to circumvent the issue. Theorem 5.1 can be viewed as a multiagent equivalent of a "state-dependent" side-payment scheme.

**5.2. Uniqueness of First-Best Nash Equilibrium**

Can we say anything about the uniqueness of the NE?

**Theorem 5.3.** If the distribution system exhibits a unique first-best solution, then the NE under AR-c (Corollary 5.1) is such that the local inventories are unique and the sum of claimed inventories for each warehouse is unique.

The nonuniqueness of pooled but claimed inventories is because of the fact that if several owners can hold inventory in the same physical space, the NE can assign each one an arbitrary fraction of the total inventory. However, there is still uniqueness in terms of the total inventory held at the warehouse (assuming nondegeneracy of the centralized problem). This
is the best we can hope for. The nonuniqueness occurs because the cost parameters (procurement and salvage) are location specific. If these costs were agent specific, then we will obtain uniqueness of both local and pooled, but claimed, inventories.

Thus, for a large class of distribution systems, with proportional cost and revenue structure, there exists an allocation mechanism that will give first-best. Furthermore, if the centralized system exhibits a unique solution, then so does the corresponding decentralized system (in the sense of Theorem 5.3). Observe that uniqueness of the centralized solution is guaranteed if the expected profit function is concave (an assumption most often used in the analysis of centralized systems in the extant literature). Because exchange of stocks in our model is based on residuals, it can be shown that if it is profitable for each retailer to satisfy its own demand first, then the profit function in our centralized system will be concave.

5.3. Numerical Examples

We now illustrate the design of an allocation mechanism to achieve the first-best solution via two numerical examples. We first present a duopoly system and then an oligopoly system.

Example 3. Consider a duopoly system with the following cost structure: \(r_n = 10, \ c_n = 1.2, \ v_n = -1, \ t_{1,2} = 1, \) and \(t_{2,1} = 2; \) that is, transshipment costs are asymmetric. Each retailer faces a demand that is uniformly distributed between \([0, 100]\). Also, let \(\beta_{n} = 1; \) that is, all consumers are willing to accept transshipped goods. The first-best solution can be computed to be \(\hat{X}^* = (77, 62).\)

Now consider the decentralized system with the fractional allocation rule (AR-f) as proposed in Theorem 5.2. Because there is no central warehouse, \(\bar{Y} = 0\) and we have
\[
\alpha_n^*(\hat{X}, \bar{D}) = \gamma_n P_n^*(\hat{X}, \bar{D}) - \left[ r_n S_n + v_n H_n - c_n X_n \right],
\]
where
\[
P_n^*(\hat{X}, \bar{D}) = \sum_{n=1}^2 \left[ r_n S_n + v_n H_n - c_n X_n \right] + \sum_{n=1}^2 \sum_{m=n}^2 (r_n - v_m - t_{m,n}) \max\{0, \min\{E_n, H_m\}\}.
\]

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Event</th>
<th>(\nu_1)</th>
<th>(\nu_2)</th>
<th>(\delta_1)</th>
<th>(\delta_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>(0 &lt; H_1 &lt; E_1)</td>
<td>10</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>R2</td>
<td>(0 &lt; E_2 &lt; H_2)</td>
<td>0</td>
<td></td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>R3</td>
<td>(0 &lt; H_3 &lt; E_3)</td>
<td></td>
<td>9</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>R4</td>
<td>(0 &lt; E_4 &lt; H_4)</td>
<td></td>
<td>0</td>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>

Let \(\gamma_1 = \gamma_2 = 0.5.\) From Theorem 5.2, the allocation rule AR-f leads to the first-best solution, \(\hat{X}^*\), if the retailers will cooperate in the shipping decision. \(\alpha_n^*(\hat{X}^*, \bar{D})\) gives the allocation of excess profits to each retailer from the shipping decision at the first-best solution. To evaluate these allocations, first recall that the excess stock and excess demand for each retailer at \(\hat{X}^*\) are given as:

\[
H_1 = \max\{77 - D_1, 0\}; \quad E_1 = \max\{D_1 - 77, 0\}, \quad H_2 = \max\{62 - D_2, 0\}; \quad E_2 = \max\{D_2 - 62, 0\}.
\]

The exchange of goods between the retailers occurs whenever one retailer has excess demand and the other has excess stock. Based on the demand realizations, this gives rise to four scenarios (denoted by R1, R2, R3, and R4) shown in Table 1; the table also lists the corresponding dual prices that we will use later in the example.

Given the expressions for \(H_n\) and \(E_n\), we evaluate \(\alpha_n^*(\hat{X}^*, \bar{D})\) in each of the four regions as shown in Table 2.

To see that these allocations may not be in core of \(\text{SAG}(\{Z_n\}, \bar{D})\), consider region R1 with \(D_1 = 75\) giving \(\alpha_1^*(\hat{X}^*, \bar{D}) = -45.\) Because the allocation is negative, it cannot be in core of \(\text{SAG}(\{Z_n\}, \bar{D})\).

Using Corollary 5.1, we construct a new allocation rule AR-c as follows:

\[
\alpha_n^*(\hat{X}, \bar{D}) = \alpha_n^*(\hat{X}, \bar{D}) + w_n(\hat{X}^*, \bar{D}),
\]
where \(w_n(\hat{X}^*, \bar{D}) = \alpha_n^*(\hat{X}^*, \bar{D}) = \alpha_n^*(\hat{X}^*, \bar{D})\) with \(\alpha_n^*(\hat{X}^*, \bar{D})\) representing allocations based on AR-d at \(\hat{X}^*\) computed using Theorem 4.1. The values of \(w_n(\cdot, \cdot)\) for each retailer are given in Table 3. By Corollary 5.1, the NE under AR-c will give the first-best solution and the allocations will also be in the core.
Table 2  Allocations Under AR-f at X^c

<table>
<thead>
<tr>
<th>Region</th>
<th>α(X^c; D)</th>
<th>α(Χ^c; D)</th>
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<tbody>
<tr>
<td>R1</td>
<td>742.5 - 10.5D</td>
<td>27.5 + 0.5D</td>
</tr>
<tr>
<td>R2</td>
<td>47.5 + 5.5D</td>
<td>-667.5 + 5.5D + D</td>
</tr>
<tr>
<td>R3</td>
<td>-128 - D</td>
<td>488 - 10D</td>
</tr>
<tr>
<td>R4</td>
<td>-753.5 + 4.5D + 5.5D</td>
<td>60.5 + 4.5D - 5.5D</td>
</tr>
</tbody>
</table>

Table 3  Side Payments Modifying the Fractional Allocation Rule

<table>
<thead>
<tr>
<th>Region</th>
<th>w1(X^c; D)</th>
<th>w2(X^c; D)</th>
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</thead>
<tbody>
<tr>
<td>R1</td>
<td>27.5 + 0.5D</td>
<td>-27.5 - 0.5D</td>
</tr>
<tr>
<td>R2</td>
<td>-47.5 + 5.5D - 5D</td>
<td>47.5 - 5.5D + 5D</td>
</tr>
<tr>
<td>R3</td>
<td>128 - D</td>
<td>-128 + D</td>
</tr>
<tr>
<td>R4</td>
<td>60.5 + 4.5D - 5.5D</td>
<td>-60.5 - 4.5D + 5.5D</td>
</tr>
</tbody>
</table>

Table 4  Demand Density

<table>
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<th>Retailer 2</th>
<th>Retailer 3</th>
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<tr>
<td>Demand</td>
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<td>200</td>
<td>150</td>
</tr>
<tr>
<td>Probability</td>
<td>0.75</td>
<td>0.25</td>
<td>1/5</td>
</tr>
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</table>

Table 5  Demand Scenarios

<table>
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<th>Demand Scenario</th>
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<th>Retailer 2</th>
<th>Retailer 3</th>
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<td>150</td>
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<td>0.10</td>
</tr>
<tr>
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<td>100</td>
<td>300</td>
<td>500</td>
<td>0.40</td>
</tr>
<tr>
<td>5</td>
<td>200</td>
<td>150</td>
<td>0</td>
<td>0.0167</td>
</tr>
<tr>
<td>6</td>
<td>200</td>
<td>150</td>
<td>500</td>
<td>0.0667</td>
</tr>
<tr>
<td>7</td>
<td>200</td>
<td>300</td>
<td>0</td>
<td>0.0333</td>
</tr>
<tr>
<td>8</td>
<td>200</td>
<td>300</td>
<td>500</td>
<td>0.1333</td>
</tr>
</tbody>
</table>

Because the first-best solution is unique, the NE is also unique.

While AR-c gives the first-best solution, it is not the only allocation rule that will give the first-best. What other allocation rules could give the first-best? One intuitive allocation would be to compensate the retailers based on a fixed (independent of demand realization) transfer price. Let $s_{nn}$ be the transfer price paid to retailer $n$ by retailer $m$ whenever $n$ ships goods to $m$. Then for this example, we find that the first-best solution is achieved when $s_{21} = $0.34 and $s_{21} = $6.04. Thus, allocation rules that give first-best need not be unique.

Example 4. Assume an oligopoly system with three identical retailers; specifically, let $r_n = 10$, $c_n = 5$, and $\gamma = 1$. Let the transportation cost $t_{ij}$ be as follows: $t_{11} = 0$, $t_{12} = t_{13} = 1$ and for all $i > 1$, $t_{ij}$ is large such that there is no transshipment between $i$ and $n$. Also, let $\beta_{ij} = 1$; that is, all consumers are willing to accept transshipped goods. Assume that each retailer faces an independent discrete demand density as described in Table 4. There are eight possible demand scenarios under this probability density as shown in Table 5.

The first-best solution is computed using the method outlined in § 3. The first-best solution is $X^c = (200, 250, 450)$ with a total first-best profit of $326.94. This solution is also unique.

Now consider the decentralized system with the fractional allocation rule (AR-f) as proposed in Theorem 5.2. Because there is no central warehouse, $\bar{Y} = 0$, the allocations $\alpha_i(x, D)$ are given by Equation (12). Let $\gamma_1 = 0.2$, $\gamma_2 = 0.3$, and $\gamma_2 = 0.5$. The allocation of surplus to each retailer for $X^c$ under AR-f is shown in Table 6. Observe that allocations to certain retailers under some demand scenarios are negative. Thus, the allocations under AR-f are not in the core of $\text{SAG}([2], \bar{D})$. Using Corollary 5.1, we will now show that a modified fractional allocation rule that is in the core of $\text{SAG}([2], \bar{D})$ and achieves the first-best solution. But, first we need to calculate the allocations based on dual prices at $X^c$. For each demand scenario, the dual prices for the shipment decision in Equation (6) are computed. The allocation to each retailer based on these prices is given by Equation (8). The results are given in Table 7.

As before, applying Corollary 5.1, the allocations $\alpha_i(x, D)$ based on the allocation rule AR-c are derived using Equation (13) where the constants $w_i(x^c, D)$ are given by Table 8 for each of the eight demand
Table 6  Allocations Based on the Fractional Allocation Rule (AR-f) for \( \bar{X} = \hat{X}^* \)

<table>
<thead>
<tr>
<th>Demand Scenario</th>
<th>( \alpha_1(\hat{X}^*, \bar{D}) )</th>
<th>( \alpha_2(\hat{X}^*, \bar{D}) )</th>
<th>( \alpha_3(\hat{X}^*, \bar{D}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-370</td>
<td>-755</td>
<td>1125</td>
</tr>
<tr>
<td>2</td>
<td>540</td>
<td>610</td>
<td>-650</td>
</tr>
<tr>
<td>3</td>
<td>-90</td>
<td>-1235</td>
<td>1825</td>
</tr>
<tr>
<td>4</td>
<td>820</td>
<td>130</td>
<td>50</td>
</tr>
<tr>
<td>5</td>
<td>-1090</td>
<td>-485</td>
<td>1575</td>
</tr>
<tr>
<td>6</td>
<td>-280</td>
<td>730</td>
<td>-450</td>
</tr>
<tr>
<td>7</td>
<td>-910</td>
<td>-1115</td>
<td>2025</td>
</tr>
<tr>
<td>8</td>
<td>-100</td>
<td>100</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 8  Allocations Based on the Modified Fractional Rule AR-c for Arbitrary \( \bar{X} \)

<table>
<thead>
<tr>
<th>Demand Scenario</th>
<th>( w_1(\hat{X}^*, \bar{D}) )</th>
<th>( w_2(\hat{X}^*, \bar{D}) )</th>
<th>( w_3(\hat{X}^*, \bar{D}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>370</td>
<td>755</td>
<td>-1125</td>
</tr>
<tr>
<td>2</td>
<td>-540</td>
<td>-610</td>
<td>1150</td>
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<td>3</td>
<td>90</td>
<td>1725</td>
<td>-1825</td>
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<tr>
<td>4</td>
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<td>5</td>
<td>1950</td>
<td>485</td>
<td>-1575</td>
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<tr>
<td>6</td>
<td>280</td>
<td>-730</td>
<td>450</td>
</tr>
<tr>
<td>7</td>
<td>910</td>
<td>1115</td>
<td>-2025</td>
</tr>
<tr>
<td>8</td>
<td>100</td>
<td>-100</td>
<td>0</td>
</tr>
</tbody>
</table>

scenarios. Once again, by Corollary 5.1, the NE under AR-c will give the first-best solution, and the allocations will also be in the core. Because the first-best solution is unique, the NE is also unique.

6. Summary and Conclusion

In this paper we have developed a framework for the analysis of a general decentralized distribution system. We have carried the formal analysis in terms of a “bare bones” model that consists of agents who have local stocks but, in addition, may decide to keep stocks at central locations (pooled inventory) to take advantage of transportation and storage costs. Each agent is an independent profit-maximizing entity who makes the sequential decisions first concerning securing inventory and then of allocating inventory to demands. We develop a cooperative (hybrid competitive-cooperative) framework for such decision making in a decentralized setting. We introduce the notion of claims that allows us to separate the ownership (with decision rights) and the location of inventories in the system. This permits each agent to decide on its stock level(s) unilaterally before observing any demand. Subsequently (ex-post—after the realization of demands), agents may cooperate to share excess stocks to satisfy unmet demand in the system.

We outline a two-step solution strategy for such a system with claims. In the first step, agents cooperatively agree on allocation rules to share the surplus profits from the exchange of stocks. Subsequently, in the second step, for a given allocation rule the agents decide on inventory levels competitively. The framework and solution strategies draw from cooperative and non-cooperative game theory. In particular, the ex post allocation policy should be such that it is a member of the “core” for every realization of demand. We show that the core, in general, is nonempty. In addition, for the ex ante inventory deployment decision, we establish conditions for the existence of a pure strategy Nash equilibrium. Next, we show that there exists an allocation mechanism that achieves the first best solution for inventory deployment and allocation and develop conditions under which this equilibrium is also unique.

To operationalize the sharing of stocks in the decentralized system, two components are essential. First, we need an information system that provides accurate data on (excess) demand and (excess) stock. Without this, there can be no exchange. Thus, we can

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say that the value of information is the excess profits generated by the exchange of stocks. Second, we need an incentive mechanism to ensure it is in the interest of every agent to participate in this exchange. Otherwise, the excess profits generated are suboptimal. Thus, in this paper we have shown the existence of an incentive mechanism that maximizes the value of excess profits generated and, hence, maximizes the value of information.

While the "bare-bones" model presented here assumes that each agent controls one single location and faces exogenous demand, the framework can be significantly generalized to allow for more complex, but realistic, situations. For example, our framework will allow for "super dealers" who control multiple locations or partnerships in which agents own "fractions" of a retailer. Some of these arrangements are common in the auto-dealership network. Our framework can also be applied in a multiproduct context when these products are partially substitutable. An obvious extension is when each retailer is stocking a different (but single) substitute product. Alternately, if each retailer stocks multiple substitute products, then each retailer's action space is a vector of quantities representing different products. If we consider each product under a retailer as a separate "location," then this multiproduct scenario is equivalent to the concept of "super agent," described above. Also, our results can be easily applied to the case of capacity allocations.

Additional situations that can be analyzed by our approach, at least partially, relate to the shift in the nature of retailing that is brought about by the growth of the Internet. The traditional retailer, who generates its own sales and manages its own inventory to satisfy those sales, is being replaced by a network of unrelated agents, each specializing in a small part of the retailing enterprise. To cite a simple example, consider the case of an e-tailer such as buy.com, a company whose main activity is to generate demand through the Internet. This demand is, in turn, satisfied by fulfillment organizations such as Ingram Entertainment Services, Inc., which usually serves many e-tailers in addition to buy.com. And while buy.com may depend on its fulfillment partners to satisfy demand under normal conditions, it may also decide (to guarantee availability) to purchase specific quantities of "hot" products (such as Sony's PlayStation 2) on its own and store it at Ingram's facilities. In the paper we refer to such inventory as "claimed" when the physical position is determined by logistical and operational considerations and may be separated from the actual "ownership." Note that other e-tailers (buy.com's competitors) may stock their own "claimed" inventory of the same product at the same location, as may Ingram itself. By the same token, buy.com may keep additional stock of the same product at different locations. Additionally, the product itself may have several relevant substitutes that may be stored at different locations, owned by different individuals, and valued differently by different customers.

A similar trend is also complicating the "sales generation" part of the transaction. Clearly, the unit sold by buy.com could have originated at a different, and independent, node of the network via "click-through" into buy.com's system, perhaps indirectly. Amazon.com's Associates program provides another example. Similarly, a given newspaper or an Internet portal may offer free advertisement services to an e-tailer in exchange for the right to own a given fraction of the resulting sales volume (or a certain number of units). As these examples amply demonstrate, the "ownership" of a given sales unit may reside at an agent who is distinct from the one who actually carries out the sales transaction. This is in addition to the fact, as seen previously, that distinct and independent agents may carry out the various functions of sales transactions, order fulfillment, and product shipments. Thus, what we have here is a network of independent agents who, make unilateral decisions considering sales generation and inventory deployment activities, but use information technology to cooperate as they match supply and demand. For such situations, the ex post part of our model, which involves optimal matching of supply and demand using a cooperative decision making framework, holds without modification. The ex ante part of the model,

however, needs to now explicitly incorporate the sales-generation activity.

To summarize, we have demonstrated the existence of allocation mechanisms that achieve the first-best solution. In general, these mechanisms may appear to be quite complex. How do simpler mechanisms (e.g., fixed transfer pricing) perform? Can they achieve first-best solution? In the duopoly example presented, we have shown that simple transfer pricing does give the first-best solution. Can this result be generalized? We explore some of these issues in a companion paper (Anupindi et al. 1999).

Finally, the analysis in this manuscript has assumed full information between all players. Under asymmetric information, however, truthful revelation of information (e.g., regarding leftover inventory and excess demand) will depend on the type of allocation policy used. In an extension of our paper, Granot and Sosić (2000) study the question of whether the agents will reveal their true residuals. Ichiishi and Radner (1999) develop conditions for truthful revelation of information in the context of a multidivisional firm with profit centers. In general, in the presence of asymmetric information, we need to study mechanisms that optimally trade off between incentives for truthful revelation and the efficiency gained from trying to achieve the centrally coordinated solution. In addition, the dual prices introduced in § 4 bear some interesting connections to the extensive literature on double auctions. We plan to pursue these issues as further research.

Appendix

Proof of Theorem 4.1. The allocation games are LP-games that can be viewed as a market games. The dual vector corresponds to competitive payoffs. The membership of the dual vector in the core for such games has been studied by Owen (1975), Shapley and Shubik (1975), and Samet and Zemel (1984), among others. □


Proof of Proposition 5.1. Consider a general duopoly system. We first write the profit function for the Centralized System as follows:

\[
\begin{align*}
    J(X_1, X_2) &= -c_1X_1 + r_1E[\min(D_1, X_1)] + v_1E[\max(0, X_1 - D_1, 0)] \\
    &\quad + c_2X_2 + r_2E[\min(D_2, X_2)] + v_2E[\max(0, X_2 - D_2, 0)] \\
    &\quad + (r_1 - v_1 - l_1)E[\max(0, \min(D_1 - X_1, X_2 - D_2))] \\
    &\quad + (r_2 - v_2 - l_2)E[\max(0, \min(D_2 - X_2, X_1 - D_1))].
\end{align*}
\]

Differentiating with respect to \( X_1 \) and \( X_2 \), respectively, we obtain:

\[
\begin{align*}
    \frac{\partial J}{\partial X_1} &= -c_1 + r_1F_1(X_1) + v_1F_1(X_1) \\
    &\quad - (r_1 - v_1 - l_1) \int_0^{X_1} \left[ F_1(X_1) + X_2 - d_2 \right] dF_2(d_2) \\
    &\quad + (r_2 - v_2 - l_2)E[\max(0, \min(D_2 - X_2, X_1 - D_1))].
\end{align*}
\]

\[
\begin{align*}
    \frac{\partial J}{\partial X_2} &= -c_2 + r_2F_2(X_2) + v_2F_2(X_2) \\
    &\quad + (r_1 - v_1 - l_1) \int_0^{X_2} \left[ F_1(X_1) + X_2 - d_2 \right] dF_2(d_2) \\
    &\quad - (r_2 - v_2 - l_2) \int_0^{X_2} \left[ F_1(X_1) + X_2 - d_2 \right] dF_2(d_2).
\end{align*}
\]

Now consider the decentralized system in which excess profits are allocated based on AR-d. The expected profit of retailer 1 for \( X \) as is as follows:

\[
\begin{align*}
    J(X_1, X_2) &= -c_1X_1 + r_1E[\min(D_1, X_1)] + v_1E[\max(0, X_1 - D_1, 0)] \\
    &\quad + (r_1 - v_1 - l_1) \int_0^{X_1} \left[ F_1(X_1) + X_2 - d_2 \right] dF_2(D_2) \\
    &\quad + (r_2 - v_2 - l_2) \int_0^{X_2} \left[ F_1(X_1) + X_2 - d_2 \right] dF_2(D_2).
\end{align*}
\]

Taking the first partial w.r.t. \( X_1 \) we write:

\[
\begin{align*}
    \frac{\partial J}{\partial X_1} &= -c_1 + r_1F_1(X_1) + v_1F_1(X_1) \\
    &\quad - (r_1 - v_1 - l_1) \int_0^{X_1} \left[ F_1(X_1) + X_2 - d_2 \right] dF_2(d_2) \\
    &\quad + (r_1 - v_1 - l_1) \int_0^{X_1} \left[ F_1(X_1) + X_2 - d_2 \right] dF_2(D_2) \\
    &\quad + (r_2 - v_2 - l_2) \int_0^{X_1} \left[ F_1(X_1) + X_2 - d_2 \right] dF_2(D_2) \\
    &\quad - (r_2 - v_2 - l_2) \int_0^{X_1} \left[ F_1(X_1) + X_2 - d_2 \right] dF_2(D_2) \\
    &\quad - (r_2 - v_2 - l_2) \int_0^{X_1} \left[ F_1(X_1) + X_2 - d_2 \right] dF_2(D_2).
\end{align*}
\]

where

\[
\begin{align*}
    \frac{\partial J}{\partial X_1} &= + K(X_1, X_2).
\end{align*}
\]
\( K(X_1, X_2) \)
\[
= (r_1 - \alpha_1 - t_1) \int_{0}^{x_2} (X_1 - D_1) \frac{f_1(X_1 + X_2 - D_1)}{D_1} dF_1(D_1) \\
- (r_1 - \alpha_1 - t_2) \int_{0}^{x_1} (X_1 - D_1) \frac{f_2(X_1 + X_2 - D_1)}{D_1} dF_2(D_1).
\]

The expected profit of retailer 2 under AR-d is written analogous to that of retailer 1 and its first partial is given by
\[
\frac{\partial f_2(X_1, X_2)}{\partial X_1} = \frac{\partial f_2(X_1, X_2)}{\partial X_2} - K(X_1, X_2). \tag{17}
\]

Because in general \( K(X_1', X_2') \neq 0 \), the equilibrium \( \hat{X}_1 = \hat{X}_2 \) and we do not get first-best. □

**Proof of Theorem 5.1.** Let the payoff function of retailer \( i \) under AR-m be \( J_i(Z_1) \) for any inventory position \( Z_1 \). Let \( \alpha_1^r(Z_1, \hat{D}) \) be an allocation of surplus to retailer \( n \) for the inventory position \( Z_1 \) such that \( \lambda_{m,n} \alpha_1^r(Z_1, \hat{D}) = w_1(Z_1, \hat{D}). \) The Nash equilibrium (NE) under AR-m is \( Z_1^{\ast\ast} \). Thus, \( \alpha_1^r(Z_1^{\ast\ast}, \hat{D}) \) is the allocation of surplus to retailer \( n \) at the NE\( Z_1^{\ast\ast}. \) We wish to implement \( Z_1^{\ast\ast}. \) If AR-m is in core of SAG(\( Z_1, \hat{D} \)), then we are done. Otherwise, consider the allocation policy AR-d, which allocates based on the dual prices. Theorem 4.1 shows that AR-d is in the core of SAG(\( Z_1, \hat{D} \)). Let \( \eta_1(Z_1^{\ast\ast}, \hat{D}) \) be the allocation based on the dual for \( Z_1^{\ast\ast}. \) Define
\[
w_1(Z_1^{\ast\ast}, \hat{D}) = \eta_1(Z_1^{\ast\ast}, \hat{D}) - \alpha_1^r(Z_1^{\ast\ast}, \hat{D}).
\]

Define the following allocation policy AR-m with allocations:
\[
\alpha_1^r(Z_1, \hat{D}) = \alpha_1^r(Z_1, \hat{D}) + w_1(Z_1^{\ast\ast}, \hat{D}).
\]

Clearly the allocations \( \alpha_1^r(Z_1^{\ast\ast}, \hat{D}) \) are in the core of SAG(\( Z_1, \hat{D} \)). Let the payoff function of retailer \( n \) under AR-m for \( Z_1^{\ast\ast} \) be \( J_i(Z_1^{\ast\ast}) \). Then,
\[
J_i(Z_1^{\ast\ast}) = J_i(Z_1) + E_1 w_1(Z_1^{\ast\ast}, \hat{D}).
\]

That is, the payoff function under AR-m and AR-m differ by a constant \( w_1(\cdot) \), which is the function of \( Z_1^{\ast\ast} \), the desired equilibrium that we wish to implement. Clearly, then the reaction functions under AR-m and AR-m are identical. Hence, the Nash equilibrium under the two are also identical. □

**Proof of Theorem 5.2.** Substituting the allocations based on AR-f in Equation (9) to derive the payoff function of retailer \( n \) for a given \( \hat{D} \) we write:
\[
P_i(Z_1, \hat{D}) = \gamma_i P_i(Z_1, \hat{D}).
\]

The expected payoff of retailer \( n \) under AR-f is then:
\[
P_i(Z_1^{\ast\ast}) = \gamma_i P_i(Z_1^{\ast\ast}).
\]

Because \( \gamma_i \) is a constant, it is straightforward to see that the set of reaction functions all retailers correspond to are the first-order conditions for the optimality of the centralized solution. Therefore, a NE solution always exists that satisfies the first-order conditions. □

**References**


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