Staying at Zero with Affine Processes
An Application to Term Structure Modelling

Alain Monfort\textsuperscript{1,2} Fulvio Pegoraro\textsuperscript{1,2}
Jean-Paul Renne\textsuperscript{2} Guillaume Roussellet\textsuperscript{1,2,3}

\textsuperscript{1}CREST

\textsuperscript{2}Banque de France

\textsuperscript{3}Dauphine University

Volatility Institute 7\textsuperscript{th} Annual Conference

All the views presented here are those of the authors and should not be associated with those of the Banque de France.
Zero lower bound (ZLB)

Several of the major central banks now face the ZLB
Stylized facts to match

- The short-term nominal rate can stay at the ZLB for several periods.
- In the meantime, longer-term yields can show substantial fluctuations [JGB yields from June 1995 to May 2014]
Closed-form pricing

- Gaussian Atsm

Positivity

- CIR
- QTSM

Can stay at 0

- Shadow rate
Closed-form pricing

- Gaussian Atsm
- CIR
- QTSM

Positivity
- Shadow rate

This Paper

Can stay at 0
Our ZLB model: a primer

We introduce a new affine process:
What we do in this paper

- We derive affine non-negative processes staying at 0 (ARG\(_0\) processes) to build a Term Structure Model which is:
  - providing positive yields for all maturities;
  - consistent with the ZLB with a short-rate experiencing prolonged periods at 0 while long-term rates still fluctuates;
  - affine: thus closed-form formulas for bond-pricing and lift-off probabilities are available.

- Empirical assessment on JGB yields (June 1995 to May 2014). Good performance of our model in terms of:
  - fitting yield levels and conditional variances;
  - calculating Risk-Neutral and Historical lift-off probabilities.
Related literature


- **Lift-off probabilities:** Bauer & Rudebusch (2013), Swanson & Williams (2013)
Contents

1 Introduction

2 The ARG$_0$ process
   - A mixture of affine distributions
   - Properties and extensions

3 The NATSM
   - Short-rate specification and the affine framework
   - Advantages of an affine framework

4 Estimation
   - State-space formulation
   - Estimation results

5 Assessing lift-off dates

6 Conclusion

7 Appendix
Defining the Gamma-Zero distribution

We construct a new distribution in two steps:

1. \( Z \sim \mathcal{P}(\lambda) \implies Z(\omega) \in \{0, 1, 2, \ldots\} \) and \( \mathbb{P}(Z = 0) = \exp(-\lambda) \).
2. We define \( X|Z \sim \gamma_Z(\mu) \), which implies:
   - If \( Z = 0 \), \( X \) is a dirac point mass at 0.
   - If \( Z > 0 \), \( X \) is gamma-distributed (continuous on \( \mathbb{R}^+ \)).

Definition

The non-negative r.v. \( X \sim \gamma_0(\lambda, \mu) \), \( \lambda > 0 \) and \( \mu > 0 \), if

\[
X | Z \sim \gamma_Z(\mu) \quad \text{with} \quad Z \sim \mathcal{P}(\lambda)
\]

\[\Rightarrow \quad \mathbb{P}(X = 0) = \mathbb{P}(Z = 0) = \exp(-\lambda).\]
A mixture of affine distributions

**A mixture distribution**

In other words, $X \sim \gamma_0(\lambda, \mu)$ if its (complicated) p.d.f. is:

$$f_X(x; \lambda, \mu) = \sum_{z=1}^{+\infty} \left[ \frac{\exp(-x/\mu) x^{z-1}}{(z-1)! \mu^z} \times \frac{\exp(-\lambda) \lambda^z}{z!} \right] 1_{\{x>0\}} + \exp(-\lambda) 1_{\{x=0\}}$$

However, simple Laplace transform:

$$\mathcal{L}_X(u; \lambda, \mu) := \mathbb{E}[\exp(uX)] = \exp\left[ \lambda \frac{u \mu}{(1 - u \mu)} \right] \quad \text{for} \quad u < \frac{1}{\mu}.$$  

$\Rightarrow$ Exponential-affine in $\lambda$. 
A mixture distribution

In other words, \( X \sim \gamma_0(\lambda, \mu) \) if its (complicated) p.d.f. is:

\[
f_X(x ; \lambda, \mu) = \sum_{z=1}^{+\infty} \left[ \exp\left(-\frac{x}{\mu}\right) \frac{x^{z-1}}{(z-1)! \mu^z} \times \frac{\exp(-\lambda)\lambda^z}{z!} \right] 1\{x>0\} + \exp(-\lambda)1\{x=0\}
\]

However, simple Laplace transform:

\[
\varphi_X(u ; \lambda, \mu) := \mathbb{E} [\exp(uX)] = \exp\left[ \lambda \frac{u\mu}{1 - u\mu} \right] \quad \text{for} \quad u < \frac{1}{\mu}.
\]

\( \implies \) Exponential-affine in \( \lambda \).
A mixture of affine distributions

A mixture distribution

In other words, $X \sim \gamma_0(\lambda, \mu)$ if its (complicated) p.d.f. is:

$$f_X(x ; \lambda, \mu) = \sum_{z=1}^{+\infty} \left[ \frac{\exp(-x/\mu) x^{z-1}}{(z-1)! \mu^z} \times \frac{\exp(-\lambda) \lambda^z}{z!} \right] \mathbf{1}_{\{x>0\}} + \exp(-\lambda) \mathbf{1}_{\{x=0\}}$$

However, simple Laplace transform:

$$\varphi_X(u ; \lambda, \mu) := \mathbb{E} [\exp(uX)] = \exp \left[ \lambda \frac{u\mu}{1 - u\mu} \right] \quad \text{for} \quad u < \frac{1}{\mu}.$$ 

$\implies$ Exponential-affine in $\lambda$. 
Main goal: Build a dynamic affine process with zero point mass.

Definition

\((X_t)\) is a ARG_0(\(\alpha, \beta, \mu\)) if \((X_{t+1}|X_t)\) is Gamma-zero distributed:

\[
(X_{t+1}|X_t) \sim \gamma_0(\alpha + \beta X_t, \mu) \quad \text{for} \quad \alpha \geq 0, \mu > 0, \beta > 0.
\]

Again, simple conditional LT, exponential-affine in \(X_t\):

\[
\varphi_{X,t}(u; \alpha, \beta, \mu) := \mathbb{E}_t[\exp(uX_{t+1})] = \exp\left(\frac{u\mu}{1 - u\mu}(\alpha + \beta X_t)\right), \quad \text{for} \quad u < \frac{1}{\mu}.
\]
A mixture of affine distributions

Summary

\[ X_t \text{ realized} \quad \alpha + \beta X_t \quad Z_{t+1} \mid X_t \sim P(\alpha + \beta X_t) \]

\[ X_{t+1} \mid Z_{t+1} \sim \gamma Z_{t+1}(\mu) \]

\( X_t \) realized

\( \text{time } t \)

\( \text{time } t + 1 \)
Interesting features and properties

Key properties:

- **Non-negative and affine process**
- **Staying at zero** with probability:

\[ P(X_{t+1} = 0|X_t = 0) = \exp(-\alpha) \neq 0. \]

\[ \alpha \neq 0 \implies \text{zero is not absorbing}. \]

\[ \text{The probability is TV in the multivariate setting}. \]

- **Affine first two conditional moments.**

Multivariate case

A multivariate VARG process can be obtained easily stacking together conditionally independent ARG processes.
Contents

1 Introduction

2 The ARG₀ process
   - A mixture of affine distributions
   - Properties and extensions

3 The NATSM
   - Short-rate specification and the affine framework
   - Advantages of an affine framework

4 Estimation
   - State-space formulation
   - Estimation results

5 Assessing lift-off dates

6 Conclusion

7 Appendix
The state of the economy is defined by a $n$-dimensional vector $X_t$. These factors follow a VARG process under $Q$.

**VARG$_\nu$ processes**

$X_t$ follows a $\text{VARG}_0(\alpha^Q, \beta^Q, \mu^Q)$ if, $\forall t$, $\forall i$:

- $Z_{i,t+1}|X_t \sim \mathcal{P}(\alpha^Q_i + \beta^Q_i' X_t)$.
- $X_{i,t+1}|Z_{i,t+1} \sim \gamma_{Z_{i,t+1}}(\mu^Q_i)$ cond. indep across $i$.

Each $X_{i,t}$ has a zero point mass.

$X_t$ has closed-form affine first two moments.
Summary

\[ \alpha_i^Q + \beta_i^{Q'} X_t > 0 \]

\[ X_t \in \mathbb{R}^n_+ \text{ realized} \]

\[ Z_{i,t+1} | X_t \sim \mathcal{P}(\alpha_i^Q + \beta_i^{Q'} X_t) \]

\[ X_{i,t+1} | Z_{i,t+1} \sim \gamma Z_{i,t+1}(\mu_i^Q) \]

\[ X_{i,t+1} \] time \( t + 1 \)
Short-rate specification

- The vector of factors $X_t$ is split into two: $X_t = (X_t^{(1)'}, X_t^{(2)'})'$

- The following structure is imposed:

$$
\begin{pmatrix}
X_t^{(1)} \\
X_t^{(2)}
\end{pmatrix} = \text{constant} + \begin{pmatrix}
\beta_{11}^Q & \beta_{12}^Q \\
0 & \beta_{22}^Q
\end{pmatrix}
\begin{pmatrix}
X_{t-1}^{(1)} \\
X_{t-1}^{(2)}
\end{pmatrix} + \xi_t^Q
$$

- The short-term rate $r_t$ is given by:

$$r_t = \delta_1' X_t^{(1)}$$

Key Properties

- $r_t$ has a zero point mass.

- $X_t^{(2)}$ appears in $Q$-expectations of future $r_t$.

  $\implies$ In the ZLB, $X_t^{(1)} = 0$ but long-term yields move with $X_t^{(2)}$. 
Pricing Formulas

The model belongs to the class of ATSM:

- Yields are affine in the factors for all maturities:

\[ R_t(h) = -\frac{1}{h} (A_h'X_t + B_h) = \bar{A}_h'X_t + \bar{B}_h. \]

- Recursive closed-form loadings formulas.

Physical \( \mathbb{P} \)-dynamics

The SDF is exp-affine with market price of risk vector \( \theta \), providing VARG \( \mathbb{P} \)-dynamics with explicit parameters.

\[ X_t|X_{t-1} \sim \text{VARG}_0(\alpha^\mathbb{P}, \beta^\mathbb{P}, \mu^\mathbb{P}) \]
Stylized facts to match (2)

Conditional volatilities: time-varying and maturity dependent.
How to treat it

- Conditional variance of yields:

\[ \mathbb{V}_t^P [R_{t+1}(h)] = \overline{A}_h' \mathbb{V}_t^P (X_{t+1}) \overline{A}_h \]

\[ = \overline{A}_h' \{ \text{diag} [\mu^P \circ \mu^P \circ (2\alpha^P + 2\beta^P X_t)] \} \overline{A}_h \]

- Time-varying and maturity-dependent.
Advantages of an affine framework

**NATSM properties**

- Yields $R_t(h)$ are non-negative;
- Long-term yields can move while $r_t = 0$ for several periods;
- Unconditional first two moments are available in closed-form;
- Conditional first two moments of yields are affine in $X_t$;
- Yields forecasts are explicitly affine in $X_t$;
Contents

1 Introduction

2 The ARG$_0$ process
   - A mixture of affine distributions
   - Properties and extensions

3 The NATSM
   - Short-rate specification and the affine framework
   - Advantages of an affine framework

4 Estimation
   - State-space formulation
   - Estimation results

5 Assessing lift-off dates

6 Conclusion

7 Appendix
State-space formulation

Observable variables

State vector \( Y_t = (R'_t, V'_t, S'_t)' \) affine in \( X_t \):

- \( R_t \): yield levels (6 maturities);
- \( V_t \): 2- and 10-y yield conditional (\( \text{EGARCH} \)) variance;
- \( S_t \): SPF for 3-m and 1-y ahead 10-y yield;

\[ \dim(X_t^{(1)}) = 1, \dim(X_t^{(2)}) = 3 \text{ and } \nu = 0; \]

Estimation technique

Affine \( \mathbb{P} \)-dynamics + affine observable variables.

\[ \Longrightarrow \text{Linear Kalman-filter-based QML.} \]
Filtered factors

![Graph showing filtered factors over dates from 1995 to 2015 for factors n°1 to n°4.](attachment:graph.png)
Estimation results

Factor loadings of yields and conditional variances

(a) Factor loadings of yields

- First factor
- Second factor
- Third factor
- Fourth factor

(b) Factor loadings of conditional variances
Fit of Conditional Variances and SPF
Fit of Yields

The following graphs display the fit of yields for different durations over time, showing observed and fitted values.
Contents

1 Introduction
2 The ARG₀ process
   - A mixture of affine distributions
   - Properties and extensions
3 The NATSM
   - Short-rate specification and the affine framework
   - Advantages of an affine framework
4 Estimation
   - State-space formulation
   - Estimation results
5 Assessing lift-off dates
6 Conclusion
7 Appendix
Lift-off probability dates under $\mathbb{P}$ and $\mathbb{Q}$

We calculate the following probabilities:

- $\mathbb{P}(r_{t+k} = 0 \mid X_t)$ and $\mathbb{Q}(r_{t+k} = 0 \mid X_t)$;
- $\mathbb{P}(r_{t+k} < 25 \text{ bps.} \mid X_t)$ and $\mathbb{Q}(r_{t+k} < 25 \text{ bps.} \mid X_t)$.

**Useful formula**

If $z \in \mathbb{R}^+$ and $\varphi_z(u)$ its Laplace transform.

$$\mathbb{P}(z = 0) = \lim_{u \to -\infty} \varphi_z(u).$$

**Next two plots:**

- **Time-series dimension**: $t$ varies ($k = 2\text{yrs and 5yrs}$).
- **Horizon dimension**: $k$ varies ($t = 11/30/07$ and $05/30/14$).
Assessing lift-off dates

2−years ahead

5−years ahead

lambda = 0

lambda = 25 bps

Dates

Probabilities

Horizon dimension of probabilities

![Graph showing horizon dimension of probabilities for different lambda values (0 and 25 bps). The x-axis represents forecast horizon, and the y-axis represents probabilities. Two lines for each lambda value are plotted for two dates: 2007-11-30 and 2014-05-30.](image-url)
Contents

1 Introduction
2 The ARG₀ process
   • A mixture of affine distributions
   • Properties and extensions
3 The NATSM
   • Short-rate specification and the affine framework
   • Advantages of an affine framework
4 Estimation
   • State-space formulation
   • Estimation results
5 Assessing lift-off dates
6 Conclusion
7 Appendix
Summary and further research

We have derived **affine non-negative processes staying at 0** and built an affine term-structure model (**NATSM**) gathering:

- a **short-rate consistent with the ZLB** experiencing periods at 0 while **long-run rates still fluctuates**;
- **closed-form formulas** for bond-pricing and lift-off probabilities.

An empirical assessment showed performance of our model for:

- **fitting yield levels and conditional variances**;
- calculating risk-neutral *and* historical **lift-off probabilities**.

**Further research:** Empirical comparison of NATSMs, derivatives pricing.
Thank you for your attention.
Contents

1 Introduction
2 The ARG₀ process
   • A mixture of affine distributions
   • Properties and extensions
3 The NATSM
   • Short-rate specification and the affine framework
   • Advantages of an affine framework
4 Estimation
   • State-space formulation
   • Estimation results
5 Assessing lift-off dates
6 Conclusion
7 Appendix
The loadings recursions are given by:

\[ R_t(h) = -\frac{1}{h}(A_h'X_t + B_h) \]

\[ A_h = -\delta + \beta^Q \left( \frac{A_{h-1} \odot \mu^Q}{1 - A_{h-1} \odot \mu^Q} \right) \]

\[ B_h = B_{h-1} + \alpha^Q \left( \frac{A_{h-1} \odot \mu^Q}{1 - A_{h-1} \odot \mu^Q} \right) \]

\( \odot \) is the element-wise product.
The historical dynamics

- The SDF is exp-affine with market price of risk vector $\theta$:

$$\frac{dP_{t,t+1}}{dQ_{t,t+1}} = \exp \left[ \theta' X_{t+1} - \psi^Q_t(\theta) \right]$$

Change of measure property

$X_t$ follows a VARG$_\nu(\alpha^P, \beta^P, \mu^P)$ process under the historical measure $P$.

$$\alpha^P_j = \frac{\alpha^Q_j}{1 - \theta_j \mu^Q_j}, \quad \beta^P_j = \frac{1}{1 - \theta_j \mu^Q_j} \beta^Q_j, \quad \mu^P_j = \frac{\mu^Q_j}{1 - \theta_j \mu^Q_j}.$$  

Rk: $\nu$ is the same under both measures.
Table: Parameter estimates

<table>
<thead>
<tr>
<th></th>
<th>P-parameters</th>
<th></th>
<th>Q-parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimates</td>
<td>Std.</td>
<td>Estimates</td>
<td>Std.</td>
</tr>
<tr>
<td>( \alpha_4 )</td>
<td>3.2455</td>
<td>0.1118</td>
<td>3.2347</td>
<td>0.1113</td>
</tr>
<tr>
<td>( \beta_{1,1} )</td>
<td>0.9663</td>
<td>0.0078</td>
<td>0.9794</td>
<td>0.0042</td>
</tr>
<tr>
<td>( \beta_{2,2} )</td>
<td>0.9978</td>
<td>0.0005</td>
<td>0.9957</td>
<td>0.0006</td>
</tr>
<tr>
<td>( \beta_{3,3} )</td>
<td>0.9486</td>
<td>0.0044</td>
<td>0.9705</td>
<td>0.0023</td>
</tr>
<tr>
<td>( \beta_{4,4} )</td>
<td>0.9967</td>
<td>0.0005</td>
<td>0.9933</td>
<td>0.0003</td>
</tr>
<tr>
<td>( \beta_{2,1} )</td>
<td>0.0308</td>
<td>0.0041</td>
<td>0.0308</td>
<td>0.0041</td>
</tr>
<tr>
<td>( \beta_{3,2} )</td>
<td>0.1094</td>
<td>0.0059</td>
<td>0.1120</td>
<td>0.0061</td>
</tr>
<tr>
<td>( \beta_{4,3} )</td>
<td>(3.88 \cdot 10^{-4})</td>
<td>(2.28 \cdot 10^{-5})</td>
<td>(3.87 \cdot 10^{-4})</td>
<td>(2.27 \cdot 10^{-5})</td>
</tr>
<tr>
<td>( \mu_1 )</td>
<td>1</td>
<td>–</td>
<td>1.0135</td>
<td>0.0040</td>
</tr>
<tr>
<td>( \mu_2 )</td>
<td>1</td>
<td>–</td>
<td>0.9980</td>
<td>0.0005</td>
</tr>
<tr>
<td>( \mu_3 )</td>
<td>1</td>
<td>–</td>
<td>1.0231</td>
<td>0.0023</td>
</tr>
<tr>
<td>( \mu_4 )</td>
<td>1</td>
<td>–</td>
<td>0.9967</td>
<td>0.0003</td>
</tr>
</tbody>
</table>

Other Parameters

| \( \delta_1 \)        | 0.0030       | 0.0003                    |
| \( \theta_1 \)        | -0.0133      | 0.0039                    | \( \theta_2 \) | 0.0020       | 0.0005                    |
| \( \theta_3 \)        | -0.0226      | 0.0022                    | \( \theta_4 \) | 0.0033       | 0.0003                    |
| \( \sigma_R \)        | 0.0407       | 0.0003                    |
| \( \sigma_V \)        | \(3 \cdot 10^{-3}\) | –                         | \( \sigma_S \) | 0.15         | –                         |
Univariate case: lift-offs formulas

- $Z \in \mathbb{R}^+$ and $\varphi_Z(u)$ its Laplace transform.

  \[ P_Z\{0\} = \lim_{u \to -\infty} \varphi_Z(u). \]

- Lift-off probabilities: $(X_t) \sim \text{ARG}_0(\alpha, \beta, \mu)$ and $\varphi_{t,h}(u_1, \ldots, u_h)$ its multi-horizon conditional Laplace transform.

  - $P(X_{t+h} = 0 \mid X_t) = \lim_{u \to -\infty} \varphi_{t,h}(0, \ldots, 0, u)$
  - $P[X_{t+1} = 0, \ldots, X_{t+h} = 0 \mid X_t] = \lim_{u \to -\infty} \varphi_{t,h}(u, \ldots, u)$
    \[ = \exp(-\alpha h - \beta X_t), \]
  - $P[X_{t+1} = 0, \ldots, X_{t+h} = 0, X_{t+h+1} > 0 \mid X_t]$
    \[ = \exp[-\alpha h - \beta X_t] \left[ 1 - \exp(-\alpha) \right], \quad h > 1. \]
Multivariate Case

- \( Z \in \mathbb{R}_+^n \) and \( \varphi_Z(u_1, \ldots, u_n) \) its Laplace transform.

\[
P_Z\{0, \ldots, 0\} = \lim_{u \to -\infty} \varphi_Z(u, \ldots, u).
\]

- **Notations:** Multi-horizon conditional LT.

\[
\varphi^P_{t,k}(u_1, \ldots, u_k) = \mathbb{E}^P \left[ \exp \left( u_1' X_{t+1} + \ldots + u_k' X_{t+k} \right) \mid X_t \right] = \exp \left[ \mathcal{A}_k' X_t + \mathcal{B}_k \right]
\]

\[
\varphi^{(h)}^P_{R,t,k}(v_1, \ldots, v_k) = \mathbb{E} \left[ \exp \left( v_1 R_{t+1}(h) + \ldots + v_k R_{t+k}(h) \right) \mid X_t \right]
\]
Lift-offs

\[ \mathbb{P} [ r_{t+k} = 0 \mid X_t ] = \lim_{u \to -\infty} \varphi^{(1)}_{R,t,k}(0, \ldots, 0, u) \]

\[ \mathbb{P} [ r_{t+1} = 0, \ldots, r_{t+k} = 0 \mid X_t ] = \lim_{u \to -\infty} \varphi^{(1)}_{R,t,k}(u, \ldots, u) = p_{r,t,k} \text{ (say)} \]

\[ \mathbb{P} [ r_{t+1} = 0, \ldots, r_{t+k-1} = 0, r_{t+k} > 0 \mid X_t ] = p_{r,t,k-1} - p_{r,t,k} \]

\[ \mathbb{P} \left[ \mathbf{v}' R^{(t+k+1)}_{t+1} (h) > \lambda \mid X_t \right] = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \text{Im} \left[ \varphi^{(h)}_{R,t,k} (i \mathbf{v} x) \exp(-i \lambda x) \right] \frac{dx}{x} \]