APPENDIX: THE MODEL [1]

1. ASSUMPTIONS

Stochastic Process: Assume $CF(t)$ is current (after tax) cash flow, paid out to security holders, with risk-neutral diffusion and jump components:

$$dCF(t) = \mu CF(t)dt + \sigma CF(t)dZ(t) \quad \text{if no jump at or prior to } t$$
$$= -kCF(t_{-}) \quad \text{if jump at } t$$

where k is the fractional loss of cash flow if a jump occurs at t.

The jump is a Poisson process with constant risk-neutral intensity λ, implying the

Probability of no jump before time t $= e^{-\lambda t}$.

Expected growth rate of cash flow $= \mathrm{E}[dCF(t)/\,CF(t)] = (\mu - \lambda k)dt$

Definitions:

Riskfree rate:	r
V(t), the value of unlevered firm at $t = 0$	$V(t) = CF(t)/(r - \mu + \lambda k)$
Payout rate (fraction of pre-jump value)	$\delta = CF(t)/V(t) = r - \mu + \lambda k$
$V(t)$, *excluding a jump*, has a risk-neutral process	$dV(t)/V(t) = gdt + \sigma dZ$
where to give a risk-neutral return r,	$g = r - \delta + \lambda k$
Combining results above, we note that	$g = \mu$
Default occurs if the diffusion value $V(t)$ falls to	V_B, or if a jump occurs, whichever happens first.

Debt is in default and receives value $(1 - \alpha)$ if the barrier V_B is hit, or $(1 - k)V(t)$ if there is a jump.

Default costs as fraction of V_B (if diffusion default)	α
Cumulative diffusion default frequency at t	$F[t;\, V, V_B]$ (or F)
First passage to V_B density	$f[t;\, V, V_B]$ (or f)

[1] This is an extension of the model first presented in Princeton Lectures in Finance (2006) https://www.princeton.edu/bcf/newsevents/events/lectures-in-finance/

Note: The arguments g and σ are suppressed in the expressions above

2. DEBT PROPERTIES

Debt is issued with infinite maturity, but bought back (e.g. through a sinking fund) at constant proportional rate of principal outstanding, implying a finite average maturity. Without loss of generality, let $t = 0$ be the current time. Debt issued at $t = 0$ has

Debt Principal	P
Debt Coupon	C
Proportional Buyback Rate	m
Debt service rate (Coupon plus Buyback)	$C + mP$

At time t > 0, debt principal, coupon, and service rate of debt issued at time 0 will be the above quantities reduced by the factor e^{-mt}.

Average life of debt absent default	$T = \int_0^\infty t\, e^{-mt} dt = 1/m$
Liquidity/Agency Cost Premium[2]	$h = h_0 + h_1 T$

Debt that is retired at time $t > 0$, is replaced by newly issued debt with the same principal and coupon, and infinite life. This debt also will be retired proportionately through time at rate m. Thus, *total* debt principal P and coupon C remain constant through time. Newly issued debt will be issued at then-current market prices, denoted by $D(V(t))$, implying an

Instantaneous debt rollover cost rate	$m(P - D(V(t)))$

3. DEBT VALUATION

Debt value will be expected cash flows under the risk neutral measure discounted exponentially at rate $r + h$:

$$D(h) = \int_0^\infty e^{-(r+h)t}(C + mP)e^{-mt}(1-F)e^{-\lambda t}dt + (1-\alpha)V_B \int_0^\infty e^{-(r+h)t}e^{-\lambda t}e^{-mt} f\, dt$$
$$+ (1-k)\int_0^\infty e^{-(r+h)t}e^{-mt}(e^{gt}V)\lambda e^{-\lambda t}(1-F)dt \quad (1)$$

[2] Huang and Huang (2003; 2012) estimate liquidity spreads by the difference between empirical spreads and spreads predicted by a structural model calibrated to default data. Myers (1977) and Leland and Toft (1996) note the positive relationship between debt maturity and agency costs. Longstaff, Mithal, and Neis (2005) estimate the liquidity spreads as 50 – 72 bps, increasing by approximately 3 bps per year maturity.

The first term is the discounted coupon plus principal payments, which decline exponentially at the rate m as debt is retired. Note that coupons are paid only if (i) the default barrier has not been reached, with probability 1 – F, and that no jump has occurred, which is with probability $e^{-\lambda t}$. The second term is discounted payoffs if the barrier is reached at time t, times the probability that a jump has not occurred. Note that e^{-mt} appears in this term and the next because current debt only has claim to fraction e^{-mt} of value. The final term is the value if the jump occurs at time t, which occurs with probability $\lambda e^{-\lambda t}$, reduced by (1 – F), the probability the boundary V_B is reached before the jump. Note that default by jump gives expected value $(1 - k)V(t)$, where the expected value of $V(t) = V e^{gt}$ and V is the current firm value. Conditional on no prior jumps, $V(t)$ grows at rate g, whereas inclusive of expected jump loss, $V(t)$ grows at rate $r - \delta$.

Integrating the first term and last terms by parts gives:

$$D(h) = \frac{C + mP}{r + m + \lambda + h}\left(1 - \int_0^\infty e^{-(r+m+\lambda+h)t} f\, dt\right) + (1-\alpha)V_B \int_0^\infty e^{-(r+m+\lambda+h)t} f\, dt$$
$$+ \frac{\lambda(1-k)V}{r + m + \lambda + h - g}\left(1 - \int_0^\infty e^{-(r+m+\lambda+h-g)t} f\, dt\right) \qquad (2)$$

We now make use of a key result on first passage times $f(t; V_0, V_B)$ where dV/V follows a log Brownian motion with drift rate g:

$$dV/V = g\,dt + \sigma dZ$$

$$q(z, V, V_B) \equiv \int_0^\infty e^{-zt} f(t; V, V_B)\,dt = \left(\frac{V}{V_B}\right)^{-y(z)},$$
where $\qquad (3)$
$$y(z) = \frac{(g - .5\sigma^2) + ((g - .5\sigma^2)^2 + 2z\sigma^2)^{0.5}}{\sigma^2}$$

Note that we have suppressed the arguments (g, σ) of the stochastic process in the definitions of h and y, which we continue to do hereafter. Recalling $g = \mu$, and we can rewrite the debt value function as

$$D(h) = \frac{C + mP}{r + m + \lambda + h}\left(1 - \left(\frac{V}{V_B}\right)^{-y1}\right) + (1-\alpha)V_B\left(\frac{V}{V_B}\right)^{-y1} + \frac{\lambda(1-k)V}{r + m + \lambda + h - g}\left(1 - \left(\frac{V}{V_B}\right)^{-y2}\right) \qquad (4)$$

where

$$y_1 = y(r+m+\lambda+h) = \frac{(g-.5\sigma^2)+[(g-.5\sigma^2)^2+2((r+m+\lambda+h)\sigma^2]^{0.5}}{\sigma^2}$$

$$y_2 = y(r+m+\lambda+h-\mu) = \frac{(g-.5\sigma^2)+[(g-.5\sigma^2)^2+2((r+m+\lambda+h-g)\sigma^2]^{0.5}}{\sigma^2}$$

4. EQUITY VALUATION

Equity holders discount cash flows without an additional risk premium, i.e. $h = 0$.[3] The value of equity in a levered firm will reflect the value of the unlevered firm V_0, plus the value of tax savings provided by deductibility of coupon payment, less the value of default costs, less the value (to shareholders) of the cash flows to debt. Bond cash flows are discounted at rate r rather than $r + h$, and will have cost to shareholders $D(0) > D(h)$. Thus equity has value

$$E = V + TS - DC - D(0),$$

where the value of default costs is denoted by DC and the value of tax savings, TS, derives from a constant cash flow τC or zero otherwise (the "simple" tax shield) or, alternatively, a constant cash flow τC when the firm is solvent (i.e. $V > V_T$) and $\tau\delta V$ otherwise (i.e. $V_T > V > V_B$). The critical (solvency) tax shield threshold V_T solves:

$$V_T \equiv \{C + m[P - D(h)\big|_{V=V_T}]\} / EBIT,$$

where the constant parameter $EBIT$ reflects an inverse earnings multiple (such as 1/12.5). The valuations for the complex tax shield are available from the authors.

The simple tax savings, accruing to shareholders and thus discounted at r, are worth:

$$TS = \int_0^\infty e^{-rt} e^{-\lambda t} \tau C(1-F)dt = \frac{\tau C}{r+\lambda}(1-\left(\frac{V}{V_B}\right)^{-y3})$$

where (5)

$$y_3 = y(r+\lambda) = \frac{(g-.5\sigma^2)+[(g-.5\sigma^2)^2+2(r+\lambda)\sigma^2]^{0.5}}{\sigma^2}$$

Default costs (incurred by default from diffusion) are given by:

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[3] Alternatively, equity holders could also discount at the riskless rate plus a risk premium. In our formulas, rate r could be viewed as the riskless rate plus such an equity premium. In this case, the bond liquidity premium h would be the incremental liquidity premium for debt relative to equity, which could in fact be negative.

$$DC = \alpha V_B \int_0^{\infty} e^{-(r+\lambda)t} f\, dt = \alpha V_B \left(\frac{V}{V_B}\right)^{-y3} \qquad (6)$$

Moreover, we consider initial and additional debt issuance costs q_I and q_R in that equity holders initially receive only $(1-q_I)D(h)$ instead of $D(h)$ from issuing debt at time 0 and additionally incur issuance (rollover) costs of the form mPq_R absent bankruptcy.

5. OPTIMAL DEFAULT

Equity value for arbitrary *V* under the simple tax shield specification is given by:

$$\begin{aligned}
E &= V + TS - DC - RC - D(0) \\
&= V + \frac{\tau C}{r+\lambda}\left(1-\left(\frac{V}{V_B}\right)^{-y3}\right) - \alpha V_B \left(\frac{V}{V_B}\right)^{-y3} - \frac{mPq_R}{r+\lambda}\left(1-\left(\frac{V}{V_B}\right)^{-y3}\right) \\
&\quad - \frac{C+mP}{r+m+\lambda}\left(1-\left(\frac{V}{V_B}\right)^{-y4}\right) - (1-\alpha)V_B\left(\frac{V}{V_B}\right)^{-y4} - \frac{\lambda(1-k)V}{r+m+\lambda-\mu}\left(1-\left(\frac{V}{V_B}\right)^{-y5}\right) \qquad (7)
\end{aligned}$$

where

$$y_4 = y(r+m+\lambda) = \frac{(g-.5\sigma^2)+[(g-.5\sigma^2)^2+2((r+m+\lambda)\sigma^2]^{0.5}}{\sigma^2}$$

$$y_5 = y(r+m+\lambda-\mu) = \frac{(g-.5\sigma^2)+[(g-.5\sigma^2)^2+2((r+m+\lambda-\mu)\sigma^2]^{0.5}}{\sigma^2}$$

Note *D*(0) is the value of debt cash flows when *h* = 0, since equity value is reduced by these cash flows but equity cash flows, unlike debt cash flows, are not discounted with an additional liquidity discount *h*. Therefore, the liquidity (or agency) discount incurred by equity holders at the time of issuing debt can be expressed as follows:

$$LD = D(0) - D(h)$$

Default occurs at equity's optimal level of *V*, which solves $dE(V)/dV|_{V=V_B} = 0$, implying:

$$V_B = \frac{\dfrac{(C+mP)y_4}{r+m+\lambda} + \dfrac{(mPq_R - \tau C)y_3}{r+\lambda}}{1+(1-\alpha)y_4 + \alpha y_3 - \dfrac{\lambda(1-k)}{r+m+\lambda-\mu}y_5}. \qquad (8)$$

Finally, multiple stages of debt issuance and upward restructuring will be available later.