Optimal Delta Hedging for Options

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Background

Presentation based on:

“Optimal Delta Hedging for Options”

John Hull and Alan White

www-2.rotman.utoronto.ca/~hull or ssrn2658343

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Hedging: The Textbook Approach

- Develop model
- Calculate partial derivatives with respect to stochastic variables
- Hedge risks by trading instruments to make partial derivatives close to zero

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Hedging Options in Practice

- Use Black-Scholes-Merton model with volatility set equal to implied volatility of the option
- Calculate Greek letters including the partial derivative wrt implied volatility (even though the model assumes volatility is constant)
- Hedge Greeks

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Justification for the Approach

- To a good approximation the value of an option is a deterministic (Black-Scholes-Merton) function of the asset price, S, and the implied volatility, σ_{imp}

$$f = f_{\text{BS}}(S, \sigma_{\text{imp}})$$

- A Taylor Series expansion shows that, if a portfolio's partial derivatives wrt S and σ_{imp} are low, there is little risk.

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Well Known Empirical Result

- An equity price and its volatility are negatively correlated (e.g. Black (1976) and Christie (1982))
- Possible reason is leverage (e.g., Geske (1979) model)
- Another possible reason is the "volatility feedback" effect

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The Minimum Variance (MV) Delta

- The MV delta is the position in the underlying that minimizes the variance of the changes in the value of the portfolio
- It is different from the normally calculated Black-Scholes delta because it takes into account the expected change in implied volatility conditional on an equity price change

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Prior Research

Researchers who have tested the performance of MV delta using a stochastic volatility or local volatility model:

- Alexander and Nogueira (2007)
- Alexander, Kaeck, and Noueira (2009)
- Alexander, Rubinov, Kalepky, and Leontsinis (2012)
- Bakshi, Cao, and Chen (1997, 2000)
- Bartlett (2006)
- Coleman, Kim, Li, and Verma (2001)
- Crépey (2004)
- Poulsen, Shenk-Hoppe, and Ewald (2009)
- Vähämaa (2004)

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Our Contribution

To determine the properties of the MV delta empirically

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The Data (2004-2015)

	Underlying	Calls ('000s)	Puts ('000s)
SPX	S&P 500	1,354	1,429
XEO	S&P 100 (Eur)	450	474
OEX	S&P 100 (Amer)	419	459
DJX	Dow Jones	625	691
Individual stocks	30 in DJX	5,445	5,829
ETFs	Commodities, gold, silver, oil, bond indices	2,123	2,422

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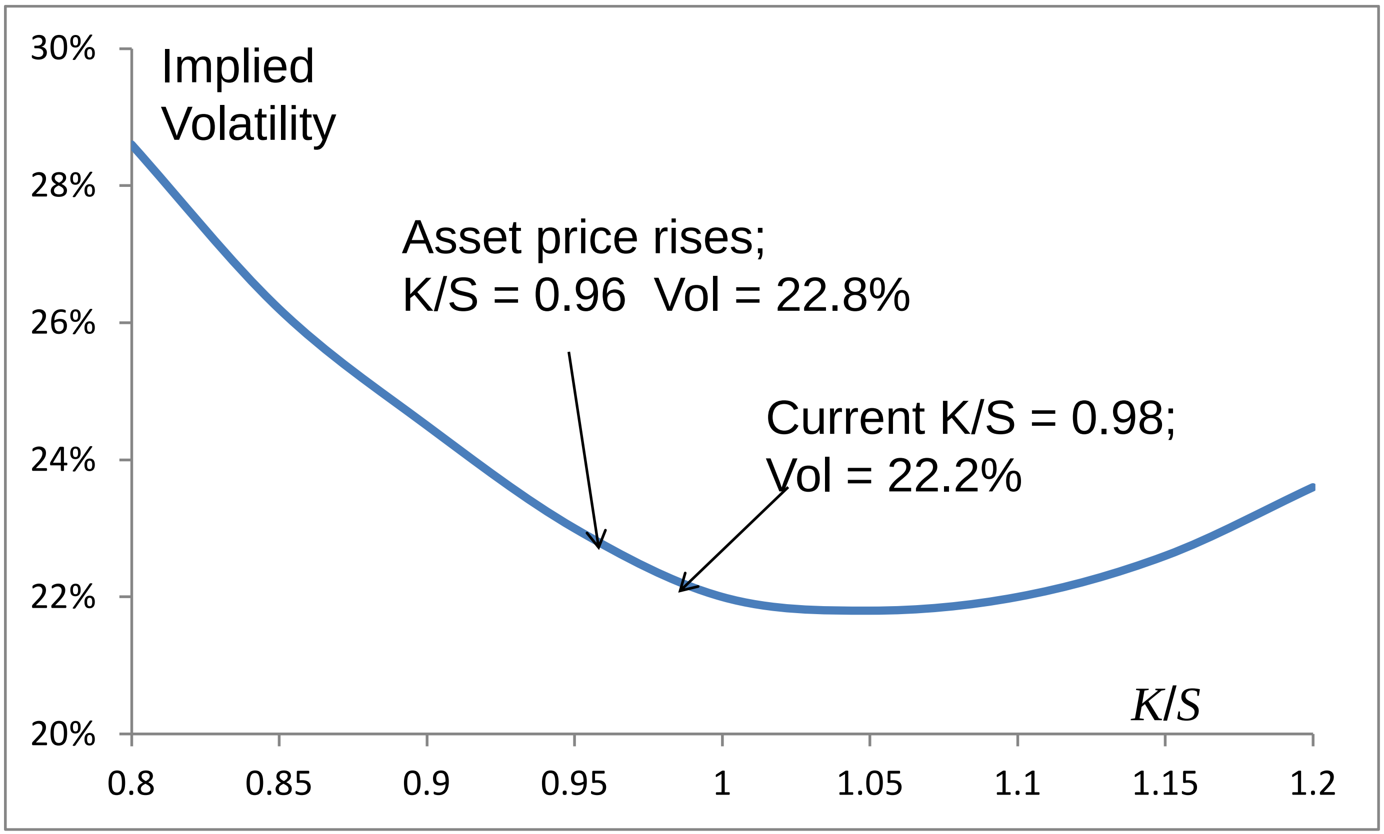
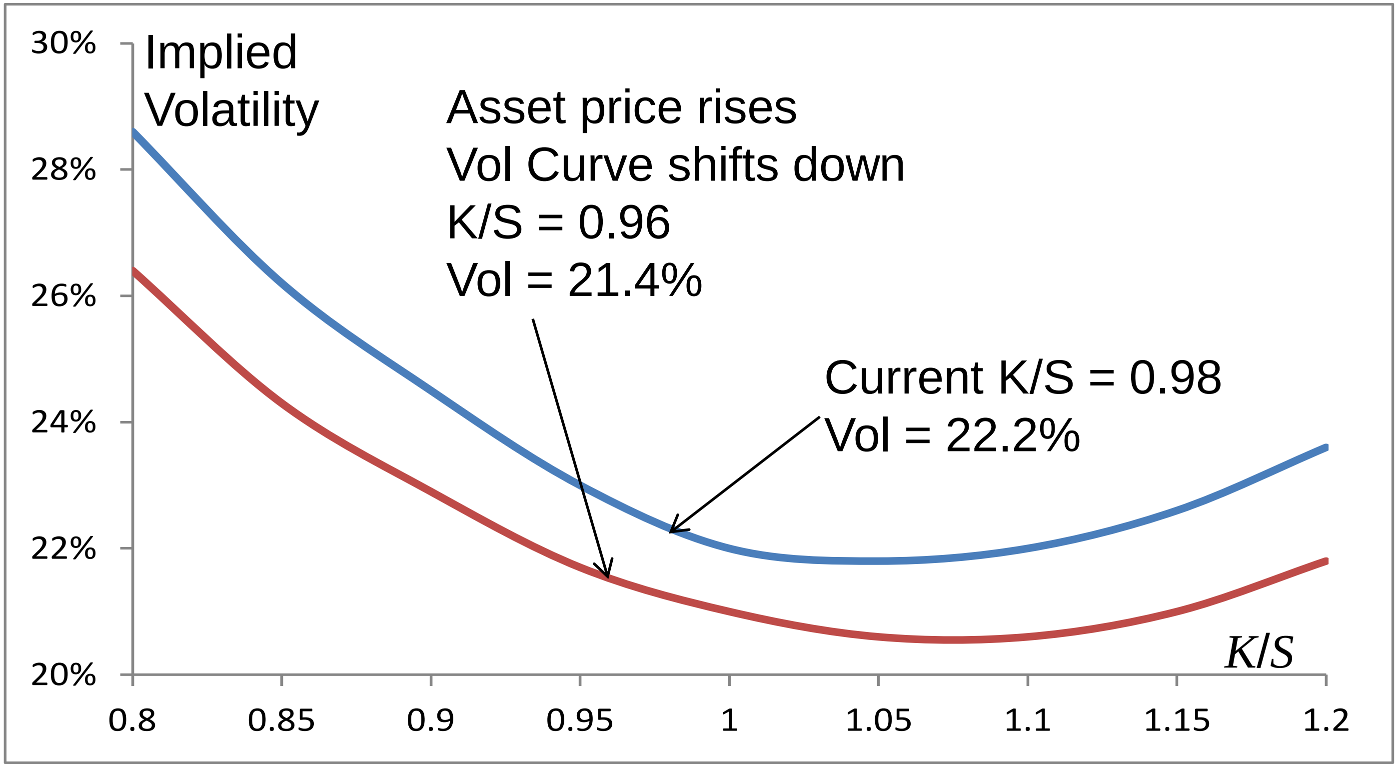
Impact of Asset Price Increase if Smile Stays the Same



But Smile Moves Down When Asset price Increases…



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Initial Empirical Exploration for European options on S&P 500

- Measure moneyness by the Black-Scholes delta, δ_{BS}
- Estimate minimum variance delta, δ_{MV}, for different levels of moneyness and option maturity using

$$\Delta f = \delta_{MV} \Delta S + \varepsilon$$

- On average δ_{MV} is about 8% less than δ_{BS}

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Minimum Variance Delta and Expected Volatility Changes

To a good approximation

$$f = f_{\mathrm{BS}}(S, \sigma_{\mathrm{imp}})$$

so that

$$\delta_{\mathrm{MV}} = \frac{\partial f_{\mathrm{BS}}}{\partial S} + \frac{\partial f_{\mathrm{BS}}}{\partial \sigma_{\mathrm{imp}}} \frac{\partial E(\sigma_{\mathrm{imp}})}{\partial S} = \delta_{\mathrm{BS}} + \nu_{\mathrm{BS}} \frac{\partial E(\sigma_{\mathrm{imp}})}{\partial S}$$

or

$$\delta_{\mathrm{MV}} - \delta_{\mathrm{BS}} = \nu_{\mathrm{BS}} \frac{\partial E(\sigma_{\mathrm{imp}})}{\partial S}$$

where ν_{BS} is the Black-Scholes vega

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The Model

- Empirically we find that $\delta_{\mathrm{MV}} - \delta_{\mathrm{BS}}$ is approximately independent of T (except through its dependence on δ_{BS})
- Scale invariance implies that it is independent of S (except through its dependence on δ_{BS})
- For a European options, the Black-Scholes vega has the approximate form

$$\nu_{\mathrm{BS}} = S\sqrt{T}G(\delta_{\mathrm{BS}})$$

  for some function G.
- It follows that, for some function H

$$\frac{\partial E(\sigma_{\mathrm{imp}})}{\partial S} = \frac{H(\delta_{\mathrm{BS}})}{S\sqrt{T}}$$

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The Model *continued*

- We find that a quadratic function provides a good approximation

$$\delta_{\mathrm{MV}} - \delta_{\mathrm{BS}} = \nu_{\mathrm{BS}} \frac{a + b\delta_{\mathrm{BS}} + c\delta_{\mathrm{BS}}^2}{S\sqrt{T}}$$

- We estimate this with

$$\Delta f = \delta_{\mathrm{BS}} \Delta S + \frac{\nu_{\mathrm{BS}}}{\sqrt{\mathrm{T}}} \frac{\Delta S}{S} (a + b\delta_{\mathrm{BS}} + c\delta_{\mathrm{BS}}^2) + \varepsilon$$

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Out of Sample Testing

- Estimate quadratic's parameters using 36 months of data
- Use estimated parameters to determine optimal hedges for the next month
- Calculate reduction in variance of hedging error relative variance of the Black-Scholes hedging error
- Results indicate that a significant improvement in hedging can be achieved (15% -30% variance reduction for calls; 9-14% for puts for out-of- to at-the-money options)
- Variance reduction increases as option moves out of the money
- Vega exposure is reduced (by 20% to 40% for calls and 8% to 12% for puts)

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Other Results

- Similar results for European and American style options on S&P100 and DJX
- Results for options on individual stocks, and options on ETFs, similar but much weaker
- Results for puts less strong than calls (put data is noisier)

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Alternative Approaches

- SABR model and local volatility model have been used in previous research to determine the minimum variance delta
- We find that our model gives much better results

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Conclusions

- Important to get as much mileage from delta as possible
- Quadratic approximation is an improvement over the use of SABR or a local volatility model for options on stock indices
- Vega exposure is reduced
- Works very well for stock indices, but less well for individual stocks
- Some support for volatility feedback effect explanation of negative correlation between asset price and volatility for bond and commodity ETFs.

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