Multiproduct-Firm Oligopoly: An Aggregative Games Approach*

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Abstract

We develop an aggregative games approach to study oligopolistic price competition with multiproduct firms. We introduce a new class of IIA demand systems, derived from discrete/continuous choice, and nesting CES and logit demands. The associated pricing game with multiproduct firms is aggregative and a firm's optimal price vector can be summarized by a uni-dimensional sufficient statistic, the ι-markup. We prove existence of equilibrium using a nested fixed-point argument, and provide conditions for equilibrium uniqueness. In equilibrium, firms may choose not to offer some products. We analyze the pricing distortions and provide monotone comparative statics. Under (nested) CES and logit demands, another aggregation property obtains: All relevant information for determining a firm's performance and competitive impact is contained in that firm's uni-dimensional type. We extend the model to non-linear pricing, quantity competition, general equilibrium, and demand systems with a nest structure. Finally, we discuss applications to merger analysis and international trade.

1 Introduction

Analyzing the behavior of multiproduct firms in oligopolistic markets appears to be of first-order importance. Multiproduct firms are endemic and play an important role in the economy.

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Even when defining products quite broadly at the NAICS 5-digit level, multiproduct firms account for 91% of total output and 41% of the total number of firms (Bernard, Redding, and Schott, 2010). Similarly, many markets are characterized by oligopolistic competition. Even at the 5-digit industry level, concentration ratios are fairly high: For instance, in U.S. manufacturing, the average NAICS 5-digit industry has a four-firm concentration ratio of 35% (Source: Census of U.S. Manufacturing, 2002). The ubiquitousness of multiproduct firms and oligopoly is reflected in the modern empirical IO literature, where oligopolistic price competition with multiproduct firms abounds (e.g., Berry, 1994; Berry, Levinsohn, and Pakes, 1995; Nevo, 2001).

In contrast to single-product firms, a multiproduct firm must choose not only how aggressive it wants to be in the market place but also how to vary its markups across products within its portfolio. In contrast to monopolistically competitive firms, an oligopolistic multiproduct firm must take self-cannibalization into account, both when setting its markups and when deciding which products to offer. Multiproduct-firm oligopoly therefore gives rise to a number of important questions: What determines the within-firm markup structure, between-firm markup differences, and the industry-wide markup level? What explains firms' scope in oligopoly? Along which dimensions are markups and product offerings distorted by oligopolistic behavior? Due to the technical difficulties discussed below, these questions have been under-researched in the existing literature. In this paper, we develop an aggregative games approach to circumvent the technical difficulties and address these and related questions. We make several contributions.

We introduce a new class of quasi-linear demand systems satisfying the Independence of Irrelevant Alternatives (IIA) axiom. This class can be derived from a discrete/continuous choice model of consumer demand, and nests standard constant elasticity of substitution (CES) and multinomial logit (MNL) demands as special cases.

We use this class of demand systems to analyze oligopolistic price competition between multiproduct firms with arbitrary firm and product heterogeneity. The associated pricing game has two important properties. First, it is aggregative in that a firm's profit depends on rivals' prices only through an industry-level aggregator that is common to all firms. Second, a firm's optimal price vector is such that, for every product in that firm's portfolio, the Lerner index multiplied by a product-level elasticity measure is equal to a firm-level sufficient statistic, called the ι-markup. That ι-markup pins down the price level of the firm. We can therefore think of the firm's maximization problem as one of choosing the right ι-markup.

These two properties allow us to prove existence of a pricing equilibrium under weak conditions using a nested fixed point argument. This approach circumvents problems that arise when attempting to apply off-the-shelf equilibrium existence theorems, such as the failure of quasi-concavity, (log-)supermodularity and upper semi-continuity of the profit functions. It also gives rise to an efficient algorithm for computing equilibrium, and allows us to derive sufficient conditions for equilibrium uniqueness.

Despite our game not being supermodular, we are able to rank equilibria from the consumers' and firms' viewpoints, and to perform comparative statics on the set of equilibria.

We explore the impact of entry, trade liberalization, and productivity and quality shocks on industry conduct and performance. Among other results, we find that a shock that makes the industry more competitive (such as a trade liberalization or the entry of new competitors) induces firms to broaden their scope in equilibrium. Intuitively, as the industry becomes more competitive, a firm starts worrying more about consumers purchasing its rivals' products, and less about cannibalizing its own sales. We also show that oligopolistic competition between multiproduct firms generates two types of welfare distortions: First, the industry-level aggregator is too low as firms are setting positive markups and, second, given the equilibrium aggregator level, some firms are inefficiently large while others are inefficiently small. Perhaps surprisingly, there are no within-firm pricing distortions.

We show that, in the special cases of (nested) CES and MNL demands, a type aggregation property obtains: All relevant information for determining a firm's performance and competitive impact is contained in that firm's uni-dimensional type. This property allows us to obtain additional predictions (e.g., about the impact of productivity and quality shocks on social welfare) that are unavailable in the general case. We also provide applications to merger analysis and international trade that exploit this property.

We develop several extensions to our framework. First, we consider multiproduct-firm pricing with a richer aggregative structure in which a firm's profit depends not only on its own prices and the industry-level aggregator but also a nest-level sub-aggregator. This allows for substitution patterns that go beyond those implied by the IIA property, and covers nested CES and MNL demands as special cases. Second, building on insights from Neary (2003, 2016), we relax the assumption of quasi-linear preferences and study multiproduct-firm oligopoly in a general equilibrium model with a continuum of sectors. This approach encompasses models that have been used in the quantitative trade literature as special cases (Atkeson and Burstein, 2008; Edmond, Midrigan, and Xu, 2015; Hottman, Redding, and Weinstein, 2016). Finally, we show how to adapt our approach to analyze non-linear pricing and quantity competition.

Our paper contributes to the relatively small literature on multiproduct-firm oligopoly pricing with horizontally differentiated products.[1] One strand of that literature focuses on proving equilibrium existence and uniqueness in multiproduct-firm oligopoly pricing games with firm and product heterogeneity and demand systems derived from discrete/continuous choice. Importantly, Caplin and Nalebuff (1991)'s powerful existence theorem for pricing games with single-product firms, the proof of which relies on establishing quasi-concavity of a firm's profit function in own price, does not extend to the case of multiproduct firms. The reason is that, even with standard MNL demand, a multiproduct firm's profit function often fails to be quasi-concave (Spady, 1984; Hanson and Martin, 1996). For this reason, the literature has focused on special cases of discrete/continuous choice demand systems, such as MNL demand (Spady, 1984; Konovalov and Sándor, 2010), CES demand (Konovalov and

[1] A separate strand of literature studies price and quantity competition between multiproduct firms selling vertically differentiated products. See, among others, Champsaur and Rochet (1989), and Johnson and Myatt (2003, 2006).

Sándor, 2010), and nested MNL demand where each firm owns a nest of products (Gallego and Wang, 2014). Our aggregative games techniques provide a unified approach to address existence and uniqueness issues in pricing games with discrete/continuous choice demand systems.

The concept of aggregative games was introduced by Selten (1970, 1973). In games with additive aggregation, each player has a fitting-in correspondence, and the set of pure-strategy Nash equilibria corresponds to the set of fixed points of the aggregate fitting-in correspondence. McManus (1962, 1964) and Selten (1970) use the aggregate fitting-in correspondence to establish existence of a Nash equilibrium in a homogeneous-goods Cournot model. Szidarovszky and Yakowitz (1977), Novshek (1985) and Kukushkin (1994) refine this approach further.[2] Our proof of equilibrium existence and our characterization of the set of equilibria also rely on the aggregate fitting-in correspondence. Corchon (1994) and Acemoglu and Jensen (2013) show that aggregative games also deliver powerful monotone comparative statics results, in the spirit of Milgrom and Roberts (1994) and Milgrom and Shannon (1994). We perform such monotone comparative statics in Section 3.3.

In recent work, Anderson, Erkal, and Piccinin (2013) adopt an aggregative games approach to study pricing games similar to ours, but restrict attention to single-product firms. They are mainly interested in long-run comparative statics with free entry and exit. Armstrong and Vickers (2016) reduce the dimensionality of a multiproduct monopolist's quantity-setting problem by confining attention to demand systems that have the property that consumer surplus is homothetic in quantities. They show that the multiproduct monopolist optimally scales down the welfare-maximizing vector of quantities by a common multiplicative factor.[3]

The remainder of the paper is organized as follows. In Section 2, we describe the class of demand systems and the multiproduct-firm pricing game. This is followed, in Section 3, by the equilibrium analysis. We prove existence of equilibrium under mild conditions and uniqueness of equilibrium under stronger conditions. We characterize the equilibrium pricing structure as well as firms' scope, provide a welfare analysis, and perform monotone comparative statics. In Section 4, we extend our analysis to non-linear pricing, quantity competition, general equilibrium, and nested demand systems. In Section 5, we specialize to the cases of (nested) CES and MNL demands and show that the type aggregation property obtains. Finally, in Section 6, we provide applications to merger analysis and trade liberalization (analyzed rigorously in the Online Appendix) and discuss the contributions and limitations of our framework.

[2]The aggregate fitting-in correspondence is not well-defined when aggregation is not additive (Cornes and Hartley, 2012). See Dubey, Haimanko, and Zapechelnyuk (2006) and Jensen (2010) for treatments of games with non-additive aggregation

[3]Their class of demand systems does not nest ours (e.g., it does no include CES-like demand with heterogeneous price elasticity parameters) nor does ours nest theirs (e.g., ours does not include linear demand). Armstrong and Vickers (2016) also extend Bergstrom and Varian (1985) to establish equilibrium existence in a Cournot oligopoly model with identical multiproduct firms.

2 The Model

Consumer demand. Consider an industry with a finite and non-empty set of differentiated products $\mathcal{N}$. The representative consumer has quasi-linear preferences. For a given price vector $p \in \mathbb{R}^{\mathcal{N}}_{++}$, consumer surplus is given by $V(p) = \log H(p)$, where $H(p) \equiv \sum_{j \in \mathcal{N}} h_j(p_j) + H^0$, h_j is a $\mathcal{C}^3$ and positive-valued function for every j in $\mathcal{N}$, and $H^0 \geq 0$ is a constant. In the following, $H(p)$ will be referred to as the aggregator.

Applying Roy's identity, we obtain the demand for product $i \in \mathcal{N}$:

$$D_i(p) = \widehat{D}_i\left(p_i, H(p)\right) = \frac{-h_i'(p_i)}{H(p)}. \tag{1}$$

We assume that $h_i' < 0$ (demand never vanishes) and $h_i'' > 0$ (products are substitutes) for every i. Note that CES demand (with $h_i(p_i) = a_i p_i^{1-\sigma}$, where $a_i > 0$ and $\sigma > 1$ are parameters) and MNL demand (with $h_i(p_i) = \mathrm{e}^{\frac{a_i - p_i}{\lambda}}$, where $a_i \in \mathbb{R}$ and $\lambda > 0$ are parameters) are both special cases of our class of demand systems. Note also that the demand system (1) has the IIA property ($\frac{\partial D_j / D_j}{\partial p_k} = 0$ for $i, j, k \in \mathcal{N}$, $k \neq i, j$). That property will allow us to greatly simplify the multiproduct-firm pricing problem in Section 3.

The demand system (1) can be rationalized in two ways. In the discrete/continuous choice micro-foundation, there is a population of consumers, each of whom first decides which product to consume, and then how much of that product to consume.[4] Each consumer receives a taste shock ε_i for every product $i \in \mathcal{N} \cup \{0\}$, where product 0 denotes the outside option. Each consumer then chooses the product i that delivers the highest indirect utility level $\log h_i(p_i) + \varepsilon_i$ (resp., $\log H^0 + \varepsilon_0$ if i is the outside option). Conditional on having chosen product $i \in \mathcal{N}$, a consumer consumes $-\left(\log h_i(p_i)\right)'$ units of that product, where we have once again applied Roy's identity to the indirect utility function $\log h_i(p_i)$. Assuming that the components of the random vector $(\varepsilon_i)_{i \in \mathcal{N} \cup \{0\}}$ are drawn i.i.d. from a type-I extreme value distribution, we can apply Holman and Marley's theorem to obtain the probability that product i is chosen:

$$\mathbb{P}_i(p) = \frac{h_i(p_i)}{\sum_{j \in \mathcal{N}} h_j(p_j) + H^0}.$$

The expected demand per consumer for product i is therefore given by

$$\underbrace{\frac{h_i(p_i)}{\sum_{j \in \mathcal{N}} h_j(p_j) + H^0}}_{\text{choice probability for product } i} \times \underbrace{\frac{-h_i'(p_i)}{h_i(p_i)}}_{\text{conditional demand for product } i} = D_i(p).$$

[4] The discrete/continuous choice approach was pioneered by Novshek and Sonnenschein (1979) and Hanemann (1984). Anderson, de Palma, and Thisse (1987) show that CES demand can be derived from discrete/continuous choice. Discrete/continuous choice models of consumer demand have been used by empirical researchers to estimate demand for electric appliances (Dubin and McFadden, 1984), soft drinks (Chan, 2006), and painkillers (Björnerstedt and Verboven, 2016). In Smith (2004), consumers first choose a supermarket, and then how much to spend at that supermarket based on the price index at that store.

A necessary and sufficient condition for $\log h_i$ to be an indirect subutility function is that h_i be log-convex, which we assume in the following. Note that the discrete/continuous choice micro-foundation allows us to interpret H^0 (or rather $\log H^0$) as the value of the outside option.

Another way of rationalizing the demand system (1) is to derive it from quasi-linear utility maximization. In Online Appendix I.2, we show that the demand system (1) is indeed quasi-linearly integrable, provided that each function h_i is log-convex.

To summarize, a demand system is a collection of $\mathcal{C}^3$, strictly positive, strictly-decreasing, and log-convex functions $(h_j)_{j\in\mathcal{N}}$, along with a non-negative scalar H^0.

Pricing game. The pricing game consists of three elements: $((h_j)_{j\in\mathcal{N}}, H^0)$ is the demand system defined above; $\mathcal{F}$, the set of firms, is a partition of $\mathcal{N}$ such that $|\mathcal{F}| \geq 2$; $(c_j)_{j\in\mathcal{N}} \in \mathbb{R}_{++}^{\mathcal{N}}$ is a profile of marginal costs. The profit of firm $f \in \mathcal{F}$ is defined as follows:[5]

$$\Pi^f(p) = \sum_{\substack{k\in f \\ p_k<\infty}} (p_k - c_k) \frac{-h_k'(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j) + H^0}, \ \forall p \in (0,\infty]^{\mathcal{N}}. \tag{2}$$

Note that we are allowing firms to set infinite prices, which essentially compactifies firms' action sets. This compactification ensures that each firm's maximization problem has a solution (provided that rival firms are not pricing all their products at infinity). The assumption we are making here is that, if $p_k = \infty$, then the firm simply does not supply product k, and therefore does not earn any profit on this product. We say that product k is active if $p_k < \infty$. Infinite prices are discussed in greater detail in Online Appendix II.3.

We study the normal-form game in which firms set their prices simultaneously, and payoff functions are given by equation (2). A pure-strategy Nash equilibrium of that normal-form game is called a pricing equilibrium.

3 Equilibrium Analysis

In this section, we provide an equilibrium analysis of the multiproduct-firm pricing game. In the first part, we adopt an aggregative games approach to prove existence of equilibrium and characterize the set of pricing equilibria. In the second part, we investigate how firm behavior is affected by changes in the aggregator. In the third part, we study the equilibrium properties, both from a positive and normative point of view, and comparative statics. Finally, we provide conditions under which the equilibrium is unique.

[5] Throughout the paper, we adopt the convention that the sum of an empty collection of real numbers is equal to zero. Note that, since h_j is monotone, $h_j(\infty) \equiv \lim_{p_j\to\infty} h_j(p_j)$ exists for every j.

3.1 Equilibrium Existence and Characterization: An Aggregative Games Approach

There are three main difficulties associated with the equilibrium existence problem. First, Π^f is not necessarily quasi-concave in $(p_j)_{j\in f}$ (Spady, 1984; Hanson and Martin, 1996). Second, Π^f is not necessarily upper semi-continuous in $(p_j)_{j\in f}$.[6] Third, as shown in Online Appendix II.2, if f is a multiproduct firm, then Π^f is neither supermodular nor log-supermodular in $(p_j)_{j\in f}$. The first two difficulties imply that standard existence theorems for compact games (such as Nash or Glicksberg's theorems) based on Kakutani's fixed-point theorem cannot be applied. The last two difficulties imply that existence theorems based on supermodularity theory and Tarski's fixed-point theorem (see Milgrom and Roberts, 1990; Vives, 1990, 2000; Topkis, 1998) have no bite. The second (and, to some extent, the third) difficulty prevents us from using Jensen (2010)'s existence theorem for aggregative games with monotone best replies.

The idea behind our existence proof is to reduce the dimensionality of the problem in two ways. First, we show that a firm's optimal price vector can be fully summarized by a uni-dimensional sufficient statistic, which is pinned down by a single equation in one unknown. Second, the pricing game is aggregative (see Selten, 1970), in that the profit of a firm depends only on its own profile of prices and the uni-dimensional sufficient statistic $H(p)$.

In the following, we present a sketch of our existence proof. We refer the reader to Appendix A for details. In this sketch, we introduce the key concepts of ι-markup, pricing function, fitting-in function and aggregate fitting-in function, which will prove useful to describe the equilibria of our pricing game, and to understand our comparative statics results.

For the sake of expositional simplicity, suppose that first-order conditions are sufficient for optimality. Ignoring the possibility of infinite prices for the time being, firm f's profit is given by

$$\Pi^f(p) = \sum_{j\in f}(p_j - c_j)\widehat{D}_j\left(p_j, H(p)\right) \equiv \widehat{\Pi}^f(p^f, H(p)),$$

where $p^f = (p_j)_{j\in f}$ is the profile of prices chosen by firm f. The first-order conditions for each firm's profit maximization problem hold at price vector $p \in \mathbb{R}^{\mathcal{N}}_{++}$ if and only if for every $f \in \mathcal{F}$ and $k \in f$,

$$0 = \frac{\partial \Pi^f}{\partial p_k} = \widehat{D}_k + (p_k - c_k)\frac{\partial \widehat{D}_k}{\partial p_k} + \frac{\partial H}{\partial p_k}\left(\sum_{j\in f}(p_j - c_j)\frac{\partial \widehat{D}_j}{\partial H}\right),$$

[6] To see this, suppose that demand is CES, and that firm $f = \{k\}$ is a single-product firm. If firm f's rivals are setting infinite prices for all their products, then $\Pi^f = (\sigma-1)(p_k - c_k)/p_k$ for every $p_k > 0$. It follows that Π^f goes to $\sigma - 1$ as p_k goes to infinity. This is strictly greater than 0, which is the profit firm f receives when it sets $p_k = \infty$. Therefore, Π^f is not upper semi-continuous in p_k. If profit functions had been defined over $(0,\infty)^{\mathcal{N}}$ instead of $(0,\infty]^{\mathcal{N}}$, then payoff functions would be upper semi-continuous, but the lack of compactness would become an issue.

$$= \widehat{D}_k \left(1 - \frac{p_k - c_k}{p_k} \left|\frac{\partial \log \widehat{D}_k}{\partial \log p_k}\right| + \frac{\frac{\partial H}{\partial p_k}}{\widehat{D}_k} \left(\sum_{j\in f}(p_j - c_j)\frac{\partial \widehat{D}_j}{\partial H}\right)\right).$$

This first-order condition can be rewritten as:

$$\frac{p_k - c_k}{p_k} \underbrace{\left|\frac{\partial \log \widehat{D}_k}{\partial \log p_k}\right|}_{=p_k \frac{h_k''(p_k)}{-h_k'(p_k)} \equiv \iota_k(p_k)} = 1 + \sum_{j\in f}(p_j - c_j) \underbrace{\frac{\frac{\partial H}{\partial p_k}}{\widehat{D}_k}\frac{\partial \widehat{D}_j}{\partial H}}_{=\widehat{D}_j}. \tag{3}$$

Condition (3) has two important properties. First, the left-hand side of that equation depends only on p_k. Second, the right-hand side is the same for every k in f. These properties are an implication of the IIA property, which, as shown by Anderson, Erkal, and Piccinin (2013, Proposition 5) implies that demand is multiplicatively separable in the aggregator, i.e., $\widehat{D}_k = d_k(p_k)\phi(H)$ for some functions d_k and ϕ.[7] We generalize our results to the entire class of demand systems that satisfy the IIA property in Section 4. There, we also show that similar properties arise when the demand system has a nested structure.

Further simplifying condition (3), we obtain that the profile of prices p is a pricing equilibrium if and only if

$$\frac{p_k - c_k}{p_k}\iota_k(p_k) = 1 + \widehat{\Pi}^f(p^f, H),\ \forall f \in \mathcal{F},\ \forall k \in f, \tag{4}$$
$$\text{and } H = \sum_{j\in\mathcal{N}} h_j(p_j) + H^0.$$

We learn two facts from equation (4). First, as already mentioned, for a given $f \in \mathcal{F}$, the right-hand side of equation (4) is independent of the identity of $k \in f$. It follows that, in any pricing equilibrium, for any $f \in \mathcal{F}$, and for all $k, l \in f$,

$$\frac{p_k - c_k}{p_k}\iota_k(p_k) = \frac{p_l - c_l}{p_l}\iota_l(p_l).$$

Put differently, there exists a scalar μ^f, which we call firm f's *ι-markup*, such that $\frac{p_k-c_k}{p_k}\iota_k(p_k) = \mu^f$ for every $k \in f$. We say that firm f's profile of prices, $(p_k)_{k\in f}$ satisfies the *common ι-markup property*.[8] Second, we see from equation (4) that firm f's equilibrium profit is equal to the value of its ι-markup minus one.[9]

[7]See Online Appendix IX for an in-depth analysis of the relationship between the aggregative games approach, the common ι-markup property (introduced below), and the IIA property.

[8]The fact that a firm optimally sets the same absolute markup (possibly adjusted by a price-sensitivity parameter) over all its products when demand is of the (nested) MNL type, and the same relative markup over all its products when demand is of the CES type, was already pointed out by Anderson, de Palma, and Thisse (1992), Konovalov and Sándor (2010), and Gallego and Wang (2014). The common ι-markup property generalizes these findings to the whole class of demand systems that can be derived from discrete/continuous choice.

[9]If demand had been defined as $D_i(p) = M(-h_i'(p))/H(p)$, where $M > 0$ is a market size parameter, then

The constant ι-markup property can be interpreted as follows. Consider a hypothetical single-product firm selling product k. Suppose that this firm behaves in a monopolistically competitive way, in the sense that it does not internalize the impact of its price on the aggregator H. Firm k therefore faces the demand $\widehat{D}_k(p_k, H)$ and, since it takes H as given, believes that the price elasticity of demand for its product is simply $|\partial \log \widehat{D}_k/\partial \log p_k|$, which is precisely $\iota_k(p_k)$. Therefore, firm k prices according to the inverse elasticity rule: $\frac{p_k-c_k}{p_k} = \frac{1}{\iota_k(p_k)}$. In our model, firm f internalizes its impact on the aggregator level as well as self-cannibalization effects. It therefore prices in a less aggressive way, according to the modified inverse elasticity rule $\frac{p_k-c_k}{p_k} = \frac{\mu^f}{\iota_k(p_k)}$, with $\mu^f > 1$. Put differently, firm f sets the price that a firm would set under monopolistic competition, if that firm believed that the price elasticity of demand is equal to $\iota_k(p_k)/\mu^f$, instead of $\iota_k(p_k)$. What is remarkable is that the ι-markup μ^f, which summarizes the impact of firm f's behavior on H, is firm-specific, rather than product-specific.

Next, we use ι-markups to reduce the dimensionality of firms' profit maximization problems. Suppose that the function $p_k \mapsto \frac{p_k-c_k}{p_k}\iota_k(p_k)$ is one-to-one for every $k \in \mathcal{N}$, and denote its inverse function by $r_k(\cdot)$. We call r_k the *pricing function* for product k. Then, using equation (4), firm f's pricing strategy can be fully described by a uni-dimensional variable, μ^f, such that

$$\mu^f = 1 + \widehat{\Pi}^f \left(\left(r_j(\mu^f) \right)_{j\in f}, H \right). \tag{5}$$

Suppose that equation (5) has a unique solution in μ^f, denoted $m^f(H)$. We call m^f firm f's *fitting-in function*. Then, the equilibrium existence problem boils down to finding an H such that

$$H = \underbrace{\sum_{f\in\mathcal{F}} \sum_{j\in f} h_j \left(r_j \left(m^f(H) \right) \right) + H^0}_{\equiv \Gamma(H)}.$$

In the parlance of aggregative games, Γ is the *aggregate fitting-in function*. The equilibrium existence problem reduces to finding a fixed point of that function. As we will see later on, the aggregative games approach is also useful to establish equilibrium uniqueness: The pricing game has a unique equilibrium if the following index condition is satisfied: $\Gamma'(H) < 1$ whenever $\Gamma(H) = H$.

This informal exposition leaves a number of questions open. Are first-order conditions sufficient for optimality? Can infinite prices be accommodated? Is the function $p_k \mapsto \frac{p_k-c_k}{p_k}\iota_k(p_k)$ one-to-one for every k? Are fitting-in functions well-defined? Does the aggregate fitting-in function have a fixed point? We need one assumption to answer all these questions in the affirmative.

Assumption 1. *For every $j \in \mathcal{N}$ and $p_j > 0$, $\iota_j'(p_j) \geq 0$ whenever $\iota_j(p_j) > 1$.*

Theorem 1. *Suppose that the demand system $((h_j)_{j\in\mathcal{N}}, H^0)$ satisfies Assumption 1. Then, the pricing game $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ has a pricing equilibrium for every $\mathcal{F}$ and $(c_j)_{j\in\mathcal{N}}$.*

firm f's equilibrium profit would be $M(\mu^f - 1)$.

The set of equilibrium aggregator levels coincides with the set of fixed points of the aggregate fitting-in function Γ. If H^* is an equilibrium aggregator level, then, in the associated equilibrium, consumer surplus is given by $\log H^*$, firm $f \in \mathcal{F}$ earns profit $m^f(H^*)-1$, and product $k \in f$ is priced at $r_k(m^f(H^*))$.

Proof. See Appendix A. □

Broadly speaking, Assumption 1 says that for every product j, ι_j, the price elasticity of the monopolistic competition demand for product j, should be non-decreasing in p_j. This condition is sometimes called Marshall's second law of demand. It clearly holds with CES and MNL demands, where $\iota_j(p_j)$ is equal to σ and p_j/λ, respectively. In Online Appendix VI.1, we provide other examples of positive, decreasing, and log-convex functions that satisfy Assumption 1, and develop a cookbook to generate such functions.[10]

The nested fixed point structure used to prove equilibrium existence also provides an efficient algorithm for computing the set of equilibria of the pricing game. In Online Appendix XIII.2, we describe this algorithm in the special cases of CES and MNL demands.

3.2 Properties of Fitting-in and Pricing Functions

In this section, we study the properties of the product-level pricing function r_k and the firm-level fitting-in function m^f, and discuss how these properties shape the behavior of firm f. (These properties are rigorously established in Appendix A.) These functions turn out to be convenient for deriving and interpreting comparative statics in Section 3.3.

For every product $k \in \mathcal{N}$, denote $\bar{\mu}_k = \lim_{p_k \to \infty} \iota_k(p_k)$, and let p_k^{mc} be the unique solution of equation $\frac{p_k - c_k}{p_k}\iota_k(p_k) = 1$. Note that p_k^{mc} is the price at which product k would be sold under monopolistic competition. $\bar{\mu}_k$ is the highest ι-markup that product k can support.

Proposition 1 (**Pricing function**). *r_k is continuous and strictly increasing on $(1, \bar{\mu}_k)$. Moreover, $\lim_{\mu^f \to 1} r_k(\mu^f) = p_k^{mc}$, $\lim_{\mu^f \to \bar{\mu}_k} r_k(\mu^f) = \infty$, and $r_k(\mu^f) = \infty$ for every $\mu^f \geq \bar{\mu}_k$.*

In words, the price of product k increases when the ι-markup μ^f increases. If μ^f approaches unity (the monopolistic competition ι-markup), then p_k approaches p_k^{mc} (the monopolistic competition price). If μ^f is above $\bar{\mu}_k$, then $\iota_k(p_k)/\mu^f$, the adjusted price elasticity of demand under monopolistic competition, is strictly lower than unity for every p_k. Therefore, firm f sets an infinite price for product k, i.e., it does not supply product k.

Next, we turn our attention to the firm-level fitting-in function m^f. For every firm f, let $\bar{\mu}^f = \max_{j \in f} \bar{\mu}_j$ denote the highest ι-markup that firm f can sustain.

Proposition 2 (**Fitting-in function**). *m^f is continuous and strictly decreasing on $(0, \infty)$. Moreover, $\lim_{H \to 0} m^f(H) = \bar{\mu}^f$, and $\lim_{H \to \infty} m^f(H) = 1$.*

[10] Our aggregative games approach to equilibrium existence relies on first-order conditions being sufficient for global optimality. In Online Appendix III, we show that Assumption 1 is the weakest assumption under which first-order conditions are indeed sufficient with multiproduct firms. When Assumption 1 does not hold, a potential games approach can be used to establish equilibrium existence (see Slade, 1994; Monderer and Shapley, 1996). See Online Appendix III.4 for details.

As competition intensifies (H increases), firm f reacts by lowering its ι-markup. As the industry approaches the monopolistic competition limit ($H \to \infty$), m^f tends to 1, the ι-markup under monopolistic competition. Combining Propositions 1 and 2, we see that, as competition intensifies, firm f lowers the prices of all its products. An immediate consequence is that the aggregate fitting-in function Γ is strictly increasing.

Propositions 1 and 2 also imply the following corollary:

Corollary 1. *As competition intensifies (i.e., as H becomes higher), each firm's set of active products (weakly) expands.*

To fix ideas, let $f = \{1, 2, \ldots, N^f\}$, and assume that products are ranked as follows: $\bar{\mu}_1 > \bar{\mu}_2 > \ldots > \bar{\mu}_{N^f}$. When competition is very soft (H close to 0), $m^f(H)$ is close to $\bar{\mu}_1$, and strictly higher than $\bar{\mu}_2$. Therefore, only product 1 is supplied. As H increases, $m^f(H)$ decreases, and eventually crosses $\bar{\mu}_2$, so that product 2 starts being supplied as well. When H approaches the monopolistic competition limit, $m^f(H)$ is strictly lower than $\bar{\mu}_{N^f}$, and firm f therefore sells all of its products.

To see the intuition, consider a firm f selling two goods, 1 and 2, and suppose that firm f can make more profit from good 1, conditional on consumers buying it, than from good 2. Suppose the firm contemplates offering good 2 in addition to good 1. Introducing a second product is profitable if and only if a sufficiently large fraction of its demand is stolen from rivals rather than from the firm's more profitable good 1. In the IIA case, the share of demand that is stolen from good 1 is proportional to the initial market share of that product, which is smaller the larger is H.[11] Put differently, because self-cannibalization is less of a concern when competition is intense, the firm offers more products.

3.3 Properties of Equilibria and Comparative Statics

Markups. Our class of demand systems can generate rich patterns of equilibrium markups within a firm's product portfolio. To see this, let us first consider the special case of CES demand (i.e., $h_j(p_j) = a_j p_j^{1-\sigma}$ for all $j \in \mathcal{N}$). In this case, $\iota_j = \sigma$ for all j, and the common ι-markup property implies that, in equilibrium, $\frac{p_j - c_j}{p_j} = \frac{\mu^f}{\sigma}$ for all $j \in f$. Firm f therefore sets the same Lerner index for all the products in its portfolio, and thus charges higher absolute markups on products that it produces less efficiently. These markup patterns are not robust to changes in the demand system. Suppose for instance that $h_j(p_j) = a_j p_j^{1-\sigma_j}$ for all $j \in \mathcal{N}$. Then, in equilibrium, $\frac{p_j - c_j}{p_j} = \frac{\mu^f}{\sigma_j}$, and firm f no longer sets the same Lerner index over all its products (unless $\sigma_i = \sigma_j$ for every $i, j \in f$). Similarly, it does not necessarily charge higher absolute markups on higher marginal cost products.

The same point can be made about the special case of MNL demand ($h_j(p_j) = \exp\left(\frac{a_j - p_j}{\lambda_j}\right)$ for all $j \in \mathcal{N}$). With common λ's, a multiproduct firm charges the same absolute markup

[11] Going beyond IIA, the key property is that the diversion ratio from product 1 to product 2 is decreasing in H.

over all its products, and sets a lower Lerner index on higher marginal cost products. Again, these properties no longer hold with heterogeneous λ's.[12] More generally, the pattern of markups within a firm's product portfolio depends on demand-side conditions, as captured by the functions $(\iota_j)_{j\in f}$, and on supply-side considerations $((c_j)_{j\in f})$.

Comparing equilibria. If we know that H^* is an equilibrium aggregator level, then we can compute consumer surplus ($\log H^*$), the profit of firm $f \in \mathcal{F}$ ($m^f(H^*) - 1$) and the price of each product $k \in f$ ($r_k(m^f(H^*))$). Moreover, Propositions 1 and 2 imply that if there are multiple equilibria, then these equilibria can be Pareto-ranked among firms, with this ranking being the reverse of consumers' ranking of equilibria:

Proposition 3. *Suppose that there are two pricing equilibria with aggregators H_1^* and $H_2^* > H_1^*$, respectively. Then, each firm $f \in \mathcal{F}$ makes a strictly larger profit in the first equilibrium (with aggregator H_1^*), whereas consumers' indirect utility is higher in the second equilibrium (with aggregator H_2^*). In addition, the set of equilibrium aggregator levels has a maximal and a minimal element.*

Proof. See Online Appendix XIV.1. □

Welfare analysis. Next, we analyze the welfare distortions arising from multiproduct-firm oligopoly pricing. An immediate observation is that firms' pricing is constrained efficient in the following sense: Firm f's equilibrium prices $(r_k(m^f(H^*)))_{k\in f}$ maximize social welfare subject to the constraint that the firm's contribution to the aggregator, H^f, is held fixed at its equilibrium value. (Note that, in the discrete/continuous choice micro-foundation, this thought experiment is equivalent to maximizing social welfare, subject to the constraint that firm f attracts the same number of consumers as it does in equilibrium.) The reason is that consumer surplus and rivals' profits are held fixed by the constraint, but firm f's prices maximize its profit by definition.

As there are no within-firm pricing distortions, this leaves us with two types of distortions. The first distortion comes from the fact that, under oligopoly, firms set positive markups. This implies that H^*, the equilibrium aggregator level, is strictly lower than the aggregator level under perfect competition ($\sum_{j\in\mathcal{N}} h_j(c_j)$). The second distortion is due to the fact that, conditional on the aggregator level H^*, some firms are contributing too much to H^*, while some others are contributing too little. This is easily seen by maximizing social welfare subject to the constraint that consumer surplus is equal to $\log H^*$. The $|\mathcal{N}|$ first-order conditions boil down to the following optimality condition:

$$\mu^* = 1 - \Lambda^* + \sum_{f\in\mathcal{F}} \widehat{\Pi}^f \left(\left(r_j(\mu^*)\right)_{j\in f}, H^* \right), \tag{6}$$

[12] Björnerstedt and Verboven (2016) analyze a merger in the Swedish market for painkillers, and find that a CES demand specification (or, in the authors' own words, a constant expenditure demand specification) with random coefficients gives rise to more plausible markup predictions than an MNL demand specification with random coefficients.

where Λ^* is the Lagrange multiplier associated with the consumer surplus constraint, and μ^* is the optimal industry-wide ι-markup.

This means that the pricing equilibrium H^* is constrained efficient if and only if $m^f(H^*) = m^g(H^*)$ for every $f, g \in \mathcal{F}$. This condition is unlikely to hold in general. When it does not hold, some firms set their ι-markups above μ^* and end up producing too little, while some other firms set their ι-markups below μ^* and therefore produce too much. Whether a given firm contributes too much or too little to the aggregator can be assessed by comparing equations (5) and (6).

Comparative statics. Although our pricing game is not supermodular, we can exploit its aggregative structure to perform comparative statics on the set of equilibria. The approach is similar to the one in Corchon (1994) and Acemoglu and Jensen (2013), and can be summarized as follows:[13] Suppose that the value of a parameter changes; study how this change affects firms' pricing and fitting-in functions, and hence, the aggregate fitting-in function; analyze how the associated shift in the aggregate fitting-in function affects the set of equilibrium aggregator levels; finally, use pricing and fitting-in functions to translate these changes in aggregator levels into changes in markups, prices, profits, and sets of active products.

Outside option / entry. We first ask how an increase in the value of the outside option H^0 or the entry of a new competitor affects the set of equilibria:[14]

Proposition 4. *Suppose that H^0 increases, or that a new competitor enters. Then, in both the equilibrium with the smallest and largest value of the aggregator H, this induces (i) a decrease in the profit of all firms, (ii) a decrease in the prices of all goods, (iii) an increase in consumer surplus, and (iv) an expansion of the set of active products for every firm.*

Proof. See Online Appendix XIV.2. □

As the outside option improves, or as entry takes place, the aggregate fitting-in function shifts upward. Since that function is strictly increasing, it follows that the lowest and highest equilibrium aggregator levels increase. The rest of the proposition follows from the monotonicity properties of the pricing and fitting-in functions (Propositions 1 and 2). The intuition behind our product portfolio expansion result was already discussed in Section 3.2: As the industry becomes more competitive, firms worry more about losing consumers to their rivals than about cannibalizing their own sales. This leads them to introduce more products, in order to increase the likelihood that one of these products will be purchased.

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[13] Acemoglu and Jensen (2013) make a number of assumptions (compactness, pseudo-concavity and upper semi-continuity) which do not hold in our framework. This prevents us from applying their results off the shelf.

[14] We study the impact of entry as follows. Suppose that firm $f^0 \in \mathcal{F}$ is initially inactive, i.e., $p_j = \infty$ for every $j \in f^0$. The set of post-entry equilibrium aggregator levels is obtained by solving the pricing game $((h_j)_{j \in \mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$. The set of pre-entry equilibrium aggregator levels is obtained by solving the pricing game $((h_j)_{j \in \mathcal{N} \setminus f^0}, H^0 + \sum_{j \in f^0} h_j(\infty), \mathcal{F} \setminus \{f^0\}, (c_j)_{j \in \mathcal{N} \setminus f^0})$.

In an international trade context with a competitive (or monopolistically competitive) fringe of importers, the parameter H^0 can be interpreted as the consumer surplus derived from foreign varieties. Trade liberalization then corresponds to an increase in H^0. According to Proposition 4, trade liberalization lowers prices and markups, and raises consumer surplus. Perhaps more surprisingly, trade liberalization induces import-competing firms to broaden their product portfolios. Similarly, the entry of a competitor induces incumbent firms to lower their prices, and, using the terminology of Johnson and Myatt (2003), to introduce "fighting brands" to preserve their market shares.[15]

Productivity / quality. Productivity and quality shocks have more ambiguous effects. In the following, we focus on productivity shocks for the sake of conciseness, keeping in mind that quality shocks have similar effects.[16] Suppose that c_k, the marginal cost of product k owned by firm f, increases. Then, firm f's fitting-in function, m^f, shifts downward. Intuitively, product k has now become less profitable, and firm f therefore has less incentives to divert sales toward that product. It therefore lowers the prices of its other products, and, hence, its ι-markup. Despite the fact that μ^f goes down, $p_k = r_k(m^f(H))$ goes up due to the direct impact of the increase in c_k. Therefore, firm f's contribution to the aggregator $(\sum_{j\in f} h_j(r_j(m^f(H))))$ may go up or down, depending on whether the decrease in h_k is offset by the increase in h_j ($j \neq k$). This means that the aggregate fitting-in function may shift upward or downward.

If Γ shifts downward (as it does under CES and MNL demands; see Section 5), then, by monotonicity of Γ, the highest and lowest equilibrium aggregator levels decrease, which lowers consumer surplus. By Proposition 1, firm f's rivals increase their ι-markups. Therefore, by Proposition 2, they end up earning higher profits, charging higher prices, and supplying fewer products. Whether firm f ends up decreasing its ι-markup (and hence, making lower profit and supplying more products) is unclear, since the direct effect of the increase in c_k may be dominated by the indirect effect of the decrease in H. If, instead, Γ shifts upward, then consumers end up benefiting from the marginal cost increase, while firm f's rivals make less profit, set lower prices, and supply more products. As for firm f, the direct effect of the increase in c_k is now reinforced by the indirect effect of the increase in H. Therefore, firm f charges a lower ι-markup, makes less profit, and supplies more products.[17]

[15] In Johnson and Myatt (2003)'s Cournot model with vertically differentiated products, fighting brands can emerge only if marginal revenue does not decrease everywhere. Under the more standard assumption of decreasing marginal revenue, an incumbent firm always reacts to entry by pruning its product line. In our framework with horizontal product differentiation, fighting brands are the rule rather than the exception.

[16] We can augment our consumer demand model by introducing quality as follows. The demand model is now given by $((a_j h_j)_{j\in\mathcal{N}}, H^0)$, where, for every j, h_j satisfies all the assumptions we have made so far, $H^0 \geq 0$, and a_j is a strictly positive scalar, which we call product j's quality. The idea, in the discrete/continuous choice micro-foundation, is that an increase in a_k raises the probability that product k is chosen $(a_k h_k/(\sum_j a_j h_j + H^0))$, but does not affect the conditional demand for product k $(d\log(a_j h_j)/dp_j = d\log h_j/dp_j)$. This is consistent with the way in which product quality or vertical product characteristics are usually introduced in CES or MNL demand systems.

[17] This discussion suggests that, in principle, a firm may benefit from being less efficient, and equilibrium consumer surplus may be (locally) increasing in the marginal cost of one of the products. In Online Appen-

3.4 Equilibrium Uniqueness

In this section, we briefly summarize our main equilibrium uniqueness results. Formal statements and proofs can be found in Online Appendix V, where we derive conditions under which the aggregate fitting-in function $\Gamma(\cdot)$ has a unique fixed point.

We first discuss uniqueness with single-product firms. Let $\rho_j \equiv h_j h_j''/(h_j')^2$ for every $j \in \mathcal{N}$. In Online Appendix V, we show that the assumption that ρ_j is non-decreasing for every j guarantees equilibrium uniqueness with single-product firms. To interpret this condition, consider a hypothetical situation in which a single-product monopolist, firm $\{j\}$, is pricing against an outside option H^0. That firm therefore faces the demand function $\widehat{D}_j(p_j, h_j(p_j) + H^0)$. In Online Appendix V.1, we show that the function $p_j \mapsto 1/\widehat{D}_j(p_j, h_j(p_j) + H^0)$ is convex for every H^0 if and only if ρ_j is non-decreasing. Caplin and Nalebuff (1991) argue that this convexity condition is "just about as weak as possible" (see the paragraph after their Proposition 3, p. 38). They show that, under this condition, single-product firms' profit functions are quasi-concave in own prices. In their framework, equilibrium existence then follows from Kakutani's fixed-point theorem. We find that, although this convexity condition is not needed to obtain equilibrium existence, it does guarantee equilibrium uniqueness.

Establishing equilibrium uniqueness with multiproduct firms is harder. In Online Appendix V, we show that the equilibrium is indeed unique under stronger variants of the assumption that ρ_j is non-decreasing. We provide a list of functional forms that do satisfy those stronger variants, and develop a cookbook to construct such functional forms. As a byproduct, we obtain equilibrium uniqueness under CES and MNL demands. Finally, we show that, regardless of the monotonicity properties of ρ_j, the equilibrium is unique, provided that firms are sufficiently inefficient (c_j sufficiently large for every j) and/or that the outside option is sufficiently attractive (H^0 sufficiently high). Intuitively, when the products in $\mathcal{N}$ are relatively unattractive compared to the outside option (either because marginal costs are high, or because the outside option delivers high consumer surplus), firms have low market shares, and, hence, little market power. Firms therefore set ι-markups close to those they would choose under monopolistic competition, and react relatively little to changes in their rivals' behavior.

4 Extensions

In this section, we extend our analysis along several dimensions. We first consider a more general class of demand systems, which includes nested CES and MNL demands as special cases. We then show how to adapt our framework to permit general equilibrium effects. Next, we study non-linear pricing. Finally, we study quantity competition with multiproduct firms. For the sake of brevity, we do not state any formal results. A formal treatment can be found in the Online Appendix.

dix XIV.3 and XIV.4, we provide examples in which these seemingly counter-intuitive comparative statics results do obtain.

Nested demand systems. In this section, demand is derived from the indirect subutility function

$$V(p) = \Psi\left(\sum_{m\in\mathcal{M}} \Phi^m\left(\sum_{j\in m} h_j(p_j)\right)\right),$$

where $\mathcal{M}$, the set of nests, is a partition of the set of products, Ψ and Φ^m ($m \in \mathcal{M}$) are smooth functions, and h_j satisfies the usual assumptions for every j. Applying Roy's identity, we obtain the demand for product $i \in n \in \mathcal{M}$:

$$D_i(p) = -h_i'(p_i)\Phi^{n\prime}\left(\sum_{j\in n} h_j(p_j)\right)\Psi'\left(\sum_{m\in\mathcal{M}} \Phi^m\left(\sum_{j\in m} h_j(p_j)\right)\right).$$

The demand system introduced in Section 2 is a special case, where $\Psi(\Phi) = \log(\Phi + H^0)$ for every $\Phi > 0$, and $\Phi^m(H) = H$ for every $H > 0$ and $m \in \mathcal{M}$. This new class of demand systems therefore provides a generalization along two lines. First, it includes the entire class of IIA demand systems as a special case (see Anderson, Erkal, and Piccinin, 2013, Proposition 5). Second, it allows for substitution patterns that go beyond IIA. Note that nested CES ($h_j(p_j) = a_j p_j^{1-\sigma}$, $\Phi^m(H) = H^\alpha$, $\Psi(\Phi) = \log(\Phi + H^0)$, with $a_j > 0$, $\sigma > 1$, $\alpha \in (0,1)$, and $H^0 \geq 0$) and nested MNL ($h_j(p_j) = \mathrm{e}^{\frac{a_j - p_j}{\lambda}}$, $\Phi^m(H) = H^\alpha$, $\Psi(\Phi) = \log(\Phi + H^0)$, with $a_j \in \mathbb{R}$, $\lambda > 0$, $\alpha \in (0,1)$, and $H^0 \geq 0$) demands are special cases.

In Online Appendix VII, we derive necessary and sufficient conditions under which these demand systems can be given discrete/continuous choice micro-foundations.[18] We show that, under the same conditions, the demand system is quasi-linearly integrable. This extension also permits a particular type of consumer heterogeneity, where the h functions are consumer-specific, and take the additively separable form $h_i(p_i, t_i) = h_i(p_i) + t_i$, where $(t_j)_{j\in\mathcal{N}}$ is the consumer's type, drawn from some smooth distribution. See the discussion at the end of Online Appendix VII.1 for details.

When studying the pricing game in Online Appendix VIII, we confine attention to the case in which the firm partition is a filtration of the nest partition, i.e., a nest can contain products owned by different firms, but a firm cannot own products in multiple nests. This assumption ensures that the pricing game retains some aggregative properties, in the sense that the only thing that matters for firm f operating in nest n is the profile of prices it is choosing, $(p_j)_{j\in f}$, the value of the nest-level sub-aggregator $H^n \equiv \sum_{j\in n} h_j$, and the value of the industry-level aggregator $\Phi = \sum_{m\in\mathcal{M}} \Phi^m$. Moreover, the common ι-markup property continues to hold. We then derive conditions on the functions Ψ, Φ^n, and h_j under which

[18]The discrete/continuous choice process is more involved than in Section 2. The consumer first observes the value of an outside option, and decides whether to take it. If he turns down the outside option, then he observes a vector of nest-specific taste shocks, drawn i.i.d. from a type-I extreme value distribution, and picks a nest. Next, he observes the value of a nest-specific outside option, and decides whether to take it. If he turns it down, then he observes a vector of product-specific taste shocks, drawn i.i.d. from a type-I extreme value distribution, and chooses the product that delivers the highest indirect utility. Finally, he decides how much of that product to consume.

first-order conditions are sufficient for global optimality. Under these conditions, the game can be solved using a nested fixed point argument similar to the one in Section 3.1. This allows us to establish equilibrium existence and uniqueness. As a byproduct, we obtain existence and uniqueness of a pricing equilibrium under nested CES and MNL demands.

Whether or not the comparative statics derived in Section 3.3 extend to this more general setting depends on the behavior of the curvature of the function Ψ. If $-\Psi''/\Psi'$ is non-increasing (as it is, e.g., when Ψ is a power function), then all the firms lower their markups and supply (weakly) more products when the industry becomes more competitive (i.e., as the aggregator Φ increases). If that condition is violated, then prices and the set of active products are both non-monotone in industry competitiveness. In Online Appendix VIII.3, we provide an example where such non-monotonicity arises.

General equilibrium. Our framework with quasi-linear preferences rules out income effects. Incorporating such income effects appears important to assess the impact of economy-wide policies. We now provide a general equilibrium extension of our framework in the spirit of Neary (2003, 2016). There is a continuum of sectors, indexed by $z \in [0,1]$. The representative consumer's indirect utility function is additively separable across sectors, as in Bertoletti and Etro (2017), and given by

$$V(p,y) = \int_{[0,1]} \Psi\left(\sum_{j \in \mathcal{N}(z)} h_j\left(\frac{p_j(z)}{y}, z\right), z\right) dz,$$

where $\mathcal{N}(z)$ is the set of products in sector z, $p_j(z)$ is the price of product j in sector z and y is consumer income. Note that the set of products $\mathcal{N}(z)$, the functions $h_j(\cdot, z)$, and the function $\Psi(\cdot, z)$ (introduced in the previous extension) are all allowed to vary across sectors. Normalizing $y \equiv 1$ and applying Roy's identity yields the demand for product i in sector z:

$$D_i(p) = \frac{\partial_1 \Psi\left(\sum_{j \in \mathcal{N}(z)} h_j\left(p_j(z), z\right), z\right)\left(-\partial_1 h_i\left(p_i(z), z\right)\right)}{\int_{[0,1]}\left(\sum_{j \in \mathcal{N}(z')} p_j(z')\left(-\partial_1 h_j\left(p_j(z'), z'\right)\right)\right)\partial_1 \Psi\left(\sum_{j \in \mathcal{N}(z')} h_j\left(p_j(z'), z'\right), z'\right) dz'},$$

where $\partial_k g$ denotes the partial derivative of g with respect to its kth argument. The numerator corresponds to demand under quasi-linear preferences while the denominator is the marginal utility of income which each firm takes as given.

There is a fixed labor supply, L. In each sector z, there is a set of firms $\mathcal{F}(z)$. The marginal cost of product $j \in \mathcal{N}(z)$ is $wc_j(z)$, where w is the economy-wide wage rate, and $c_j(z)$ the product's labor requirement.

An equilibrium is a wage rate w and a profile of prices $p(\cdot)$ such that, in every sector z, the profile of prices forms a Nash equilibrium, and the labor market clears. In Online Appendix X, we derive conditions under which an equilibrium exists. As a byproduct, we establish existence and uniqueness of equilibrium for the case of nested CES demand structures, where

$h_j = a_j p_j^{1-\sigma}$ and Ψ is either a power function or proportional to the logarithm. Such demand structures have recently been used in quantitative oligopoly models of international macroeconomics and trade, including Atkeson and Burstein (2008), Edmond, Midrigan, and Xu (2015), and Hottman, Redding, and Weinstein (2016).

Non-linear pricing. We have assumed throughout that firms compete in linear prices. We now show that our methodology can also be usefully applied to study non-linear pricing, implicitly assuming that demand has been derived from discrete/continuous choice. To this end, suppose that firms can charge two-part tariffs: For every $j \in \mathcal{N}$, p_j (resp., F_j) denotes the variable (resp., fixed) part of the two-part tariff contract for product j. Since conditional demand is the same for all consumers, firms find it optimal to set all variable parts equal to marginal cost (thereby maximizing joint surplus conditional on consumers purchasing the good), and compete on the fixed parts.

When all firms set the variable parts equal to marginal costs, the consumer's indirect utility conditional on choosing product j (net of the taste shock and income) is $\log h_j(c_j) - F_j$, and his conditional demand for product j is $-h_j'(c_j)/h_j(c_j)$. Firm f's profit is therefore given by:

$$\Pi^f = \sum_{k \in f} \left(F_k - c_k \frac{-h_k'(c_k)}{h_k(c_k)} \right) \frac{h_k(c_k)\, \mathrm{e}^{-F_k}}{\sum_{j \in \mathcal{N}} h_j(c_j)\, \mathrm{e}^{-F_j} + H^0}.$$

These are the payoff functions of the pricing game $((\widetilde{h}_j)_{j \in \mathcal{N}}, H^0, \mathcal{F}, (\widetilde{c}_j)_{j \in \mathcal{N}})$ with linear tariffs, where, for every $j \in \mathcal{N}$, $\widetilde{h}_j(F_j) = h_j(c_j)\exp(-F_j)$, and $\widetilde{c}_j = c_j(-h_j'(c_j))/h_j(c_j)$. Put differently, the pricing game with two-part tariffs is formally equivalent to a pricing game with MNL demand and linear pricing. We know that the MNL pricing game has a unique equilibrium (see Section 3.4).

An immediate observation is that all available products always end up being sold under non-linear pricing, whereas, as discussed before, this is not necessarily the case under linear pricing. This comes from the fact that the non-linear pricing game is equivalent to a linear pricing game with MNL demand. Since MNL products are such that $\bar{\mu}_j = \infty$ (recall that $\widetilde{\iota}_j(F_j) = F_j$), such products are always supplied. The intuition is that a firm is better able to extract the additive taste shock ε_j under non-linear pricing than it is under linear pricing.

Quantity competition. The techniques developed in this paper can also be used to solve quantity competition games with differentiated products and multiproduct firms. In the following, we briefly describe how our aggregative games approach can be adapted, and refer the reader to Online Appendix XI for more details. Suppose that the inverse demand for product i can be written as $P_i(x_i, H) = h_i'(x_i)/H$, where $H = \sum_{j \in \mathcal{N}} h_j(x_j)$ is the aggregator, and x_j is the output of product j. Taking first-order conditions, it is possible to show that an additive form of the common ι-markup property must hold in equilibrium: For every firm f, there exists μ^f such that $\frac{P_k - c_k}{P_k} + \iota_k(x_k) = \mu^f$ for all $k \in f$, where ι_k is still the elasticity of h_k'. Inverting the ι-markup equation yields an *output function* $\chi_k(\mu^f, H)$ for every product

k. In contrast to the pricing function r_k, that function now depends not only on μ^f, but also on H. We can then use the ι-markup μ^f to reduce firm f's profile of first-order conditions to a unique equation

$$\mu^f = \frac{1}{H} \sum_{j \in f} \chi_j(\mu^f, H) h_j'(\chi_j(\mu^f, H)),$$

which uniquely pins down firm f's *fitting-in function*, $m^f(H)$. Interestingly, it is still the case that $m^f(H)$ is decreasing in H. This means that the set of active products continues to expand as the industry becomes more competitive under quantity competition. We provide a worked-out example in Online Appendix XI.11.

Combining output and fitting-in functions allows us to define the *aggregate fitting-in function*:

$$\Gamma(H) = \sum_{f \in \mathcal{F}} \sum_{j \in f} h_j \left(\chi_j(m^f(H), H) \right).$$

We then derive conditions under which first-order conditions are sufficient for optimality, and Γ has a unique fixed point. Under these conditions, there is a unique equilibrium. These conditions hold, e.g., under CES demand.

5 Type Aggregation: (Nested) CES and MNL Demands

In this section, we analyze multiproduct-firm pricing when demand is either CES or MNL. We show that another aggregation property, called *type aggregation*, obtains: Even though firms may differ in multiple dimensions (number of products, qualities and marginal costs), all the information relevant for determining a firm's fitting-in function can be summarized by its uni-dimensional "type." This type-aggregation property allows us to derive clear-cut comparative statics. The property will also prove useful when we apply our framework to merger analysis (by allowing us to summarize merger-specific synergies with a uni-dimensional sufficient statistic) and international trade (by providing a theoretically sound measure of firm-level productivity). As shown in Online Appendix XIII.2, type aggregation also makes it easier to compute pricing equilibria. Although the results in this section are stated in the case without nests, we show in Online Appendix XIII.1 that they continue to hold under nested CES or MNL demands under the assumption that each firm owns an entire nest of products.

In the CES case, let $h_k(p_k) = a_k p_k^{1-\sigma}$ for every $k \in \mathcal{N}$, where $a_k > 0$ is the quality of product k, and $\sigma > 1$ is the elasticity of substitution. In the MNL case, let $h_k(p_k) = \exp\left(\frac{a_k - p_k}{\lambda}\right)$ for every $k \in \mathcal{N}$, where $a_k \in \mathbb{R}$ is the quality of product k, and $\lambda > 0$ is a price sensitivity parameter. Any pricing game based on these h functions satisfies the conditions for existence and uniqueness of equilibrium provided in Section 3.4.

Let $T^f = \sum_{k \in f} a_k c_k^{1-\sigma}$ in the CES case, and $T^f = \sum_{k \in f} \exp\left(\frac{a_k - c_k}{\lambda}\right)$ in the MNL case. Note that T^f would be equal to firm f's contribution to the aggregator H (and thus to consumer surplus) if that firm were to price competitively (i.e., set $p_k = c_k$ for all $k \in f$). We call T^f firm f's type. Simplifying equation (5), which pins down firm f's fitting-in function, we obtain:[19]

$$\text{(CES)} \qquad \mu^f \left(1 - \frac{\sigma - 1}{\sigma}\frac{T^f}{H}\left(1 - \frac{\mu^f}{\sigma}\right)^{\sigma-1}\right) = 1, \tag{7}$$

$$\text{(MNL)} \qquad \mu^f \left(1 - \frac{T^f}{H}\,\mathrm{e}^{-\mu^f}\right) = 1. \tag{8}$$

Equation (7) (resp., (8)) implicitly defines a function $m(T^f/H)$. Firm f's fitting-in function is simply $H \mapsto m(T^f/H)$. An immediate implication is that firms f and g have the same fitting-in function if and only if they have the same type ($T^f = T^g$). Note that (for fixed T^f) the CES markup equation converges pointwise to the MNL markup equation as σ goes to infinity. A pricing game with MNL demand can therefore be viewed as a limiting case of pricing games with CES demand.

Next, we claim, that if firms f and g have the same type, then their contributions to the aggregator are the same. We introduce the following notation. Under CES demand, for a given aggregator level H, $s_k = a_k p_k^{1-\sigma}/H$ is the market share (in value) of product k. Under MNL demand, market shares are defined in volume: $s_k = \mathrm{e}^{\frac{a_k - p_k}{\lambda}}/H$. Firm f's market share is $s^f = \sum_{k\in f} s_k$. It can then be shown that

$$\text{(CES)} \quad s^f = \frac{T^f}{H}\left(1 - \frac{m\left(\frac{T^f}{H}\right)}{\sigma}\right)^{\sigma-1} \equiv S\left(\frac{T^f}{H}\right),$$

$$\text{(MNL)} \quad s^f = \frac{T^f}{H}\,\mathrm{e}^{-m\left(\frac{T^f}{H}\right)} \equiv S\left(\frac{T^f}{H}\right).$$

Firm f's market share function is $H \mapsto S(T^f/H)$. Therefore, firms f and g share the same market share function if and only if $T^f = T^g$. Put differently, firm f and g's contributions to the aggregator are identical if and only if they have the same type. Note also that, as σ tends to infinity, the definition of market shares in the CES case converges pointwise to the one in the MNL case.

Recall that H is an equilibrium aggregator level if and only if $\Gamma(H)/H = 1$. Under CES and MNL demands, this condition is equivalent to market shares (including the outside option's market share) adding up to 1. Finally, recall that firm f's equilibrium profit is equal to its equilibrium ι-markup minus one: $\Pi^f = m(T^f/H) - 1 \equiv \pi(T^f/H)$.

We summarize these findings in the following proposition:

Proposition 5. *Consider a pricing game with CES or MNL demands. If firm f is replaced by firm g such that $T^g = T^f$, then the equilibrium value of the aggregator is unaffected, and*

[19] Recall that μ^f is proportional to the Lerner index under CES demand, and proportional to the absolute markup under MNL demand.

firm g ends up charging the same markup, earning the same profit, and commanding the same market share as firm f.

Proposition 5 implies that, for any multiproduct firm f, there always exists a "type-equivalent" single-product firm.[20]

We also obtain the following comparative statics:

Proposition 6. *In a pricing game with CES or MNL demands, m', S', $\pi' > 0$. Moreover, for $f \neq g$ in $\mathcal{F}$, equilibrium consumer surplus and social welfare are increasing in T^f, firm f's equilibrium ι-markup, market share and profit are increasing in T^f, and firm g's equilibrium ι-markup, market share and profit are decreasing in T^f.*

Proof. See Online Appendix XIII.4. □

The first part of the proposition says that a firm charges a higher markup, commands a larger market share, and makes a larger profit if it has more products, if it is more productive, if it sells higher-quality products (higher T^f), or if it operates in a less competitive environment (lower H). Recall from Section 3.3 that a reduction in marginal cost c_k or an increase in quality a_k generally has an ambiguous effect on consumer surplus and firms' equilibrium behavior and profit. Under CES and MNL demands, clear-cut comparative statics obtain. Interestingly, the result that an increase in T^f always raises social welfare is in contrast to standard results in homogeneous-goods Cournot models (Lahiri and Ono, 1988; Zhao, 2001), or in models of price or quantity competition with differentiated products and linear demand (Wang and Zhao, 2007), where a reduction in a firm's marginal cost lowers social welfare if that firm initially has a low market share.[21]

6 Discussion, Applications and Conclusion

Contribution. The main contribution of this paper consists in developing a tractable approach to multiproduct-firm oligopoly under price competition. The aggregative structure of the pricing game and the common ι-markup property deliver simple, yet powerful existence, uniqueness and characterization results. Our approach gives rise to a computationally efficient algorithm, and to a simple decomposition of the welfare distortions in multiproduct-firm oligopoly. Monotone comparative statics results allow us to make predictions on how markups and firm scope vary with the competitive environment. Under (nested) CES and MNL demands, type aggregation obtains; that is, multidimensional firm heterogeneity can be mapped into a single-dimensional type. In extensions, we adapt the framework to analyze non-linear

[20] To see this, fix $T^f > 0$, and define firm $\hat{f}$ as a firm selling only one product with quality $\hat{a} = T^f$ (resp. $\hat{a} = \lambda \log T^f + 1$) in the CES (resp. MNL) case and marginal cost $\hat{c} = 1$. Then, $T^f = T^{\hat{f}}$, and firms f and $\hat{f}$ are therefore equivalent in the sense of Proposition 5.

[21] Wang and Zhao (2007) claim that, with MNL demand, a reduction in a (single-product) firm's marginal cost can lower social welfare. Proposition 6 shows that this statement is incorrect.

pricing, quantity competition, and general equilibrium. We also study multiproduct-firm pricing with a richer aggregative structure in which a firm's profit depends not only on its own prices and the market-level aggregator but also a nest-level sub-aggregator. As a secondary contribution, the paper provides a complete characterization of the class of demand systems that can be derived from (multi-stage) discrete/continuous choice with i.i.d. Gumbel taste shocks.

Compared to pure discrete choice demand systems (such as mixed/nested MNL), our framework brings in an additional degree of flexibility: Consumers can choose a variable quantity of their chosen product. This dimension was found to be important in applications to soft drinks, painkillers, and other industries (Smith, 2004; Chan, 2006; Björnerstedt and Verboven, 2016). The framework imposes only minimal restrictions on the shape of the "conditional" demand function (i.e., $-h_i'/h_i$), which allows us to go beyond the special cases of MNL and CES demands. As we have shown, this additional flexibility permits richer patterns of markups across products within the same firm.

Applications. Our framework can be used to address important questions in industrial organization and international trade. One such question that has received much attention in the international trade literature relates to how multiproduct firms react to an increase in competition (e.g., due to entry or trade liberalization) by adjusting their product range.[22] A common finding in this literature is that firms respond to a trade liberalization by refocusing on their core competencies, i.e., by shrinking their product ranges.[23] In models with CES demand and product-level fixed costs, this is due to the fact that more intense competition reduces variable profits on all products, and therefore makes it harder to cover fixed costs. In models with linear demand, more intense competition chokes out the demand for products sold at a high price. In our model, firms respond to an increase in competition by introducing "fighting brands" (Johnson and Myatt, 2003). The intuition is that, when competition is more intense, firms have to worry more about losing market shares to rival products rather than their own, implying that self-cannibalization matters less.[24]

The key normative questions in merger analysis relate to the welfare effects of mergers and the optimal merger approval policy. When addressing these questions, the theoretical literature, including Farrell and Shapiro (1990), McAfee and Williams (1992), and Nocke and Whinston (2010, 2013), has largely relied on the (single-product) homogeneous-goods Cournot model. Yet, almost all mergers involve multiproduct firms. In Online Appendix XV,

[22]In the international trade literature, this question is usually addressed in models of monopolistic competition, under either CES demand (Bernard, Redding, and Schott, 2010, 2011; Nocke and Yeaple, 2014) or linear demand (Dhingra, 2013; Mayer, Melitz, and Ottaviano, 2014). An exception is Eckel and Neary (2010) who study (identical) multiproduct firms in a Cournot model with linear demand, which Eckel, Iacovone, Javorcik, and Neary (2015) extend to heterogeneous firms.

[23]In Nocke and Yeaple (2014), the prediction depends on whether the firm sells only domestically or not. In Qiu and Zhou (2013), high-productivity firms respond to trade liberalization by expanding their scope if the variety-introduction cost rises sufficiently steeply with the number of varieties.

[24]In Online Appendix XII, we provide a more thorough discussion of the mechanisms underlying the contrasting predictions.

we show how our aggregative-games approach to multiproduct firms can be used to re-visit some of the questions addressed in that literature. In the case of (nested) CES and MNL demands, the type aggregation property greatly simplifies the multidimensional nature of potential merger-specific synergies (e.g., reduction in marginal costs, quality improvements, changes in firm scope) since all that matters for the post-merger outcome is the merged entity's single-dimensional post-merger type. In the Online Appendix, we provide necessary and sufficient conditions for a merger to increase consumer surplus (thereby extending Farrell and Shapiro, 1990) and aggregate surplus. We also analyze a merger's "external effect" (Farrell and Shapiro, 1990) and show how it depends on the distribution of pre-merger market shares. Finally, we show that Nocke and Whinston (2010)'s result on the dynamic optimality of a myopic merger approval policy carries over to multiproduct-firm oligopoly pricing with (nested) CES and MNL demands.

As in the Cournot model, the merger M between firms f and g is CS-nondecreasing (i.e., does not lower consumer surplus) if and only if

$$S\left(\frac{T^M}{H^*}\right) \geq S\left(\frac{T^f}{H^*}\right) + S\left(\frac{T^g}{H^*}\right),$$

where T^M is the post-merger type, T^f and T^g are the pre-merger types, H^* is the pre-merger aggregator level, and S is the market share function defined in Section 5. In words, at the pre-merger aggregator level, the merged firm wants to contribute more to the aggregator than the merger partners did jointly before the merger. For this condition to be satisfied, there must be merger-specific synergies, that is, $T^M > T^f + T^g$. This condition can also be used to show that a CS-nondecreasing merger is privately profitable. As $S' > 0$, there exists a unique cutoff type, $\hat{T}^M$, such that the merger is CS-nondecreasing if and only if $T^M \geq \hat{T}^M$. An important property is that $\hat{T}^M$ is decreasing in H^*. Intuitively, the market power effect of the merger is weaker, the more competitive is the industry before the merger. This implies a sign-preserving complementarity in the consumer surplus effect of two disjoint mergers: A merger that is CS-nondecreasing given current market structure remains CS-nondecreasing after another CS-nondecreasing merger has been approved. Using similar arguments as in Nocke and Whinston (2010), this implies that an antitrust authority that approves only mergers that are CS-nondecreasing at the time of approval will never have ex post regret.

A classic question in international trade is how a trade liberalization affects firm- and industry-level productivity and domestic welfare. As mentioned above, most of the recent trade literature involving multiproduct firms has addressed this question in models of monopolistic competition.[25] In Online Appendix XVI, we apply our framework to re-visit this question for the cases of (nested) CES and MNL demands. Among other things, we show the formal equivalence of the domestic welfare effect of a unilateral trade liberalization and the external effect of a merger. In particular, the domestic welfare effect is more likely to be negative if the market share of domestic firms is higher and the domestic industry is

[25] Eckel and Neary (2010) and Eckel, Iacovone, Javorcik, and Neary (2015) are recent exceptions.

more concentrated. In contrast, under monopolistic competition with CES, MNL or linear demands, such a unilateral trade liberalization would have an unambiguously positive effect on domestic welfare.

Limitations. Throughout, we have maintained the assumption that production costs are linear in output, as is commonly assumed in the theoretical IO literature. Product-level fixed costs are important ingredients in many models of international trade. They could be accommodated by considering a two-stage game in which firms first decide on what products to supply, and then compete in prices.[26] The equilibrium analysis in the present paper applies to the second stage of that game. Existence of a (mixed-strategy) subgame-perfect equilibrium of the two-stage game follows immediately as the first stage is a finite game. However, the first-stage product choice is likely to generate a multiplicity of equilibria.

Introducing non-constant marginal costs but maintaining the assumption of additive separability of the cost function across products is straightforward. The only difference to our analysis above is that the pricing function would now depend not only on the ι-markup but also on the aggregator, as in the case of quantity competition discussed in Section 4. Relaxing the assumption of additive separability, thereby allowing for (dis-)economies of scope, would complicate the analysis. While it is easy to show that the common ι-markup property would carry over, it would be harder to back out prices from ι-markups as the marginal cost of a product would now depend on all of the firm's prices.

The biggest limitation of the aggregative games approach to pricing games relates to the constraints it necessarily imposes on the class of demand systems that can be accommodated.

Note first that our class of demand systems does not nest linear demand. While it is true that linear demand gives rise to an aggregative pricing game, that demand system is not differentiable, implying that an approach based on first-order conditions is not valid. Indeed, Cumbul and Virag (2017) show that this non-differentiability can generate a continuum of pricing equilibria with asymmetric (single-product) firms. Under price competition, the linear demand system is thus much less well-behaved than commonly perceived.[27]

By construction, the aggregative games approach to multiproduct-firm pricing does not allow us to address questions related to spatial competition as cross-price effects work only through the aggregator. However, some form of spatial competition can be accommodated by introducing nests, as we do in Section 4: All products in the same nest are "local" competitors while all products in other nests are "global" competitors.

The modern empirical IO literature has emphasized the importance of allowing for rich

[26] This is the approach taken by Shaked and Sutton (1990) and Dobson and Waterson (1996) for the case of linear demand and symmetric products. Using a demand system derived from discrete/continuous choice with identical products, Anderson and de Palma (1992, 2006) consider a three-stage game where firms first make entry decisions, then decide on how many products to offer, and then compete in prices. In contrast to Shaked and Sutton (1990) and Dobson and Waterson (1996), they confine attention to symmetric equilibria.

[27] A feature of linear demand that is not shared by the class of demand systems considered in this paper is the presence of a "choke price." In Online Appendix IV we show that such a choke price can be accommodated in our framework by allowing the conditional demand $-h_i'/h_i$ to vanish when p_i is sufficiently high.

substitution patterns that go beyond those implied by the IIA property. Such flexible substitution patterns are typically obtained by using a mixed MNL demand system (as in Berry, Levinsohn, and Pakes, 1995; Nevo, 2001) or a nested MNL demand system (as in Goldberg, 1995; Verboven, 1996; Goldberg and Verboven, 2001). While our approach does not allow us to handle random coefficients in general,[28] it is able to accommodate nested demand systems (such as nested MNL or nested CES) under some restriction on the ownership partition of the set of products, as shown in Section 4.

Trade-offs for empirical work. The limitations described above should be weighed against the benefits our approach delivers. First, the aggregative games approach yields existence and uniqueness of equilibrium – whereas no such result is available for multiproduct-firm pricing games with mixed MNL demands – which is useful both for estimation and counterfactuals. Second, the nested fixed point structure inherent to aggregative games gives rise to an efficient algorithm for computing equilibrium, which is helpful both for empirical and computational work. Third, the continuous dimension of the discrete/continuous choice process generates rich patterns of markups within a firm's product portfolio. In light of these trade-offs, the comparative advantage of our approach is in empirical applications to (i) industries in which consumers demand a variable amount of their chosen product, (ii) computationally complex dynamic models in which the pricing game needs to be solved for multiple times, and (iii) international trade where researchers typically work at a more aggregate level at which flexible substitution patterns may appear less important. We expect the approach to be useful also for antitrust practitioners who may value both equilibrium uniqueness for counterfactual analysis and the computational tractability.

A Appendix: Proof of Theorem 1

We first state the following preliminary technical lemma:

Lemma A. *Let h be a $\mathcal{C}^3$, strictly decreasing and log-convex function from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$. Then:*

(a) $\lim_{p\to\infty} ph'(p) = \lim_{p\to\infty} h'(p) = 0$.

Assume in addition that h satisfies Assumption 1, and define $\gamma(p) = h'^2(p)/h''(p)$, $\rho(p) = h(p)/\gamma(p)$, and $\iota(p) = ph''(p)/(-h'(p))$. Then:

(b) There exists a unique scalar $\underline{p}$ such that for every $p > 0$, $\iota(p) > 1$ if and only if $p > \underline{p}$. Moreover, $\iota'(p) \geq 0$ for all $p > \underline{p}$.

(c) $\bar{\mu} \equiv \lim_{p\to\infty} \iota(p) > 1$.

[28] Recall from Section 4, however, that certain types of (random) consumer heterogeneity can be accommodated.

(d) *For every* $p > \underline{p}$, $\gamma'(p) < 0$.

(e) $\lim_{p\to\infty} \gamma(p) = 0$.

(f) *If* $\lim_{p\to\infty} h(p) = 0$ *and* $\bar{\mu} < \infty$, *then* $\lim_{p\to\infty} \rho(p) = \frac{\bar{\mu}}{\bar{\mu}-1}$.

Proof. See Online Appendix II.1. □

We can now prove Theorem 1. Let $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ be a pricing game satisfying Assumption 1. The following lemma ensures that each firm sets at least one finite price in any equilibrium.

Lemma B. *In any Nash equilibrium* $(p_j^*)_{j\in\mathcal{N}}$, *for every firm* $f \in \mathcal{F}$, *there exists* $k \in f$ *such that* $p_k^* < \infty$.

Proof. This follows from profit function (2): If firm f sets $p_j^* = \infty$ for all $j \in f$, then it makes zero profit; if instead it sets $p_k \in (c_k, \infty)$ for some $k \in f$, then its profit is strictly positive. □

Next, we show that first-order conditions (appropriately generalized to handle infinite prices) are necessary and sufficient for global optimality. To this end, fix a firm $f \in \mathcal{F}$ and a price vector for firm f's rivals $(p_j)_{j\in\mathcal{N}\setminus f}$, and denote $H^{0\prime} = \sum_{j\in\mathcal{N}\setminus f} h_j(p_j) + H^0$. Suppose that at least one of the prices set by firm f's rivals is finite, so that $H^{0\prime} > 0$, and define

$$G^f((p_j)_{j\in f}, H^{0\prime}) = \sum_{\substack{k\in f \\ p_k<\infty}} (p_k - c_k)\frac{-h_k'(p_k)}{\sum_{j\in f} h_j(p_j) + H^{0\prime}}. \tag{9}$$

Note that $G^f((p_j)_{j\in f}, H^{0\prime})$ is the profit of firm f when it sets price vector $(p_j)_{j\in f}$ and its rivals set price vector $(p_j)_{j\in\mathcal{N}\setminus f}$. We study the following maximization problem:

$$\max_{(p_j)_{j\in f}\in(0,\infty]^f} G^f\left((p_j)_{j\in f}, H^{0\prime}\right). \tag{10}$$

We prove the following lemma:

Lemma C. *Maximization problem* (10) *has a solution. Moreover, if* $(p_j)_{j\in f}$ *solves that maximization problem, then* $p_j \geq c_j$ *for all* $j \in f$, *and* $p_k < \infty$ *for some* $k \in f$.

Proof. The observation that firm f sets a least one finite price follows from the argument in the proof of Lemma B. Next, we show that firm f does not set any price below marginal costs. Let $(p_j)_{j\in f} \in (0,\infty]^f$. Suppose that $p_k < c_k$ for some k in f, and let $\tilde{p}_j \equiv \max(c_j, p_j)$ for every j in f. When firm f deviates from $(p_j)_{j\in f}$ to $(\tilde{p}_j)_{j\in f}$, it stops making losses on products j such that $p_j < c_j$, and it raises the demand for products j such that $p_j \geq c_j$. Therefore, $G^f((p_j)_{j\in f}, H^{0\prime}) < G^f((\tilde{p}_j)_{j\in f}, H^{0\prime})$, and $(p_j)_{j\in f}$ is not a solution of maximization problem (10).

Next, we show that maximization problem (10) has a solution. Assume without loss of generality that $f = \{1, \ldots, n\}$. For every $k \in f$ and $x_k \in [0,1]$, define

$$\phi_k(x_k) = \begin{cases} c_k + \frac{x_k}{1-x_k} & \text{if } x_k < 1, \\ \infty & \text{if } x_k = 1. \end{cases}$$

Note that $\lim_{x_k \to 1} \phi_k(x_k) = \phi_k(1) = \infty$. For every $x \in [0,1]^n$, define $\phi(x) = (\phi_1(x_1), \ldots, \phi_n(x_n))$. ϕ is a bijection from $[0,1]^n$ to $\prod_{k=1}^n [c_k, \infty]$. Finally, let

$$\Psi(x) = G^f\left(\phi(x), H^{0\prime}\right), \quad \forall x \in [0,1]^n.$$

Since ϕ is a bijection, the maximization problem $\max_{x\in[0,1]^n} \Psi(x)$ has a solution if and only if maximization problem (10) has a solution. All we need to do now is show that Ψ is continuous on $[0,1]^n$.

Clearly, Ψ is continuous at every point x such that $x_k < 1$ for every $1 \le k \le n$. Next, let x such that $x_k = 1$ for some $1 \le k \le n$. To fix ideas, suppose that $x_k = 1$ for all $1 \le k \le K$, and that $x_k < 1$ for all $K+1 \le k \le n$, where $K \ge 1$. Then,

$$\begin{aligned}
\lim_{\tilde{x}\to x} \Psi(\tilde{x}) &= \lim_{\tilde{x}\to x} \sum_{k=1}^n (\phi_k(\tilde{x}_k) - c_k) \frac{-h_k'(\phi_k(\tilde{x}_k))}{\sum_{j=1}^n h_j(\phi_j(\tilde{x}_j)) + H^{0\prime}}, \\
&= \frac{\sum_{k=1}^n \lim_{\tilde{x}_k\to x_k} (\phi_k(\tilde{x}_k) - c_k)(-h_k'(\phi_k(\tilde{x}_k)))}{\sum_{j=1}^n \lim_{\tilde{x}_j\to x_j} h_j(\phi_j(\tilde{x}_j)) + H^{0\prime}}, \\
&= \frac{0 + \sum_{k=K+1}^n (\phi_k(x_k) - c_k)(-h_k'(\phi_k(x_k)))}{\sum_{j=1}^K h_j(\infty) + \sum_{j=K+1}^n h_j(\phi_j(x_j)) + H^{0\prime}}, \\
&= \Psi(x),
\end{aligned}$$

where the third line follows by Lemma A-(a). Therefore, Ψ is continuous. Combining this with the fact that $[0,1]^n$ is compact implies that the maximization problem $\max_{x\in[0,1]^n} \Psi(x)$ has a solution. □

The next step is to solve the firm's maximization problem using first-order conditions. Since the objective function is not necessarily differentiable at infinite prices, we need to generalize the definition of first-order conditions to account for that. Note first that, if all the products in $f' \subsetneq f$ are priced at infinity, then the profit function $G^f(\cdot, H^{0\prime})$ is still $\mathcal{C}^2$ in $(p_j)_{j\in f\setminus f'} \in \mathbb{R}_{++}^{f\setminus f'}$, as can be seen by inspecting equation (9). Next, we slightly abuse notation, by denoting $\left(p_k, (p_j)_{j\in f\setminus\{k\}}\right)$ the price vector with k-th component p_k, and with other components given by $(p_j)_{j\in f\setminus\{k\}}$. We generalize first-order conditions as follows:

Definition A. *We say that the generalized first-order conditions of maximization problem* (10) *hold at price vector* $(\tilde{p}_j)_{j\in f} \in (0,\infty]^f$ *if for every* $k \in f$,

(a) $\frac{\partial G^f}{\partial p_k}((\tilde{p}_j)_{j\in f}, H^{0\prime}) = 0$ *whenever* $\tilde{p}_k < \infty$, *and*

(b) $G^f\left((\tilde{p}_j)_{j\in f}, H^{0\prime}\right) \geq G^f\left(\left(p_k, (\tilde{p}_j)_{j\in f\setminus\{k\}}\right), H^{0\prime}\right)$ *for every* $p_k \in \mathbb{R}_{++}$ *whenever* $\tilde{p}_k = \infty$.

It is obvious that generalized first-order conditions are necessary for optimality:

Lemma D. *If* $(p_j)_{j\in f} \in (0,\infty]^f$ *solves maximization problem* (10)*, then the generalized first-order conditions are satisfied at price profile* $(p_j)_{j\in f}$.

Next, we want to show that, if the generalized first-order conditions hold at a price vector, then this price vector satisfies a generalized version of the common ι-markup property introduced in Section 3.1. To define this generalized common ι-markup property, we first need to establish a few facts about the functions $\nu_k : p_k \mapsto \frac{p_k - c_k}{p_k}\iota_k(p_k)$. Let p_k^{mc} be the unique solution of the equation $\nu_k(p_k) = 1$. Product k would be priced at p_k^{mc} under monopolistic competition. We prove the following lemma:

Lemma E. *For every* $k \in f$ *and* $\mu^f \in (1, \bar{\mu}_k)$*, the equation* $\nu_k(p_k) = \mu^f$ *has a unique solution in the interval* $(0,\infty)$*, denoted* $r_k(\mu^f)$.[29] *If* $\mu^f \geq \bar{\mu}_k$*, then that equation does not have a solution. Moreover,* $r_k(\cdot)$ *is strictly increasing and* $\mathcal{C}^1$ *on* $(1, \bar{\mu}^f)$*, and satisfies* $\lim_{\mu^f\to 1} r_k(\mu^f) = p_k^{mc}$*,* $\lim_{\mu^f\to\bar{\mu}_k} r_k(\mu^f) = \infty$*, and*

$$r_k'(\mu^f) = \frac{\gamma_k\left(r_k(\mu^f)\right)}{\mu^f\left(-\gamma_k'\left(r_k(\mu^f)\right)\right) - (\mu^f - 1)\left(-h_k'\left(r_k(\mu^f)\right)\right)} > 0. \tag{11}$$

Proof. We first argue that p_k^{mc} is well-defined. To see this, note that ν_k is continuous and, by Lemma A, $\nu_k(p_k) < 1$ for every $p_k < \max(\underline{p}_k, c_k)$, $\lim_{p_k\to\infty}\nu_k(p_k) = \bar{\mu}_k > 1$, and ν_k is strictly increasing on $(\max(\underline{p}_k, c_k), \infty)$. Therefore, p_k^{mc} is well-defined, and $p_k^{mc} > \max(\underline{p}_k, c_k)$. The same line of argument implies that, for every $\mu^f > 1$, the equation $\nu_k(p_k) = \mu^f$ has at most one solution in the interval $(0,\infty)$, and that any solution must be strictly greater than $\max(\underline{p}_k, c_k)$. Moreover, since ν_k is strictly increasing on $(\max(\underline{p}_k, c_k), \infty)$ and $\sup_{p_k > \max(\underline{p}_k, c_k)}\nu_k(p_k) = \bar{\mu}_k$, no solution exists if $\mu^f \geq \bar{\mu}_k$.

Next, we argue that, if $\mu^f \in (1, \bar{\mu}_k)$, then the solution $r_k(\mu^f)$ exists and has the properties stated in the Lemma. Since $p_k^{mc} > \max(\underline{p}_k, c_k)$, it follows from Lemma A-(b) that the function ι_k is non-decreasing on (p_k^{mc}, ∞), and that $\nu_k'(p_k) > 0$ for every $p_k > p_k^{mc}$. By the inverse function theorem, ν_k establishes a $\mathcal{C}^1$-diffeomorphism from (p_k^{mc}, ∞) to $\nu_k\left((p_k^{mc}, \infty)\right)$, and the inverse function $r_k(\cdot)$ satisfies $r_k'(\mu^f) = 1/\nu_k'(r_k(\mu^f))$. Note that

$$\nu_k'(p_k) = \left(\frac{-(p_k - c_k)h_k'(p_k)}{\gamma_k(p_k)}\right)' = \frac{-h_k' - (p_k - c_k)h_k'' + \gamma_k'(p_k - c_k)\frac{h_k'}{\gamma_k}}{\gamma_k} = \frac{(\nu_k - 1)h_k' - \nu_k\gamma_k'}{\gamma_k}.$$

This proves equation (11). Since ν_k is strictly increasing,

$$\nu_k\left((p_k^{mc}, \infty)\right) = \left(\lim_{p_k\to p_k^{mc}}\nu_k(p_k), \lim_{p_k\to\infty}\nu_k(p_k)\right) = (1, \bar{\mu}_k). \qquad \square$$

[29] $\bar{\mu}_k$ was defined in Lemma A as $\bar{\mu}_k = \lim_{p_k\to\infty}\iota_k(p_k)$. γ_k was defined there as $\gamma_k = h_k'^2/h_k''$.

We extend the function r_k by continuity as follows: $r_k(1) = p_k^{mc}$ and $r_k(\mu^f) = \infty$ for every $\mu^f \geq \bar{\mu}_k$. We can now generalize the common ι-markup property to price vectors with infinite components:

Definition B. *We say that the price vector* $(p_j)_{j\in f} \in (0,\infty]^f$ *satisfies the common* ι*-markup property if there exists a scalar* $\mu^f \geq 1$*, called the* ι*-markup, such that* $p_k = r_k(\mu^f)$ *for every* $k \in f$*.*

For every $k \in \mathcal{N}$, extend γ_k by continuity at infinity: $\gamma_k(\infty) = 0$ (see Lemma A-(e)). Let $\bar{\mu}^f = \max_{j\in f} \bar{\mu}_j$. The following lemma allows us to simplify first-order conditions considerably:

Lemma F. *Suppose that the generalized first-order conditions for maximization problem* (10) *hold at price vector* $(p_j)_{j\in f} \in (0,\infty]^f$*. Then,* $(p_j)_{j\in f}$ *satisfies the common* ι*-markup property. The corresponding* ι*-markup,* μ^f*, solves the following equation on interval* $(1, \bar{\mu}^f)$*:*

$$\mu^f = 1 + \mu^f \frac{\sum_{j\in f} \gamma_j\left(r_j(\mu^f)\right)}{\sum_{j\in f} h_j\left(r_j(\mu^f)\right) + H^{0\prime}}. \tag{12}$$

In addition, $G^f\left((p_j)_{j\in f}, H^{0\prime}\right) = \mu^f - 1$*.*

Proof. Assume without loss of generality that $f = \{1,\dots,n\}$, and let $f' = \{k \in f : \ p_k < \infty\}$. Clearly, $f' \neq \emptyset$, because if this set were empty, then the firm could obtain a strictly positive profit by pricing, say, product 1, at some finite price $p_1 > c_1$, which would violate condition (b) in Definition A. Assume without loss of generality that $f' = \{1,\dots,K\}$, where $1 \leq K \leq n$. Taking the first-order condition for product $i \in f'$, and simplifying as we did in Section 3.1, we obtain:

$$\frac{\partial G^f}{\partial p_i} = D_i \times \left(1 - \nu_i(p_i) + G^f\left((p_j)_{1\leq j\leq n}, H^{0\prime}\right)\right) = 0. \tag{13}$$

It follows that $\nu_i(p_i) = \nu_j(p_j) \equiv \mu^f > 1$ for every $1 \leq i, j \leq K$. By Lemma E, this means that $p_i = r_i(\mu^f)$ and $\mu^f < \bar{\mu}_i$ for every $1 \leq i \leq K$. Moreover, μ^f satisfies

$$\mu^f = 1 + G^f\left((p_j)_{1\leq j\leq n}, H^{0\prime}\right).$$

Next, we claim that, for every $j \geq K+1$, $r_j\left(\mu^f\right) = \infty$, or, equivalently, $\bar{\mu}_j \leq \mu^f$. Assume for a contradiction, that, for some $j \geq K+1$, $\bar{\mu}_j > \mu^f$. To fix ideas, assume that this j is equal to $K+1$. Let $\tilde{G}^f(x)$ and $\tilde{D}_{K+1}(x)$ be the profit of firm f and the demand for product $K+1$ at price vector $(p_1,\dots,p_K,x,\infty,\dots,\infty)$. Note that $\tilde{G}^f(x)$ tends to $G^f((p_j)_{j\in f}, H^{0\prime}) = \mu^f - 1$ as x goes to infinity (see the proof of Lemma C). Using the expression of marginal profit given in equation (13), we see that, for every $x \in \mathbb{R}_{++}$,

$$\tilde{G}^{f\prime}(x) = \tilde{D}_{K+1}(x)\left(1 - \underbrace{\nu_{K+1}(x)}_{\xrightarrow[x\to\infty]{} \bar{\mu}_{K+1}} + \underbrace{\tilde{G}^f(x)}_{\xrightarrow[x\to\infty]{} \mu^f - 1}\right).$$

Since, by assumption, $\bar{\mu}_{K+1} > \mu^f$, this implies that $\tilde{G}^{f\prime}(x) < 0$ for x sufficiently high. Therefore, there exists $\tilde{p}_{K+1} \in \mathbb{R}_{++}$ such that

$$\tilde{G}^f(\tilde{p}_{K+1}) > \tilde{G}^f(\infty) = G^f\left((p_1, \ldots, p_K, \infty, \ldots, \infty), H^{0\prime}\right),$$

which contradicts condition (b) in Definition A. It follows that $r_j\left(\mu^f\right) = \infty$ for all $j \geq K+1$,

Note that μ^f satisfies

$$\begin{aligned}
\mu^f &= 1 + \frac{\sum_{j=1}^K (p_j - c_j)(-h_j'(p_j))}{\sum_{j\in f} h_j(p_j) + H^{0\prime}}, \\
&= 1 + \frac{\sum_{j=1}^K \frac{p_j - c_j}{p_j} \iota_j(p_j)\gamma_j(p_j)}{\sum_{j\in f} h_j(p_j) + H^{0\prime}}, \\
&= 1 + \mu^f \frac{\sum_{j=1}^K \gamma_j(p_j)}{\sum_{j\in f} h_j(p_j) + H^{0\prime}}, \\
&= 1 + \mu^f \frac{\sum_{j\in f} \gamma_j(r_j(\mu^f))}{\sum_{j\in f} h_j(r_j(\mu^f)) + H^{0\prime}}, \text{ by Lemma A-(e)},
\end{aligned}$$

which is equation (12). □

All we need to do now is study equation (12):

Lemma G. *Equation* (12) *has a unique solution on interval* $(1, \bar{\mu}^f)$.

Proof. We first show that a solution exists. By Lemma C, maximization problem (10) has a solution $p^* = (p_j^*)_{j\in f}$. By Lemma D, p^* satisfies the generalized first-order conditions. Hence, by Lemma F, p^* satisfies the common ι-markup property, and the corresponding ι-markup $\mu^{f*} \in (1, \bar{\mu}^f)$ is a solution of equation (12).

Define

$$\phi : \mu^f \in (1, \bar{\mu}^f) \mapsto (\mu^f - 1)\left(\sum_{j\in f} h_j\left(r_j(\mu^f)\right) + H^{0\prime}\right) - \mu^f \sum_{j\in f} \gamma_j\left(r_j(\mu^f)\right).$$

Note that μ^f solves equation (12) if and only if $\phi(\mu^f) = 0$. We now argue that ϕ is strictly increasing. To see this, note that ϕ is continuous on $(1, \bar{\mu}^f)$, and $\mathcal{C}^1$ on $(1, \bar{\mu}^f) \setminus \{\bar{\mu}_j\}_{j\in f}$ by Lemma E. All we need to do is show that $\phi'(\mu^f) > 0$ for every $\mu^f \in (1, \bar{\mu}^f) \setminus \{\bar{\mu}_j\}_{j\in f}$. Fix such a μ^f, and let f' be the set of j's in f such that $\bar{\mu}_j > \mu^f$. Then, since $\gamma_j(\infty) = 0$ for every j (Lemma A-(e)),

$$\begin{aligned}
\phi'(\mu^f) &= H^{0\prime} + \sum_{j\in f\setminus f'} h_j(\infty) + \sum_{j\in f'} (h_j - \gamma_j) + (\mu^f - 1)\left(\sum_{j\in f'} r_j' h_j'\right) - \mu^f \left(\sum_{j\in f'} r_j' \gamma_j'\right), \\
&= H^{0\prime} + \sum_{j\in f\setminus f'} h_j(\infty) + \sum_{j\in f'} (h_j - \gamma_j) + \sum_{j\in f'} \underbrace{r_j'\left(\mu^f(-\gamma_j') - (\mu^f - 1)(-h_j')\right)}_{=\gamma_j \text{ by Lemma E}},
\end{aligned}$$

$$= H^{0\prime} + \sum_{j \in f} h_j > 0.$$

Hence, ϕ is strictly increasing, and equation (12) has a unique solution. $\square$

Combining Lemmas C–G allows us to conclude the analysis of maximization problem (10):

Lemma H. *Maximization problem* (10) *has a unique solution. The generalized first-order conditions associated with this maximization problem are necessary and sufficient for global optimality. The optimal price vector (which contains at least one finite component) satisfies the common ι-markup property, and the corresponding ι-markup, μ^{f*}, is the unique solution of equation* (12)*. The maximized value of the objective function is $\mu^{f*} - 1$.*

Proof. Let $(p_j^*)_{j \in f}$ be a solution of maximization problem (10). By Lemma C, such a $(p_j^*)_{j \in f}$ exists and $p_k^* < \infty$ for some $k \in f$. By Lemma D, $(p_j^*)_{j \in f}$ satisfies the generalized first-order conditions. Therefore, by Lemma F, $(p_j^*)_{j \in f}$ satisfies the common ι-markup property, and the corresponding μ^f solves equation (12). By Lemma G, this equation has a unique solution, which we denote μ^{f*}. Therefore, $(p_j^*)_{j \in f} = \left(r_j(\mu^{f*})\right)_{j \in f}$, and maximization problem (10) has a unique solution. The fact that the maximized value of the objective function is $\mu^{f*} - 1$ follows from Lemma F.

Conversely, assume that the generalized first-order conditions hold at price vector $(\tilde{p}_j)_{j \in f}$. Then, by Lemmas F and G, $(\tilde{p}_j)_{j \in f} = \left(r_j(\mu^{f*})\right)_{j \in f} = (p_j^*)_{j \in f}$. It follows that generalized first-order conditions are sufficient for global optimality. $\square$

We now turn our attention to the equilibrium existence problem. The price vector $p \in (0, \infty]^{\mathcal{N}}$ is a Nash equilibrium if and only if, for every $f \in \mathcal{F}$, $(p_j)_{j \in f}$ maximizes $G^f \left(\cdot, \sum_{j \in \mathcal{N} \setminus f} h_j(p_j) + H^0 \right)$. From Lemma B, each firm sets at least one finite price in any Nash equilibrium. Hence, $\sum_{j \in \mathcal{N} \setminus f} h_j(p_j) + H^0 > 0$ for every f, and we can apply Lemma H: There exists a pricing equilibrium if and only if there exists a profile of ι-markups $\left(\mu^f\right)_{f \in \mathcal{F}} \in \prod_{f \in \mathcal{F}} (1, \bar{\mu}^f)$ such that

$$\mu^f = 1 + \mu^f \frac{\sum_{j \in f} \gamma_j \left(r_j(\mu^f) \right)}{\sum_{g \in \mathcal{F}} \sum_{j \in g} h_j \left(r_j(\mu^g) \right) + H^0}, \quad \forall f \in \mathcal{F}.$$

This is, in turn, equivalent to finding an aggregator level $H > 0$ and a profile of ι-markups $\left(\mu^f\right)_{f \in \mathcal{F}} \in \prod_{f \in \mathcal{F}} (1, \bar{\mu}^f)$ such that $H = \sum_{g \in \mathcal{F}} \sum_{j \in g} h_j \left(r_j(\mu^g) \right) + H^0$ and for all $f \in \mathcal{F}$,

$$\mu^f = 1 + \mu^f \frac{\sum_{j \in f} \gamma_j \left(r_j(\mu^f) \right)}{H}. \tag{14}$$

Our approach to equilibrium existence consists in showing that this nested fixed point problem has a solution. We start by studying the inner fixed point problem:

Lemma I. *For every $f \in \mathcal{F}$, for every $H > 0$, equation* (14) *has a unique solution in μ^f on the interval $(1, \bar{\mu}^f)$, denoted $m^f(H)$.*
The function $m^f(.)$ is continuous, strictly decreasing, and satisfies $\lim_{H\to\infty} m^f(H) = 1$ and $\lim_{H\to 0^+} m^f(H) = \bar{\mu}^f$.

Proof. μ^f solves equation (14) if and only if

$$\psi(\mu^f, H) \equiv \frac{\mu^f - 1}{\mu^f} - \frac{\sum_{j\in f} \gamma_j \left(r_j(\mu^f)\right)}{H} = 0.$$

Note that $\psi(\cdot,\cdot)$ is continuous on $(1,\infty) \times \mathbb{R}_{++}$, and that, by Lemmas A-(d) and E, $\psi(\cdot, H)$ is strictly increasing for every $H > 0$. Moreover, for every H, $\lim_{\mu^f\to 1} \psi(\mu^f, H) < 0$, and, by Lemma A-(e), $\lim_{\mu^f\to\bar{\mu}^f} \psi(\mu^f, H) > 0$. Hence, there exists a unique $\mu^f \in (1, \bar{\mu}^f)$ such that $\psi(\mu^f, H) = 0$. The fact that ψ is strictly increasing in μ^f and H implies that the solution, $m^f(H)$, is strictly decreasing in H.

Let $\hat{H} > 0$. We now show that $\lim_{H\to\hat{H}^-} m^f(H) = m^f(\hat{H})$. To this end, let $(H^n)_{n\geq 0}$ be a strictly increasing sequence such that $H^n \underset{n\to\infty}{\longrightarrow} \hat{H}$. Then, the sequence $(m^f(H^n))_{n\geq 0}$ is strictly decreasing. Therefore, that sequence has a limit, which we denote $\mu^f \in (1, \bar{\mu}^f)$. For every $n \geq 0$, we have that $\psi(m^f(H^n), H^n) = 0$. Taking limits, and using the continuity of ψ, we obtain that $\psi(\mu^f, \hat{H}) = 0$. Hence, $\mu^f = m^f(\hat{H})$, and $\lim_{H\to\hat{H}^-} m^f(H) = m^f(\hat{H})$. The same argument implies that $\lim_{H\to\hat{H}^+} m^f(H) = m^f(\hat{H})$. By monotonicity of m^f, it follows that m^f is continuous.

By monotonicity, the limits $\lim_{H\to\infty} m^f(H)$ and $\lim_{H\to 0^+} m^f(H)$ exist and satisfy $1 \leq \lim_{H\to\infty} m^f(H) < \lim_{H\to 0^+} m^f(H) \leq \bar{\mu}^f$. Taking the limit as H tends to infinity in equation $\psi(m^f(H), H) = 0$ immediately implies that $\lim_{H\to\infty} m^f(H) = 1$. If $\lim_{H\to 0^+} m^f(H) < \bar{\mu}^f$, then taking the limit as H tends to 0^+ in equation $\psi(m^f(H), H) = 1$ gives us the contradiction $-\infty = 0$. Hence, $\lim_{H\to 0^+} m^f(H) = \bar{\mu}^f$. □

We can now take care of the outer fixed-point problem, which, by Lemma I consists in finding an $H > 0$ such that $\Omega(H) = 1$, where

$$\Omega(H) \equiv \frac{1}{H}\left(H^0 + \sum_{f\in\mathcal{F}} \sum_{k\in f} h_k(r_k(m^f(H)))\right). \tag{15}$$

The following lemma guarantees that the outer fixed-point problem has a solution:

Lemma J. *There exists $H^* > 0$ such that $\Omega(H^*) = 1$.*

Proof. By Lemmas E and I, Ω is continuous. In addition, when H goes to ∞, the numerator of Ω goes to $H^0 + \sum_{f\in\mathcal{F}} \sum_{k\in f} h_k\left(p_k^{mc}\right)$, which is finite. Hence, $\lim_{H\to\infty} \Omega(H) = 0$. If we show that Ω is strictly greater than 1 in the neighborhood of 0^+, then we can apply the intermediate value theorem to obtain the existence of H^*.

Assume first that $h_j(\infty) > 0$ for some j in $\mathcal{N}$. Since h_j is decreasing, $h_j(p_j) \geq h_j(\infty)$ for all $p_j > 0$. Hence,

$$\Omega(H) \geq \frac{h_j(\infty)}{H} \underset{H \to 0^+}{\longrightarrow} \infty.$$

The same reasoning implies that $\lim_{H \to 0^+} \Omega(H) = \infty$ if $H^0 > 0$.

Next, assume that $H^0 = 0$ and $\lim_{p_k \to \infty} h_k(p_k) = 0$ for all $k \in \mathcal{N}$. For every $f \in \mathcal{F}$, we define the threshold $\underline{H}^{f\prime} > 0$ as follows. If $\bar{\mu}_k = \bar{\mu}^f$ for all $k \in f$, then let $\underline{H}^{f\prime} = 1$. If $\bar{\mu}_k < \bar{\mu}^f$ for some $k \in f$, then, since $\lim_{H \to 0^+} m^f(H) = \bar{\mu}^f$ and by monotonicity of m^f, there exists $\hat{H}^f > 0$ such that $m^f(H) > \max(\{\bar{\mu}_k\}_{k \in f} \backslash \{\bar{\mu}^f\})$ whenever $H < \hat{H}^f$. In that case, let $\underline{H}^{f\prime} \equiv \hat{H}^f$. Having done that for every $f \in \mathcal{F}$, let $\underline{H}' = \min_{f \in \mathcal{F}} \underline{H}^{f\prime}$. Then, for every $H \in (0, \underline{H}')$,

$$\Omega(H) = \frac{1}{H} \sum_{f \in \mathcal{F}} \sum_{\substack{j \in f \\ \bar{\mu}_j = \bar{\mu}^f}} h_j\left(r_j\left(m^f(H)\right)\right).$$

We partition the set of firms into two subsets: $\mathcal{F}'$ and $\mathcal{F}''$, where $\mathcal{F}' = \left\{f \in \mathcal{F} : \ \bar{\mu}^f = \infty\right\}$, and $\mathcal{F}'' = \mathcal{F} \backslash \mathcal{F}'$.

Let $f \in \mathcal{F}''$. By Lemma A-(f), $\lim_{p_k \to \infty} \rho_k(p_k) = \frac{\bar{\mu}^f}{\bar{\mu}^f - 1}$ for every $k \in f$ such that $\bar{\mu}_k = \bar{\mu}^f$.[30] In addition, by Lemmas E and I, for every $k \in f$, $r_k\left(m^f(H)\right) \underset{H \to 0^+}{\longrightarrow} \infty$. Therefore, there exists $\underline{H}^{f\prime\prime} > 0$ such that

$$\rho_k\left(r_k\left(m^f(H)\right)\right) \geq \frac{\bar{\mu}^f}{\bar{\mu}^f - 1}\left(1 - \frac{1}{2|\mathcal{F}|}\right), \quad \forall H < \underline{H}^{f\prime\prime},\ \forall k \in f \text{ s.t. } \bar{\mu}_k = \bar{\mu}^f.$$

Let $\underline{H}'' = \min_{f \in \mathcal{F}''} \underline{H}^{f\prime\prime}$ (or any strictly positive real number if $\mathcal{F}''$ is empty), and $\underline{H} = \min\left(\underline{H}', \underline{H}''\right)$. For every $H < \underline{H}$,

$$\begin{aligned}
\Omega(H) &= \frac{1}{H}\left(\sum_{f \in \mathcal{F}'} \sum_{\substack{k \in f \\ \bar{\mu}_k = \bar{\mu}^f}} h_k\left(r_k\left(m^f(H)\right)\right) + \sum_{f \in \mathcal{F}''} \sum_{\substack{k \in f \\ \bar{\mu}_k = \bar{\mu}^f}} h_k\left(r_k\left(m^f(H)\right)\right)\right), \\
&\geq \frac{1}{H}\left(\sum_{f \in \mathcal{F}'} \sum_{\substack{k \in f \\ \bar{\mu}_k = \bar{\mu}^f}} \gamma_k\left(r_k\left(m^f(H)\right)\right) + \sum_{f \in \mathcal{F}''} \sum_{\substack{k \in f \\ \bar{\mu}_k = \bar{\mu}^f}} \gamma_k\left(r_k\left(m^f(H)\right)\right)\rho_k\left(r_k\left(m^f(H)\right)\right)\right), \\
&\geq \sum_{f \in \mathcal{F}'} \frac{1}{H} \sum_{\substack{k \in f \\ \bar{\mu}_k = \bar{\mu}^f}} \gamma_k\left(r_k\left(m^f(H)\right)\right) + \sum_{f \in \mathcal{F}''} \frac{\bar{\mu}^f}{\bar{\mu}^f - 1}\left(1 - \frac{1}{2|\mathcal{F}|}\right) \frac{1}{H} \sum_{\substack{k \in f \\ \bar{\mu}_k = \bar{\mu}^f}} \gamma_k\left(r_k\left(m^f(H)\right)\right), \\
&= \sum_{f \in \mathcal{F}'} \frac{\sum_{k \in f} \gamma_k\left(r_k\left(m^f(H)\right)\right)}{H} + \sum_{f \in \mathcal{F}''} \frac{\bar{\mu}^f}{\bar{\mu}^f - 1}\left(1 - \frac{1}{2|\mathcal{F}|}\right) \frac{\sum_{k \in f} \gamma_k\left(r_k\left(m^f(H)\right)\right)}{H},
\end{aligned}$$

[30] ρ_k was defined in Lemma A as $\rho_k = h_k / \gamma_k$.

$$= \sum_{f\in\mathcal{F}'} \frac{m^f(H)-1}{m^f(H)} + \sum_{f\in\mathcal{F}''} \frac{m^f(H)-1}{m^f(H)} \frac{\bar{\mu}^f}{\bar{\mu}^f-1}\left(1-\frac{1}{2|\mathcal{F}|}\right), \text{ using equation (14),}$$

where the second line follows by log-convexity ($h_k \geq \gamma_k$ for all k). When H goes to 0^+, the right-hand side term on the last line goes to

$$|\mathcal{F}'| + |\mathcal{F}''|\left(1-\frac{1}{2|\mathcal{F}|}\right) \geq |\mathcal{F}| - \frac{1}{2},$$

which is strictly greater than 1. Therefore, $\Omega(H) > 1$ when H is small enough. □

This concludes the proof of Theorem 1.

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