Conic Option Pricing

Dilip B. Madan

Robert H. Smith School of Business

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Conic Finance

- Conic Finance deals with a world in which the space of tradeable claims is not closed under negation.
- Specifically, the terms for trading cash flows depend on the direction of trade.
- As a consequence the law of one price is abandonned and replaced by upper and lower prices at which one may buy or sell.
- However, the space of tradeable claims remains a convex cone.
- It is just not a subspace that is also closed under negation.

No Arbitrage and the Cone of Acceptable Risks

- In the absence of arbitrage the zero cost tradeable claims fail to meet the cone of nonnegative cash flows.
- There then exist measures separating zero cost tradeable claims from the nonnegative cash flows.
- Acceptable risks have positive valuations under the separating measures and zero cost tradeable claims have nonpositive valuations.
- Denote by $\mathcal{M}$ the set of separating measures defining the Acceptable Risks $\mathcal{A}$ as those with positive valuation for all measures in $\mathcal{M}$.
- The dual objects $\mathcal{A}$ and $\mathcal{M}$ introduced in Artzner, Delbaen, Eber and Heath (1999) are at the core of Conic Finance.

Acceptable Risks, Upper and Lower Prices

- The upper/lower price is the supremum/infimum of all valuations using the measures $\mathcal{M}$ defining risk acceptability.
- As a consequence the upper price is a convex function while the lower price is a concave function.
- We thus maximize the lower price of cash flows held and minimize the upper price of cash flows promised.
- Conic finance thus delivers new objective functions for risk management and hedging
- Minimizing the difference between the upper and lower price for example, is a good objective termed capital minimization in Carr, M. and Vicente Alvarez (2011).

Upper and Lower Prices and Distorted Expectations

- When risk acceptability is defined solely by the risk distribution function and we ask for the upper and lower prices to be additive across comonotone risks then both prices are recast as distorted expectations.
- Specifically, there is a concave distribution function $\Psi(u)$ defined on the unit interval such that the lower price is the expectation under the distorted distribution function $\Psi(F(x))$ for original distribution function F.
- Similarly there is a convex distribution function $\widehat{\Psi}(u)$ such that the upper price is the expectation under the distorted distribution function $\widehat{\Psi}(F(x))$.

Distorted Expectations and the Supporting Measures

- The set of measures $\mathcal{M}$ approving acceptability is all measures Q such that for all sets A and original probability P

$$\begin{aligned} Q(A) &\leq \Psi(P(A)) \\ \widehat{\Psi}(P(A)) &\leq Q(A) \end{aligned}$$

- For the two sets to coincide we must have that

$$\widehat{\Psi}(u) = 1 - \Psi(1-u).$$

Conic Option Pricing

- We now apply these ideas to Option Pricing by first evaluating upper and lower prices for options as suitably distorted expectations.
- The distribution function being distorted is the physical distribution function for the stock at the option maturity.
- Unlike classical risk neutral valuation we work completely under the physical measure.
- We have to describe the construction of the physical distribution function at the option maturity.
- We then have to describe the selection of the specific distortion to be applied.

Physical Distribution Function for Daily Returns

- We recognize that prices must move to afford returns and the moves must be surprises occuring at surprise times.
- Hence we model daily returns by pure jump processes with price changes occuring at unpredictable Poisson times.
- Given the large number of price changes involved we employ a limit law that hence must be self decomposable.
- There are then infinitely many jumps in any interval and we use the simple self decomposable law given by the variance gamma model, M. Carr and Chang (1998), M. and Seneta (1990).

Physical Distribution Function at Option Maturity

- For any self decomposable law X and any constant $c < 1$ there exists by definition an independent random variable $X^{(c)}$ such that

$$X \stackrel{(d)}{=} cX + X^{(c)}.$$

- X equals in distribution, shaved X plus independent.
- Eberlein and M. (2010) showed that a good way to go to a longer horizon is to run the shaved component as an i.i.d. process and to scale the independent component.
- Specifically for horizon of H days we write

$$X_H \stackrel{(d)}{=} (cX)_H + H^{\gamma} X^{(c)}.$$

- We use $c = 1/2$, $\gamma = 1/2$ and approximate X_H by another variance gamma.

Selection of Parametric Distortion

- Cherny and M. (2009) introduced the distortion named minmaxvar with parameter γ

$$\Psi^{(\gamma)}(u) = 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}.$$

- Distorted expectations reweight outcomes at quantiles u by $\Psi^{(\gamma)\prime}(u)$.
- For minmaxvar the reweighting goes to infinity as u tends to zero and it goes to zero as u tends to one.
- Hence we have both risk aversion and absence of gain enticement.

Selection of Distortion Parameter

- No arbitrage requires that prices of traded claims be above and below the lower and upper valuations.
- This is accomplished when the risk neutral distribution function $F_{RN}(x)$ and the physical distribution function $F(x)$ satisfy the inequality
$$\widehat{\Psi}^{(\gamma)}\left(F(x)\right) \leq F_{RN}(x) \leq \Psi^{(\gamma)}\left(F(x)\right).$$
- The parameter γ is selected to be the smallest value meeting this inequality for a range of values for x.

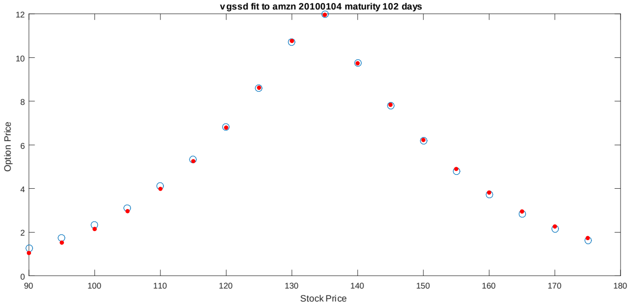
Distortion Parameter for AMZN January 4, 2010

- For the maturity of 0.27945 or 102 days the risk neutral variance gamma parameters were

$$
\begin{aligned}
\sigma &= 0.2259 \\
\nu &= 0.2506 \\
\theta &= -0.2218.
\end{aligned}
$$

Graph of Fit

- We present a graph of the fit of the risk neutral variance gamma model to the market data.



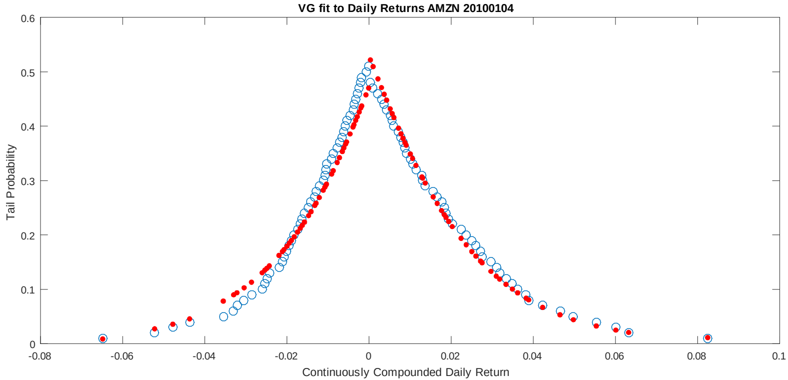
Physical Variance Gamma for Daily Returns

- The physical parameters fit to the past year of daily returns by matching observed and variance gamma model tail probabilities were

$$\begin{aligned} \sigma &= 0.0276 \\ \nu &= 0.6104 \\ \theta &= 0.0037. \end{aligned}$$

Graph of Variance Gamma fit for Daily Returns

- We present the observed and model physical tail probabilities for *AMZN* on January 4, 2010.



Variance Gamma Parameters for the Option Maturity

- The variance gamma parameters for the longer horizon of 102 days obtained on shaving by half the daily return and square root scaling the independent component are

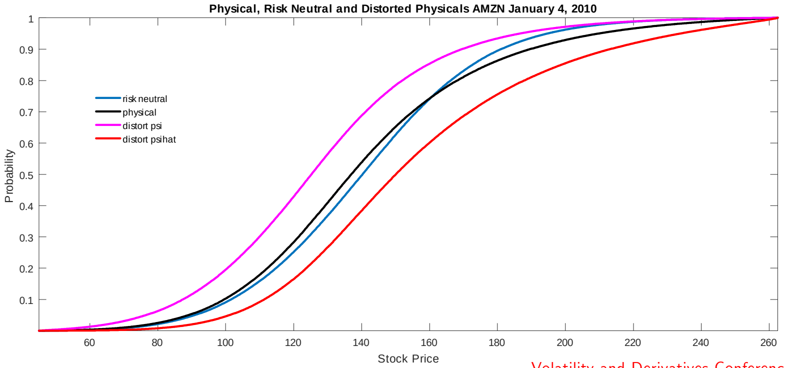
$$\begin{aligned} \sigma &= 0.2715 \\ \nu &= 0.3958 \\ \theta &= -0.0379. \end{aligned}$$

- We observe here that θ has changed sign due to the skewness effects on the independent component that is then scaled to preserve shape.

Graph of Distortion Stress Calibration

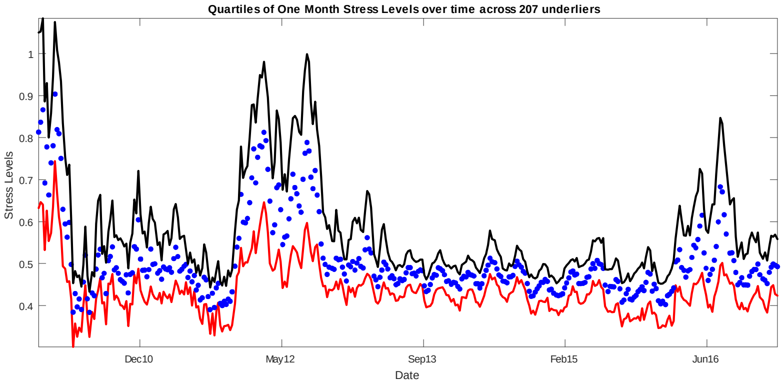
- We show the physical, risk neutral and distorted physical densities. The stress level accomplishing the required domination was

$$\gamma = 0.2433.$$



Stress Levels Through Time

- We present quartiles across 207 underliers of the stress levels through time.



- The lower prices of puts and calls are as follows.

$$L(P) = \int_0^K \widehat{\Psi}^{(\gamma)}\left(F_P\left(\ln\left(\frac{s}{S_0}\right)\right)\right)ds$$
$$L(C) = \int_K^\infty \widehat{\Psi}^{(\gamma)}\left(\widehat{F}_P\left(\ln\left(\frac{s}{S_0}\right)\right)\right)ds$$

- The upper prices are as follows.

$$U(P) = \int_0^K \Psi^{(\gamma)}\left(F_P\left(\ln\left(\frac{s}{S_0}\right)\right)\right)ds$$
$$U(C) = \int_K^\infty \Psi^{(\gamma)}\left(\widehat{F}_P\left(\ln\left(\frac{s}{S_0}\right)\right)\right)ds$$

From two prices to mid quotes

- Given lower and upper prices L, U we construct mid quotes M using a return balance principle.
- Buy at M and sell at U, V times and buy at L sell at M, W times then

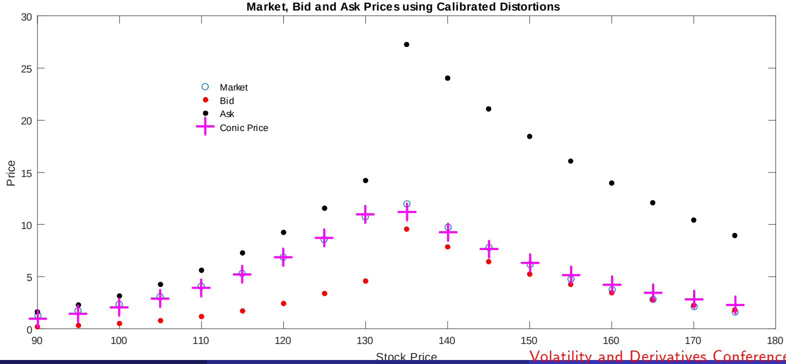
$$\left(\frac{U}{M}\right)^{V} = \left(\frac{M}{L}\right)^{W}$$

- As a consequence

$$M = U^{\alpha} L^{1-\alpha}.$$

Mid Quote Calibration

- For options on *AMZN* maturing in 102 days on January 4, 2010 we present the lower and upper prudent valuations stress level of 0.2433.
- Also shown are the candidate mid-quotes using an upper valuation weight of 0.1490 for calls.
- For puts the upper valuation weight was 0.7720.



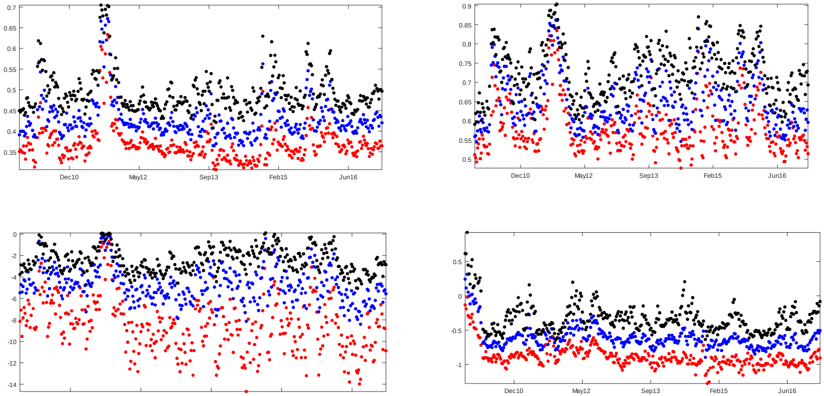
Generalized weighting scheme

- Preliminary investigations with a fixed weight on the upper valuation for the two sides delivered a satisfactory result for the puts but a poorer performance for calls.
- The upside on out of the money calls can be quite active requiring a lowering of the weight on the upper valuation.
- We therefore extended the model to also allow the weight to depend on the moneyness.
- The weight on the upper valuation is then modeled as

$$\begin{aligned}\omega_c &= \alpha_c + \beta_c \ln(K/F) \\ \omega_p &= \alpha_p + \beta_p \ln(K/F),\end{aligned}$$

- for calls and puts with strike K and the forward at F.

- We present the quartiles across assets each day for the alpha and beta coefficients.



Hedging Issues

- The upper lower valuation spreads should narrow post hedges.
- To investigate this we allowed for delta hedging and hedging by also holding a contract paying the squared return.
- Suppose we have some hedging assets with payoffs H allowing us to access the cash flows

$$Y = C + x'H$$

Post Hedge Valuation Bounds

- At the selected distortion one has by selection of the stress level that the market price Y lies above and below its lower and upper valuation

$$L(Y) \leq w(Y) \leq U(Y).$$

- Denote by $w(C)$ the market price of C and by $w(H)$ the market price of H.
- We then have that

$$L(Y) \leq w(C) + x'w(H) \leq U(Y)$$

- and so

$$L(Y) - x'w(H) \leq w(C) \leq U(Y) - x'w(H)$$

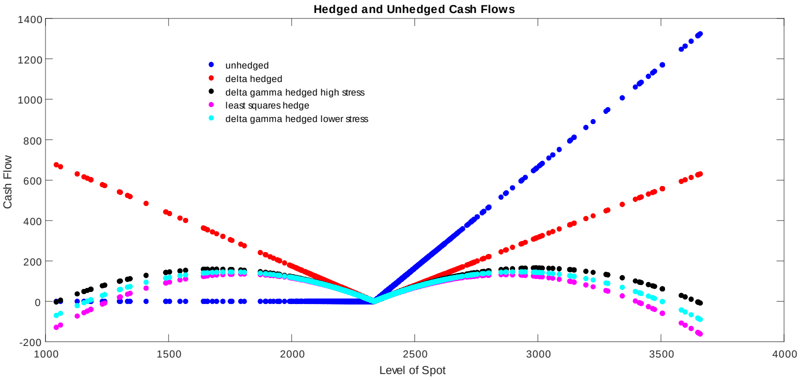
- and the post hedge lower and upper valuations are just net of the cost of the hedge.

Example of Post Hedge Valuation Bounds

- Consider a call option on the S&P 500 index on January 29, 2015 with a maturity of 22 days.
- The stress level employed was 0.6883.
- We consider two hedges, a single delta at initiation and a second hedge that also allows one to earn the squared return over the interim to the maturity.
- The hedge positions are determined to minimize the capital required in each case.

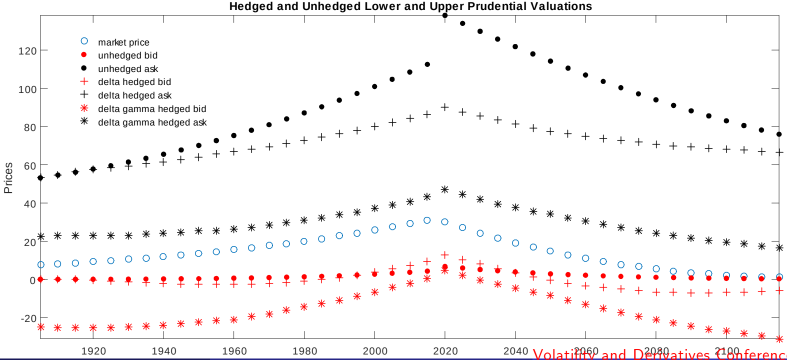
Graph of Post Hedge Cash Flows

- We present the hedge cash flows Y in the two capital minimizing hedges for various stress levels along with the least squares hedge.
- All cash flows have been shifted to be zero at the strike.



Graph of Post Hedge Upper and Lower Valuations

- We present the associated hedged upper and lower valuations.
- The hedge costs for the delta hedge recognize that the position is in a forward contract to sell stock at the call option strike.
- The squared return is priced using the calibrated variance gamma risk neutral model.



Conclusion I

- Physical probabilities for equity underliers for return horizons matching option maturities are distorted sufficiently to have the lower and upper prudential valuations of two price economies straddle the market prices of options.
- A return balance principle models transaction prices as a geometric average of the extremal valuations.
- Strike sensitive upper valuation weighting models are calibrated to market data for some 203 underliers over seven years.
- It is observed that one may explain option prices based on sufficiently distorted physical measures for extremal valuations geometrically averaged to construct candidate mid quotes.

Conclusion II

- Post hedge extremal valuations lower the spread between upper and lower valuations.
- The hedging design advocated is that of capital minimization defined as the difference between the upper and lower valuations.
- Such hedges are immune to cost of hedge issues.
- Furthermore, they work on both sides of the super and sub replication problems here generalized to formal concepts of risk acceptability.
- The generalization provides a dispersion metric with parametric control over the required aggresiveness for the hedge.