FAEETS OF THE KNAPSACK POLYTOPE
FROM MINIMAL COVERS

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Abstract. In this paper we give easily computable best upper and lower bounds on the coefficients of facets of the knapsack polytope associated with minimal covers. For some coefficients the upper bounds are equal to the lower bounds; for the others the two bounds differ by 1. We give a necessary and sufficient condition for all lower bounds to be equal to the corresponding upper bounds, i.e., for the facet associated with the given minimal cover to be unique. Also, we define a partial order on the set of minimal covers and show that all facets associated with minimal covers can be obtained from weak covers; but that each facet obtainable from several ordered minimal covers is easiest to compute from the strongest one.

Further, we characterize the class of all facets associated with minimal covers, and show that the facets obtainable by Padberg’s sequential lifting procedure are precisely those members of the class which have integer coefficients for a certain right-hand side. We then give a procedure for generating the other (simultaneously) lifted facets, which have fractional coefficients for the same right-hand side.

1. Introduction and preliminaries. This paper deals with facets of the knapsack polytope, i.e., of the convex hull of 0–1 points satisfying a knapsack inequality. Though we build on the results of [1], [12] and [16], the paper is self-contained. Our main objective is to complete a characterization as possible of the class of facets associated with certain subsets of the index set, called minimal covers.

After defining a few concepts (§ 1), we review some earlier results on the sequential lifting of facets and their implications. In § 2, we give best upper and lower bounds on the coefficients of sequentially lifted facets. The bounds can be computed from two simple formulae. For an easily identifiable subset of the coefficients, the lower bounds are equal to the upper bounds, i.e., the coefficients themselves are explicitly given; while for the remaining coefficients, the lower bounds differ from the upper bounds by 1.

In § 3, we discuss some uniqueness and dominance properties of sequentially lifted facets. A necessary and sufficient condition is given for all lower bounds on the coefficients to be equal to the corresponding upper bounds, i.e., for the facet associated with the given minimal cover to be unique. Also, minimal covers are partially ordered by the weaker/stronger relation, and it is shown that all facets associated with minimal covers can be obtained from the weak (= weakest) covers. On the other hand, a facet obtainable from several ordered minimal covers is easiest to compute from the strongest among these covers.

Finally, in § 4 we give a characterization of the entire class of facets associated with minimal covers. First, the upper and lower bounds established in § 2 for

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sequentially lifted facets are extended to “simultaneously” lifted facets. Then it is shown that the facets associated with minimal covers are in a one-to-one correspondence with the vertices of a polyhedral set in the subspace of those variables for which the upper and lower bounds on the coefficients are not equal (i.e., differ by 1). The polyhedron in question is defined by a set of linear inequalities with coefficients equal to 0 or 1. We show that the sequentially lifted facets are precisely those corresponding to integer vertices of the polyhedron, whereas the other facets associated with minimal covers, which cannot be obtained by sequential lifting, are associated with the fractional vertices. Finally, a subclass of easily computable facets with fractional coefficients is characterized.

Some of the results of this paper were presented at the NATO Advanced Study Institute on Combinatorial Programming, Versailles, September 1974 [3].

We start by introducing the notation and terminology of [1].

Consider the inequality

\[ \sum_{j \in N} a_j x_j \leq a_0 \]

where \(a_0, a_j\) are positive integers and \(x_j = 0\) or \(1, j \in N = \{1, \ldots, n\}\).

The knapsack polytope \(P\) associated with (1) is the convex hull of 0–1 points satisfying (1), i.e.,

\[ P = \operatorname{conv} \left\{ x \in \{0, 1\}^n \mid \sum_{j \in N} a_j x_j \leq a_0 \right\}. \]

An inequality

\[ \sum_{j \in N} a_j x_j \leq a_0 \]

is said to be a facet of \(P\) if it is satisfied by every \(x \in P\), and satisfied with equality by exactly \(d - 1\) affinely independent points \(x \in P\), where \(d\) is the dimension of \(P\).

Throughout this paper we will assume that \(d = n\), which is true if and only if \(a_i = a_0, \forall i \in N\).

A set \(S \subseteq N\) is called a cover for (1), viz. for \(P\), if

\[(i)\quad \sum_{j \in S} a_j > a_0.\]

A cover is called minimal if

\[(ii)\quad \sum_{j \in S \setminus \{i\}} a_j = a_0, \quad \forall i \in S.\]

The set \(E(S) = S \cup S'\), where

\[ S' = \{ j \in N - S \mid a_j \geq a_0, \forall k \in S \}, \]

is called the extension of \(S\) (to \(N\)).

A minimal cover \(S\) is called strong, if

\[(iii)\quad \text{either } E(S) = N \text{ or } \sum_{j \in S - \{i\}} a_j + a_i \leq a_0.\]
where
\[ a_i = \max_{j \in S} a_{ij}, \quad a_{ij} = \max_{j \in N-E(S)} a_{ij}. \]

In other words, \( S \) is strong if there exists no minimal cover of the same size as \( S \), whose extension strictly contains that of \( S \).

Throughout this paper we will assume that \( N \), and the covers \( S \) to be considered, are ordered so that \( i < j \Rightarrow a_i \leq a_j \).

In [4] it was shown that, if \( \mathcal{S} \) denotes the set of minimal covers for (1), the set of inequalities
\[
\sum_{j \in E(S)} x_j \leq |S| - 1, \quad S \in \mathcal{S},
\]
termed canonical, is equivalent to (1) in the sense of having the same 0–1 solution set. For related developments see [5], [6].

In [1], [7] and [14], necessary and sufficient conditions were given for a canonical inequality to be a facet of \( P \). Furthermore, in [1] a sufficient condition was given for an inequality of the form
\[
\sum_{j \in E(S)} \pi_j x_j \leq |S| - 1,
\]
where \( S \) is a strong cover and where the coefficients \( \pi_j \) (not necessarily 0 or 1) are given by an explicit formula, to be a facet of \( P \). These easily computable facets of \( P \) are special cases of the class to be presently discussed, and which we call "sequentially lifted" facets.

For any subset \( V \subset N, |V| = v \), we denote
\[
P_V = \text{conv} \left\{ x \in \{0, 1\}^v \mid \sum_{j \in V} a_j x_j \leq a_0 \right\}.
\]

It is known (see, for instance, [1], [12], [14]) that if \( S \) is a minimal cover for (1), then the inequality
\[
\sum_{j \in S} x_j \leq |S| - 1
\]
is a facet of \( P_0 \). Furthermore, facets of lower-than-\( n \)-dimensional polytopes which are projections of \( P \) can be "lifted" into \( n \)-space so as to yield facets of \( P \). A sequential procedure for accomplishing this was given by Padberg, whose result was first established for the node-packing polytope [11], then extended to 0–1 programming polytopes with positive coefficients [12]. For extensions of this procedure to other classes of problems see [10], [15]; for a generalization to a nonsequential lifting procedure for facets of arbitrary 0–1 polytopes, see [16]. For other related developments see [8], [9], [13].

**Theorem 1** (Padberg [12]). Let \( S \) be a minimal cover for (1), let \( N - S = \{j_1, \cdots, j_p\} \) be arbitrarily ordered (\( p = |N - S| \)), and consider the sequence of
knapsack problems \( K_i \), defined recursively as

\[
z_i = \max \sum_{j \in S} x_j + \sum_{j \neq j_1}^{j_i} \beta_i x_j,
\]

\[
\sum_{j \in S} a_j x_j + \sum_{j \neq j_1}^{j_i} a_j x_j \leq a_0 - a_i,
\]

\( x_j = 0 \) or \( 1 \), \( j \in S \cup \{j_1, \ldots, j_{i-1}\} \)

for \( i = 1, \ldots, p \) (where summation over the empty set is \( 0 \)), with the coefficients \( \beta_j \) defined by

\[
\beta_j = |S| - 1 - z_n, \quad j = j_1, \ldots, j_{i-1}.
\]

Then the inequality

\[
\sum_{j \in S} x_j + \sum_{j \in N-S} \beta_j x_j \leq |S| - 1
\]

is a facet of \( P \).

The procedure defined in Theorem 1 will be called sequential lifting, and the facets of \( P \) obtained by this procedure will be called sequentially lifted facets. The coefficients of a facet obtained by sequential lifting depend on the ordering of \( N-S \). Thus the same minimal cover \( S \) may yield as many as \( |N-S|! \) facets of \( P \), though the number of distinct (sequentially lifted) facets is usually much smaller than this number, as will be shown later.

Calculating the coefficients \( \beta_j \) of sequentially lifted facets requires the solution of a sequence of knapsack problems. Furthermore, the value of the coefficients depends on the sequence in which the indices \( j \in N-S \) are considered. For each \( i \in N-S \), let \( \beta'_i \) denote the value of \( \beta_i \) when \( i = j_1 \), i.e., when \( i \) is taken to be the first index in \( N-S \). In other words,

\[
\beta'_i = |S| - 1 - z'_i,
\]

where \( z'_i \) is the value of the knapsack problem \( K'_i \):

\[
\max \left\{ \sum_{j \in S} x_j \left| \sum_{j \in S} a_j x_j \leq a_0 - a_i, x_j = 0 \text{ or } 1, j \in S \right. \right\}
\]

The problem \( K'_i \) has a simpler structure than the subsequent knapsack problems which have to be solved to find the other coefficients of the sequence, and due to this structure the general solution to \( K'_i \) can be given explicitly, i.e., the coefficients \( \beta'_i \) can be calculated trivially, as shown in the following proposition. Let \( S_h \) denote the set of the first \( h \) elements of \( S \) (where \( S \) is ordered according to decreasing values of \( a_j, j \in S \)).

**Proposition 1.** For all minimal covers \( S \) and all \( i \in N-S \), \( \beta'_i = h \), where \( h \) is defined by

\[
\sum_{j \in S - S_{h+1}} a_j \leq a_0 - a_i < \sum_{j \in S - S_h} a_j.
\]
**Proof.** The solution \( \bar{x} \) to \( K'_i \), given by

\[
\bar{x}_j = \begin{cases} 
1, & j \in S - S_{h+1}, \\
0, & j \in S_{h+1},
\end{cases}
\]

is feasible by the definition of \( h \) and yields \( z'_i = |S| - (h + 1) \), hence

\[
\beta'_i = |S| - 1 - z'_i = h,
\]

as in the proposition.

To see that \( \bar{x} \) is optimal, note that from the definition of \( h \) and the ordering of \( S \), there is no feasible solution with more than \( |S| - (h + 1) \) variables equal to 1. Q.E.D.

Since each coefficient \( \beta_{n_i} \) and each knapsack problem \( K_{n_i} \) depends on the sequence in which the coefficients \( \beta_{n_i}, h < i \), were calculated, we will denote by \( K_{n_i}(Q) \) the knapsack problem \( K_{n_i} \) associated with a given sequence \( Q = \{j_1, \ldots, j_{n-1}\} \), i.e., the knapsack problem used for calculating \( \beta_{n_i} \) right after the sequence of coefficients indexed by \( Q \); and we will denote by \( z_{n_i}(Q) \) and \( \beta_{n_i}(Q) \), respectively, the values \( z_{n_i} \) and \( \beta_{n_i} \) obtained by solving \( K_{n_i}(Q) \). Here \( Q \) is a subset of \( N - S \), and the underline denotes a particular ordering. Subsets of an ordered set will be assumed to preserve the ordering, unless otherwise specified. In this notation, \( K'_i = K_i(\emptyset) \) and \( \beta'_i = \beta_i(\emptyset) \).

**Proposition 2.** For any sequence \( Q = \{j_1, \ldots, j_p\} \), \( Q \subseteq N - S \), and any subsequence \( Q_k = \{j_1, \ldots, j_k\} \), \( 0 \leq k < p \),

\[
\beta_{n_i}(Q) \leq \beta_{n_i}(Q_k), \quad i = p + 1, \ldots, n.
\]

**Proof.** Any feasible solution to \( K_{n_i}(Q_k) \) is a feasible solution to \( K_{n_i}(Q) \); hence \( z_{n_i}(Q) \leq z_{n_i}(Q_k) \), i.e., \( \beta_{n_i}(Q) \leq \beta_{n_i}(Q_k) \). Q.E.D.

In particular, for \( Q = \emptyset \), Proposition 2 yields

**Remark.** For any sequence \( Q, Q \subseteq N - S \), and any \( i \in N - S \cup Q \), \( \beta_i(Q) \leq \beta'_i \).

**Proposition 3.** For any sequence \( Q, Q \subseteq N - S \), and any \( i, h \in N - S \cup Q \),

\[
a_h \geq a_i \Rightarrow \beta_{h}(Q) \geq \beta_{i}(Q).
\]

**Proof.** If \( a_h \geq a_i \), then any feasible solution to \( K_h(Q) \) is a feasible solution to \( K_i(Q) \); hence \( z_i(Q) \geq z_h(Q) \) and therefore \( \beta_i(Q) \geq \beta_h(Q) \). Q.E.D.

Remark 1 above generalizes to arbitrary valid inequalities (not necessarily sequentially lifted facets), with coefficients equal to 1 for \( j \in S \).

**Proposition 4.** If

\[
\sum_{j \in S} x_j + \sum_{j \in N - S} \alpha_j x_j \leq |S| - 1
\]

is a valid inequality for \( P \), then \( \alpha_j \leq \beta'_j, \forall j \in N - S \).

**Proof.** This follows from Theorem 1. Assume that for some \( i \in N - S \), \( \alpha_i > \beta'_i \). From Theorem 1, \( \beta'_i = |S| - 1 - z'_i \), where \( z'_i \) is the value of the knapsack problem \( K'_i \)

\[
\max \left\{ \sum_{j \in S} x_j \mid \sum_{j \in S} a_j x_j \leq a_0 - a_i, \ x_j = 0 \text{ or } 1, j \in S \right\}.
\]
Let \( \bar{x} \in R^{|\mathcal{S}|} \) be an optimal solution to \( K' \). Then \( x \in R^n \), defined by
\[
\bar{x}_j = \begin{cases} 
\text{x}_j, & j \in \mathcal{S} \\
1, & j = i, \\
0, & j \in N - S \cup \{i\},
\end{cases}
\]
satisfies (1) but violates the inequality in the proposition, since
\[
\sum_{j \in \mathcal{S}} \bar{x}_j + \sum_{j \in N - S} \alpha_j \bar{x}_j = z'_i + \alpha_i > |\mathcal{S}| - 1.
\]
Q.E.D.

As shown above, the calculation of every sequentially lifted facet requires the solution of a sequence of knapsack problems \( K_i \) (one for each coefficient), only the first one of which can be solved trivially. On the other hand, there exists an important class of valid inequalities whose members are in a one-to-one correspondence with the minimal covers of (1) (see [1]), and are trivially easy to compute. Furthermore, when the coefficients \( a_j, j \in N - S \), satisfy an additional requirement, the corresponding inequality is one of the sequentially lifted facets of \( P \). This class of inequalities is defined as follows.

**Theorem 2** (Balas [1]). Let \( S \) be a minimal cover for (1), \( E(S) \) the extension of \( S \), and \( S_h \) the set of the first \( h \) elements of \( S \), \( h = 1, \ldots, |S| \). Let \( N \) be partitioned into \( N_0, N_1, \ldots, N_q, q = |S| - 1 \), where
\[
N_0 = N - E(S), \quad N_1 = E(S) - \bigcup_{h=2}^q N_h,
\]
\[
N_h = \left\{ i \in E(S) \mid \sum_{j \in S_h} a_{ij} \leq a_i < \sum_{j \in S_{h+1}} a_j \right\}, \quad h = 2, \ldots, q,
\]
and define
\[
\pi_j = h, \quad \forall j \in N_h, \quad h = 0, 1, \ldots, q.
\]
Then the inequality
\[
\sum_{j \in \mathcal{S}} x_j + \sum_{j \in N - S} \pi_j x_j \leq |\mathcal{S}| - 1
\]
is satisfied by all \( x \in P \). Furthermore, if
\[
\sum_{j \in S_{h+1} - S_h} a_{ij} \leq a_i - a_{ih}, \quad \forall i \in N_h, \quad h = 0, 1, \ldots, q,
\]
then (5) is a facet of \( P \).

**Remark 2.** Condition (6) implies that \( S \) is a strong cover.

The coefficients \( \pi_j \) of Theorem 2 are related in an interesting and meaningful way to the knapsack problems \( K_i \) that one has to solve in order to calculate the sequentially lifted facets of Theorem 1. This relationship will be explored in the next section and used to characterize sequentially lifted facets.

We have defined three different coefficients for each \( j \in N - S \): \( \beta_j, \beta'_j, \pi_j \). From the definitions, each of the three coefficients \( \beta, \beta'_j, \pi_j \) is integer for all \( j \in N - S \), for any sequentially lifted facet. The \( \beta_j \) are the actual coefficients of a facet obtained from \( S \) by sequential lifting, while the \( \beta'_j \) and \( \pi_j \) are auxiliary coefficients that will be used to characterize the \( \beta_j \). The coefficients \( \beta_j \) depend both
on the set $S$ and on the sequence in which they are calculated. The calculation of each $\beta_i$, $i \in N - S$, requires the solution of a knapsack problem of the form $K_i$. On the other hand, the auxiliary coefficients $\beta_j'$ and $\pi_j$ depend only on the set $S$, i.e., are independent of the sequence in which they are calculated. Also, their calculation is trivial, since they are given explicitly by formulae (2) and (3).

2. **Sequentially lifted facets.** In this section we give tight bounds on the coefficients $\beta_j$ in terms of the easily computable auxiliary coefficients $\pi_j$, $\beta_j'$.

**Lemma 1.** For $j \in Q \subset N - S$, let $\gamma_j$ be arbitrary real numbers, and let

$$Q' = \{j \in Q|\gamma_j \leq \pi_j\},$$

where $\pi_j$ are the coefficients defined in Theorem 2.

Then for any $i \in N - S \cup Q$, the knapsack problem $G_i$:

$$\max \sum_{j \in S} x_j + \sum_{j \in Q} \gamma_j x_j,$$

$$\sum_{j \in S} a_j x_j + \sum_{j \in Q} a_j x_j \leq a_0 - a_i,$$

$$x_j = 0 \text{ or } 1, \quad \forall j \in S \cup Q,$$

has an optimal solution $\bar{x}$ such that

$$\bar{x}_j = \begin{cases} 0, & j \in S_h, \\
1, & j \in S - S_h, \end{cases}$$

for some integer $h$, $1 \leq h \leq |S| - 1$, and

$$\bar{x}_j = 0, \quad \forall j \in Q'.$$

**Proof.** (i) First we show that with every optimal solution to $G_i$ which violates (7) we can associate an optimal solution which satisfies (7).

Let $\bar{x}$ be an optimal solution to $G_i$ which violates (7), and let $k$ be the largest element of $S$ such that $\bar{x}_k = 0$. Let $S_k$ denote, as before, the set of the first $k$ elements of $S$, and let $S_k = \{j \in S|\bar{x}_j = 1\}$, $|S_k| = k_0$. By hypothesis, $S_k \neq \emptyset$, i.e., $k_0 > 0$. Since $(h, i \in S, h < i) \Rightarrow a_h \geq a_i$, it follows that $\bar{x}$ defined by

$$\bar{x}_j = \begin{cases} 0, & j \in S_k - k_0, \\
1, & j \in S - S_k - k_0, \\
\bar{x}_j, & j \in Q, \end{cases}$$

is a feasible solution to $G_i$; and since $\sum_{j \in S} \bar{x}_j = \sum_{j \in S} \bar{x}_j$, $\bar{x}$ is also optimal. Clearly, $\bar{x}$ satisfies (7) with $h = k - k_0$.

(ii) Next we show that with every optimal solution to $G_i$ which satisfies (7) but violates (8) one can associate an optimal solution which satisfies both (7) and (8).

Let $\bar{x}$ be an optimal solution to $G_i$ which satisfies (7) but not (8), and let $\bar{Q}' = \{j \in Q'|\bar{x}_j = 1\}$.

From the definition of the coefficients $\pi_j$,

$$\pi_j \equiv \sum_{k \in S_{\bar{j}}} a_k, \quad \forall j \in Q$$
and thus

\[ \sum_{j \in \bar{O}'} a_j \geq \sum_{j \in \bar{O}'} \left( \sum_{k \in S_{\bar{\pi}_j}} a_k \right) \geq \sum_{k \in S_p} a_k \]

where \( p = \sum_{j \in \bar{O}'} \bar{\pi}_j \).

We claim that \( p \leq h - 1 \), where \( h \) is the same as in (7). Indeed, if \( p \geq h \), then

\[ \sum_{j \in S \cup \bar{O}} a_j \hat{x}_j \geq \sum_{j \in \bar{O}} a_j + \sum_{j \in S - S_a} a_j \]

\[ \geq \sum_{j \in S_p} a_j + \sum_{j \in S - S_a} a_j \quad \text{(from (9))} \]

\[ \geq \sum_{j \in S} a_j \quad \text{(since \( p \geq h \))} \]

\[ > a_0 \quad \text{(since \( S \) is a cover).} \]

Then denoting the set of the last \( p \) indices in \( S_h \) by \( S(h, p) \), i.e. \( S(h, p) = S_a - S_a - p \), from (9) we have

\[ \sum_{j \in \bar{O}'} a_j \geq \sum_{j \in S(h, p)} a_j. \]

(10)

Now define \( \tilde{x} \) by

\[ \tilde{x}_j = \begin{cases} 0, & j \in \bar{O}', \\ 1, & j \in S(h, p), \\ \tilde{x}_j, & \text{otherwise.} \end{cases} \]

Clearly, \( \tilde{x} \) satisfies (7) and (8). From (10) and the fact that \( \tilde{x} \) is feasible, it follows that \( \tilde{x} \) is feasible; and since the objective function coefficient of each \( x_p \), \( j \in S(h, p) \) is equal to 1, with \( p = \sum_{j \in \bar{O}'} \bar{\pi}_j \), \( \tilde{x} \) has the same objective value as \( \tilde{x} \), i.e., is optimal. Q.E.D.

Lemma 1 suggests that the coefficients \( \pi_i \) may play a crucial role in solving the knapsack problems \( K_i \), and hence in calculating the coefficients \( \beta_i \). This is indeed the case, as it will soon be seen; and, as one may guess from Theorem 2, the meaning of a given \( \pi_i \) depends on whether condition (6) is or is not satisfied for that particular \( i \in N - S \). In view of this, we partition \( N - S \) into two subsets \( I \) and \( J \), where \( I \) is the set of those \( i \in N - S \) for which the condition

\[ \sum_{j \in S - S_a + 1} a_j \geq a_0 - a_i \]

is satisfied, while \( J \) is the set of those \( i \in N - S \) for which (6) is not satisfied. From the definition of a strong cover and the fact that \( \pi_i = 0 \) for \( i \in N - E(S) \), we have the following

**Remark 3.** If \( S \) is strong, \([N - E(S)] \subseteq I \).

The next theorem characterizes the coefficients \( \beta_i \) of sequentially lifted facets in terms of the explicitly given coefficients \( \beta_i \) and \( \pi_i \):

**Theorem 3.** For any sequentially lifted facet of \( P \),

\[ \beta_i = \begin{cases} \pi_i, & i \in I, \\ \pi_i + 1, & i \in J. \end{cases} \]

(11)
and

\[ \beta_i = \begin{cases} 
\pi_i, & i \in I, \\
\pi_i + 1, & i \in J.
\end{cases} \]

Furthermore, for any sequence \( Q, Q \subseteq N - S, \) and any \( i \in N - S \cup Q, \) \( \beta_i(Q) \) depends only on those \( \beta_j, j \in Q, \) such that \( \beta_j = \pi_j + 1, \) i.e.,

\[ \beta_i(Q) = \beta_i(\hat{Q}), \]

where \( \hat{Q} = \{ j \in Q | \beta_i = \pi_i + 1 \} . \)

Proof. (a) We first prove (11). From the definition of \( \pi_i, \)

\[ \sum_{j \in S_{\pi_i}} a_j \leq a_i \]

and, since \( S \) is a cover,

\[ \sum_{j \in S - S_{\pi_i}} a_i > a_0 - a_i. \]

From this and (6), we have for \( i \in I \)

\[ \sum_{j \in S - S_{\pi_i} + 1} a_i \leq a_0 - a_i < \sum_{j \in S - S_{\pi_i}} a_i \]

and therefore, from Proposition 1, \( \pi_i = \beta'_i. \)

For \( i \in J, \) condition (6), is violated, i.e.,

\[ a_0 - a_i < \sum_{j \in S - S_{\pi_i} + 1} a_i. \]

From the definition of \( \pi_i, \) we have

\[ \sum_{j \in S_{\pi_i} + 1} a_i > a_i \]

and since \( S \) is a minimal cover,

\[ \sum_{j \in S - S_{\pi_i} + 2} a_i < a_0 - a_i. \]

From (14) and (15), in view of Proposition 1, we then have \( \beta'_i = \pi_i + 1. \) This completes the proof of (11).

(b) Next we show that for any sequence used in calculating the coefficients \( \beta_i, j \in N - S, \) one has \( \beta_j = \pi_j, \) \( \forall i \in I. \)

Since \( \beta_i \leq \beta'_i \) from Proposition 2, and \( \beta'_i = \pi_i, i \in I, \) from (11), we have to prove only \( \beta_i \leq \pi_i. \) Suppose this is false, and for some \( i \in I \) and some sequence \( Q = \{ j_1, \ldots, j_q \}, Q \subseteq N - S, \) we have \( \beta_i(Q) \equiv \pi_i - 1. \) Denoting \( Q \cup \{ i \} = Q' \) and \( S \cup Q' = V, \) the inequality

\[ \sum_{j \in S} x_j + \sum_{j \in Q'} \beta_j x_j \equiv |S| - 1 \]

is a facet of \( P_v. \)

Now replace the sequence \( \{ j_1, \ldots, j_q, i \} \) by \( \{ i, j_1, \ldots, j_q \} , \) and let

\[ \sum_{j \in S} x_j + \sum_{j \in Q'} \tilde{\beta}_j x_j \equiv |S| - 1 \]
be the facet of $P_V$ obtained by using this latter sequence in the lifting procedure.

Since $i \in I$, and $i$ is now the first element of the sequence, $\beta_i = \beta'_i = \pi_i$, as shown above under (a). Hence, according to the assumption, $\beta_i \geq \beta_i + 1$. However, since $\beta_i \leq \pi_i$, from Lemma 1 the solution to each of the knapsack problems yielding the coefficients $\beta_j, j = j_1, \cdots, j_\alpha$, remains unchanged if the variable $x_i$ is deleted; hence it is the same as the solution to the corresponding knapsack problem yielding the coefficient $\beta_j$. It then follows that $\beta_i \geq \beta_i + 1$ and $\beta_i = \beta_i, \forall j \in Q$, i.e., the inequality (17) strictly dominates (16). But this contradicts the fact that both inequalities are facets of $P_V$, thus proving that $\beta_i = \pi_i, \forall i \in I$.

(γ) We now show that for any sequence used in computing the coefficients $\beta_j, j \in N \setminus S$, one has $\beta_j \geq \pi_j, \forall i \in J$. For $i \in J$, condition (6), is violated; i.e.,

$$(14') a_i > a_0 - \sum_{j \in S \setminus S_{n+1}} a_j.$$

Consider now the $(n + 1)$-dimensional knapsack polytope

$$P' = \text{conv}\left\{ x \in \{0, 1\}^{n+1} \mid \sum_{j=1}^{n+1} a_j x_j \leq a_0 \right\},$$

obtained from $P$ by introducing the variable $x_{n+1}$ with coefficient

$$(18) a_{n+1} = a_0 - \sum_{j \in S \setminus S_{n+1}} a_j.$$

Clearly, $S$ is a minimal cover for $P'$, and the coefficients $\pi_i$ and $\beta'_i, i \in N \setminus S$, defined by Theorem 2 and Proposition 1 respectively, are the same for $P'$ as they are for $P$, since they depend only on the coefficients $a_i, j \in S \cup \{i\}$. Further, by the choice of $a_{n+1}$, condition (6), is satisfied for $i = n + 1$, and thus $\beta'_{n+1} = \pi_{n+1}$ from (11), while $\beta_{n+1} = \pi_{n+1}$, as shown under (β). Also, from the definitions, $\pi_{n+1} = \pi_i$. Finally, since $a_i > a_{n+1}$ from (14') and (18), it follows from Proposition 3 that $\beta_i(Q) \geq \beta_{n+1}(Q) = \pi_i$ for any sequence $Q$. This proves that $\beta_i \geq \pi_i, \forall i \in J$.

Since on the other hand $\beta_i \leq \beta'_i, \forall i \in N \setminus S$, from Proposition 2, while $\beta'_i = \pi_i + 1, \forall i \in J$, from (11), and since all the coefficients in question are integer, $\beta_i = \pi_i$ or $\pi_i + 1, \forall i \in J$. This completes the proof of (12).

(δ) Finally, (13) follows from Lemma 1. Indeed, $\beta_i(Q) = |S| - 1 - z_i$, where $z_i$ is the value of an optimal solution to the knapsack problem $K_i$:

$$\max \sum_{i \in S} x_i + \sum_{j \in O} \beta_j x_j,$$

$$\sum_{i \in S} a_i x_i + \sum_{j \in O} a_j x_j \leq a_0 - a_i,$$

$$x_j = 0 \text{ or } 1, \quad j \in S \cup Q.$$

Since $K_i$ is of the same form as the problem $G_i$ of Lemma 1, its optimal objective function value remains unchanged if the columns $j \in O$ such that $\beta_j \leq \pi_j$ are deleted, i.e., if $Q$ is replaced by $Q$. Q.E.D.

**Remark 4.**

$$\pi_i = \beta_i = \beta'_i, \quad i \in I.$$

$$\pi_i \leq \beta_i \leq \pi_i + 1, \quad i \in J.$$
Theorem 3 has important consequences for the computational effort required to calculate facets of the knapsack polytope obtained from minimal covers.

First, the coefficients $\beta'_i$ and $\pi_i$ are given by the direct formulae (2) and (3), which define the two sets of $|S|$ intervals

$$
(2') \quad \left( a_0 - \sum_{j \in S - S_h} a_j, \quad a_0 - \sum_{j \in S - S_{h+1}} a_j \right), \quad h = 0, 1, \cdots, |S| - 1.
$$

and

$$
(3') \quad \left( \sum_{j \in S_h} a_j, \sum_{j \in S_{h+1}} a_j \right), \quad h = 0, 1, \cdots, |S| - 1.
$$

respectively. Then $\beta'_i$ and $\pi_i$ are found by identifying the interval of (2') and (3') respectively, to which $a_i$ belongs. The set $I$ for which condition (6), holds is precisely the set of those $i \in N - S$ for which the two intervals containing $a_i$ have the same index $h$; and for this set $\beta_i$ is obtained simply by taking $\beta_i = \pi_i$. As for the set $J$, Theorem 3 gives two pieces of information which facilitate the solution of the problems $K_i, i \in J$. One is the knowledge that the value $z_i$ of the optimum is either $|S| - \pi_i - 1$, or $|S| - \pi_i - 2$ (since $\beta_i$ is either $\pi_i$ or $\pi_i + 1$). This reduces the task of solving $K_i$ to checking whether it has a feasible solution with value $|S| - \pi_i - 1$. The other one is the fact that all those columns $j$ such that $\beta_j = \pi_j$ can be removed from $K_i$. This keeps down the size of the problems to be solved.

Also, given a minimal cover $S$ and a sequentially lifted facet $F$ of $P$ obtained from $S$, the only changes in the sequence used to obtain $F$ that have to be considered in order to generate all sequentially lifted facets obtainable from $S$ are changes in the position of those coefficients $j \in J$ such that $\beta_j = \pi_j + 1$.

An easily verifiable sufficient condition for $\beta_i = \pi_i + 1$ to hold for some $i \in J$ is that the value of the linear program $\bar{K}_i$ associated with $K_i$ is less than $|S| - \pi_i - 1$. If, to simplify notation, we denote by $\beta_j$ all the coefficients of the objective function of $K_i$ (i.e., for $j \in S, \beta_j = -1$), then $\bar{K}_i$ becomes

$$
\bar{z}_i = \max \left\{ \sum_{j \in S \cup O} \beta_j x_j \left| \sum_{j \in S \cup O} a_j x_j \leq a_0 - a_i, \ 0 \leq x_j \leq 1, \ j \in S \cup O \right. \right\}.
$$

Let $S \cup O$ be reindexed according to decreasing $\beta_j / a_j$, i.e., $S \cup O = \{j_1, \cdots, j_r\}$, where $r = |S \cup O|$, and

$$
i, k \in \{1, \cdots, r\}, \quad i < k \Rightarrow (\beta_{j_i} / a_{j_i}) \geq (\beta_{j_k} / a_{j_k}),
$$

and let $p \in \{1, \cdots, r\}$ be defined by

$$
\sum_{j = j_1}^{j_p} a_j \leq a_0 - a_i < \sum_{j = j_1}^{j_p+1} a_j.
$$

Then

$$
\bar{z}_i = \sum_{j = j_1}^{j_p} \beta_j + \frac{\beta_{j_1+1} a_{j_1+1}}{\beta_{j_1+1}} \left( a_0 - a_i - \sum_{j = j_1}^{j_p} \beta_j \right).
$$
and if
\[ (19) \quad \bar{z}_i < |S| - \pi_i - 1, \]
then \( \beta_i = \pi_i + 1. \)

Example. Consider the inequality
\[ 9x_1 + 8x_2 + 7x_3 + 6x_4 + 5x_5 + 4x_6 + 4x_7 + 4x_8 + 3x_9 + 2x_{11} + x_{12} \leq 13 \]
and its minimal cover \( S = \{6, 7, 8, 9\}. \)

The intervals defined by (2') and (3') for \( h = 0, 1, 2, 3 \) are shown in the following table:

<table>
<thead>
<tr>
<th>( h )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2')</td>
<td>(-3, 1]</td>
<td>(1, 5]</td>
<td>(5, 9]</td>
<td>(9, 13]</td>
</tr>
<tr>
<td>(3')</td>
<td>[0, 4]</td>
<td>[4, 8]</td>
<td>[8, 12]</td>
<td>[12, 16]</td>
</tr>
</tbody>
</table>

Fitting the coefficients \( a_i, \ i \in N - S, \) into the intervals (2') yields the values \( \beta_i' = h, \) and doing the same for the intervals (3') gives the values \( \pi_i = h: \)
\[ \beta_i' = 2, \quad i = 1, 2, 3, 4; \quad \beta_i' = 1, \quad i = 5, 10, 11; \quad \beta_i' = 0, \quad i = 12; \]
\[ \pi_i = 2, \quad i = 1, 2; \quad \pi_i = 1, \quad i = 3, 4, 5; \quad \pi_i = 0, \quad i = 10, 11, 12. \]

Hence \( \beta_i = \beta_i' = \pi_i \) for \( i \in I = \{1, 2, 5, 12\}, \) and these coefficients remain unchanged for any sequence. Also, we have \( \pi_i \equiv \beta_i' \equiv \pi_i + 1 \) for \( i \in J = \{3, 4, 10, 11\}. \)

To get all the sequentially lifted facets associated with \( S, \) we will only have to consider different sequences of the indices in \( J, \) while the sequence of the indices in \( I \) makes no difference. Furthermore, of the \( 4! = 24 \) possible sequences of \( J, \) we will have to consider only 3.

Starting with \( J_1 = \{3, 4, 10, 11\}, \) we have \( \beta_3 = \beta_4 = 2. \) To obtain \( \beta_4, \) which is either 1 or 2, we check whether condition (19) holds. We have \( S \cup Q = \{3, 6, 7, 8, 9\}, J_1 \cap J_2 = \emptyset, |S| = 4, \pi_4 = 1, \) and \( 2 \geq 4 - 1 - 1, \) i.e., (19) does not hold. Further, \( x_3 = 1, x_i = 0, j \neq 3, \) is a solution to \( K_4, \) with \( z_4 = z_5 = 2, \) and hence \( \beta_4 = \pi_4 = 1. \) Thus the index 4 need not be introduced into the sets \( Q \) used in \( K_i \) for the subsequent indices \( i. \)

To obtain \( \beta_{10}, \) which is either 0 or 1, we check (19). Since \( S \cup Q = \{3, 6, 7, 8, 9\} \) again, \( j_1 = j_2 = 3, z_1 = 2.75, \pi_{10} = 0, \) hence \( \bar{z}_{10} < |S| - \pi_{10} - 1 = 3. \) Therefore \( \beta_{10} = \pi_{10} + 1 = 1. \)

For \( \beta_{11}, \) which is either 0 or 1, we check (19) with \( Q = \{3, 10\}, \) \( S \cup Q = \{10, 3, 6, 7, 8, 9\}, j_1 = 10, j_2 = j_3 = 3, \) and \( z_{11} = 3.25 \geq |S| - \pi_{11} - 1, \) i.e., (19) does not hold. We also find \( z_{11} = 3, \) hence \( \beta_{11} = \pi_{11} = 0, \) and the index 11 need not be added to \( Q. \)

We have thus obtained the facet
\[ 2x_1 + 2x_2 + 2x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \leq 3. \]

Next we try the sequence obtained from \( J_1 \) by interchanging 10 and 11, but \( \beta_{11} \) remains the same as before, and hence so does \( \beta_{10}. \) Further, we do not have to
interchange 4 with any of its successors, since $\beta_4 = \pi_4$ and therefore the change could not possibly affect the value of $\beta_{10}$ or $\beta_{11}$ (see last statement of Theorem 3). This exhausts all sequences starting with 3.

Next we interchange 3 and 4; i.e., consider the sequence $J_2 = \{4, 3, 10, 11\}$. We have $\beta_4 = \beta'_4 = 2$. Proceeding as above, we find that $\beta_3 = \pi_3 = 1$, and the index 3 need not be introduced in $Q$. Further, $\beta_{10} = \pi_{10} = 0$, and $\beta_{11} = \pi_{11} = 0$, and neither 10 nor 11 have to be put into $Q$. Thus the sequence $J_2$ yields the facet

$$2x_1 + 2x_2 + x_3 + 2x_4 + x_5 + x_6 + x_7 + x_8 + x_9 \leq 3.$$ 

Since $\beta_i = \pi_i$ for both $i = 3$ and $i = 10$, neither 3 nor 10 needs to be interchanged with any successor; so that $J_2$ exhausts all sequences starting with 4.

Finally, we consider the sequence $J_3 = \{11, 3, 4, 10\}$. We obtain $\beta_{11} = \beta_{11} = 1$, $\beta_3 = \pi_3 = 1$, $\beta_4 = \pi_4 = 1$, $\beta_{10} = \pi_{10} = 0$, i.e., the facet

$$2x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{11} \leq 3$$

and since $\beta_i = \pi_i$, $i = 3, 4, 10$, no new facets can be obtained by interchanging any of the last 3 indices of $J_3$.

Thus the three inequalities listed above are the only distinct sequentially lifted facets obtainable from the minimal cover $S$.

3. Uniqueness and dominance properties. Next we establish a converse of Theorem 2, which implies an interesting uniqueness property of the facets for which condition (6) is satisfied.

**Theorem 4.** Let $S$ be a minimal cover for (1), and consider the inequality

$$\sum_{j \in S} x_j + \sum_{j \in N - S} \alpha_j x_j \leq |S| - 1.$$ 

Then the following three statements are true if any two of them are true:

(a) the inequality (20) is a facet of $P$;

(b) $\alpha_j = \pi_j$, $j \in N - S$, with $\pi_j$ defined by (3), (4);

(c) condition (6) holds.

**Proof.** From Theorem 2, (b) and (c) imply (a). Suppose now that (a) and (c) are true. Since (20) is a facet, it is a valid inequality for $P$, and from Proposition 4, $\alpha_j \leq \beta_j$, $\forall j \in N - S$. Since condition (6) holds, for any sequentially lifted facet of the form

$$\sum_{j \in S} x_j + \sum_{j \in N - S} \beta_j x_j \leq |S| - 1$$

one has, from Theorem 3, $\beta_j = \beta'_j = \pi_j$, $\forall j \in N - S$ (since (6) implies that $I = N - S$, $J = \emptyset$). Hence both (20) and (21) are valid inequalities, with $\alpha_j \leq \beta_j$, $j \in N - S$. Since (20) is supposed to be a facet, it cannot be strictly dominated by (21), hence $\alpha_j = \beta_j = \pi_j$, $\forall j \in N - S$. Thus (a) and (c) implies (b).

Assume now that (a) and (b) are true, but (c) is not; then (6) is violated, say, for $i_\ast \in N - S$, i.e., $i_\ast \not\in J$ in the notation of Theorem 3. If we now generate a sequentially lifted facet of the form (21), with $i_\ast$ first in the sequence, from Theorem 3 $\beta_{i_\ast} = \beta'_{i_\ast} = \pi_{i_\ast} + 1$, and for all $j \in N - S \cup \{i_\ast\}$, $\beta_j \geq \pi_j$. But then (21)
strictly dominates (20), contrary to the assumption that (20) is a facet. Hence (a) and (b) imply (c). Q.E.D.

We recall that condition (6) implies that $S$ is a strong cover.

**Corollary 4.1.** If condition (6) holds, then the inequality

\[ \sum_{i \in S} x_i + \sum_{j \in N - S} \pi_j x_j \leq |S| - 1 \]

where the coefficients $\pi_i$ are defined by (3), (4), is the unique facet of $P$ which can be obtained from $S$ by sequential lifting; furthermore, (5) is the unique facet of $P$ having coefficients equal to 1 for all $j \in S$ and a right-hand side of $|S| - 1$.

**Proof.** If (6) holds, then $I = N - S$ and from Theorem 3, and $\beta_j = \pi_j$, $\forall j \in N - S$, for any facet obtained from $S$ by sequential lifting; hence the first statement. The second statement follows from the fact that (a) and (c) of Theorem 4 imply (b). Q.E.D.

**Corollary 4.2.** If any two of the three statements of Theorem 4 are true, then for any $N' \subset N - S$, the inequality

\[ \sum_{i \in S} x_i + \sum_{j \in N'} \pi_j x_j \leq |S| - 1 \]

is a facet of $P_V$, where $V = S \cup N'$.

**Proof.** If any two statements of Theorem 4 are true, then (a) and (b) are true. But then (a) and (b) are also true for $j \in N'$, hence the corollary follows. Q.E.D.

In the next Theorem and throughout this paper, $\subset$ means strict inclusion.

**Theorem 5.** Let $S$ and $\tilde{S}$ be two distinct minimal covers such that $|S| = |\tilde{S}|$ and $E(S) \subset E(\tilde{S})$. For $j \in N - S$, let $\beta_j$ and $\pi_j$ be the coefficients defined by (2) and (3) respectively, relative to $S$; and for $j \in N - \tilde{S}$, let $\beta_j$ and $\tilde{\pi}_j$ be the corresponding coefficients relative to $\tilde{S}$. Further, for $j \in S$ define $\beta_j = \pi_j = 1$, and for $j \in S$ define $\beta_j = \tilde{\pi}_j = 1$. Then

\[
\begin{align*}
(22) & \quad \pi_i = \tilde{\pi}_i = \beta_i = \beta_i = 1, \quad i \in S \cup (E(S) \cap \tilde{S}), \\
(23) & \quad \pi_i + 1 = \tilde{\pi}_i = \beta_i = \beta_i = 1, \quad i \in \tilde{S}\backslash E(S),
\end{align*}
\]

and

\[
(24) \quad \pi_i \equiv \tilde{\pi}_i \equiv \beta_i \equiv \beta_i \equiv \pi_i + 1 \quad i \in N - S \cup \tilde{S}.
\]

**Proof.** (a) Since $S \subset E(S)$, the index set for which (22) is claimed to hold can be partitioned into three subsets:

\[
S \cup (E(S) \cap \tilde{S}) = (S \cap \tilde{S}) \cup (S\backslash \tilde{S}) \cup ([\tilde{S}\backslash S) \cap E(S)].
\]

For $i \in S \cap \tilde{S}$, (22) follows from the definitions.

Now let $i \in S \backslash \tilde{S}$. Then $\beta_i = \pi_i = 1$. We claim that $\beta_i = 1$, i.e., (using Proposition 1) that

\[
\sum_{j \in S \backslash \tilde{S}_i} a_j \equiv a_0 \equiv a_i < \sum_{j \in \tilde{S} \backslash \tilde{S}_i} a_j.
\]

Suppose this is not true; then $\beta_i = 0$ or $\beta_i \equiv 2$. If $\beta_i = 0$, then

\[
\sum_{j \in S \backslash \tilde{S}_i} a_j \equiv a_0 \equiv a_i
\]
i.e., \((\bar{S} - \tilde{S}) \cup \{i\}\) is not a cover; which contradicts the fact that \(\bar{S}\) is a cover and \(i \in E(\bar{S})\). If, on the other hand, \(\beta_i^* \geq 2\), then
\[
a_0 - a_i < \sum_{j \in \bar{S} - \tilde{S}} a_j,
\]
i.e., \((\bar{S} - \tilde{S}) \cup \{i\}\) is a cover (of cardinality \(|S| - 1\)). But \(S \subseteq E(\bar{S})\) implies that if \(S = \{j_1, \ldots, j_s\}\) and \(\bar{S} = \{i_1, \ldots, i_s\}\), with \(s = |S| = |\bar{S}|\), then
\[
a_{j_p} \geq a_{i_p}, \quad p = 1, \ldots, s,
\]
and hence \(S - \{j_s\}\) is also a cover. But this obviously contradicts the minimality of the cover \(S\).

Further, we claim that \(\pi_i = 1\), i.e., using (3), (4),
\[
\sum_{j \in S} a_j \geq a_i < \sum_{j \in S - S_1} a_j.
\]

The first inequality holds by the fact that \(i \in S \setminus \tilde{S}\) implies \(a_i \geq a_j \forall j \in \tilde{S}\). If the second inequality is false, then \((\bar{S} - \tilde{S}) \cup \{i\}\) is a cover of cardinality \(|S| - 1\), which is impossible, as shown above. This proves (22) for \(i \in S \setminus \tilde{S}\).

Finally, let \(i \in (\tilde{S} \setminus S) \cap E(\bar{S})\). Then \(\beta_i^* = \pi_i = 1\). We claim that \(\beta_i^* = 1\), i.e.,
\[
\sum_{j \in S - S_2} a_j \geq a_i < \sum_{j \in S - S_1} a_j
\]

Since \(i \in E(\bar{S})\), \(a_i \geq a_{j_1}\), where \(j_1\) is the first element of \(S\). On the other hand \(a_i \leq a_{j_1}\), since \(i \in \tilde{S}\) and \(E(\bar{S}) \subseteq E(\tilde{S})\). Hence \(a_i = a_{j_1}\), and therefore the second inequality of (26) holds since \(S\) is a cover, while the first inequality holds because \(S\) is minimal.

We also claim that \(\pi_i = 1\), i.e.,
\[
\sum_{j \in S_1} a_j \leq a_i < \sum_{j \in S_2} a_j
\]
and again this follows from the fact that \(a_i = a_{j_1}\), since
\[
\sum_{j \in S_1} a_j = a_{j_1} < \sum_{j \in S_2} a_j
\]

This completes the proof of (22).

(b) To prove (23), we note that for \(i \in \bar{S}\), \(\beta_i^* = \pi_i = 1\) by definition. We claim that \(\beta_i^* = 1\), i.e., that (26) holds for \(i \in \bar{S} \setminus E(\bar{S})\). Indeed, \(i \in E(\bar{S})\) implies that \(a_i < a_{j_1}\), where again \(j_1\) is the first element of \(S\). Then the first inequality of (26) holds since the cover \(S\) is minimal, while the second one holds because \((\bar{S} - S_1) \cup \{i\}\) is contained in \(E(\tilde{S})\); and any set of \(|S|\) elements of \(E(\tilde{S})\) is obviously a cover.

Finally, \(\pi_i = 0\) since \(a_i < a_{j_1}\) and \(i \notin S\). This proves (23).

(c) Now let \(i \in N - S \cup \tilde{S}\). From Proposition 1, \(\beta_i^* = h\) and \(\tilde{\beta}_i^* = k\), where \(h\) and \(k\) are the smallest integers satisfying
\[
\sum_{j \in S - S_{h + 1}} a_j \leq a_0 - a_i \quad \text{and} \quad \sum_{j \in \tilde{S} - S_{k + 1}} a_j \leq a_0 - a_i
\]
respectively. From the fact that \(E(S) \subseteq E(\tilde{S})\), and from (25), it then follows that \(k \leq h\), i.e., \(\tilde{\beta}_i^* \leq \beta_i^*\), \(\forall i \in N - S \cup \tilde{S}\).
On the other hand, from the definition (3), (4), \( \pi_i = h \) and \( \bar{\pi}_i = k \), where \( h \) and \( k \) are the greatest integers satisfying
\[
\sum_{j \in S_h} a_j \leq a_i \quad \text{and} \quad \sum_{j \in S_k} a_j \leq a_i
\]
respectively. Therefore \( k \geq h \), i.e., \( \pi_i \geq \bar{\pi}_i \), \( \forall i \in N - S \cup \bar{S} \). Further, from Remark 4, \( \bar{\pi}_i \leq \beta'_i \) and \( \beta'_i \leq \pi_i + 1 \), which completes the proof of (24). Q.E.D.

**Corollary 5.1.** For any two minimal covers \( S, \bar{S} \) such that \( |S| = |ar{S}| \) and \( E(S) \subseteq E(\bar{S}) \),
\[
\{ i \in N \mid \beta'_i = \pi_i \} \subseteq \{ i \in N \mid \bar{\beta}'_i = \bar{\pi}_i \},
\]
and
\[
\{ i \in N \mid \beta'_i = \pi_i + 1 \} \supset \{ i \in N \mid \bar{\beta}'_i = \bar{\pi}_i + 1 \}.
\]

**Proof.** Weak inclusion follows in both cases from the fact that (24) in fact holds for all \( i \in N \), as is easily seen from (22), (23). Strict inclusion follows in both cases from the weak inclusion and (23), since \( \bar{S} \setminus E(S) \neq \emptyset \) always holds in view of \( E(S) \subseteq E(\bar{S}) \). Q.E.D.

Corollary 5.1 shows that, given two minimal covers \( S, \bar{S} \) defined as in the theorem, the facets obtainable by sequential lifting from \( \bar{S} \) are easier to calculate than those obtainable from \( S \), since a greater number of their coefficients \( \bar{\beta}_i \) is obtainable from the relation \( \bar{\pi}_i \geq \bar{\beta}_i \geq \beta'_i \) (where \( \bar{\pi}_i \) and \( \beta'_i \) are readily available), than is the case with the coefficients \( \beta_i \) of the facets obtainable from \( S \). On the other hand, as the next theorem shows, all the facets obtainable by sequential lifting from \( \bar{S} \) are also obtainable by sequential lifting from \( S \), whereas the converse is not true, in general.

Let \( \mathcal{S}_k \) be the family of minimal covers \( S \) of size \( |S| = k \) for (1). We have called strong a member \( S \) of the family such that \( E(S) \subsetneq E(T) \), \( \forall T \in \mathcal{S}_k \). Now let us call weak a minimal cover \( S \in \mathcal{S}_k \) such that \( E(S) \supsetneq E(T) \), \( \forall T \in \mathcal{S}_k \). Furthermore, we can define a partial order on \( \mathcal{S}_k \), by saying that, for any two minimal covers \( S, T \) of the same cardinality, \( S \) is weaker than \( T \) (\( T \) is stronger than \( S \)) if \( E(S) \subsetneq E(T) \).

**Theorem 6.** All the sequentially lifted facets of \( P \) obtainable from minimal covers of a given cardinality \( k \) can be obtained from the weak covers of cardinality \( k \).

**Proof.** Let \( S \) be a weak cover of (1) of cardinality \( k \), and let \( \bar{S} \neq S \) be any minimal cover of the same cardinality but stronger than \( S \). Further, let
\[
(27) \quad \sum_{j \in N} \bar{\beta}_j x_j \leq |S| - 1
\]
be a sequentially lifted facet of (1), obtained from \( \bar{S} \). We will show that (27) can also be obtained from \( S \).

Let
\[
\sum_{j \in N} \beta_j x_j \leq |S| - 1
\]
be a sequentially lifted facet obtained from \( S \) by first calculating the coefficients \( \beta_j \) for \( j \in \bar{S} \) (in arbitrary order), then the coefficients \( \beta_j, j \in N - S \cup \bar{S} \), in the same order as they were calculated for (27). We claim that \( \beta_j = \bar{\beta}_j \), \( \forall j \in N \).
From Theorems 3 and 5, \( \beta_j = \beta_j' = 1 \), \( \forall j \in S \cup (E(S) \cap \tilde{S}) \). Next we show that for \( j \in \tilde{S} \setminus E(S) \) one also has

\begin{equation}
\label{28}
\beta_j = \beta_j' = 1.
\end{equation}

If \(|\tilde{S} \setminus E(S)| = 1\), then \( \beta_j = \beta_j' = 1 \) from Theorem 5. Otherwise, we prove the statement by induction on the set \( \tilde{S} \setminus E(S) = \{j_1, \ldots, j_p\} \). Suppose (28) holds for \( j = j_1, \ldots, j_p \), \( p < q \). Then \( \beta_{ip+1} \) is calculated by solving the knapsack problem \( K_{ip+1} \):

\[ z_{ip+1} = \max \sum_{j \in S} x_j + \sum_{j=1}^{ip} \beta_j x_j, \]

\[ \sum_{j \in S} a_j x_j + \sum_{j=1}^{ip} a_j x_j = a_0 - a_{ip+1}, \]

\[ x_j = 0 \text{ or } 1, \quad j \in S \cup \{j_1, \ldots, j_p\}. \]

We claim that \( z_{ip+1} = |S| - 2 \). To show this, we note that every subset of cardinality \(|S| = k\) of the set \( S \cup \tilde{S} \) is a minimal cover. Therefore \( j_{p+1} \), together with any subset \( \tilde{S} \) of \( k - 1 \) elements of \( S \cup \{j_1, \ldots, j_p\} \), is a minimal cover. Hence in any optimal solution to \( K_{ip+1}, x_j = 1 \) for exactly \(|\tilde{S}| - 1 = |S| - 2\) indices \( j \in S \cup \{j_1, \ldots, j_p\} \), i.e., \( z_{ip+1} = |S| - 2 \). But then \( \beta_{ip+1} = |S| - 1 - z_{ip+1} = 1 \), i.e., if (28) holds for \( j = j_1, \ldots, j_p \), then it also holds for \( j = j_{p+1} \).

We have shown that \( \beta_j = \beta_j' \) for \( j \in S \cup \tilde{S} \). But then the sequence of knapsack problems to be solved in order to calculate the remaining coefficients \( \beta_j, j \in N - S \cup \tilde{S} \), for the facet associated with \( \tilde{S} \), is exactly the same as the corresponding sequence for the coefficients \( \beta_j \) of the facet associated with \( S \). Therefore \( \beta_j = \beta_j', j \in N - S \cup \tilde{S} \). Q.E.D.

The converse of Theorem 6 is not true in general, as illustrated by the following example. Consider the inequality

\[ 8x_1 + 3x_2 + 3x_3 + 3x_4 + 3x_5 + 2x_6 \leq 10 \]

and the two minimal covers of cardinality 4, \( S = \{2, 3, 4, 5\} \) and \( \tilde{S} = \{3, 4, 5, 6\} \).

The only facet that one can obtain from \( \tilde{S} \) by sequential lifting is

\[ 2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3. \]

The same facet is obtained from \( S \) if the indices \( j \in N - S \) are considered in the sequence \( \{6, 1\} \). If, however, we reverse the sequence, we obtain the facet

\[ 3x_1 + x_2 + x_3 + x_4 + x_5 \leq 3 \]

which cannot be recovered from \( \tilde{S} \).

Theorem 6 shows that all facets obtainable by sequential lifting can be obtained by using the weak covers only. On the other hand, Theorem 5 and its corollary show that the stronger a given minimal cover, the easier it is to calculate its coefficients. This means that in spite of the possibility of using weak covers only, it is computationally preferable to obtain each facet from the strongest cover that can yield it; i.e., to start with the strong covers of a given cardinality, and consider the remaining minimal covers of the same cardinality in decreasing order of
strength. The formulae of Theorem 5 contain the necessary information for avoiding duplication of calculations and repetitions of facets.

4. Simultaneously lifted facets. The class of sequentially lifted facets associated with a given minimal cover $S$ does not exhaust the set of all facets having coefficients equal to 1 for $j \in S$ (for a right-hand side of $|S| - 1$), as illustrated by the following example.

Consider the inequality

$$5x_1 + 3x_2 + 3x_3 + 3x_4 + 2x_5 + 2x_6 + 2x_7 + 2x_8 \leq 6$$

and the minimal cover $S = \{5, 6, 7, 8\}$. By sequential lifting one obtains from $S$ the three facets

$$3x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \leq 3,$$

$$3x_1 + x_2 + 2x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \leq 3,$$

and

$$3x_1 + x_2 + x_3 + 2x_4 + x_5 + x_6 + x_7 + x_8 \leq 3.$$

But the inequality

$$3x_1 + \frac{3}{2}x_2 + \frac{3}{2}x_3 + \frac{3}{2}x_4 + x_5 + x_6 + x_7 + x_8 \leq 3$$

is also a facet of $P$ having a coefficient of 1 for each $j \in S$, for a right-hand side of $3 = |S| - 1$, which however cannot be obtained by sequential lifting.

Facets of $P$ which have coefficients equal to 1 for all elements of some minimal cover $S$, with a righthand side of $|S| - 1$, will be called lifted facets, or facets associated with $S$, and will be represented in the general form

$$\sum_{j \in S} x_j + \sum_{j \in N - S} \alpha_j x_j \leq |S| - 1.$$  \hspace{1cm} (29)

Facets of the form (29) which can be obtained by sequential lifting from some minimal cover $T$ of cardinality $|T| = |S|$ (possibly $T \neq S$), will be termed, as before, sequentially lifted facets; those which cannot be obtained by sequential lifting, will be called simultaneously lifted facets.

In this section we characterize the class of facets of the knapsack polytope associated with minimal covers, and give a procedure for generating the simultaneously lifted facets, which can be viewed as a generalization of Padberg's procedure for obtaining the sequentially lifted facets. At the same time, the characterization given here is a specialization to the case of the knapsack polytope of a more general result concerning facets of the convex hull of an arbitrary set of 0–1 points [16].

Next we generalize one of the properties of sequentially lifted facets, proved in Theorem 3, to arbitrary lifted facets. Here $\beta_j'$ and $\pi_j$ are defined as before.

**Theorem 7.** For any lifted facet of the form (29),

$$\pi_j \leq \alpha_j \leq \beta_j', \quad j \in N - S.$$

**Proof.** The second inequality follows from Theorem 1, as shown in Proposition 4. To prove the first one, define $\alpha_j = 1, j \in S$, and suppose that $\alpha_i < \pi_i$ for some
\( i \in N - S \). Then consider the knapsack problem \( \tilde{K}_i \)

\[
\tilde{z}_i = \max \sum_{j \in N - \{i\}} a_j x_j,
\]

\[
\sum_{j \in N - \{i\}} a_j x_j \leq a_0 - a_i,
\]

\( x_j = 0 \) or \( 1 \), \( j \in N - \{i\} \).

Since \( S \) is a cover and \( a_j, a_i \leq \sum_{j \in S_n} a_j \) at most \(|S| - \pi_i - 1\) of the variables \( x_j, j \in S \), can be 1 in any feasible solution; and since \( a_j = 1, \forall j \in S \), while \( S \) is ordered according to decreasing \( a_i \), it follows that the maximum \( \tilde{z}_i \) of \( \tilde{K}_i \) is always attained for some \( \tilde{x} \in R^{n-1} \) such that \( \tilde{x}_j = 0, j \in S_n \). Further, since \( \tilde{x} \) is feasible for \( \tilde{K}_i \),

\[
a_0 \geq \sum_{j \in N - \{i\} - S_n} a_j \tilde{x}_j + a_i
\]

\[
\geq \sum_{j \in N - \{i\} - S_n} a_j \tilde{x}_j + \sum_{j \in S_n} a_j \quad \left( \text{from } a_i \geq \sum_{j \in S_n} a_j \right)
\]

\[
= \sum_{j \in N} a_j \tilde{x}_j
\]

where

\[
\tilde{x}_j = \begin{cases} \tilde{x}_j, & j \in N - \{i\} - S_n, \\ 1, & j \in S_n, \\ 0, & j = i; \end{cases}
\]

in other words \( \tilde{x} \) is a feasible solution to (1). Since (29) is a valid inequality, we have

\[
|S| - 1 \geq \sum_{j \in N} a_j \tilde{x}_j
\]

\[
= \sum_{j \in N - \{i\}} a_j \tilde{x}_j + \sum_{j \in S_n} a_j \quad \text{(since } \tilde{x}_j = 0, \forall j \in S_n) 
\]

\[
= \tilde{z}_i + \pi_i \quad \text{(since } a_j = 1, \forall j \in S). 
\]

But then

\[
\max \left\{ \pi_i x_i + \sum_{j \in N - \{i\}} a_j x_j | x \in P \right\} = \max \{ \pi_i + \tilde{z}_i, \sigma \} \leq |S| - 1,
\]

where

\[
\sigma = \max \left\{ \sum_{j \in N - \{i\}} a_j x_j | x \in P_{N - \{i\}} \right\},
\]

i.e.,

\[
\pi_i x_i + \sum_{j \in N - \{i\}} a_j x_j \leq |S| - 1
\]

is satisfied by all \( x \in P \), which contradicts the assumption that (29), with \( \alpha_i < \pi_i \), is a facet of \( P \). Q.E.D.
Partitioning again \( N - S \) into \( I \) and \( J \), where \( i \in I \) if (6), holds and \( i \in J \) otherwise (see § 2), we have

**Corollary 7.1.** For any lifted facet of the form (29),

\[
\alpha_i = \pi_i = \beta'_i, \quad \forall i \in I.
\]

Next we generalize the coefficients \( \beta_i, \beta'_i \) and \( \pi_i \) introduced in § 2 in a way that allows us to use their properties for the nonsequential (simultaneous) lifting of facets.

Given an inequality (1) and a minimal cover \( S \) for (1), let

\[
\mathcal{M}(N-S) = \left\{ M \leq N - S \left| \sum_{j \in M} a_j \equiv a_0 \right. \right\}.
\]

The elements of \( \mathcal{M}(N-S) \) are in one-to-one correspondence with the 0–1 points in \( R^{N-S} \) satisfying

\[
\sum_{j \in N-S} a_j x_j \equiv a_0,
\]

namely each such 0–1 point \( x \) is of the form

\[
x_j = \begin{cases} 1, & j \in M, \\ 0, & j \in N - S \cup M, \end{cases}
\]

for some \( M \in \mathcal{M}(N-S) \).

With each \( M \in \mathcal{M}(N-S) \), we associate a “set variable” \( x_M \) with coefficient

\[
a_M = \sum_{j \in M} a_j.
\]

Then the “set coefficients” \( \beta_M, \beta'_M \) and \( \pi_M \) are defined in relation to \( x_M \) in the same way as the coefficients \( \beta_i, \beta'_i, \pi_i \) were defined in relation to \( x_i \). Thus, \( \beta'_M = |S| - 1 - z'_M \), where \( z'_M \) is the optimal objective function value of the knapsack problem \( K'_M \):

\[
z'_M = \max \left\{ \sum_{j \in S} x_j \left| \sum_{j \in S} a_j x_j \equiv a_0 - a_M, x_j = 0 \ or \ 1, j \in S \right. \right\};
\]

or, alternatively, \( \beta'_M \) is given by (2), with \( M \) replacing \( i \). Similarly, \( \pi_M \) is given by (3), with \( i \) replaced by \( M \). Finally, the coefficients \( \beta_M \) are obtained by solving a sequence of knapsack problems \( K_M \), for a collection of pairwise disjoint sets \( M \subset N - S \). Clearly, the properties of the coefficients \( \beta_i, \beta'_i, \pi_i \) are carried over to their “set” counterparts. In particular, the coefficients \( \beta_M \) depend on \( S \), as well as on the sequence in which the disjoint subsets \( M \subset N - S \) are considered; whereas \( \beta'_M \) and \( \pi_M \) depend only on \( S \). Also,

\[
\pi_M \equiv \beta_M \equiv \beta'_M, \quad \forall M \in \mathcal{M}(N-S).
\]

The “set coefficients” \( \beta_M, \beta'_M \) and \( \pi_M \) can be viewed as functions defined on \( \mathcal{M}(N-S) \). The next theorem then says that the function \( \pi_M \) is superadditive on \( \mathcal{M}(N-S) \), whereas \( \beta'_M \) has a similar, but weaker property.
Theorem 8. For every $M \in \mathcal{M}(N-S)$, $|M| \geq 2$, and every proper subset $M_1 \subset M$,

\begin{equation}
\pi_{M_1} + \pi_{M-M_1} \leq \pi_M
\end{equation}

and

\begin{equation}
\pi_{M_1} + \beta'_{M-M_1} \leq \beta'_M.
\end{equation}

Proof. Let $M - M_1 = M_2$ and $h_i = \pi_{M_i}$ for $i = 1, 2$. From the definition (3) of the coefficients $\pi$ and the fact that $a_M = a_{M_1} + a_{M_2}$, we have

$$a_M = \sum_{j \in S_{h_1}} a_j + \sum_{j \in S_{h_2}} a_j \geq \sum_{j \in S_{h_1+h_2}} a_j$$

where $S_h$ is, as before, the set of the first $h$ elements of $S$. Hence $\pi_M \geq h_1 + h_2$, which proves (31).

Now let $h = \pi_{M_1}$ and $k = \beta'_{M_1}$. From the definitions (2) and (3) of the coefficients $\beta'$ and $\pi$ respectively,

$$a_M = a_{M_1} + a_{M_2} \geq \left( \sum_{j \in S_h} a_j \right) + \left( a_0 - \sum_{j \in S - S_h} a_j \right) \geq a_0 - \sum_{j \in S - S_h} a_j.$$

Hence $\beta'_M \geq h + k$, which proves (32). Q.E.D.

Now let

$$\mathcal{M}(J) = \left\{ M \subseteq J \mid \sum_{j \in M} a_j \leq a_0 \right\},$$

where $J$, as before, is the set of those $j \in N - S$ for which condition (6), does not hold.

Corollary 8.1. For any $M \in \mathcal{M}(J)$, $|M| \geq 2$,

$$\beta'_M - \sum_{j \in M} \pi_j \geq 1.$$

Proof. For any $i \in M$,

$$\beta'_M \geq \pi_i + \sum_{j \in M \setminus \{i\}} \pi_j \quad \text{(from (32))}$$

$$\geq \pi_i + \pi_{M \setminus \{i\}} \quad \text{(since $i \in J$)}$$

$$\geq \sum_{j \in M} \pi_j + 1 \quad \text{(from (31)). Q.E.D.$$}

The next theorem characterizes the class of all facets associated with minimal covers.

Theorem 9. Let $S$ be a minimal cover for (1). Then the inequality

\begin{equation}
\sum_{j \in S} x_j + \sum_{j \in N-S} \alpha_j x_j \leq |S| - 1
\end{equation}

is a facet of $P$ if and only if

\begin{itemize}
  \item[(i)] $\alpha_j = \begin{cases} \pi_j, & j \in I, \\
                 \pi_j + \delta_j, & j \in J. \end{cases}$
\end{itemize}
with $0 \leq \delta_j \leq 1, j \in J$; and

(ii) the vector $\delta \in \mathbb{R}^{|J|}, \delta = (\delta_j)$, is a vertex of the polyhedral set

$$T = \left\{ \delta \in \mathbb{R}^{|J|} \left| \sum_{j \in M} \delta_j \leq \beta'_M - \sum_{j \in M} \pi_j, M \in \mathcal{M}(J) \right. \right\}.$$  

Proof. (a) If (29) is a facet of $P$, then from Theorem 7, $\pi_j \leq \alpha_j \leq \beta'_j, \forall j \in N - S$, while from Theorem 3, $\beta'_j = \pi_j, \forall j \in I$, and $\beta'_j = \pi_j + 1, \forall j \in J$; thus (i) holds.

(b) Next we note that there is a one-to-one correspondence between points in general, and vertices in particular, of $T$, and the convex polyhedral set defined by $\alpha_j = \pi_j + \delta_j, j \in J$, and

$$\sum_{j \in M} \alpha_j \leq \beta'_M, \quad M \in \mathcal{M}(J).$$

Indeed, (33) holds if and only if $\delta \in T$, where $\delta_j = \alpha_j - \pi_j, j \in J$; and a given inequality of (33) is tight if and only if the corresponding inequality of $T$ is tight.

(c) We now show that (29) is a valid inequality for $P$ (where valid means satisfied by all $x \in P$), such that $\alpha_j = \pi_j, \forall j \in I$, if and only if (33) holds.

With every $M \in \mathcal{M}(J)$, we associate the set

$$X(M) = \{x \in \text{vert } P \mid x_j = 1, j \in M\}$$

where vert $P$ is the set of vertices of $P$. Clearly, the collection of sets $X(M)$, for all $M \in \mathcal{M}(J)$, is a partition of vert $P$, since the empty set is always contained in $\mathcal{M}(J)$.

We show that (29) is satisfied by all $x \in X(M)$, for a given $M \in \mathcal{M}(J)$, if and only if (33) holds for the same $M$. If $M = \emptyset$ the statement is trivially true; therefore let $M \neq \emptyset$. Since $\alpha_j = \pi_j, \forall j \in I$, (29) is satisfied by all $x \in X(M)$ if and only if

$$\max_{x \in X(M)} \left\{ \sum_{j \in S} x_j + \sum_{j \in I} \pi_j x_j \right\} + \sum_{j \in M} \alpha_j \leq |S| - 1.$$  

Denoting

$$z_M = \max_{x \in X(M)} \left\{ \sum_{j \in S} x_j + \sum_{j \in I} \pi_j x_j \right\},$$

we have

$$z_M = \max \left\{ \sum_{j \in S} x_j + \sum_{j \in I} \pi_j x_j \left| \sum_{j \in S \cup I} a_j x_j \leq a_0 - a_M, x_j = 0 \text{ or } 1, j \in S \cup I \right. \right\}$$

$$= \max \left\{ \sum_{j \in S} x_j \left| \sum_{j \in S} a_j x_j \leq a_0 - a_M, x_j = 0 \text{ or } 1, j \in S \right. \right\} = z_M',$$  

since according to Lemma 1 of § 2, the value of the maximum in the above knapsack problem is not affected by the deletion of the columns $j \in I$.

But then from (34),

$$\sum_{j \in M} \alpha_j \leq |S| - 1 - z_M' - \beta'_M.$$  

This proves that (29), with $\alpha_j = \pi_j, \forall j \in I$, is a valid inequality if and only if (33)-holds; i.e., if and only if $\delta \in T$, where $\delta_j = \alpha_j - \pi_j, j \in J$. 
(d) Before proceeding further, we mention that from Corollary 8.1,
\[ \beta_M = \sum_{i \in M} \pi_i \geq 1, \quad \forall M \in \mathcal{M}(J) \]
and for \(|M| = 1\), this inequality is tight. Thus the constraints of \(T\) include the inequalities \(\delta_i \leq 1\). Further, since all the right-hand sides are strictly positive, \(T\) is clearly full-dimensional. Hence \(\delta \in T\) is a vertex of \(T\) if and only if
\[ \sum_{i \in M_i} \delta_i = \beta'_{M_i} - \sum_{i \in M_i} \pi_i \]
for \(|J|\) linearly independent sets \(M_i\). Here a collection of \(|J|\) sets \(M_i\) is called linearly independent, if the \(|J|\) vectors \(u(M_i) \in R^{|J|}\), \(i \in J\), defined by \(u_i(M_j) = 1\) if \(j \in M_i\), \(u_i(M_j) = 0\) otherwise, are linearly independent.

(c) Now suppose (29) is a facet of \(P\). Then (i) holds, as shown under (a). Further, there exists a set of \(n\) linearly independent vectors \(x' \in \text{vert } P\), such that
\[ \sum_{j \in S} x'_j + \sum_{j \in J} \pi_j x'_j + \sum_{j \in J} \alpha_j x'_j = |S| - 1, \quad i = 1, \cdots, n, \]
or, denoting \(M_i = \{j \in J | x'_j = 1\}\),
\[ \sum_{j \in M_i} \alpha_j = |S| - 1 - \left( \sum_{j \in S} x'_j + \sum_{j \in J} \pi_j x'_j \right) \geq |S| - 1 - \beta'_{M_i} = \beta'_{M_i}, \quad i = 1, \cdots, n, \]
where we have used (35) to obtain the inequality.

This, together with (33), yields
\[ \sum_{j \in M_i} \alpha_j = \beta'_{M_i}, \quad i = 1, \cdots, n. \tag{36} \]

The system (36) is of rank \(|J|\), i.e., the collection of sets \(M_i, i = 1, \cdots, n\), contains \(|J|\) linearly independent members; for if not, then the \(n \times n\) matrix \(X\) whose \(i\)th row is \(x'\) contains an \(n \times |J|\) submatrix of rank less than \(|J|\), which contradicts the assumption that the rows of \(X\) are linearly independent. From the correspondence described under (b), the inequalities of \(T\) associated with the \(|J|\) linearly independent sets \(M_i\), for which (36) holds are also satisfied with equality; hence \(\delta \in R^{|J|}\), defined by \(\delta_j = \alpha_j - \pi_j, j \in J\), is a vertex of \(T\), as shown above under (d).

Therefore, if (29) is a facet of \(P\), then (ii) holds as well as (i).

(f) Suppose now that (i) and (ii) are true. Then (29) is a valid inequality, as shown under (c). To show that it is a facet of \(P\), we will display \(n\) linearly independent vertices \(x', i = 1, \cdots, n\) of \(P\), which satisfy (29) with equality.

Since \(\delta\) is a vertex of \(T\), there exists a collection of \(|J|\) linearly independent sets \(M_i \subset J, i = 1, \cdots, |J|\), such that
\[ \sum_{j \in M_i} \delta_j = \beta'_{M_i} - \sum_{j \in M_i} \pi_j, \]
i.e.,
\[ \sum_{j \in M_i} \alpha_j = \beta'_{M_i}, \quad i = 1, \cdots, |J|. \]
For each \( i \in J \), let \( y^i \) be an optimal solution to the knapsack problem (35), and let \( x^i \) be defined by

\[
x^i_j = \begin{cases} 
1, & j \in M_i, \\
0, & j \in J - M_i, \\
y^i_j, & j \in S \cup I.
\end{cases}
\]

Clearly \( x^i \in P, i = 1, \ldots, |J| \). If the \( |J| \times n \) matrix whose rows are \( x^i, i = 1, \ldots, |J| \), is denoted by \( B \), it can be partitioned (possibly after column permutations) so that

\[
B = (B_1, B_2)
\]

where \( B_2 \) is the \( |I| \times |I| \) submatrix whose columns are indexed by \( J \). Since the \( |J| \) sets \( M_i \) are linearly independent, \( B_2 \) is nonsingular.

Consider now \( |N - J| = |S \cup I| \) linearly independent vectors \( x^i \in P, i = |J| + 1, \ldots, n \), such that \( x^i_j = 0, \forall j \in J \), and the inequality

\[
\sum_{j \in S} x^i_j + \sum_{j \in I} \pi_j x^i_j \leq |S| - 1
\]

is tight for \( x^i, i = |J| + 1, \ldots, n \). The existence of such vectors follows from the fact that (37) is a facet of \( P_V \), where \( V = S \cup I \). If the \( (n - |J|) \times n \) matrix whose rows are the vectors \( x^i, i = |J| + 1, \ldots, n \), is denoted by \( C \), it can be partitioned (after column permutations, if necessary) so that

\[
C = (C_1, 0)
\]

where the zero matrix is of order \( (n - |J|) \times |J| \), while the \( (n - |J|) \times (n - |J|) \) matrix \( C_1 \) is nonsingular.

Then the \( n \times n \) matrix

\[
D = \begin{pmatrix} B_1 & B_2 \\ C_1 & 0 \end{pmatrix}
\]

is nonsingular, i.e., the \( n \) points \( x^i \in P \) which correspond the rows of \( D \) are linearly independent, and they all satisfy (29) with equality. Hence (29) is a facet of \( P \). Q.E.D.

A direct consequence of Theorem 9 is

**Remark 5.** \( \delta_j \geq 0, \forall j \in J \), for every vertex \( \delta \) of \( T \).

The facets characterized in Theorem 9 are a special case of the simultaneously lifted facets of the convex hull of an arbitrary 0-1 point, characterized in Theorem 7 of [16]. Theorem 9 is also a close relative of Theorems 4.1 and 4.2 of [2], which characterize facets of disjunctive programs.

**Corollary 9.1.** The inequality (29), where \( S \) is a minimal cover for (1) and \( \alpha_j \) satisfies (i), (ii), is a sequentially lifted facet if and only if \( \delta_j = 0 \) or \( 1, j \in J \).

**Proof.** The “only if” part is a direct consequence of Theorem 3. To show the “if” part, we note that \( \delta_j = 0 \) or \( 1 \) implies that \( \alpha_j = \pi_j \) or \( \beta_j \), \( j \in J \).
Now let

$$\sum_{j \in S} x_j + \sum_{j \in N - S} \beta_j x_j \equiv |S| - 1$$

be a sequentially lifted facet which has a maximal number of coefficients \(\beta\), such that \(\beta_j = \alpha_j\), and let this number be \(k\). Further, let \(N - S = \{i_1, \ldots, i_p\}\) be ordered so that \(\alpha_j = \beta_j\) for \(j = i_1, \ldots, i_k\). Since \(\alpha_j = \beta_j\) for all \(j \in I, I = \{i_{k+1}, \ldots, i_p\}\)\),

since \(\alpha_i = \beta_i\), \(i = i_{k+1}, \ldots, i_p\), either \(\alpha_i < \beta_i\), \(i = i_{k+1}, \ldots, i_p\), or \(\alpha_i > \beta_i\) for some \(i_a \in \{i_{k+1}, \ldots, i_p\}\). In the first case, (29) is strictly dominated by (38), which contradicts the fact that it is a facet. In the second case, \(\alpha_i = \pi_{i_a} + 1 = \beta_{i_a}\), but the coefficient \(\beta_{i_a}\) can be obtained by solving

$$z_{i_a}^* = \max \left\{ \sum_{j \in S} x_j + \sum_{j = i_a}^{i_b} \beta_j x_j \sum_{j \in S} a_j x_j + \sum_{j = i_a}^{i_b} a_j x_j \equiv a_0 - a_{i_a}, x_j = 0 \text{ or } 1, \forall j \right\}$$

and hence there exists a sequentially lifted facet of the form (38), with \(\beta_j = \alpha_j\), \(j = 1, \ldots, i_k\), contrary to the assumption. Q.E.D.

Theorem 9 and its Corollary imply that all facets with integer coefficients obtainable from a given minimal cover can be generated by sequential lifting; whereas the facets unobtainable by sequential lifting, if any, are those with fractional coefficients. These can be generated by finding the fractional vertices of the polyhedron \(T\), i.e., the fractional basic solutions of the system defining \(T\). In many cases there exists a more straightforward way of identifying these "fractional" lifted facets, as will be shown in the next Theorem. But first we discuss an example.

**Example.** Consider again the inequality

$$9x_1 + 8x_2 + 7x_3 + 6x_4 + 5x_5 + 4x_6 + 4x_7 + 4x_8 + 4x_9 + 3x_{10} + 2x_{11} + x_{12} \equiv 13$$

of the example of §2, and the minimal cover \(S = \{6, 7, 8, 9\}\).

From §2, we have \(I = \{1, 2, 5, 12\}\), hence

$$\alpha_i = \pi_i = 2, \quad i = 1, 2; \quad \alpha_5 = \pi_5 = 1, \quad \text{and} \quad \alpha_{12} = \pi_{12} = 0.$$  

Further, \(J = \{3, 4, 10, 11\}\), and \(\pi_i = 1, i = 3, 4; \pi_i = 0, i = 10, 11\); i.e., \(\alpha_3 = 1 + \delta_3, \quad \alpha_4 = 1 + \delta_4, \quad \alpha_{10} = 0 + \delta_{10}, \quad \text{and} \quad \alpha_{11} = 0 + \delta_{11}\), where \(0 = \delta_{10} \equiv 1, i = 3, 4, 10, 11,\)

In order to define the polyhedron \(T\) whose vertices are the vectors \(\delta\) which yield lifted facets, we have to calculate the coefficients \(\beta_M^J\) for each \(M \in M(J)\), namely for \(\{3\}, \{4\}, \{10\}, \{11\}, \{3, 4\}, \{3, 10\}, \{3, 11\}, \{3, 10, 11\}, \{4, 10\}, \{4, 11\}, \{4, 10, 11\}\). These coefficients are

$$\beta_3^J = 2, \quad \beta_4^J = 2, \quad \beta_{10}^J = 1, \quad \beta_{11}^J = 1, \quad \beta_{\{3, 4\}}^J = 3, \quad \beta_{\{3, 10\}}^J = 3, \quad \beta_{\{3, 11\}}^J = 3, \quad \beta_{\{3, 10, 11\}}^J = 3, \quad \beta_{\{4, 10\}}^J = 2, \quad \beta_{\{4, 11\}}^J = 2, \quad \beta_{\{4, 10, 11\}}^J = 1,$$

and since the inequalities defined by \(\{3, 10\}, \{3, 10, 11\}, \{4, 10, 11\}\) (which happen
to be those with $\beta_M - \sum_{j \in M} \pi_j > 1$ are easily seen to be redundant. $T$ is defined by

$$
\delta_3 + \delta_4 \leq 1 \\
\delta_3 + \delta_{11} \leq 1 \\
\delta_4 + \delta_{10} \leq 1 \\
\delta_4 + \delta_{11} \leq 1 \\
\delta_{10} + \delta_{11} \leq 1 \\
\delta_j \leq 1, \quad j = 3, 4, 10, 11.
$$

The vertices $\delta^i = (\delta^i_3, \delta^i_4, \delta^i_{10}, \delta^i_{11})$ of $T$ and the corresponding values of $\alpha^i = (\alpha^i_3, \alpha^i_4, \alpha^i_{10}, \alpha^i_{11})$ are

$$
\delta^1 = (1, 0, 1, 0), \quad \alpha^1 = (2, 1, 1, 0), \\
\delta^2 = (0, 1, 0, 0), \quad \alpha^2 = (1, 2, 0, 0), \\
\delta^3 = (0, 0, 0, 1), \quad \alpha^3 = (1, 1, 0, 1), \\
\delta^4 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \alpha^4 = \left(\frac{3}{3}, \frac{3}{3}, \frac{3}{3}, \frac{3}{3}\right).
$$

The three integer vertices $\delta^i$, $i = 1, 2, 3$, of $T$, yield the three sequentially lifted facets found for this example in §2; while the fractional vertex $\delta^4$ yields the "fractional" lifted facet

$$
2x_1 + 2x_2 + \frac{1}{2}x_3 + \frac{1}{3}x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \frac{1}{2}x_{10} + \frac{1}{2}x_{11} \leq 3.
$$

These are the only facets obtainable by lifting from the minimal cover $S = \{6, 7, 8, 9\}$.

From Theorem 9, given a minimal cover $S$, all the facets of $P$ obtainable from $S$ by nonsequential (i.e., simultaneous) lifting, are given by the set of fractional vertices of $T$. In view of the special structure of the constraint set defining $T$, it is natural to look for some direct way of finding such fractional vertices. This is the object of the next Theorem.

For any minimal cover $S$, let $J$ be, as before, the set of those indices $i \in N - S$ for which (6), does not hold.

**Theorem 10.** Let $S$ be a minimal cover for (1), and suppose there exists an integer $k$, $1 \leq k \leq |J| - 1$, such that

(i) $T$ has a vertex $\delta$ of the form

$$
\delta_j = \begin{cases} 
1, & j \in K, \\
0, & j \in J - K,
\end{cases}
$$

for every subset $K \subset J$ of cardinality $|K| = k$;

(ii) $|M| \leq k + 1$ for all $M \in \mathcal{M}(J)$.

Then every $\delta \in R^{|J|}$ of the form

$$
\delta_j = \begin{cases} 
1, & \text{for } k - h \text{ indices } j \in J, \\
\frac{h}{h+1}, & \text{for all other } j \in J,
\end{cases}
$$

where $0 \leq h \leq k$, is a vertex of $T$. 
Proof. (a) Let \( \delta \in R^J \) be defined by (39). For \( M \in \mathcal{M}(J) \) such that \( |M| \leq k \),

\[
\sum_{j \in M} \delta_j \leq |M| \quad \text{(since } \delta_j \leq 1, \forall j \in J) \]

\[
\leq \beta'_M - \sum_{j \in M} \pi_j
\]

since from (i), \( T \) has a vertex \( \tilde{\delta} \) with \( \tilde{\delta}_j = 1 \) for \( j \in M \) and thus

\[
|M| = \sum_{j \in M} \tilde{\delta}_j \leq \beta'_M - \sum_{j \in M} \pi_j.
\]

For \( M \in \mathcal{M}(J) \) such that \( |M| = k + 1 \),

\[
\sum_{j \in M} \delta_j = k - h + (|M| - k + h) \cdot \frac{h}{h + 1} \quad \text{(from (39))}
\]

\[
= k
\]

\[
= \beta'_M - \sum_{j \in M} \pi_j \quad \text{(from (i)).}
\]

Thus \( \delta \) satisfies all the inequalities defining \( T \), i.e., \( \delta \in T \).

(b) To show that \( \delta \) defined by (1) is a vertex of \( T \) for any \( h \) satisfying \( 0 \leq h \leq k \), we note that for \( h = 0 \) this is trivially true from (i). Next we show that for any \( h \) satisfying \( 1 \leq h \leq k \) there exists a collection of \( |J| \) linearly independent sets \( M_i \in \mathcal{M}(J), i = 1, \ldots, |J| \), such that the corresponding inequality of \( T \) is tight.

Consider the \( |J| \times |J| \) matrix

\[
B = \begin{pmatrix} I & 0 \\ E & F \end{pmatrix}
\]

where \( I \) is the identity of order \((k - 1) \times (k - 1)\), \( 0 \) is a zero matrix, \( E \) has all of its entries equal to 1, and \( F \) is a \((|J| - k + 1) \times (|J| - k + 1)\) square matrix of the form

\[
F = \begin{pmatrix}
\begin{array}{ccccccccccc}
1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 0 & 1 \\
\end{array}
\end{pmatrix}
\]

Since \( F \) can be partitioned according to

\[
F = \begin{pmatrix} F_1 & F_2 \\ 0 & F_3 \end{pmatrix}
\]

where \( F_3 \) is the \((h + 2) \times (h + 2)\) circulant matrix with \( h + 1 \) ones in each row and each column, and \( F_3 \) is upper triangular, \( F \) is clearly nonsingular, and hence so is \( B \). Therefore the rows of \( B \) are linearly independent, and so are the sets \( M_i \), where each \( M_i \) is the index set of the nonzero entries of row \( i \) of \( B \).
It remains to be shown that $M_i \in \mathcal{M}(J)$ for $i = 1, \cdots, |J|$. Suppose this is false for some $r \in \{1, \cdots, |J|\}$, and let $M_r = \{j_1, \cdots, j_{k+1}\}$. Then
\[
\sum_{j=1}^{j_{k+1}} a_j > a_0,
\]
i.e.,
\[
\sum_{j=1}^{j_k} a_j > a_0 - a_{k+1}.
\]
(40)

Now let
\[
\sum_{j \in S} x_j + \sum_{j=1}^{j_k} \beta_j x_j \equiv |S| - 1
\]
be a facet of $P$ obtained by sequential lifting, using the sequence $J' = \{j_1, \cdots, j_k\}$. Further, let
\[
\beta_{k+1}(J') = |S| - 1 - z_{k+1}(J'),
\]
where
\[
z_{k+1}(J') = \max \left\{ \sum_{j \in S} x_j + \sum_{j \in J'} \beta_j x_j \left| \sum_{j \in S \cup J'} a_j x_j \equiv a_0 - a_{k+1}, x_j = 0 \text{ or } 1, j \in S \cup J' \right. \right\}
\]
and let $\bar{x}$ be a solution for which the maximum is attained. From (40), $\bar{x}_j = 0$ for at least one $j \in \{j_1, \cdots, j_k\}$; hence there exists a subsequence $J'' = \{i_1, \cdots, i_p\}$, $p < k$, of $J'$, such that
\[
\beta_{k+1}(J'') = \beta_{k+1}(J').
\]
(41)

Since $|M_r| = k+1$ and $|J \cup \{j_{k+1}\}| \leq k$, from (i) we have
\[
\beta_{k+1}(J') = \pi_{j_{k+1}} + 1 \quad \text{and} \quad \beta_{k+1}(J') = \pi_p,
\]
which contradicts (41). This proves that $M_i \in \mathcal{M}(J)$, $i = 1, \cdots, |J|$. Q.E.D.

When conditions (i) and (ii) of Theorem 10 are satisfied, then all the inequalities of the form (29), with
\[
\alpha_j = \begin{cases} 
\pi_j, & j \in I, \\
\pi_j + 1, & \text{for } k - h \text{ indices } j \in J, \\
\pi_j + \frac{h}{h + 1}, & \text{for all other } j \in J,
\end{cases}
\]
for $h = 0, 1, \cdots, k$, are facets of $P$. The sufficient conditions (i), (ii) require a certain symmetry of $T$ relative to $J$.

Remark 6. Theorem 10 can be generalized by
(a) replacing $J$ with some arbitrary $J' \subset J$, and setting $a_j = 0, j \in J - J'$ in (39);
(b) removing condition (ii) and replacing "$0 \leq h \leq k$" by "$0 \leq h \leq i$, where $i$ is the greatest integer satisfying
\[
\frac{k + (|M| - 1)i}{i + 1} \leq \beta_M - \sum_{j \in M} \pi_j
\]
for all $M \in \mathcal{M}(J)$ such that $|M| \geq k + 2$".
Example. Consider the inequality
\[ 3x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 + 2x_6 + 2x_7 \leq 6 \]
and the minimal cover \( S = \{1, 2, 3\} \). We have \( J = \{4, 5, 6, 7\} \), with \( \beta_j = 1 \) and \( \pi_j = 0 \), \( \forall j \in J \), while \( T \) is defined by
\[
\begin{align*}
\delta_4 + \delta_5 + \delta_6 & \leq 2 \\
\delta_4 + \delta_5 + \delta_7 & \leq 2 \\
\delta_4 + \delta_6 + \delta_7 & \leq 2 \\
\delta_5 + \delta_6 + \delta_7 & \leq 2 \\
\delta_j & \leq 1, \quad j = 4, 5, 6, 7.
\end{align*}
\]
Since all vectors \( \delta \) of the form
\[
(42) \quad \delta_j = \begin{cases} 
1 & \text{for 2 indices } j \in \{4, 5, 6, 7\}, \\
0 & \text{for all other } j \in \{4, 5, 6, 7\}
\end{cases}
\]
are vertices of \( T \), \( T \) displays the required symmetry relative to \( J \), with \( k = 2 \). Also, the largest \( M \in M(J) \) is of cardinality \( k + 1 = 3 \). Hence, besides the vectors \( \delta \) of the form (41), all \( \delta \) of the following form are vertices of \( T \):
\[
(43) \quad \delta_j = \begin{cases} 
1 & \text{for any one } j \in \{4, 5, 6, 7\}, \\
\frac{1}{2} & \text{for the other three } j \in \{4, 5, 6, 7\}
\end{cases}
\]
and
\[
(44) \quad \delta_j = \frac{1}{3}, \quad j = 4, 5, 6, 7.
\]
Three representative facets corresponding to \( \delta \) of the form (42), (43) and (44) are
\[
(42') \quad x_1 + x_2 + x_3 + x_4 + x_5 \leq 2
\]
\[
(43') \quad x_1 + x_2 + x_3 + x_4 + \frac{1}{3} x_5 + \frac{1}{3} x_6 + \frac{1}{3} x_7 \leq 2
\]
and
\[
(44') \quad x_1 + x_2 + x_3 + \frac{1}{3} x_4 + \frac{1}{3} x_5 + \frac{1}{3} x_6 + \frac{1}{3} x_7 \leq 2.
\]
respectively. Any other facet associated with \( S \) can be obtained by interchanging the coefficients of \( x_j, j = 4, 5, 6 \) and 7, in one of the above three inequalities.

REFERENCES


