Optimal Delta Hedging for Options

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Derivatives and Volatility Conference,
NYU Stern, April 2017
Background

Presentation based on:
“Optimal Delta Hedging for Options”
John Hull and Alan White
www-2.rotman.utoronto.ca/~hull or ssrn2658343
Hedging: The Textbook Approach

- Develop model
- Calculate partial derivatives with respect to stochastic variables
- Hedge risks by trading instruments to make partial derivatives close to zero
Hedging Options in Practice

- Use Black-Scholes-Merton model with volatility set equal to implied volatility of the option
- Calculate Greek letters including the partial derivative wrt implied volatility (even though the model assumes volatility is constant)
- Hedge Greeks
Justification for the Approach

To a good approximation the value of an option is a deterministic (Black-Scholes-Merton) function of the asset price, $S$, and the implied volatility, $\sigma_{imp}$.

$$f = f_{BS}(S, \sigma_{imp})$$

A Taylor Series expansion shows that, if a portfolio’s partial derivatives wrt $S$ and $\sigma_{imp}$ are low, there is little risk.
Well Known Empirical Result

- An equity price and its volatility are negatively correlated (e.g. Black (1976) and Christie (1982))
- Possible reason is leverage (e.g., Geske (1979) model)
- Another possible reason is the “volatility feedback” effect
The Minimum Variance (MV) Delta

The MV delta is the position in the underlying that minimizes the variance of the changes in the value of the portfolio.

It is different from the normally calculated Black-Scholes delta because it takes into account the expected change in implied volatility conditional on an equity price change.
Prior Research

Researchers who have tested the performance of MV delta using a stochastic volatility or local volatility model:

- Alexander and Nogueira (2007)
- Alexander, Kaeck, and Noueira (2009)
- Alexander, Rubinov, Kalepky, and Leontsinis (2012)
- Bakshi, Cao, and Chen (1997, 2000)
- Bartlett (2006)
- Coleman, Kim, Li, and Verma (2001)
- Crépey (2004)
- Poulsen, Shenk-Hoppe, and Ewald (2009)
- Vähämaa (2004)
Our Contribution

To determine the properties of the MV delta empirically
## The Data (2004-2015)

<table>
<thead>
<tr>
<th>Underlying</th>
<th>Calls ('000s)</th>
<th>Puts ('000s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPX S&amp;P 500</td>
<td>1,354</td>
<td>1,429</td>
</tr>
<tr>
<td>XEO S&amp;P 100 (Eur)</td>
<td>450</td>
<td>474</td>
</tr>
<tr>
<td>OEX S&amp;P 100 (Amer)</td>
<td>419</td>
<td>459</td>
</tr>
<tr>
<td>DJX Dow Jones</td>
<td>625</td>
<td>691</td>
</tr>
<tr>
<td>Individual stocks</td>
<td>5,445</td>
<td>5,829</td>
</tr>
<tr>
<td>DJX 30 in DJX</td>
<td>5,445</td>
<td>5,829</td>
</tr>
<tr>
<td>ETFs Commodities, gold, silver, oil, bond indices</td>
<td>2,123</td>
<td>2,422</td>
</tr>
</tbody>
</table>
Impact of Asset Price Increase if Smile Stays the Same

Asset price rises; $K/S = 0.96$  Vol = 22.8%

Current $K/S = 0.98$;  Vol = 22.2%

Implied Volatility

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But Smile Moves Down When Asset price Increases...

Asset price rises
Vol Curve shifts down
K/S = 0.96
Vol = 21.4%

Current K/S = 0.98
Vol = 22.2%
Initial Empirical Exploration for European options on S&P 500

- Measure moneyness by the Black-Scholes delta, $\delta_{BS}$
- Estimate minimum variance delta, $\delta_{MV}$, for different levels of moneyness and option maturity using
  \[ \Delta f = \delta_{MV} \Delta S + \varepsilon \]
- On average $\delta_{MV}$ is about 8% less than $\delta_{BS}$
**Minimum Variance Delta and Expected Volatility Changes**

To a good approximation

\[ f = f_{BS}(S, \sigma_{imp}) \]

so that

\[ \delta_{MV} = \frac{\partial f_{BS}}{\partial S} + \frac{\partial f_{BS}}{\partial \sigma_{imp}} \frac{\partial E(\sigma_{imp})}{\partial S} = \delta_{BS} + \nu_{BS} \frac{\partial E(\sigma_{imp})}{\partial S} \]

or

\[ \delta_{MV} - \delta_{BS} = \nu_{BS} \frac{\partial E(\sigma_{imp})}{\partial S} \]

where \( \nu_{BS} \) is the Black-Scholes vega

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Empirically we find that $\delta_{MV} - \delta_{BS}$ is approximately independent of $T$ (except through its dependence on $\delta_{BS}$).

Scale invariance implies that it is independent of $S$ (except through its dependence on $\delta_{BS}$).

For a European options, the Black-Scholes vega has the approximate form

$$v_{BS} = S\sqrt{T}G(\delta_{BS})$$

for some function $G$.

It follows that, for some function $H$

$$\frac{\partial E(\sigma_{imp})}{\partial S} = \frac{H(\delta_{BS})}{S\sqrt{T}}$$
The Model continued

We find that a quadratic function provides a good approximation

\[ \delta_{MV} - \delta_{BS} = \nu_{BS} \frac{a + b\delta_{BS} + c\delta_{BS}^2}{S\sqrt{T}} \]

We estimate this with

\[ \Delta f = \delta_{BS}\Delta S + \nu_{BS} \frac{\Delta S}{\sqrt{T}} \frac{\Delta S}{S} (a + b\delta_{BS} + c\delta_{BS}^2) + \epsilon \]
Out of Sample Testing

- Estimate quadratic’s parameters using 36 months of data
- Use estimated parameters to determine optimal hedges for the next month
- Calculate reduction in variance of hedging error relative variance of the Black-Scholes hedging error
- Results indicate that a significant improvement in hedging can be achieved (15% -30% variance reduction for calls; 9-14% for puts for out-of- to at-the-money options)
- Variance reduction increases as option moves out of the money
- Vega exposure is reduced (by 20% to 40% for calls and 8% to 12% for puts)
Other Results

- Similar results for European and American style options on S&P100 and DJX
- Results for options on individual stocks, and options on ETFs, similar but much weaker
- Results for puts less strong than calls (put data is noisier)
Alternative Approaches

- SABR model and local volatility model have been used in previous research to determine the minimum variance delta.
- We find that our model gives much better results.
Conclusions

- Important to get as much mileage from delta as possible
- Quadratic approximation is an improvement over the use of SABR or a local volatility model for options on stock indices
- Vega exposure is reduced
- Works very well for stock indices, but less well for individual stocks
- Some support for volatility feedback effect explanation of negative correlation between asset price and volatility for bond and commodity ETFs.