Robust Economic Implications of Nonlinear Pricing Kernels

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Abstract

We derive new bounds and distance measures for stochastic discount factors (SDFs) that generalize the original variance bounds and distance of Hansen and Jagannathan (1991, 1997) and higher moment bounds of Snow (1991). These generalized measures are suitable to analyze nonlinearities in asset pricing models and trading strategies. They imply nonlinear admissible SDFs that provide a more robust discounting than the positive linear SDFs used in many asset pricing empirical applications. We illustrate the empirical usefulness of these new discount factors by revisiting the admissibility of consumption-based asset pricing models, examining the information structure embedded in industry and Fama and French portfolios, and evaluating the performance of hedge funds.

Keywords: Stochastic Discount Factors, Information-Theoretic Bounds, Robustness, Minimum Contrast Estimators, Implicit Utility Maximizing Weights.

JEL Classification: C1,C5,G1

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1 Introduction

Observing asset returns should provide information about how future cash flows are discounted. This is the fundamental insight of Hansen and Jagannathan (HJ, 1991), who derived a minimum variance stochastic discount factor (SDF) bound. This SDF variance bound proved extremely useful for diagnosing asset pricing models, return predictability, variance spanning tests, and performance measurement. The SDF is obtained by a linear projection of the observed payoffs of a set of basis securities, with the goal of pricing exactly these assets. In Hansen and Jagannathan (1997), they extend the approach to measure the distance of misspecified asset pricing models with respect to the family of such SDFs that correctly price the basis assets. In both papers, they measure distance with a quadratic norm that involves only the first two moments of observed payoffs or returns.

While very useful, pricing kernels obtained by linear projections may not be informative enough to diagnose asset pricing models. Snow (1991) proposed SDF bounds that go beyond the HJ quadratic norm and integrate information about moments of returns higher than the variance. Snow (1991) derives bounds on the δ^{th} norm of a SDF m and shows that setting a bound on the δ^{th} moment of m can be done by creating an option on a portfolio of the payoffs x of the primitive assets such that the option payoff z raised to the $(\rho - 1)$ power correctly prices the asset payoffs: $E[z^{\rho-1}x] = E[q]$, with $\frac{1}{\delta} + \frac{1}{\rho} = 1$. Note that this imposes a positivity constraint on m. The methodology used by Snow (1991) imposes a further constraint on the norm $(\delta > 1)$.[1] These two constraints taken together often introduce pricing errors in the primitive assets, which is at odds with the original idea of HJ to derive the SDF bounds. Our first contribution is to propose a family of positive SDFs that correctly prices the primitive assets and that incorporates information about moments of returns higher than the variance.

Given a set of basis assets payoffs, we minimize general convex functions of SDFs called Minimum Discrepancy (MD) measures (Corcoran, 1998) in order to obtain a projected nonlinear SDF that prices exactly the selected basis assets[2]. We derive new bounds for the SDF, called information bounds, that naturally extend the original HJ variance bounds and the extended bounds proposed by Snow (1991). A well-known example of such discrepancy measures is the Kullback-Leibler information criterion (KLIC). Stutzer (1995) diagnosed

[1] The $\delta > 1$ constraint comes from the Holder inequality methodology used by Snow (1991) to solve the problem. Bounds on the δ^{th} norm of m $inf E[m^\delta]^{\frac{1}{\delta}}$ are formed using the ρ^{th} moments of asset returns $E(\pi(p)) \leq E[m^\delta]^{\frac{1}{\delta}} E[p^{+\rho}]^{\frac{1}{\rho}}$, with $\frac{1}{\delta} + \frac{1}{\rho} = 1$, with $\delta > 1$.

[2] Minimum Discrepancy measures of the Cressie Read (1984) family have been recently adopted in the econometric literature as one step alternatives to Generalized Method of Moments (GMM) estimators (see Newey and Smith (2004), Kitamura (2006))

asset pricing models with information functions implied by this criterion[3]. We choose a family of discrepancy functions that admits as particular cases the HJ (1991) quadratic criterion and the KLIC, but offers other information criteria that have different implications for diagnosing models and assessing their pricing properties.

The fact that the primitive assets are not well priced raises an important issue about the validity of the obtained bounds. For example, take the HJ variance bound with positivity constraint. If the implied SDF does not price well the primitive assets this bound will be below the minimum variance HJ bound, obtained without the positivity constraint. Our methodology provides a way to obtain a positive SDF with a variance bound that is closest to the minimum variance HJ bound but with the advantage that it prices the basis assets without error.

The solutions for these SDFs are obtained through dual problems that are easier to solve than the primal problems and offer a nice economic interpretation. Each primal minimum discrepancy problem corresponds to a dual optimal portfolio problem, with the maximization of a specific utility function in the Hyperbolic Absolute Risk Aversion (HARA) family. This duality has been stressed in HJ (1991) where maximizing the Sharpe ratio in the space of excess returns corresponds to finding a minimum variance in the space of SDFs[4]. The first-order conditions for these HARA optimization problems imply SDFs that are nonlinear and positive, directly generalizing the linear SDF in HJ (1991) with positivity constraints[5]. It also captures Snow (1991), who focused his analysis on specific moments on the space of SDFs, as a particular case. Our approach considers combinations of moments of SDFs.

Our implied nonlinear SDFs are related to a number of previous studies that feature nonlinear SDFs. Bansal and Viswanathan (1993) propose a neural network approach to construct a nonlinear stochastic discount factor that embeds specifications by Kraus and Litzenberger (1976) and Glosten and Jagannathan (1994). Our approach provides a family of SDFs given by different hyperbolic functions of basis assets returns implied by portfolio problems. In Dittmar (2002), who also analyzes nonlinear pricing kernels, preferences re-

[3]Stutzer (1996) used the same information theoretical approach based on the entropy measure to extract canonical probabilities, that is, risk-neutral probabilities that price consistently a set of options using as basis assets the underlying asset or adding to it other traded options. Our methodology naturally generalizes his approach to discrepancy measures other than the entropy measure.

[4]Our approach encompasses the exponential tilting (ET) criterion of Stutzer (1995) and its corresponding optimum portfolio of a CARA investor, as well as the empirical likelihood (EL) criterion and its corresponding log utility maximizing portfolio, denominated growth portfolio by Bansal and Lehmann (1997).

[5]HJ (1991) consider two variance bounds, the first for the family of all admissible SDFs, the second restricting that family to consider only strictly positive SDFs. We generalize the second HJ problem because we are interested in implications compatible with arbitrage-free economies.

strict the definition of the pricing kernel. Under the assumption of decreasing absolute risk aversion, he finds that a cubic pricing kernel is able to best describe a cross-section of industry portfolios. Our nonparametric approach embeds such cubic nonlinearities implicitly. Although not based on preferences, our pricing kernels are also consistent with dual HARA utility functions that can exhibit decreasing absolute risk aversion and decreasing absolute prudence[6]. Boyle et al. (2008) obtain robust prices for derivative securities based on SDFs that cause minimum perturbations on prices of derivatives payoffs. Our methodology, if used to price derivatives will provide pricing intervals based on the SDFs implied by our chosen family of discrepancies.

From an econometric point of view, our implied SDFs are closely related to the so called implied probabilities that appear in MD estimation problems. They are a set of probabilities optimally chosen (on a discrepancy sense) such that they reweight the sampling empirical probabilities in order to make the estimated model to satisfy a set of imposed moment conditions. While Back and Brown (1993) derived implied probabilities for GMM estimators, Owen (1988) provide implied probabilities for EL, Kitamura and Stutzer (1997) for ET, Imbens et al. (1998) for Cressie Read, and Brown and Newey (2002) for Generalized Empirical Likelihood (GEL) estimators. Smith (2004) shows how those probabilities can be used to obtain efficient moment estimation as in Brown and Newey (1998). Our SDFs represent a novel contribution to this literature since they can be seen as a scaled nonparametric version of GEL implied probabilities constrained to be strictly positive under the whole Cressie Read family[7].

A significant literature aims at sharpening the variance bounds by conditioning on information available to economic agents. Gallant, Hansen and Tauchen (1990) derive an optimal variance bound when the first two conditional moments are known, while Bekaert and Liu (2004) propose an optimally-scaled bound which is valid even when the first and second conditional moments are misspecified. Chabi-Yo (2008) introduces higher moments of returns in volatility bounds by finding the SDFs that are linear functions of payoffs and squared payoffs (volatility contracts), linking these bounds to skewness and kurtosis

[6]Dittmar (2002) starts with an approximation of an unknown marginal utility function by a Taylor series expansion but restricts the polynomial terms in the expansion by imposing decreasing absolute prudence (Kimball, 1993). Therefore, the risk factor obtains endogenously from preference assumptions and is a sole function of aggregate wealth. Our SDFs come from solutions to dual optimal HARA portfolio problems that endogenously determine aggregate wealth as a linear combination of a predetermined set of basis assets. These solutions potentially satisfy the desirable properties of decreasing absolute risk aversion (Arditti, 1967) and decreasing absolute prudence.

[7]For the particular case of Euclidean Likelihood estimators, Antoine, Bonnal and Renault (2007) propose an alternative simple shrinkage of implied probabilities towards sampling empirical probabilities to guarantee that they become positive.

of returns[8]. In contrast, the discrepancy measures we propose in this paper put weights on all moments of the distribution of returns in the dual optimization problem. Moreover, by considering a family of discrepancy measures, we add robustness to our diagnosis since each discrepancy puts different weights on the moments of returns. Another strand of literature uses conditioning information to evaluate the performance of managed portfolios. A representative example is the study by Farnsworth et al. (2002). They consider several parametric models, both linear and nonlinear, to measure the investment performance of fund managers. We extend the literature on conditional performance measurement by producing conditional measures that take into account all conditional moments of the SDFs.

Conditional approaches have the potential advantage of having thinner-tailed conditional distributions that control better the effect of extreme observations on the moments of the asset returns. However, our generalized discrepancy measures, even taken unconditionally, are better able to capture the effect of these extreme observations because they account for higher moments in the unconditional distribution of returns. This is especially important when evaluating performance of managed portfolios since private information on which fund managers condition their trades is unobservable. In this case only the potentially fatter-tailed unconditional returns are observable. Our unconditional SDFs will account for the effect of this unobservable information.

We propose three empirical applications to illustrate the usefulness of our approach. The first involves diagnosing asset pricing models. We revisit the admissibility of consumption-based asset pricing models (CCAPMs). We start by showing that for a large number of information bounds, the traditional CCAPM with power utility only becomes admissible with high risk-aversion coefficients (of the order of 30), reinforcing the classical results of HJ[9].

We also analyze the polynomial version of Chapman's (1997) CCAPM considering two different sets of basis assets returns. First, we adopt a traditional set of basis assets (the one-year Treasury bill and the S&P 500) and conclude that the Chapman's model lies in the feasible region for a large number of discrepancies, especially when larger weights are attributed to higher moments of returns. However, it is inadmissible for discrepancies in the neighborhood of the KLIC criterion[10]. Then, we calibrate Chapman's SDF taking as basis

[8]Balduzzi and Kallal (1997) add risk variables in addition to asset returns and obtain more stringent bounds than the HJ bounds. Kan and Zhou (2006) tighten the HJ bound by assuming that the pricing kernel is a reduced-form function of a finite set of state variables.

[9]The HJ bound showed clearly the inadmissibility of the basic CCAPM model with power utility for reasonably low values of the risk aversion parameter, but more elaborate versions with non-separable utility (Heaton, 1995), incomplete markets (Constantinides and Duffie, 1996) or polynomial pricing kernels (Chapman, 1997) lied inside the feasible region defined by the bound.

[10]In this paper, we do not provide statistical tests based on the probability distribution of the information

assets the returns on the three Fama and French (FF) factors. The HJ SDF with positivity constraint implies large pricing errors for the FF factors. It means that the diagnosis about the admissibility of Chapman's model will be tainted. Our bounds identify nonlinear SDFs that perfectly price the Fama and French factors and imply higher volatilities than the HJ SDF with positivity constraint, which imposes a more stringent test on the admissibility of the Chapman's model.

In a second application, we examine the information structure of industry portfolios and Fama and French portfolios through the lens of our family of nonlinear SDFs. Lewellen, Nagel, and Shanken (2009) suggest that industry portfolios should be used in addition to Fama and French portfolios to make more robust inference about asset pricing models. We generate information frontiers based on these two sets of assets. For different discrepancies in our family, industry portfolios consistently bring non-redundant information over and above Fama and French portfolios under different ranges of SDF means. In particular, for minimum-discrepancy estimators putting more weight on higher moments of returns, industry portfolios generate frontiers above the frontiers generated by Fama and French portfolios for an important range of SDF means.

In the last application, we evaluate the performance of hedge funds with the nonlinear SDFs implied by the minimum-discrepancy estimators[11]. Hedge funds use dynamic trading strategies that produce returns exhibiting nonlinear patterns. To capture nonlinearities and measure the alpha performance of the funds, Agarwal and Naik (2004) use a linear regression in which they introduce the returns on a portfolio of options along with the other usual risk factors.[12] Diez and Garcia (2009) estimate and test the presence of option-like nonlinearities in hedge fund returns and determine whether hedge funds provide value to investors. We exploit the generality of our family of nonlinear SDFs to evaluate the performance of hedge funds and compare it to these two approaches. Our analysis accounts explicitly for higher moments of returns induced by option-like strategies. Moreover, an important feature of our discrepancy-based approach is the possibility to capture more complex nonlinearities since options portfolios can be included as basis assets. We find that the various SDFs almost always agree on the performance of the best and worst fund indices,

bounds. For the asymptotic distribution of our information bounds see Almeida and Garcia (2010); for the asymptotic distribution of the HJ bounds see Hansen, Heaton and Luttmer (1995), and Kan and Robotti (2007) for finite sample properties.

[11]The stochastic discount factor approach has been widely used to measure the performance of managed portfolios. With this approach, abnormal performance is measured by the expected product of a portfolio's returns and a stochastic discount factor. The evaluation can proceed unconditionally or conditionally to a set of lagged instruments. See for instance, Ferson and Siegel (2003), Farnsworth et at. (2002), and Bailey, Li and Zhang (2004).

[12]See also Mitchell and Pulvino (2001) and Fung and Hsieh (2001) for characterizing hedge fund returns as returns from option-based trading strategies.

but that they differ in the ranking of the intermediate ones. Alpha valuations obtained with the implied nonlinear SDFs are not always in line with performances exhibited when introducing option factors linearly.

Our multiplicity of discrepancy measures allows us to conduct a robustness analysis on the performance of hedge funds. In particular we search for a value of the parameter indexing our family of discrepancy measures that produces a zero performance of the fund, if any. In another robustness exercise, we use an estimator that averages across a range of HARA functions, and solves the portfolio problem of this averaging function to obtain the corresponding nonlinear SDF. Therefore, by combining functions, we extend the original family, since a linear combination of HARA functions is not a HARA function.

The rest of the paper is organized as follows. In section 2, we describe how the minimum discrepancy SDFs are derived in a variety of contexts. Section 2.2 makes explicit the corresponding dual optimal portfolio problems. Section 2.4 extends the analysis to conditional settings. Section 2.5 considers two extensions of our nonparametric methodology: Minimum-discrepancey estimators of parametric models and assessment of model misspecifications in the spirit of Hansen and Jagannathan (1997). In Section 3, we present three empirical applications. First, we illustrate how the minimum discrepancy SDFs can be used to provide robust diagnostics of asset pricing models. Then, we explore the differences between various minimum discrepancy bounds to identify information contents in Fama-French and industry portfolios. In the last empirical example, we analyze the problem of performance evaluation of hedge funds. We describe our nonparametric approach to performance measurement and contrast it with the parametric approaches used in the literature. Section 4 concludes.

2 Minimum Discrepancy Stochastic Discount Factors

We adopt the same setting as Hansen and Jagannathan (1991). Let R denote the vector of returns of basis assets whose realizations are given by a time series $\{R_i\}_{i=1,\dots,T}$ in a K-dimensional space. First, we are looking for admissible SDFs that are functions of these returns.[13] We make use of the definition of an admissible SDF as a set of moment conditions, the Euler equations. For an arbitrary admissible SDF m the Euler equation holds for all K basis assets returns:

$$E(mR) = 1_K, \tag{1}$$

[13] We are not making a complete markets assumption, in the sense that there exists an infinite number of SDFs (or pricing kernels) that are admissible with respect to that set of basis assets (Duffie, 2001).

where 1_K represents a K-dimensional vector of ones. On its sample form, for a SDF with mean a, the Euler equations become:

$$\frac{1}{T}\sum_{i=1}^{T} m_i \left(R_i - \frac{1}{a}1_K\right) = 0_K. \tag{2}$$

where 0_K is a K-dimensional vector of zeros.

The next step is to restrict the set of admissible SDFs. Hansen and Jagannathan (1991) find an admissible linear SDF with minimum variance, obtained by minimizing a quadratic function in the space of admissible SDFs. Instead, given a convex and homogeneous discrepancy function ϕ, we search for a Minimum Discrepancy (MD) SDF that solves the following minimization problem in the same space of admissible SDFs:

$$\begin{gathered}\hat{m}_{MD} = \underset{\{m_1,\dots m_T\}}{arg\ min}\ \tfrac{1}{T}\textstyle\sum_{i=1}^{T}\phi(m_i),\\ \text{subject to } \tfrac{1}{T}\textstyle\sum_{i=1}^{T} m_i\left(R_i - \tfrac{1}{a}1_K\right) = 0_K,\ \tfrac{1}{T}\textstyle\sum_{i=1}^{T} m_i = a, m_i > 0\ \forall i.\end{gathered} \tag{3}$$

In this optimization problem, restrictions to the space of admissible SDFs come directly from the general discrepancy function ϕ[14]. The conditions $\sum_{i=1}^{T} m_i\left(R_i - \frac{1}{a}1_K\right) = 0_K$ and $\frac{1}{T}\sum_{i=1}^{T} m_i = a$ must be obeyed by any admissible SDF m with mean a. In addition, we explicitly impose a positivity constraint to guarantee that the implied MD SDF is compatible with absence of arbitrage in an extended economy containing derivatives of basis assets (Chen and Knez, 1996).

This minimization problem is based on the space of discrete SDFs with dimension T (the dimension of the sample of data), which can become impractical. According to Borwein and Lewis (1991), the minimization problem can be solved in a generally much smaller dimensional space by using the following dual problem:

$$\hat{\lambda} = \underset{\alpha\in\Re,\lambda\in\Lambda}{arg\ sup}\ a*\alpha - \sum_{i=1}^{T}\frac{1}{T}\phi^{*,+}\left(\alpha + \lambda'\left(R_i - \frac{1}{a}1_K\right)\right), \tag{4}$$

where $\Lambda \subseteq R^K$ and $\phi^{*,+}$ denotes the convex conjugate of ϕ restricted to the positive real line:

$$\phi^{*+}(z) = \underset{w>0}{sup}\ zw - \phi(w). \tag{5}$$

Note that any convex discrepancy function can be chosen to arrive at empirical estimates of these minimum discrepancy SDFs. We choose the Cressie-Read (1984) family of

[14]Restrictions on the space of SDFs other than those imposed by variance have been analyzed by Snow (1991), Stutzer (1995), Bansal and Lehmann (1997), Bernardo and Ledoit (2000), and Cerny (2003). Our approach contains all these papers as particular cases.

discrepancies defined as:

$$\phi^{\gamma}(m) = \frac{(m)^{\gamma+1} - a^{\gamma+1}}{\gamma(\gamma+1)}, \gamma \in \Re, \tag{6}$$

where each fixed value of γ implies one specific discrepancy function. This family embeds as particular cases the SDFs derived by HJ (1991), Snow (1991), Stutzer (1995), Bansal and Lehmann (1997) and Cerny (2003). In addition, it offers a nice economic motivation to our information theoretic minimization problems since they are equivalent to dual HARA utility maximization problems.[15]

Before we make explicit this equivalence, we show in the next sub-section how to solve the dual problem and recover the restricted admissible SDF. We then derive SDF frontiers to parallel the minimum-variance frontier of HJ (1991). We conclude this section with several extensions to the basic methodology.

2.1 Finding the Optimal Admissible Minimum-Discrepancy (MD) SDF

In the following theorem, we provide the solution method to obtain an admissible SDF that obeys the restrictions imposed by the discrepancy function. After solving the dual problem, we recover the MD SDF by using the first-order conditions of this maximization problem.

Theorem 1. *Let the discrepancy function belong to the class of Cressie Read functions: $\phi(m) = \frac{m^{\gamma+1} - a^{\gamma+1}}{\gamma(\gamma+1)}$ with $\gamma \in \Re$. In this case, the optimization problem in equation (4) specializes to:*

$$\hat{\lambda_{\gamma}} = \underset{\lambda \in \Lambda_{CR}}{arg\ sup} \frac{1}{T} \sum_{i=1}^{T} \left(\frac{a^{\gamma+1}}{\gamma+1} - \frac{1}{\gamma+1} \left(a^{\gamma} + \gamma\lambda' \left(R_i - \frac{1}{a} 1_K \right) \right)^{\left(\frac{\gamma+1}{\gamma}\right)} \right), \tag{7}$$

where $\Lambda_{CR} = \{\lambda \in R^K,\ s.t.\ for\ i = 1, ..., T : \left(1 + \gamma\lambda' \left(R_i - \frac{1}{a} 1_K\right)\right) > 0\}$.

Proof: See Appendix A.

For each choice of γ we obtain a distinct set of estimates for λ ($\hat{\lambda_{\gamma}}$) that will lead to a different MD stochastic discount factor ($\hat{m}^{\gamma}_{MD}$). The MD SDF $\hat{m}^{\gamma}_{MD}$ is recovered via the first-order conditions of Equation (7) with respect to λ, evaluated at the optimal Lagrange Multipliers $\hat{\lambda}$ that solve (7):

$$\frac{1}{T} \sum_{i=1}^{T} \left(a^{\gamma} + \gamma \hat{\lambda}'_{\gamma} \left(R_i - \frac{1}{a} 1_K \right) \right)^{\frac{1}{\gamma}} \left(R_i - \frac{1}{a} 1_K \right) = 0_K \tag{8}$$

By comparing Equation (8) to Equation (3) we obtain the MD SDF:

[15] In section 2.5, we will further support the choice of this family on econometric grounds.

$$\hat{m}^i_{MD} = a \frac{\left(a^\gamma + \gamma \hat{\lambda}'_\gamma \left(R_i - \frac{1}{a} 1_K\right)\right)^{\frac{1}{\gamma}}}{\frac{1}{T} \sum_{i=1}^{T} \left(a^\gamma + \gamma \hat{\lambda}'_\gamma \left(R_i - \frac{1}{a} 1_K\right)\right)^{\frac{1}{\gamma}}}, i = 1, ..., T. \tag{9}$$

Note that the population form for the SDF solving the MD bound problem will be a hyperbolic function of the original returns R:

$$\hat{m}_{MD}(R) = a \frac{\left(a^\gamma + \gamma \hat{\lambda}'_\gamma \left(R - \frac{1}{a} 1_K\right)\right)^{\frac{1}{\gamma}}}{E\left[\left(a^\gamma + \gamma \hat{\lambda}'_\gamma \left(R - \frac{1}{a} 1_K\right)\right)^{\frac{1}{\gamma}}\right]}. \tag{10}$$

2.2 Interpretation as an Optimal Portfolio Problem

Problem (7) has an interesting economic interpretation as an optimal portfolio problem. The solution for the MD bound for each Cressie Read estimator will correspond to an optimal portfolio problem based on the following HARA-type utility function

$$u(W) = -\frac{1}{\gamma + 1} \left(a^\gamma - \gamma W\right)^{\left(\frac{\gamma+1}{\gamma}\right)}, \tag{11}$$

with $a > 0$ and W such that $a^\gamma - \gamma W > 0$, which guarantees that function u is well defined for an arbitrary γ, is concave and strictly increasing.

Specific values of γ will specialize the optimal portfolio problems to widely adopted utility functions. A value of -1 will correspond to a logarithmic utility function, 0 to the exponential, and 1 to quadratic utility[16]. The corresponding SDFs will be:

$$\hat{m}_{\{\gamma=-1\}}(R) = \mu * \frac{1}{\left(\frac{1}{a} - \hat{\lambda}' \left(R - \frac{1}{a} 1_K\right)\right)}; \tag{12}$$

$$\hat{m}_{\{\gamma=0\}}(R) = \mu * e^{\hat{\lambda}' \left(R - \frac{1}{a} 1_K\right)}; \tag{13}$$

$$\hat{m}_{\{\gamma=1\}}(R) = \mu * \left(a + \hat{\lambda}' \left(R - \frac{1}{a} 1_K\right)\right). \tag{14}$$

where, in each case, μ is such that the SDF mean equals a.

Stutzer (1995) proposed an interpretation for the exponential case based on a standard two-period model of optimal portfolio choices (see Huang and Litzenberger (1988)). We extend this interpretation to the whole Cressie-Read family. Suppose an investor distributes

[16] For $\gamma = 1$, similarly to HJ, the optimal SDF will be a linear function of excess returns. Note that the Lagrange Multipliers $\lambda_{\gamma=1}$ are restricted via the optimization domain (Λ_{CR}) to guarantee a positive SDF, exactly as in HJ with positivity constraint.

his/her initial wealth W_0 putting λ_j units of wealth on the risky asset R_j and the remaining $W_0 - \sum_{j=1}^{K} \lambda_j$ in a risk-free asset paying $r_f = \frac{1}{a}$. Terminal wealth is then $W = W_0 * r_f + \sum_{j=1}^{K} \lambda_j * (R_j - r_f)$. Assume in addition that this investor maximizes the HARA utility function provided above in equation (11), solving the following optimal portfolio problem:

$$\Omega = \underset{\lambda \in \Lambda}{sup} E\left(u(W)\right), \tag{15}$$

where $\Lambda = \{\lambda : 1 - \gamma W(\lambda) > 0\}$. Note that by scaling the original vector λ to be $\tilde{\lambda} = \frac{-\lambda}{\left(a^{\gamma} - \frac{\gamma W_0}{a}\right)}$, we can decompose the utility function in $u(W) = u(W_0 * r_f) * \left(a^{\gamma} + \gamma \tilde{\lambda}\left(R - \frac{1}{a} 1_K\right)\right)^{\left(\frac{\gamma+1}{\gamma}\right)}$. This decomposition essentially shows that solving the optimality problem in (7) will measure the gain achieved when switching from a total allocation of wealth to the risk-free asset paying r_f to an optimal (in the utility u sense) diversified allocation that includes both risky assets and the risk-free asset.

2.3 Minimum Discrepancy SDF Frontier

To complete our characterization of MD SDFs, we provide an operational algorithm to obtain such variables when there is no risk-free asset in the space of returns. Similarly to HJ, the idea is to propose a grid of possible meaningful values for the SDF mean, say fixing a set $A = \{a_1, a_2, ..., a_J\}$, and to solve the optimization problem in (7) for each $a_l \in A$, obtaining a corresponding optimal weights vector $\hat{\lambda}_{\gamma}(a_l)$ for each SDF mean. The SDF frontier is given by the following expression:

$$I_P(a_l, \gamma) = \frac{a_l^{\gamma+1}}{1+\gamma} + \frac{1}{T} \sum_{i=1}^{T} -\frac{1}{\gamma+1} \left(a_l^{\gamma} + \gamma \hat{\lambda}_{\gamma}(a_l)' \left(R_i - \frac{1_K}{a_l}\right)\right)^{\left(\frac{\gamma+1}{\gamma}\right)}, l = 1, 2, ..., J. \tag{16}$$

Alternatively, we can go back to the basic definition of the bound as a minimum discrepancy problem, and write the solution by first obtaining the implied MD SDFs appearing in Equation (9) $\hat{m}^i_{MD,a_l}$, and substitute it in the sampled divergence function ϕ, obtaining the MD SDF frontier[17]:

$$I_{MD}(a_l, \gamma) = \frac{1}{T} \sum_{i=1}^{T} \frac{(\hat{m}^i_{MD,a_l})^{\gamma+1} - a_l^{\gamma+1}}{\gamma(\gamma+1)}, l = 1, 2, ..., J. \tag{17}$$

[17]Expressions in (16) and (17) should be equivalent, but this may not be the case in some empirical applications where part of the implied MD SDFs could potentially become negative, if not restricted to be positive. In such cases forcing a positive solution could generate different solutions for the primal and dual problems, with larger MD primal values. See Borwein and Lewis (1991) for some mathematical examples.

2.3.1 HJ with Positivity Constraint as a Particular Case

When we choose $\gamma = 1$ on the Cressie Read family, the discrepancy function becomes $\phi(m) = \frac{m^2 - a^2}{2}$, and we are solving the following MD SDF bound:

$$\begin{aligned} \hat{m}_{MD(\gamma=1)} &= \underset{\{m_1, \ldots m_T\}}{arg\ min} \frac{1}{T} \sum_{i=1}^{T} \frac{m_i^2 - a^2}{2}, \\ \text{subject to} & \frac{1}{T} \sum_{i=1}^{T} m_i \left(R_i - \frac{1}{a} 1_K \right) = 0,\ \frac{1}{T} \sum_{i=1}^{T} m_i = a, m_i > 0\ \forall i. \end{aligned} \tag{18}$$

for SDFs with a fixed mean value equal to a.

This equation represents, apart from a normalization factor of $\frac{1}{2}$, the HJ (1991) variance bound with a positivity constraint.

2.3.2 Snow (1991) Moment Specific Approach as a Particular Case

Snow (1991) solved moment specific problems of the type $\underset{m>0}{inf} E[m^{\delta}]^{\frac{1}{\delta}}$, for $\delta > 1$, where m is an admissible SDF. They correspond, apart from an affine transformation, to Cressie Read discrepancies where $\gamma > 0$. However, they do not include important cases like ET ($\gamma = 0$) and EL ($\gamma = -1$) whose discrepancies are respectively $\phi(m) = mln(m)$ and $\phi(m) = -ln(m)$. Moreover, Cressie Read discrepancies with $\gamma \leq 0$ include interesting cases of minimization of SDF moments with power smaller than one and also the special cases of discrepancies considering negative moments of SDFs whose Taylor expansions reveal combinations of SDF moments. As stressed before, in contrast to Snow (1991), our approach gives a robust treatment to diagnostics of models and trading strategies, and in addition deals with discrepancies that consider combinations of moments of SDFs that are economically motivated. As will be clear in the empirical examples, the implied SDFs from Snow (1991) in most cases present pricing errors for primitive assets, while SDFs that are implied by minimization of Cressie Read discrepancies with $\gamma \leq 0$ in general price primitive assets without pricing errors, perfectly satisfying the moment conditions implied by the Euler equations.

2.4 Conditioning Information

An important extension of our methodology considers conditioning information when solving the MD bounds and obtaining the corresponding implied SDFs. This can be naturally implemented by introducing conditional expectations on Equation (1)

$$E(m_{t+1} R_{t+1} | I_t) = 1_K, \tag{19}$$

where I_t is the information set at time t.

Bekaert and Liu (BL, 2004) discuss three types of variance bounds using conditioning information: the naive approach, the Gallant, Hansen and Tauchen (GHT, 1990) efficient conditioning bound, and their optimally scaled bound (OS). The three approaches use the concept of a scaled return $\tilde{R}_{t+1} = z_t' R_{t+1}$, where the scale is given by an I_t-measurable K-dimensional random variable z_t, with K being the dimension of the basis assets return vector. The idea is to assume that the conditional expectation in (19) could be substituted by an infinity of unconditional expectations of the type:

$$E(m_{t+1} z_t R_{t+1}) = 1_K' z_t. \tag{20}$$

The BL methodology consists in finding the instrument z_t that maximizes the HJ volatility bound for the scaled returns $\tilde{R}$. BL show that the OS bound is superior to the naive approach since it finds the optimal scaling instrument instead of testing all possible combinations of instrumental I_t-measurable variables. They also show that the OS bound coincides with the GHT bound when the first and second conditional moments of returns are known. Moreover, their approach delivers a valid bound even when a misspecification of the conditional moments invalidates the GHT bound. Due to the superiority of BL's approach we dedicate this section to deriving, from a theoretical point of view, optimally scaling Minimum Discrepancy bounds that consider all Cressie Read MD functions instead of a variance bound. The naive approach is obtained as a by-product of our calculations, being a particular case of the OS case.

Let the vector y_t represent the set of (Markovian) conditioning variables of the economy such that $I_t = \sigma(y_t)$, where $\sigma(.)$ represents the $\sigma-$algebra generated by (.). Given an instrument z_t, consider the one-dimensional scaled payoff space $P_{z_t} = \{\alpha z_t' R_{t+1}, \alpha \in \Re\}$. It should be clear that the infinitely many I_t-measurable instruments z_t define a family of scaled payoff spaces indexed by z_t. Note that for each member in the Cressie Read family (γ), there is a MD bound associated with each scaling vector z_t, which only depends on the unconditional moments of the scaled return $z_t R_{t+1}$. A small but important difference from the approach in Section 2 is that scaled returns $z_t R_{t+1}$have prices $1_K' z_t$ instead of 1. To deal with this case, we have to solve a slightly more general version of Equation (4) that deals with prices different from unity (see Borwein and Lewis (1991)):

$$\hat{\lambda} = \underset{\alpha \in \Re, \lambda \in \Lambda}{arg\ sup}\ a * \alpha + \sum_{t=1}^{T} \lambda 1_K' z_{t-1} - \frac{1}{T} \phi^{*,+} \left(\alpha + \lambda \left(z_{t-1}' R_t - \frac{1}{a} \right) \right), \tag{21}$$

where $\Lambda \subseteq R^K$. Note that this is a one-dimensional problem in the Lagrange Multiplier λ since the scaled return is a transformation from $\Re^k$ to $\Re$. Applying this equation to a member of the Cressie Read family $\phi(m) = \frac{m^{\gamma+1} - a^{\gamma+1}}{\gamma(\gamma+1)}$ we obtain:

$$\hat{\lambda}_\gamma(z) = \underset{\lambda \in \Lambda_{CR}}{arg\ sup} \frac{1}{T} \sum_{t=1}^{T} \left(\frac{a^{\gamma+1}}{\gamma+1} + \lambda 1_K' z_{t-1} - \frac{1}{\gamma+1} \left(a^\gamma + \gamma\lambda \left(z_{t-1}' R_t - \frac{1}{a} \right) \right)^{\left(\frac{\gamma+1}{\gamma}\right)} \right), \quad (22)$$

where $\Lambda_{CR} = \{\lambda \in R,\ \text{s.t. for } i = 1, ..., T : \left(1 + \gamma\lambda \left(z_{i-1}' R_i - \frac{1}{a}\right)\right) > 0\}$.

The first-order conditions of this problem with respect to λ indicate that the MD SDF for a fixed instrument z_t is:

$$\hat{m}_{MD}^t(z) = a \frac{\left(a^\gamma + \gamma\hat{\lambda}_\gamma(z)\left(z_{t-1} R_t - \frac{1}{a}\right)\right)^{\frac{1}{\gamma}}}{\frac{1}{T}\sum_{t=1}^{T}\left(a^\gamma + \gamma\hat{\lambda}_\gamma(z)\left(z_{t-1} R_t - \frac{1}{a}\right)\right)^{\frac{1}{\gamma}}}, t = 1, ..., T. \quad (23)$$

According to Equation (17), the corresponding MD naive bound is given by:

$$I_{MD}(a, \gamma, z_t) = \frac{1}{T}\sum_{t=1}^{T} \frac{(\hat{m}_{MD}^t(z))^{\gamma+1} - a^{\gamma+1}}{\gamma(\gamma+1)}. \quad (24)$$

We define then the optimally scaled MD bound to be:

$$I_{OSB}(a, \gamma) = \sup_{z_t} I_{MD}(a, \gamma, z_t) \quad (25)$$

Note that we are not able to explicitly derive the optimal instrument as BL since there are no analytical formulas for the Lagrange multiplier $\hat{\lambda}_\gamma(z)$ that depend on the particular instrument z_t.[18]

2.5 Extensions of the Basic Minimum-Discrepancy SDF Methodology

The problem of finding a minimum discrepancy SDF in section 2 can be cast in a more general setting where the moment conditions include a candidate parametric asset pricing model as follows: $E[g(z_i, \beta)] = 0, i = 1, ..., T$, where $\{z_i\}_{i=1,...,T}$ represents a stationary and ergodic series of random vectors, $g(., \beta)$ is a vector in R^m, and β is an unknown vector of parameters on a set $B \in R^K$. In such a generalized context, MD Estimators seek to estimate β by finding a set of probabilities π that will reweight the sample empirical probabilities $\frac{1}{T}$ to satisfy the moment conditions exactly (see Corcoran (1998)). The discrepancy between the empirical probabilities $\frac{1}{T}$ and π is measured through a convex divergence function ϕ.[19]

[18] However, if we decide to approximate the Lagrange multiplier by a parametric function of z, we could obtain an explicit solution for the optimal instrument. We leave this topic for future research.

[19] See Kitamura (2006) for the formulation of the population problem.

The parameter vector β and the probability measure π are estimated by:

$$\left\{\hat{\beta}_{MD}, \hat{\pi}_{MD}\right\} = \underset{\beta\in B, \{\pi_1,\ldots\pi_T\}}{arg\ min} \frac{1}{T}\sum_{i=1}^{T}\phi(T\pi_i), \text{ subj. to} \sum_{i=1}^{T}\pi_i g(z_i,\beta) = 0_K, \ \sum_{i=1}^{T}\pi_i = 1 \quad (26)$$

By solving this problem one essentially obtains a point estimate $\hat{\beta}_{MD}$ for the vector of parameters that satisfies the moment conditions and minimizes the discrepancy ϕ between the implied probabilities $\hat{\pi}_{MD}$ and the uniform weights $\frac{1}{T}$. Moreover, these implied probabilities (Back and Brown (1993)) can be used to obtain an efficient estimation of moment conditions and probability distribution functions via bootstrapping schemes (Brown and Newey (1998)).

Newey and Smith (2004) showed that under the Cressie Read (1984) family of discrepancies, given by $\phi(\pi) = \frac{(\pi)^{\gamma+1}-1}{\gamma(\gamma+1)}$, the solution of the optimization problem (26) can be obtained by looking at its dual version:[20]

$$\begin{aligned}\left\{\hat{\beta}_{GEL}, \hat{\lambda}_{GEL}\right\} &= \underset{\beta\in B}{arg\ min} \underset{\lambda\in\Lambda(\beta)}{sup} \tfrac{1}{T}\textstyle\sum_{i=1}^{T} -\tfrac{1}{\gamma+1}(1+\gamma\lambda' g(z_i,\beta))^{\left(\frac{\gamma+1}{\gamma}\right)} \\ &= \underset{\beta\in B}{arg\ min} \underset{\lambda\in\Lambda(\beta)}{sup} \textstyle\sum_{i=1}^{T} M(\lambda' g(z_i,\beta))\end{aligned} \quad (27)$$

Theorem 2.2 in Newey and Smith (2004) states that the solutions to problems (26) and (27) are coincident under the Cressie Read family, meaning that $\hat{\beta}_{MD} = \hat{\beta}_{GEL}$. Moreover, the theorem shows how to recover the implied probabilities $\hat{\pi}_{MD}$ via the first derivatives of the function M, evaluated at the optimal set of parameters $\hat{\beta}_{GEL}$ and Lagrange Multipliers $\hat{\lambda}_{GEL}$ that solve (27):

$$\hat{\pi}^i_{MD} = \frac{M_1(\hat{\lambda}'_{GEL} g(z_i, \hat{\beta}_{GEL}))}{\sum_{j=1}^{T} M_1(\hat{\lambda}'_{GEL} g(z_j, \hat{\beta}_{GEL}))} \quad (28)$$

with $M_1(v) = \frac{dM(v)}{dv}$.

The dual optimization problem for MD Cressie Read estimators defines the important class of Generalized Empirical Likelihood (GEL) estimators proposed by Smith (1997). This class encompasses a large set of one-step estimators that are alternative to the two-step GMM estimator, including EL (Owen (1988), Imbens (1997)), ET (Kitamura and Stutzer (1997)), and Continuously Updating Estimator (CUE, Hansen, Heaton, and Yaron (1996)), among others.

Our MD SDFs from Section 2 are obtained by making the parametric model to be a constant and noting that each SDF m with mean a can be associated with a corresponding

[20] In fact, any convex discrepancy function defines a Generalized Minimum Contrast estimator whose solution can be obtained through its dual portfolio-type problem (see Kitamura (2006)).

risk-neutral probability measure $\pi(m)$ (Cochrane, 2000), via a simple linear transformation $\pi(m) = \frac{m}{T*a}$, assuming that the risk-free rate is constant and equal to $\frac{1}{a}$, and that the discrete physical probabilities are equal to the empirical probability $\frac{1}{T}$. The risk neutral probabilities $\pi(m)$ are precisely the implied probabilities by Back and Brown (1993).

Instead of assuming that there exists a true parametric model, we can adopt the point of view of Hansen and Jagannathan (1997) and argue that all models are approximations and therefore are misspecified. Hansen and Jagannathan (1997) compare misspecified asset pricing models based on least-square projections on a family of admissible stochastic discount factors. Almeida and Garcia (2010) extend their fundamental contribution by considering MD projections where misspecification is measured by convex functions that can explicitly take into account combinations of moments of asset returns.

Given a proxy asset pricing model y, and a convex discrepancy function ϕ, the idea posed by the MD problem is to find a positive admissible SDF which is as close as possible to y in the ϕ discrepancy sense:

$$\delta_{MD}(\theta) = \min_{m \in L^2} E\{\phi(1 + m - y(\theta))\} \text{ subject to } E(mx) = q, \;\; , m > 0. \tag{29}$$

This problem should be of interest when the underlying primitive securities include assets with non-Gaussian returns.

These two extensions complete the generalization of the approach developed by Hansen and Jagannathan (1991, 1997) with a quadratic objective function.

3 Empirical Applications of the Minimum-Discrepancy SDF

We have explained how to generalize HJ volatility bounds to information bounds accounting for higher moments of the return distribution of basis assets. We now need to show that this generalization matters in important ways in the main financial applications of the HJ bounds. Among the most prominent ones is the use of the bounds to verify admissibility of asset pricing models. Therefore, as a motivating example, we start by diagnosing consumption-based asset pricing models, first diagnosing the canonical CCAPM as in Hansen and Jagannathan (1991), and then examining the model of Chapman (1997) with a polynomial SDF in consumption growth. In a second application, based on a variety of information bounds, we analyze how the information embedded in industry portfolios differs from the information in Fama-French factors. Finally, since we recover a non-parametric SDF taking into account higher moments of returns, it seems appropriate to use it to evaluate the performance of hedge funds, since their return distributions exhibit both skewness and kurtosis.

3.1 Diagnosing Asset Pricing Models

Hansen and Jagannathan (1991) illustrated the usefulness of their volatility bound by plotting the mean-variance pairs of the canonical consumption-based asset pricing model of Breeden (1979) for various values of the preference parameters. The CCAPM SDF is given by:

$$m_t = \beta \left(\frac{C_t}{C_{t-1}} \right)^{-\alpha}. \tag{30}$$

Given a time series of consumption growth rates, one can compute the values of the mean and the variance of the SDF given values for the parameters β and α. Hansen and Jagannathan (1991) used the annual (1891- 1985) time-series data on stocks and bonds of Campbell and Shiller (1988). We use a similar dataset updated to 2009 available on Shiller's website[21] to illustrate their main point reaffirming the equity premium puzzle. The dataset contains S&P 500 returns and both one-year nominal and real U.S. interest rates. In our empirical applications we adopt S&P 500 returns representing stocks and either one-year nominal or real U.S. interest rates representing the risk-free asset, depending on the particular application[22]. The average value for the nominal one-year rate was 4.67% and 1.91% for the corresponding real rate. If adopted as the risk-free asset, those two rates will imply SDF means equal to 0.9813 and 0.9554, respectively. Figure 2 plots the annual time series of those two interest rates from 1890 to 2009.

When testing the CCAPM, the MD values for the consumption model are obtained by using Equation (17). We verify that for a large range of CR discrepancy functions ($\gamma \in [-5, -3, -1, 0, 1, 1.5]$), the classic result that a high value of the risk aversion parameter is needed to place the model in the admissible region is maintained. In other words, even for a nonlinear SDF of the market return, the equity premium puzzle remains. This is consistent with a number of results obtained in the literature. For instance, adopting equity and risk-free data, Snow (1991) identified that frontiers for moments of the SDF smaller than two become more restrictive (than variance bounds) with respect to the CCAPM model. Based on Shiller's dataset over the period 1890-1995, Julliard and Ghosh (2010) adopted entropic estimators to estimate the CCAPM under disaster risk, obtaining a coefficient of risk aversion of 32. Almeida and Garcia (2010) analyze in details the estimation of the CCAPM model under a whole range of CR estimators, obtaining risk-aversion coefficients

[21] http://www.econ.yale.edu/ shiller/data.htm.

[22] For the diagnostic of the CCAPM models below we adopt the real interest rate as the risk-free asset. On the other hand, when analyzing the HJ SDF with positivity constraint with Fama and French factors as basis assets, we adopt both real and nominal rates to define an acceptable region for SDF means.

higher than 30 for all adopted estimators.

Our next example considers, in the spirit of Chapman (1997), a consumption-based model with polynomial consumption-growth terms. Under a fixed set of calibrated parameters, we compare the HJ variance bound to several MD bounds.

The stochastic discount factor of the nonlinear consumption model is given by:

$$m_t^{chap} = \theta_0 P_0(x_t) + \theta_1 P_1(x_t) + \theta_2 P_2(x_t), \tag{31}$$

where x_t is time t consumption growth, and P_i are the Legendre polynomials:

$$\begin{aligned} P_0(x) &= \sqrt{(\frac{1}{2})}, \\ P_1(x) &= \sqrt{(\frac{3}{2})}x, \\ P_2(x_t) &= \sqrt{(\frac{5}{2})}0.5(3x^2 - 1). \end{aligned}$$

We choose values of the parameters ($\theta_0 = 6.88$,$\theta_1 = -6.8$,$\theta_2 = 5.5$) that make the consumption model admissible according to the HJ variance bound. In the top panel of Table 1, we can see that the SDF variance is within the HJ frontier with a ratio of variance to the HJ bound of 1.0303. The SDF mean is equal to 0.982 what corresponds to an average annual risk-free rate of 1.83%, and skewness and kurtosis values indicate that the SDF distribution is non-normal. Since the HJ diagnosis of such model puts zero weight on skewness and kurtosis, the variance bound may be insufficient to declare the model admissible or not.[23].

In the bottom panel, we compute the ratios of the discrepancies of the Chapman SDF to the CR bounds for various values of the Cressie Read discrepancy parameter γ. First, we can observe that these ratios achieve higher values (greater than one) for larger absolute values of the discrepancy parameter γ. These high ratio values under high CR discrepancy parameters indicate that the consumption model becomes more easily admissible under CR bounds that weight more heavily higher moments of market returns (see Figure 1). On the other hand, for low values of the discrepancy parameter, specially for the ones close to zero, the consumption model becomes slightly non-admissible, achieving discrepancy ratios around 0.985. CR bounds that have a discrepancy parameter close to zero have a similar behavior to the entropic bounds obtained by Stutzer (1995). They generate SDFs that are approximately exponential on the basis assets returns putting low but non negligible weights on skewness and kurtosis of market returns. In summary, we see that when higher

[23]In Figure 1 we can see that the CR estimator with $\gamma = 1$ that corresponds to HJ with positivity constraint puts zero weight on skewness and kurtosis.

moments are not taken into account at all, corresponding to the HJ case, the consumption model is slightly admissible, when higher moments are strongly taken into account (CR estimators with high γ) the model becomes strongly admissible and for intermediate cases where skewness and kurtosis matter in a moderate way (estimators similar to the ET estimator), the model becomes slightly non-admissible.

The two examples above illustrate that accounting for higher moments of the distribution of returns with the CR discrepancy measures can change the diagnosis based on the HJ frontier. Ultimately the answer will depend on the basis assets returns, the discrepancy parameters, and the asset pricing model data and parameters. Moreover, the examples illustrate the importance of a more robust diagnostic. For instance, given the results above, one may be comforted in adopting the Chapman's CCAPM since it appears admissible under a large sub-family of Cressie Read discrepancy bounds, at least when basis assets are the Treasury one year rate and a stock index.

Our last example considers model diagnostics when the HJ linear SDF with positivity constraint (HJ SDF w.p.c.) entails mispricing of the basis assets. Apparently it is strange to imagine that the HJ SDF w.p.c. will not perfectly price a set of basis assets. However, as shown by HJ (1991), given a certain SDF mean it is not always possible to find a positive admissible SDF that is a linear combination of returns. In such a case, diagnosing asset pricing models based on the variance bound implied by the HJ SDF w.p.c. may not be appropriate. The main reason is that depending on the magnitude of the pricing errors, the HJ SDF w.p.c. may produce a volatility smaller than that obtained from an unrestricted HJ bound, which sets the minimum that any admissible SDF should satisfy. Therefore, an asset pricing model may enter in the feasible region while not being actually admissible. In this context, we will show that admissible SDFs implied by the minimum discrepancy measures will provide more reliable variance bounds than the HJ bound w.p.c.

To illustrate the issue cited above, we construct variance bounds based on the annual time series of the three Fama and French factors (size, book to market, and market) from 1927 to 2004. In order to construct an economically meaningful variance bound, we make use of the average nominal and real U.S. interest rates between 1927 and 2004 to give us approximate limits for the region of acceptable SDF means. The average nominal rate was 4.65% and the average real rate 1.49%. If either one of those two rates is adopted as the risk-free asset, we obtain the following interval for SDF means: [0.955, 0.985]. Therefore, we generate variance frontiers for SDF means between 0.95 and 0.985, considering a large number of estimators: CR($\gamma = -2$), CR($\gamma = -1.5$), CR($\gamma = -1$) (EL estimator), CR($\gamma = 0$) (KLIC), CR($\gamma = 0.3$), CR($\gamma = 0.5$), traditional HJ (1991), CR($\gamma = 1$) (HJ w.p.c.), and CR($\gamma = 2$).

Table 2 presents the pricing errors and variances of the different implied SDFs when we solve the MD problems for a fixed SDF mean equal to 0.985. In this table the SDF mean is compatible with a risk-free rate equal to 1.49%, precisely the average real rate from 1927 to 2004[24].

From the table we observe that the pricing errors increase with γ. In particular, as the discrepancy measures get close to the HJ SDF w.p.c. ($\gamma = 1$) pricing errors are very high, while they decrease down to zero for estimators close to the KLIC criterion ($\gamma = 0$). In particular, the SDFs in the region analyzed by Snow (1991) (corresponding to Cressie Read γ's equal to 0.5, 1 and 2) present the largest pricing errors. On the other hand, for values of $\gamma < 0$, all the implied SDFs present practically zero pricing errors. The clear contrast between SDFs analyzed by Snow (1991) and those in the CR family that we are first presenting in this work represents one of our important contributions in this paper. First because our analysis allows for a deeper understanding on the issues that involve the higher moment bounds proposed by Snow (1991). Second, and most importantly, we improve Snow's work by providing new bounds that take into account higher order moments of SDFs based on implied SDFs that perfectly price the basis assets returns.

The results in Table 2 extend to different SDF means. Figure 3 presents the pricing errors (in basis points) obtained from the Euler Equations applied to each estimator when pricing the market (RM), size (SMB), and book to market (HML) returns, for all SDF means[25]. Note that for values of $\gamma \leq 0$, the implied SDFs again present practically zero pricing errors. In contrast, the HJ SDF w.p.c. presents large pricing errors for SDF means higher than 0.97. Those errors occur because a nonnegative SDF that is a linear combination of returns does not exist in such cases. The consequence is a volatility for the implied HJ SDF w.p.c. that is much smaller than the minimum volatility suggested by the HJ traditional bound (see Figure 4).

Figure 4 illustrates the small variance issue when the pricing errors are large. For the sake of clarity we present only the variance of four of the estimators above[26]: ET, CR($\gamma = 0.3$), HJ, and HJ with positivity constraint. For each estimator (apart from HJ) and fixed SDF mean we obtain the implied SDFs from the first-order conditions of the discrepancy problems solved above, and calculate the variance of the implied SDFs. Note that for SDF means larger than 0.97 the HJ SDF w.p.c. breaks and its variance becomes smaller than the minimum acceptable variance. Normally, the HJ bound w.p.c. should

[24]Pricing errors for the HJ SDF are not reported since it prices the basis assets by construction.

[25]We do not include the estimator CR($\gamma = 2$) since its pricing errors are too large (for most SDF means) and would distort the results. Those errors are available upon request.

[26]The estimator CR($\gamma = -1.5$) presents variance larger than 0.4 for all considered SDF means. CR($\gamma = 0.5$) has a very similar variance behavior to CR($\gamma = 0.3$).

be above the HJ bound. Here we observe the contrary, which means that forcing the linear SDF to be positive produces large pricing errors on the basis assets and reduces the variance. Relying on the HJ bound w.p.c. to diagnose models in this region of SDF means would be an issue. In such cases, where the restricted HJ SDF does not work, adopting the traditional HJ bounds could be an option but it is important to stress that they are based on SDFs that in general achieve negative values in at least one state of nature. In constrast, our CR estimators present variances higher than the traditional HJ bound and either small or null pricing error depending on the discrepancy parameter γ.

Now, suppose that a researcher is willing to adopt a sharper bound than the traditional HJ bound to diagnose asset pricing models. How should he/she proceed? We will suggest to make use of CR estimators to generate the most conservative variance bound that is sharper than HJ. For this reason we will make use of a number of important observations with respect to the CR SDFs. First, and most importantly, note that the CR implied SDFs are a continuous function of γ. This allows us to approximate the variance bound by trying to choose the CR SDF with γ as close as possible to one that at the same time presents small or null pricing errors. Table 2 and Figures 3 and 4 indicate that either the KLIC SDF ($\gamma = 0$) or the CR($\gamma = 0.3$) appear to be appropriate candidates to generate reliable variance bounds. Those SDFs are positive, approximately admissible, and offer bounds that are sharper than the traditional HJ bounds. This example indicates that our methodology can also be adopted to improve variance bounds, especially in the presence of pricing errors on the Euler equations of basis assets returns.

3.2 Do industry portfolios add non-redundant pricing information to Fama-French Factors?

Lewellen Nagel, and Shanken (LNS, 2009) suggest that when basis assets have a very strong factor structure, as it is the case for FF factors, they help (linear) asset pricing models to achieve positive results in diagnostic tests more frequently than desirable. LNS (2009) criticize a number of asset pricing models and suggest several prescriptions to improve the way those models are tested. One of these suggestions is to include as test assets not only the Fama-French portfolios but also other portfolios that would help in mitigating the strong factor structure of the FF portfolios. In particular, they show that industry portfolios have a covariance structure sufficiently different from the FF portfolios. Therefore making use of industry portfolios as basis assets makes it more difficult for asset pricing models to achieve the admissible region in HJ variance bounds and distance.

LNS (2009) indirectly showed how important it is to understand the relative information gain of including new basis assets in diagnosing asset pricing models, especially when the former set of basis assets has a strong factor structure. Given that information in their case was measured by variances and covariances, a natural extension of this idea is to consider more general ways of measuring information when comparing pairs of sets of basis assets. Our discrepancy measures provide a precise way of measuring more general forms of information, since these discrepancies give different weights to moments of returns when finding the implied admissible SDF that achieves the MD frontier. Therefore, we analyze the relative information gain of including industry portfolios as basis assets, instead or in addition to the FF portfolios. We will measure information by the frontiers obtained with the discrepancy measures.

For a fixed discrepancy function in the Cressie Read family, we compare pairs of MD frontiers obtained with two different sets of basis assets. The first set includes only the FF factors and the second set includes only the industry portfolios. As each MD measure puts a different set of weights on moments of returns, the corresponding implied SDFs are obtained through distinct nonlinear functions of the basis assets. By contrast, for the HJ methodology, the implied SDF is a linear function of the basis assets.

Figure 5 shows pairs (FF and Industry) of frontiers of information for six different estimators: HJ, and Cressie Read (CR) with curvature parameters γ of -3, -1, 0, 1, and 3. In each panel of the figure, there is a solid line (FF frontier) and a dashed line (industry frontier). As the number of admissible SDFs is reduced when the frontier gets higher, we can interpret higher frontiers as containing more information than lower frontiers, especially if the goal is to diagnose asset pricing models. Note that the HJ picture (middle right panel) indicates that there is a region (SDF means between 0.995 and 0.998) where the industry portfolios produce a higher variance bound than the FF portfolios. This result supports the suggestion of LNS (2009). In a relevant region of the SDF mean, testing an asset pricing model with industry portfolios would be more challenging. It is even more the case for the two CR estimators with extreme γ's (-3,3) since the information gain increases significantly in magnitude and for a larger set of the SDF means. These extreme estimators are the ones that put more weight on skewness and kurtosis (see Figure 1). We can therefore conclude that considering the information contained in higher moments of returns makes industry portfolios even more informative.

In Figure 6 we adopt a more direct measure of the information gain. We plot the ratio of the frontier obtained when both FF factors and industry portfolios are used as basis assets, over the frontier obtained when only FF factors are basis assets. Note that for most estimators, the ratios are above 1.8. It means that including the industry portfolios as additional basis assets to the FF factors make these frontiers go up by around 80%. Also, the highest gains for most estimators appear precisely in the region of SDF means between

0.992 and 0.999.

3.3 Performance Evaluation of Hedge Funds

Chen and Knez (1996) have rationalized a stochastic discount factor approach to assess the performance of mutual funds. The stochastic discount factor can be based on parametric asset pricing models such as the CAPM, APT or consumption-based models[27] or rely on reference portfolios as in Hansen and Jagannathan (1991). A main drawback of the first approach is that the measures do not assign zero performance to the reference portfolios, contrary to the second approach that correctly prices, by construction, the basis portfolios used for finding the minimum variance SDF. However, the data-based SDF approach has potentially an infinite number of measures that can be obtained if markets or information is limited[28]. Indeed, the minimum-variance SDF is just one possible measure that in fact may not be appropriate for evaluating the performance of hedge funds since their returns exhibit non-normalities and nonlinearities, as we argued earlier. The proposed Cressie-Read family to capture these nonlinear patterns restricts somewhat the number of possible measures[29].

To detect the presence of nonlinearities in hedge fund returns, most papers have used option-like functions in an otherwise linear regression on several risk factors[30]. Extending earlier work by Fung and Hsieh (2001) and Mitchell and Pulvino (2001), Agarwal and Naik (2004) show that a wide range of equity-oriented hedge fund strategies exhibit a relationship with option-based risk factors that consist of returns obtained by buying, and selling one month later, liquid put and call options on the Standard & Poor's (S&P) 500 index. An important improvement with the discrepancy-based SDF approach is to include more elaborate nonlinearities. First, the approach entails nonlinear exposures to all included factors while the former option-based approach is limited to a few factors. Second,in the discrepancy-based approach, one can include options portfolios as reference portfolios, adding nonlinearities on these option-like exposures.

The SDF approach has the distinct advantage of valuing these complex nonlinearities without relying on a particular asset pricing model. Indeed, the performance of a hedge

[27] See in particular Farnworth et al. (2002) and the numerous references therein.

[28] See on this point a recent paper by Ahn, Cao and Chrétien (2009) that proposes a bounds approach to limit the number of performance measures.

[29] Of course there is a continuous domain of γ, the parameter of the discrepancy function, but for each γ it identifies a unique solution. In the last section we discuss possible choices among these solutions.

[30] Glosten and Jagganathan (1994) have suggested to include such option-like functions as basis functions to characterize a potentially nonlinear SDF. For performance evaluation they use the Black-Scholes model to value these option-like functions.

fund, that we will designate by α_{HF}, will be given by:

$$\alpha_{HF} = E[m_{CR}R_{HF} - 1], \tag{32}$$

where R_{HF} refers to the returns on a particular hedge fund and m_{CR} to the SDF series implied by a particular Cressie-Read discrepancy measure. We should emphasize that the expected discounted returns are evaluated unconditionally. In other words, we want to evaluate an average performance over a certain period without accounting for some potential public information that could have helped produce the hedge fund returns. In mutual fund performance evaluation a conditional evaluation is most often performed to verify that a positive performance is not simply the reflection of some publicly available information and is really attributable to the ability or superior information of the manager[31].

Several reasons explain our choice of an unconditional measure. First and foremost, our primary goal is to illustrate the usefulness of our discrepancy-based SDF for evaluating the performance of portfolios exhibiting returns with potentially high skewness and kurtosis. Whether the return distributions are conditional or unconditional is of secondary importance for this purpose. A second point, consistent with our goal, is that we will apply our methodology to a set of hedge fund indices and not to individual hedge funds. This will allow us to compare the results obtained with our methodology to previous studies of hedge fund nonlinear returns that have essentially been conducted with indices. A third reason comes from the model-free and nonparametric nature of our approach. When an asset pricing model is at the center of the performance evaluation it may be justified to account for economic or financial factors deemed important but left out of the model. In our approach we choose the number of risk factors that appear to affect hedge fund returns and include them in our reference portfolios. Our discrepancy function explores linear and nonlinear exposure to these factors[32].

[31] Dahlquist and Soderlind (1999) provide a thorough analysis of the SDF approach suggested by Chen and Knez (1996) with conditional and unconditional assessments of performance.

[32] Other more econometric or statistical reasons may be invoked to use an unconditional approach. Dahlquist and Soderlind (1999) point out difficulties associated with conditional tests. They report that the power of the tests decreases with the conditioning information and that numerical problems are encountered when one wants to maintain the positivity of the SDF in a conditional setting. Second, the conditional approach refers to managed portfolios but the conditioning information is always public since the true private information, which is in fact used to actually manage the portfolios, is unobservable. Therefore, with respect to this private information, we observe the portfolio returns always unconditionally. It means in particular that this absence of conditioning will generate fatter tails (see Garcia, Renault and Tsafack, 2007). Our unconditional SDFs are incorporating higher moments, which helps better capture the effect of this unobservable private information.

3.3.1 Stochastic Discount Factors Implied by Risk Factors

Previous studies that have characterized hedge fund returns have typically used a number of linear and nonlinear (option return portfolios) factors. Hasanhodzic and Lo (2007) show that five factors provide a reasonable set of risk exposures for a typical hedge fund. Therefore, we include the following five factors: i) CRSP value-weighted NYSE, AMEX, and NASDAQ combined index as a stock market measure; ii) equally weighted portfolio of British, German and Japanese one-month eurocurrency deposits to capture any exposure to an exchange rate (FX) factor; iii) Lehman U.S. Corporate AA Intermediate Bond Index to capture bond market risk; iv) return of the Lehman U.S. Corporate BAA Intermediate Bond Index in excess of the return on the Lehman U.S. Treasury index to capture a credit risk factor; and v) Goldman Sachs Commodity index[33]. For the nonlinear exposures to risk factors, we include four option factors used in Agarwal and Naik (2004), one out-of-the-money put factor (OTM put), one out-of-the-money call factor (OTM call), one at-the-money put factor (ATM put) and one at-the-money call factor (ATM call)[34].

The first step in our performance evaluation experiment is to compute the stochastic discount factor. In order to obtain the implied SDF for different values of γ in our CR discrepancy function, we must solve an optimal portfolio problem under the particular utility function defined by a particular value of γ on the dual space. We choose to estimate SDFs for $\gamma \in \{-3, -1, 0, 1, 3\}$, a set that includes the HJ linear SDF with positivity constraint ($\gamma = 1$), a pair of SDFs that give mild weights to skewness and kurtosis ($\gamma = -1, 0$), and a pair of SDFs that give large weights to skewness and kurtosis ($\gamma = -3, 3$) (see Figure 1). Negative γ's imply SDFs that are more volatile and produce larger peaks than positive γ's, compatible with the fact that risk aversion, on each state, is an increasing function of $gamma$.

Table 3 presents the optimal weights given by each Cressie Read estimator to the five risk factors described above. Numbers between parentheses represent the percentage of the optimal portfolio allocated to each factor. Interestingly, in absolute values, the optimal weights for most CR estimators given to the Credit Risk factor are the highest among all risk factors, followed by the Bond factor. Those two factors cover around 85% of the optimal portfolio allocations of all estimators. Note that, according to the table, all the CR estimators are actually heavily selling U.S. Treasury and Corporate bonds. In addition, commodities represent only tiny portions of their optimal portfolios, while the S&P index

[33]Our sample is comprised of monthly time-series from January 1996 to March 2004. These dates are determined by considerations on the hedge fund data that we explain in section 3.3.2.

[34]For a discussion of the option features in the characterization of hedge fund returns see Diez and Garcia (2009).

is bought (around 5% of the portfolios allocations), and the weighs are not too different on the currency factor. While investors with γs of -3 , -1 and 0 will buy the currency index, the ones with higher γs will sell it.

Table 4 presents the optimal weights given by each CR estimator once the option factors are added to the five risk factors. Overall, the weights attributed to the original risk factors are not very distinct from the no-options case. However, put options appear to be especially important in the optimal portfolios, being responsible for around 4% of the total allocation of all estimators, except for $\gamma = 1$. For this estimator, which the HJ mean-variance estimator with positivity constraint, options do not appear to be important since they represent overall only 1% of the portfolio. This is consistent with the idea that once kurtosis and skewness are significantly weighted in the utility function, then options become more important in the optimal portfolio (see Appendix B).

In principle, all SDFs implied from our HARA optimization problems should be admissible. However, the imposition of a positivity constraint may introduce pricing errors in the Euler equations of some basis assets returns. This was already the case for the HJ SDF with positivity constraint that presented large pricing errors when pricing the returns on the three Fama and French factors (see Section 3.1). Here the problem of pricing errors is exarcebated due to a larger number of basis assets (five as opposed to three), and especially when we introduce returns on calls and puts, that in addition to being very volatile and of large magnitude, contribute to an even larger number of basis assets.

Tables 5 and 6 present the pricing errors for the five different implied SDFs ($\gamma \in \{-3, -1, 0, 1, 3\}$) when the basis assets are respectively the five initially chosen risk factors (stock, currency, bond, credit and commodity), and those same factors with the inclusion of the four option factors (ATM call, OTM call, ATM put, OTM put). Note that the two SDFs with negative γ present zero pricing errors in both tables for all basis assets, implying that they are in fact admissible SDFs under both groups of basis assets. The KLIC SDF ($\gamma = 0$) has zero pricing error when the risk assets don't include option factors and, once option factors are included, it presents a pricing error of one basis point for the two put option returns. This indicates that it is an admissible SDF for the first group of risk assets and that its pricing errors are negligible once option factors are included in the set of risk assets. On the other hand, the implied SDFs with positive γ's present considerable pricing errors even when options are not included as risk factors, with errors around 10 to 20 basis points. And when options are included as factors (see Table 6), both implied SDFs ($\gamma = 1, 3$) present very large errors when pricing the option factors (errors ranging from 100 to 1000 basis points). In particular, the HJ SDF with positivity constraint ($\gamma = 1$) prices the two put option returns with errors of -278 and -421 basis points respectively for

ATM and OTM returns. Those errors indicate that we should be cautious when analyzing the performance of hedge funds with the HJ SDF with positivity constraint, and more generally with implied SDFs with large positive γ's.

Figures 7 and 8 exhibit the sample paths of most SDFs implied by the optimal portfolio weights[35], with and without the option factors. The estimators for $\gamma = 0$ and $\gamma = -1$ in Figure 7 provide very similar patterns for the dynamics of the SDFs with coinciding peaks and troughs, but the variance obtained for $\gamma = -1$ is higher than the variance of the $\gamma = 0$ SDF. The option factors change the SDFs significantly only at specific points in time associated with particular market conditions. These differences are more subdued with the HJ and $\gamma = 1$ estimators as shown in Figure 8.

We summarize the relative information gain for the various estimators when options are included in Figure 9. It is observed that the relative gain is a declining function of the CR curvature parameter, which can be interpreted as an average risk-aversion coefficient in the HARA utility function[36]. Also, the relative gains increase with the mean of the SDFs.

While previous pictures showed historical sample paths of implied SDFs, it will be interesting to also observe SDFs as a function of aggregate wealth. This is possible if we use the two-period optimal portfolio interpretation of subsection 2.2. There, final wealth is obtained as a linear combination of an investment of $W_0 - \sum_j \lambda_{CR,j}^{opt}(\gamma)$ units in the risk free asset (with return r_f), and $\lambda_{CR,j}^{opt}(\gamma)$ units in the j^{th} risky asset (either bond, credit, commodity, stock, or currency)[37]. W_0 represents initial wealth, and $\lambda_{CR}^{opt}(\gamma)$ is the optimal portfolio vector obtained from each specific Cressie Read estimator ($\gamma = -1, 0, 1, 3$). Figure 10 shows these Cressie Read implied SDFs. Observe that the SDFs exhibit hyperbolic shapes as a function of wealth, precisely as described by the theoretical relationship in Equation (10). Dittmar (2002) has shown that nonlinearities on SDFs are very important to explain the cross-section of returns for industrial portfolios, and in accordance to his results, we find implied CR SDFs that are highly nonlinear under some specific discrepancies ($\gamma = -1, 0$). On the other hand, the SDF obtained with $\gamma = 1$ is a non-negative SDF which is a linear function of the risk factors, that is, the HJ SDF with positivity constraint. Note also that all the implied SDFs are strictly positive and decrease with wealth satisfying an important economic property: decreasing absolute risk aversion. It means that under the particular set of risk factors adopted here, our implied nonlinear SDFs are admissible and

[35]We don't present the CR SDF with γ=-3 because it has a very similar shape to the one with γ=-1, except for some slightly higher peaks.

[36]By also observing the time series of each implied SDF, it is noticed that the SDFs variances also decline with risk aversion.

[37]We have also obtained the implied SDFs as functions of wealth when options are included as factors. Due to the similarity to the SDFs without options we decided not to include them.

are globally decreasing functions of wealth.

Figure 10 also reveals that SDFs with smaller γ's are defined for larger ranges of final wealth. This is related to how the optimal portfolio solutions produce more volatile returns for smaller γ's, while for higher γ's, the positivity restriction constraints the solutions to be less volatile. Implied SDFs coming from smaller gammas also achieve more extreme values, especially for bad events.

3.3.2 Computing the Alphas of Hedge Fund Indices

We now use these SDFs to assess the performance of several indices of hedge funds built from the TASS database. As of 2004, the last year of our sampling period, it provided monthly returns and net asset value data on 4,606 funds beginning in February 1977. For building these hedge fund indexes, our sample starts in January 1996 and ends in March 2004[38]. The individual funds are classified into nine categories: 1) convertible arbitrage; 2) fixed-income arbitrage; 3) event driven; 4) equity market neutral; 5) long-short equity; 6) global macro; 7) emerging markets; 8) dedicated short bias; 9) managed futures.[39].

To obtain the α_{HF} performance measure defined in Equation (32) we use the respective implied SDFs and the returns on the hedge fund indices. In Table 7, we report the average alphas associated with the various estimators, with and without the option factors. For comparison we also report the performance evaluation corresponding to a linear model of the risk factors, a model where the Agarwal and Naik (2004) put and call option factors are added to the five risk factors that enter linearly, and the nonlinear model in Diez and Garcia (2009). In the latter the option factors are estimated using a threshold approach and valued according to the Black and Scholes model.

In general, the estimators agree on the two extreme categories, the best (convertible arbitrage, C1) and the worst (managed futures, C9), but exhibit more variation across the other categories. However the results are relatively robust regarding the positive or negative assessment of performance. The presence of options can change this sign from positive to negative as in the long-short equity hedge (C5). It suggests that if nonlinearities are present they may just reflect the use of derivatives that hedge funds pay for on the market and not the timing strategies of hedge fund managers.

For some categories such as equity market neutral (C4), we notice an important difference between the assessment of the CR estimators and all the previous estimators. It is consistently negative for the CR estimators, while it is slightly positive for the other

[38]The starting date is chosen to obtain a reasonable representation for all categories of funds. The end date is set so we can compare our results to previous studies that have introduced nonlinearities in the form of options.

[39]For a description of the typical strategies followed in each category, see the book of L'habitant (2004)

estimators except for the high volatility one[40].

3.3.3 Robustness

We have shown how different values of γ could affect the conclusions of our analysis. In this section we want to focus on the robustness concepts that can be invoked to address this question. First, it seems natural to start when evaluating performance to determine for which value of γ, if any, the measure goes to zero. If the performance stays positive for high positive γ's, one can be pretty confident that even very risk averse investors will be ready to buy this portfolio. A second approach will be to build a robust measure of performance with a SDF that results from averaging across a range of HARA functions before solving the portfolio optimization problem.

(*a*) Searching for Zero Performance We want to determine whether any value of γ can be found that will reduce the performance value to zero. We conduct this analysis for a range of values for γ between -8 and 5[41] and by considering the five risk factors defined in the previous section. The variation will be enough to give us information about the performance behavior of the various categories of funds. We plot the results of the analysis in Figure 11. It comes out very clearly that some categories (C1, C3, and C7) exhibit a positive performance across the board. However, while convertible arbitrage (C1) appears very stable, the emerging markets (C7) category is much more variable, getting close to zero as γ increases. Similarly, one can clearly identify the bad performance categories. Equity market neutral (C4), dedicated short bias (C8) and managed futures (C9) are staying below zero. Equity market neutral is an interesting category since the finding says that it may be neutral in terms of correlation with the market but not neutral to movements in the risk factors that affect higher moments of the return distribution (see Patton (2008)). The remaining categories involve a change of sign in the average performance. Also the performance line of two categories can cross. This requires a more refined measure of robustness.

Given this range of performance values, one can also determine if one category uniformly dominates another category, or if the minimum or the maximum of one category is

[40]Adopting distinct concepts to define neutrality (i.e. "mean", "correlation", "variance", "tail" and "complete"), Patton (2008) showed that around one quarter of market neutral hedge funds are not neutral with respect to market risks. Similarly in spirit, by capturing higher-order risks on market returns, our CR estimators deliver negative risk adjusted performance alphas for the equity market neutral category suggesting non-neutrality for those funds.

[41]For values of γ above 1, we are introducing some pricing errors for the reference portfolios by imposing positivity constraints on the SDF. This region of the parameter space is still interesting to explore to build measures corresponding to high risk aversion.

below or above another category. This is the robustness approach emphasized in Ahn, Cao and Chrétien (2009) for evaluating the performance of mutual funds. Their work can be characterized as a robust Chen and Knez (1996) approach that involves a quadratic norm to bound the performance measure. They interpret different admissible SDFs (the ones that price the basis assets) as the marginal utility of different classes of investors[42]. Since our values of γ index the curvature of a HARA utility function, similarly to them, we can associate a value of γ with an average risk aversion over the sample period of performance evaluation. However, our performance measures are designed for evaluating portfolios that involve return distributions with non negligible higher moments such as hedge funds.

(b) **A Robust Measure of Performance** As a possible robust measure of performance, we suggest the use of an estimator that averages across a range of HARA functions, and solves the portfolio optimization problem of this averaging function. This should imply a SDF that takes into account different γ curvatures (or degrees of risk aversion in our interpretation). In addition, it emphasizes the fact that our methodology is not limited to the Cressie Read family, since a linear combination of HARA functions is not a HARA function. Using the 5 initial risk factors adopted in the Hedge Funds experiment, we solve this problem for an average of all HARA functions with $\gamma \in \{-6, -2, -1.1, 0.1, 2, 4\}$:

$$\underset{\lambda \in \Lambda_{CR_{aver}}}{sup} \frac{1}{6} \sum_{j=1}^{6} E\left[-\frac{1}{\gamma_j + 1} (1 + \gamma_j * \lambda'(R - r_f))^{\frac{\gamma_j + 1}{\gamma_j}} \right] \tag{33}$$

where $\Lambda_{CR_{aver}} = \{\lambda \in R^K, \text{ s.t. for } i = 1, ..., T : \underset{j}{min} \left(1 + \gamma_j \lambda' \left(R_i - r_f\right)\right) > 0\}$. Figure 12 presents the implied SDF. Note that it is less volatile than the implied Cressie Read SDFs (see Figures 7 and 8) in both good and bad states. This could be expected since it is solving an averaging problem. It is important to make clear that this SDF is not an average of the previously obtained SDFs but a new SDF implied from a completely different portfolio problem. Moreover, calculating the convex conjugate of the negative of the function appearing in Equation (33), we could obtain the exact discrepancy function that we are minimizing in this averaging case[43].

Interestingly, for some categories of Hedge Funds, the average estimator produces performance values that differ from the average of the performances obtained by each corresponding Cressie Read estimator for $\gamma \in \{-6, -2, -1.1, 0.1, 2, 4\}$. In particular, for the

[42]Note however that due to their focus in obtaining robust intervals of performance for managed funds, their algorithm does not provide the admissible SDFs themselves.

[43]The discrepancy function of the average estimator does not present a closed-form formula since it is a nonlinear function of the zero of a nonlinear equation implied by the convex conjugation problem. The calculation is numerically performed in Matlab and is available from the authors upon request.

convertible arbitrage (C1) funds the average estimator and the average of Cressie Read estimators almost coincide. While the average estimator suggests an annualized performance of 3.74%, the average of CR estimators produces 3.67%. The proximity in the values is an additional indication that the positive performance of this category is robust. On the other hand, for the Equity market neutral (C4) category the average estimator is more optimistic than the average of CR estimators. While the average of CR estimators gives a -2.03% annualized performance, the average estimator produces -0.66%, a value that is higher than any individual CR estimator performance obtained for C4 (see Figure 11). This example indicates that making use of the average estimator can be especially useful for categories whose performance exhibit too much variation across different CR estimators. This suggests also a closer look at the results of category C9, that of managed futures. By looking at Figure 11 its performance exhibits a large variability of negative values across CR estimators. In this case, the average of the CR estimators for $\gamma \in \{-6, -2, -1.1, 0.1, 2, 4\}$ gives a performance of -4.17%, while the average estimator gives -6.46% a much worse number. Like this average estimator we can produce other functions to aggregate Cressie Read discrepancies under a unified metric, as long as the final discrepancy function is convex. Alternatively, this aggregation of metrics can be proposed directly in the dual space (as we have done with the average estimator), as long as the final utility function is concave.

4 Conclusion

We extend results on stochastic discount factor variance bounds of Hansen and Jagannathan (1991) by proposing more general Minimum Discrepancy (MD) bounds based on the minimization of discrepancy convex functions. Solutions to these MD problems naturally imply nonparametric and nonlinear SDFs that take into account higher moments of the distributions of assets returns. We relate the problem of finding general MD bounds to that of solving an optimal portfolio problem, therefore generalizing the duality between SDF frontiers and optimal portfolio frontiers put forward in Hansen and Jagannathan (1991). When specializing to the Cressie Read family of discrepancies, our bounds are obtained as solutions to optimal portfolio problems based on HARA utility functions. We point to the special cases corresponding to the logarithmic, exponential and quadratic utility functions.

We showed how the implied SDFs can be used in three empirical applications. First, we use the new SDF frontiers to bring a new perspective on diagnosing asset pricing models. Then, building on a discussion in Lewellen, Nagel and Shanken (2009) we use our discrepancy measures to analyze if industry portfolios contain nonredundant pricing information when compared to Fama-French factors. In a third application, we analyzed hedge fund performance evaluation with different metrics based on our implied SDFs. Our results indicate that this new class of higher-order SDF frontiers has a strong potential to be used in a large number of financial problems, specially those involving assets with nonlinear payoffs.

In this paper, we have voluntarily left aside the important issue of estimating the parameters of the asset pricing models under scrutiny and limited ourselves to a diagnosis as in the original paper of Hansen and Jagannathan (1991). In Almeida and Garcia (2010), we are assessing specification errors in stochastic discount factor models with our new metrics to generalize the quadratic-norm evaluation methodology developed in Hansen and Jagannathan (1997). Given the general formulation of the discrepancy problem presented in Section 2.5, where the moment conditions involved a β vector of model parameters, this generalization appears as a natural extension of the SDF frontier investigation conducted in the current paper.

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Appendix A - Proof of Theorem 1

Under the Cressie Read family, the convex conjugate of ϕ is given by:

$$\phi^{*+}(z) = \frac{(\gamma z)^{\frac{\gamma+1}{\gamma}}}{\gamma+1} + \frac{a^{\gamma+1}}{\gamma(\gamma+1)} \tag{34}$$

The optimization problem becomes:

$$\hat{\lambda} = \underset{\alpha\in\Re,\lambda\in\Lambda}{arg\ sup}\ a*\alpha - \sum_{i=1}^{T}\frac{1}{T}\frac{\left(\gamma\left(\alpha+\lambda'\left(R_i-\frac{1}{a}1_K\right)\right)\right)^{\frac{\gamma+1}{\gamma}}}{\gamma+1} - \frac{a^{\gamma+1}}{\gamma(\gamma+1)}, \tag{35}$$

In order to concentrate α out of the optimization problem in equation (35) let $\Gamma(\alpha) = a*\alpha - \frac{(\gamma\alpha)^{\frac{\gamma+1}{\gamma}}}{\gamma+1}$. Then the optimal concentrated α should solve:

$$\frac{d\Gamma(\alpha)}{d\alpha} = 0 \Rightarrow \hat{\alpha} = \frac{a^{\gamma}}{\gamma} \tag{36}$$

Substituting $\hat{\alpha}$ on Equation (35) gives the desired result.

Appendix B - Taylor Expansion of the HARA Utility Function Implied by the Cressie Read Estimators

For simplicity let us assume that there is only one risky asset with return R. According to the optimal portfolio interpretation section (subsection 2.2), the utility function that is maximized to obtain the solution of the Cressie Read Bounds and their implied SDFs is given by:

$$u(v) = -\frac{1}{\gamma+1}(1-\gamma v)^{\frac{\gamma+1}{\gamma}} \tag{37}$$

where $v = \lambda*(R-\frac{1}{a})$, and a represents the SDF mean. The solution of the HARA portfolio problem gives the optimal lambdas λ_{opt} that will be used to define the Cressie Read bound and the corresponding implied SDF, both obtained at $v_0 = \lambda_{opt} * E[(R-\frac{1}{a})]$.

Now, we are interested in performing a Taylor expansion around the optimal λ-scaled expected excess return of the risky asset $v0$ that will represent the aggregate risky in the economy. The goal is to analyze how the coefficient of risk aversion γ will affect the weights given to skewness and kurtosis in the specific solutions of our HARA-utility problems. To that end, we use the corresponding second, third, and fourth derivatives of u in a fourth order Taylor expansion, and take expected values of both sides:

$$\begin{aligned}&E[u(v)] \approx u(v0) + \tfrac{1}{2}u_2(v0)\lambda_{opt}^2 * E(R-E(R))^2 + \\ &\tfrac{1}{6}u_3(v0)\lambda_{opt}^3 * E(R-E(R))^3 + \tfrac{1}{24}u_4(v0)\lambda_{opt}^4 * E(R-E(R))^4\end{aligned} \tag{38}$$

Those derivatives are respectively given by:

$$u_2(v) = -(1-\gamma v)^{-1+\frac{1}{\gamma}} \tag{39}$$

$$u_3(v) = (1-\gamma)(1-\gamma v)^{-2+\frac{1}{\gamma}} \tag{40}$$

$$u_4(v) = -(1-\gamma)(1-2\gamma)(1-\gamma v)^{-3+\frac{1}{\gamma}} \tag{41}$$

Looking at the third derivative of u we see that skewness could be weighed negatively for Cressie Read estimators with $\gamma > 1$. However, according to the Taylor expansion, the optimal lambda gives an extra degree of flexibility for the sign of the third moment. For instance, a negative lambda for estimators with $\gamma > 1$ will provide a positive weight to skewness. This flexibility guarantees that for the whole range of γs our utility function can potentially satisfy the concept of decreasing absolute risk aversion from Arditti (1967) ($\lambda_{opt}^3 u_3(v0) > 0$).

In Figure 1 we provide pictures with the sensitivity of our estimators to skewness and kurtosis. They plot the third and fourth derivatives as functions of γ. Note that we chose optimal lambdas compatible with decreasing absolute risk aversion. The derivative functions are depicted for small positive, zero, and small negative $v0$, which corresponds to the lambda-scaled expected excess return of the risky asset. As in principle $v0$ may achieve any arbitrary value being a solution to the HARA portfolio problem, it becomes clear the richness with which the CR estimators can weight skewness and kurtosis.

Note that skewness weights are always positive and they increase when γ goes away from the quadratic case ($\gamma = 1$). For instance, EL ($\gamma = -1$) puts higher weights than ET ($\gamma = 0$), CUE (CR($\gamma = 1$)) gives zero weight, and CR($\gamma = 3$) gives weights comparable to EL ones. Regarding the fourth derivative, except for the region of $0.5 < \gamma < 1$, kurtosis is a non-positive and concave function of γ indicating that all CR estimators outside that region satisfy the concept of decreasing absolute prudence proposed by Kimball (1993) ($\lambda_{opt}^4 u_4 < 0$). Limiting cases including the quadratic utility (CUE, $\gamma = 1$) and the cubic utility (CR $\gamma = 0.5$) put zero weight to kurtosis. Note that Cressie Read estimators with positive γ's give more (negative) weight to kurtosis than the corresponding estimators with negative γ's. For instance, EL ($\gamma = -1$) weights kurtosis on the interval [-12,-10], while CR($\gamma = 3$) weights it on the interval [-20,-10], for the particular values of $v0$ that we chose.

Table 1: Information Bounds and Consumption Models.

The first panel on this table presents statistics for a Chapman's (1997) type polynomial SDF $m_{Chap}(c_t) = \sum_{i=0}^{2} \theta_i * P_i(c_t)$, where c_t is consumption growth at time t, and P_i is the i_{th} normalized Legendre polynomial. Parameters values are ($\theta_0 = 6.88$,$\theta_1 = -6.8$,$\theta_2 = 5.5$). The HJ Bound is obtained from returns on the S&P 500 and a risk-free rate as in Campbell and Shiller (1989). Data is annual over the period 1891 to 2004. In the second panel, the first column indicates different discrepancy functions of the Cressie Read family $\phi(\pi) = \frac{\pi^{\gamma+1}-1}{(\gamma+1)\gamma}$. The second column presents CR bounds based on returns from S&P 500 and a risk-free rate as in Campbell and Shiller (1989). Data is annual over the period 1891 to 2004. Cressie Read estimators solve dual utility maximization problems based on HARA functions parameterized by γ. The third column presents the discrepancy of the polynomial CCAPM model described in the previous panel. For each fixed discrepancy (γ), model discrepancy is obtained by normalizing the polynomial SDF $\pi_{Chap} = \frac{m_{Chap}}{E(m_{Chap})}$, and calculating $\phi(\pi_{Chap})$.

	CCAPM SDF Descriptive Statistics				
Mean	**Variance**	**Skewness**	**Kurtosis**	**HJ Bound**	**Ratio Variance/HJ Bound**
0.982	0.0818	0.75	4.63	0.0794	**1.0303**

CR γ	**CR Bound**	**Model Discrepancy**	**Ratio**
-3	0.0354	0.0575	1.623
-2	0.0385	0.0480	1.245
-1	0.0406	0.0428	1.054
-0.5	0.0419	0.0414	0.988
-0.3	0.0416	0.0410	0.986
-0.1	0.0413	0.0407	0.985
0	0.0419	0.0413	0.986
0.1	0.0409	0.0404	0.988
0.3	0.0405	0.0403	0.995
0.5	0.0401	0.0403	1.005
0.8	0.0392	0.0404	1.031
1	0.0401	0.0406	1.013
1.2	0.0396	0.0408	1.030
1.5	0.0388	0.0414	1.067

Table 2: Pricing Errors for Implied SDFs when basis assets are the Fama and French Factors. The first column indicates different discrepancy functions of the Cressie Read family $\phi(\pi) = \frac{\pi^{\gamma+1}-1}{(\gamma+1)\gamma}$. The second column presents the variance of Cressie Read implied SDFs from the first-order conditions of the dual HARA problems with the three Fama and French factors (market, size, and book to market) as basis assets. Data is annual over the period 1927 to 2004. The last three columns present pricing errors (in bps) achieved by each CR implied SDF when pricing each Fama and French Factor. The SDF mean is equal to 0.985 which corresponds to an average real interest equal to 1.49%.

CR γ	Variance	Pricing Error Market	Pricing Error SMB	Pricing Error HML
-2	0.942	0.0	0.0	0.0
-1.5	0.839	0.0	0.0	0.0
-1	0.681	0.0	-0.1	0.0
0	0.398	0.2	0	0
0.3	0.360	5.0	0.3	2.1
0.5	0.344	8.1	0.5	3.3
1	0.245	154.2	109.2	140.6
2	0.084	502.3	74.8	271.2

Table 3: Optimal Portfolio Weights for Cressie Read Estimators.
Risk factors are composed by monthly returns over the period 1996:1 to 2004:3. The Bond and Credit risk factors are represented respectively by the Lehman U.S. Corporate AA and BAA Intermediate Bond Indexes. The stock market factor is represented by the CRSP value-weighted NYSE, AMEX, and NASDAQ combined index. The FX factor is represented by an equally-weighted portfolio of British, German and Japanese one-month Eurocurrency deposits. The commodity factor is captured by the Goldman Sachs Commodity Index. Cressie Read estimators solve HARA utility maximization problems whose portfolios are linear combinations of the listed risk factors. A fixed SDF mean equal to 0.9962 is adopted.

CR Estimators	Risk Factors				
	Bond	**Credit**	**Commodity**	**Stocks**	**Currency**
Cressie Read ($\gamma = -3$)	-20.0(47.9%)	-14.4(34.5%)	-0.7(1.6%)	1.8(4.3%)	4.9(11.7%)
Cressie Read ($\gamma = -1$)	-35.8(43.7%)	-34.6 (42.3%)	-0.7(0.9%)	3.7(4.5%)	7.1(8.6%)
Cressie Read ($\gamma = 0$)	-39.8 (37.8%)	-56.2 (53.5%)	0.1 (0.1%)	5.7 (5.4%)	3.4(3.2%)
Cressie Read ($\gamma = 1$)	-23.3(29.9%)	-44.4(56.8%)	1.2(1.6%)	3.8(4.8%)	-5.4 (6.9%)
Cressie Read ($\gamma = 3$)	-7.9(28.0%)	-15.1(53.8%)	0.9(3.1%)	1.9(6.6%)	-2.4(8.5%)

Table 4: Optimal Portfolio Weights for CR Estimators with Option Factors.
Risk factors are composed by monthly returns over the period 1996:1 to 2004:3. The Bond and Credit risk factors are represented respectively by the Lehman U.S. Corporate AA and BAA Intermediate Bond Indexes. The stock market factor is represented by the CRSP value-weighted NYSE, AMEX, and NASDAQ combined index. The FX factor is represented by an equally-weighted portfolio of British, German and Japanese one-month Eurocurrency deposits. The commodity factor is captured by the Goldman Sachs Commodity Index. The four option factors come from Agarwal and Naik (2004) and are represented by two call options (C.) and two put options (P.), each one either at-the-money (ATM) or out-of-the-money (OTM). Cressie Read estimators solve HARA utility maximization problems whose portfolios are linear combinations of the listed risk factors. A fixed SDF mean equal to 0.9962 is adopted.

Estim.	Risk Factors								
	Bnd	**Cred.**	**Comm.**	**Stoc.**	**Curr.**	**ATM C.**	**OTM C.**	**ATM P.**	**OTM P.**
CR (-3)	-16.4(33.4%)	-21.6 (44.0%)	-1.4(2.9%)	3.2(6.5%)	3.4(7.0%)	0.7 (1.4%)	-0.8(1.6%)	0.8(1.6%)	-0.8(1.6%)
CR (-1)	-32.7(33.4%)	-45.7(46.7%)	-1.0(1.0%)	8.9(9.1%)	3.6(3.7%)	0.9(0.9%)	-1.2(1.3%)	1.9(2.0%)	-1.8(1.9%)
CR (0)	-36.7(29.4%)	-65.6(52.6%)	0.2(0.1%)	12.8(10.2%)	3.2(2.5%)	0.4(0.3%)	-0.9(0.7%)	2.6(2.1%)	-2.6(2.1%)
CR (1)	-18.8(27.2%)	-41.3(59.8%)	0.6(0.9%)	6.2(8.9%)	-1.5(2.2%)	0.1(0.1%)	-0.5(0.6%)	-0.2 (0.2%)	-0.0 (0.1%)
CR (3)	-7.1(27.0%)	-16.5 (63.3%)	0.3(1.1%)	1.1(4.0%)	0.1(0.2%)	0.0 (0.1%)	-0.2(0.8%)	-0.5(2.0%)	0.4(1.4%)

Table 5: Pricing Errors for implied SDFs.

This table presents pricing errors (in basis points) achieved by implied SDFs when pricing five risk factors adopted as basis assets. Risk factors are composed by monthly returns over the period 1996:1 to 2004:3. The Bond and Credit risk factors are represented respectively by the Lehman U.S. Corporate AA and BAA Intermediate Bond Indexes. The stock market factor is represented by the CRSP value-weighted NYSE, AMEX, and NASDAQ combined index. The FX factor is represented by an equally-weighted portfolio of British, German and Japanese one-month Eurocurrency deposits. The commodity factor is captured by the Goldman Sachs Commodity Index. Cressie Read estimators solve HARA utility maximization problems whose portfolios are linear combinations of the listed risk factors. A fixed SDF mean equal to 0.9962 is adopted.

CR Estimators	Risk Factors				
	Bond	**Credit**	**Commodity**	**Stocks**	**Currency**
Cressie Read ($\gamma = -3$)	0	0	0	0	0
Cressie Read ($\gamma = -1$)	0	0	0	0	0
Cressie Read ($\gamma = 0$)	0	0	0	0	0
Cressie Read ($\gamma = 1$)	-8	-9	-18	28	21
Cressie Read ($\gamma = 3$)	-22	-29	-31	11	2

Table 6: Pricing Errors for implied SDFs with Option Factors.

This table presents pricing errors (in basis points) achieved by implied SDFs when pricing nine risk factors adopted as basis assets. Risk factors are composed by monthly returns over the period 1996:1 to 2004:3. The Bond and Credit risk factors are represented respectively by the Lehman U.S. Corporate AA and BAA Intermediate Bond Indexes. The stock market factor is represented by the CRSP value-weighted NYSE, AMEX, and NASDAQ combined index. The FX factor is represented by an equally-weighted portfolio of British, German and Japanese one-month Eurocurrency deposits. The commodity factor is captured by the Goldman Sachs Commodity Index. The four option factors come from Agarwal and Naik (2004) and are represented by two call options (C.) and two put options (P.), each one either at-the-money (ATM) or out-of-the-money (OTM). Cressie Read estimators solve HARA utility maximization problems whose portfolios are linear combinations of the listed risk factors. A fixed SDF mean equal to 0.9962 is adopted.

Estim.	Risk Factors								
	Bnd	**Cred.**	**Comm.**	**Stoc.**	**Curr.**	**ATM C.**	**OTM C.**	**ATM P.**	**OTM P.**
CR (-3)	0	0	0	0	0	0	0	0	0
CR (-1)	0	0	0	0	0	0	0	0	0
CR (0)	0	0	0	0	0	0	0	1	1
CR (1)	-9	-13	-16	15	10	136	141	-278	-421
CR (3)	-22	-23	-13	39	-1	-99	-191	-803	-985

Table 7: Hedge Funds Performance Evaluation under Different Estimators.
This table contains the alphas of different Hedge Fund categories (listed in Appendix A) obtained with implied SDFs from different Cressie Read estimators. Risk factors are composed by monthly returns over the period 1996:1 to 2004:3. The Bond and Credit risk factors are represented respectively by the Lehman U.S. Corporate AA and BAA Intermediate Bond Indexes. The stock market factor is represented by the CRSP value-weighted NYSE, AMEX, and NASDAQ combined index. The FX factor is represented by an equally-weighted portfolio of British, German and Japanese one-month eurocurrency deposits. The commodity factor is captured by the Goldman Sachs Commodity Index. The four option factors come from Agarwal and Naik (2004) and are represented by two call options (C.) and two put options (P.), each one either at-the-money (ATM) or out-of-the-money (OTM). Cressie Read estimators solve HARA utility maximization problems whose portfolios are linear combinations of the listed risk factors. A fixed SDF mean equal to 0.9962 is adopted.

CR Estimators	Hedge Funds Categories								
	C1	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**	**C9**
Cressie Read ($\gamma = -3$)	3.89	0.88	1.71	-2.65	1.18	0.05	3.37	-2.23	-1.93
CR ($\gamma = -3$) with Options	4.04	2.46	2.89	-2.43	0.23	-1.27	3.81	-3.11	-4.87
Cressie Read ($\gamma = -1$)	3.77	0.84	1.75	-2.19	0.66	0.23	2.22	-0.48	-2.71
CR ($\gamma = -1$) with Options	3.93	2.00	2.46	-2.22	-0.28	-0.45	2.05	-1.09	-4.28
Cressie Read ($\gamma = 0$)	4.29	1.00	2.02	-2.11	0.14	-0.19	2.29	-0.48	-4.26
CR ($\gamma = 0$) with Options	4.06	1.71	1.78	-2.15	-1.05	-0.28	0.38	0.63	-4.30
Hansen and Jagannathan	4.55	2.04	2.43	-1.52	-0.10	0.03	2.46	0.08	-5.39
HJ with Options	3.92	2.66	1.19	-1.58	-1.62	0.06	-0.71	2.58	-4.36
Cressie Read ($\gamma = 1$)	4.39	0.92	2.39	-1.09	1.54	-1.09	3.31	-4.16	-5.79
CR ($\gamma = 1$) with Options	3.35	1.10	1.06	-1.44	-0.80	-1.15	-0.41	-1.01	-4.71
Cressie Read ($\gamma = 3$)	3.60	0.27	1.62	-0.54	0.86	-3.10	0.45	-2.70	-6.71
CR ($\gamma = 3$) with Options	3.58	0.10	1.90	-0.74	2.15	-2.66	2.55	-5.82	-6.70
Previous Estimators	Hedge Funds Categories								
	C1	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**	**C9**
Linear Factor Model	3.72	0.35	1.66	0.02	0.49	-1.51	-0.50	1.10	-4.15
Diez and Garcia $\sigma = 5\%$	5.72	2.09	4.71	2.25	2.84	1.29	8.18	-0.82	-8.10
Diez and Garcia $\sigma = 15\%$	4.78	0.57	2.24	0.51	0.93	-1.32	1.90	0.79	-6.04
Diez and Garcia $\sigma = 25\%$	2.44	-2.67	-1.45	-1.40	-1.91	-7.63	-4.78	3.71	-1.73
Agarwal and Naik	3.58	0.40	1.59	0.22	0.98	-1.58	-1.03	1.30	-3.96

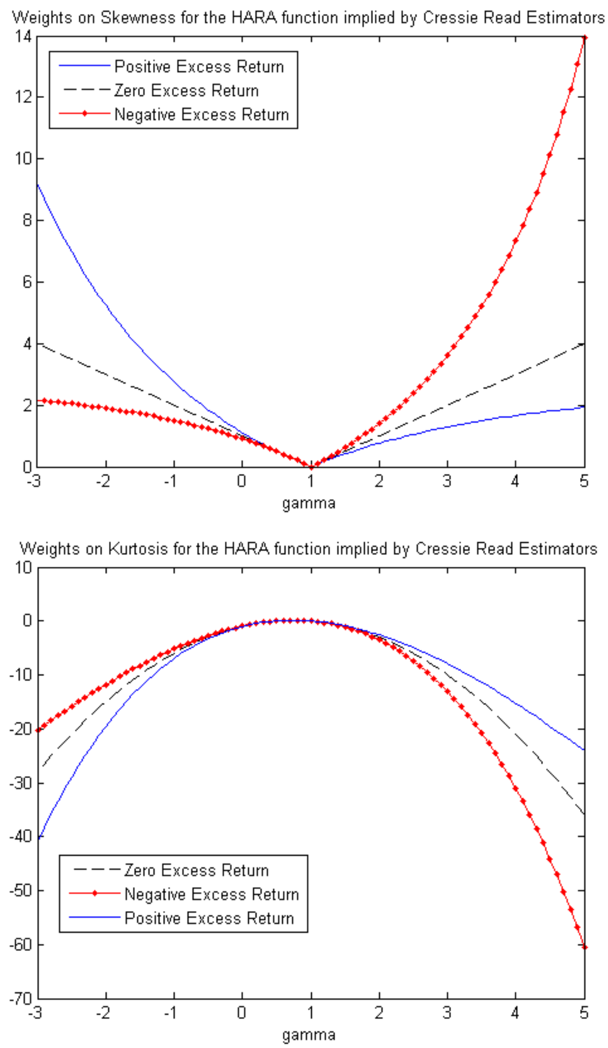

Figure 1: Skewness and Kurtosis Weights on Cressie Read Estimators
This picture presents the third and fourth derivatives of the HARA function $-\frac{1}{\gamma+1}(1+\gamma v)^{\frac{\gamma+1}{\gamma}}$ evaluated at an arbitrary value of v. Based on a Taylor expansion argument, by parameterizing $v = \lambda(R - \frac{1}{a})$ where R is a vector of returns, a is the SDF mean, and λ is a vector of weights, the third and fourth derivatives of the HARA function will indicate the weights given to skewness and kurtosis of the linear combination of returns.

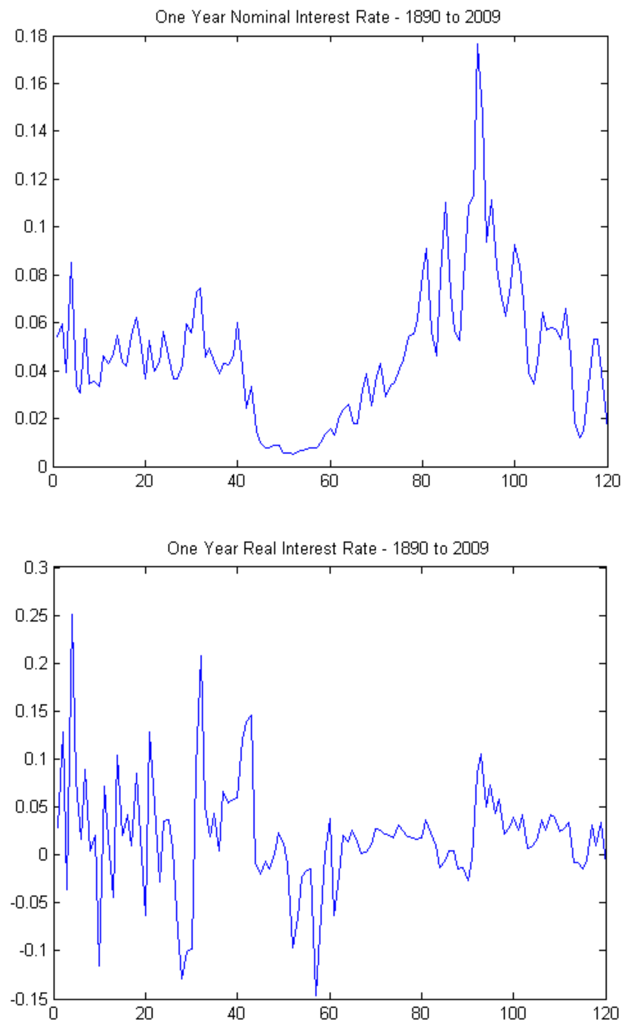

Figure 2: Nominal and Real U.S. One-Year Interest Rate
This picture presents both nominal and real U.S. one-year interest rates from 1890 to 2009 updated from Campbell and Shiller's (1988) dataset.

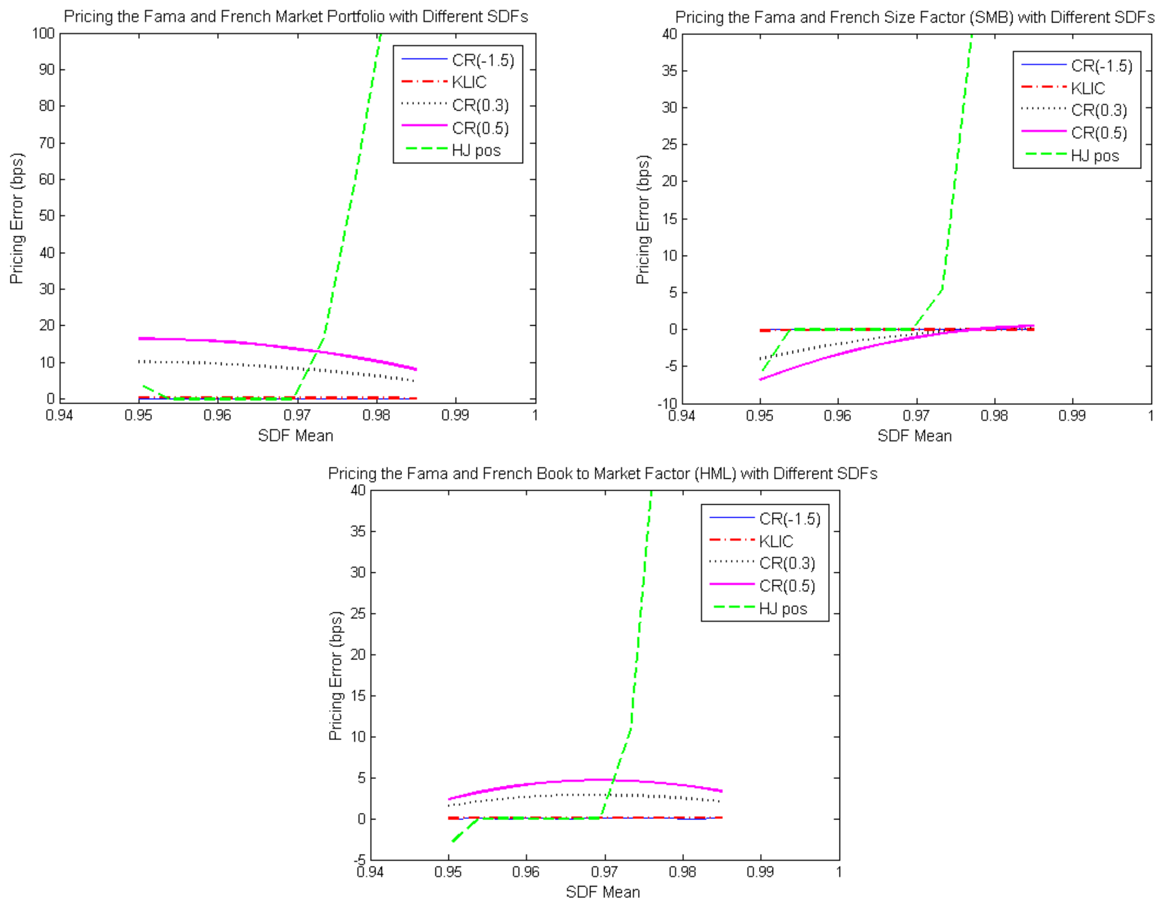

Figure 3: Euler Equation Errors for Different Estimators with Fama and French Factors as Basis Assets
This picture presents pricing errors obtained for the estimators CR($\gamma = -1.5$), KLIC, CR($\gamma = 0.3$), CR($\gamma = 0.5$), and CR($\gamma = 1$) (HJ w.p.c.). The SDFs are obtained from first-order conditions of HARA utility maximization problems when the basis assets are the three Fama and French Factors (market, size, and book to market). Fama and French Factors data are annually from 1927 to 2004.

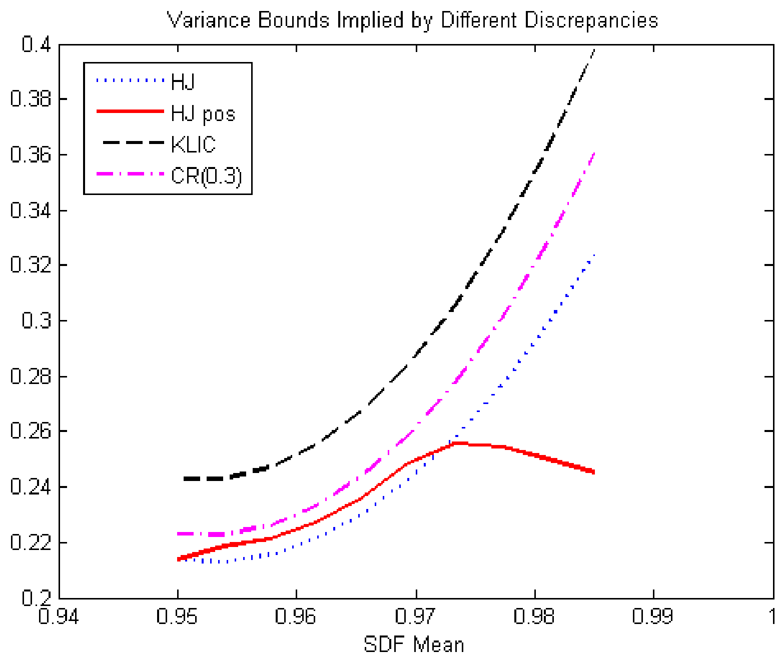

Figure 4: Sharpening Variance Bounds with Fama and French Factors as Basis Assets

This picture presents variance bounds derived for the following estimators: HJ, HJ with positivity constraint, KLIC, and CR($\gamma = 0.3$). The KLIC, CR($\gamma = 0.3$), and the HJ SDF with positivity constraint are obtained from first-order conditions of HARA utility maximization problems when the basis assets are the three Fama and French Factors (market, size, and book to market). Fama and French Factors data are annually from 1927 to 2004.

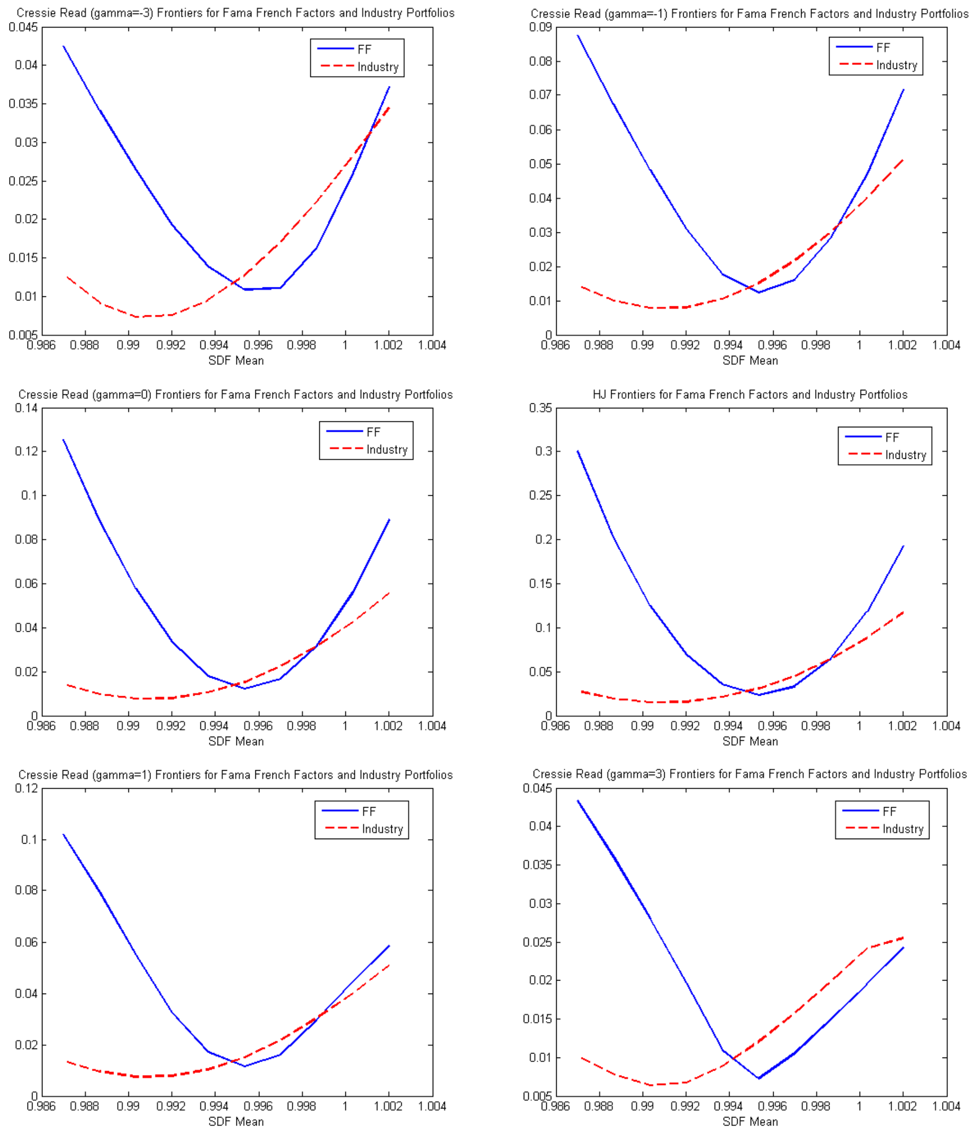

Figure 5: Comparing Information available in Fama French Factors and Industry Portfolios.
This picture presents CR bounds based on monthly returns on Fama-French factors and ten industry portfolios over the period 1963:1 to 2008:12. The solid line is obtained with the three Fama-French factors. The dotted line is obtained with the ten industry portfolios.

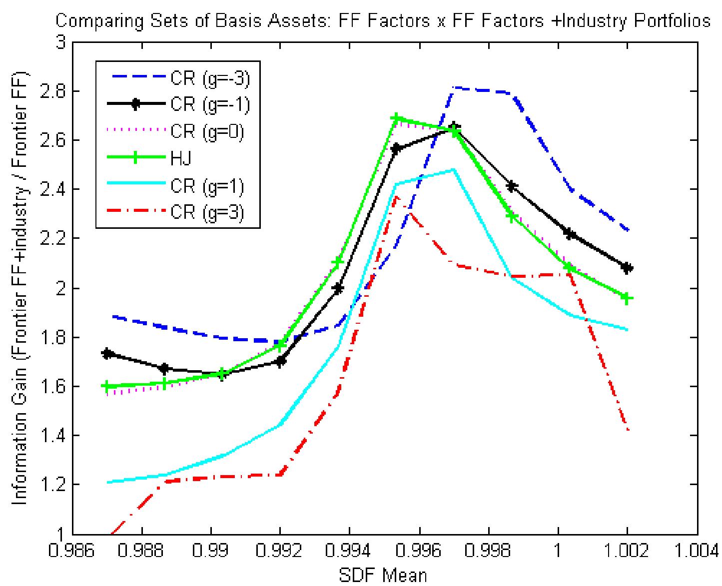

Figure 6: Information Gain when Industry Portfolios are Included as Basis Assets

This picture presents ratios of HJ and CR frontiers based on monthly returns on Fama and French factors and ten industry portfolios over the period 1963:1 to 2008:12. Each γ indicates a different discrepancy function on the Cressie Read family $\phi(\pi) = \frac{\pi^{\gamma+1}-1}{(\gamma+1)\gamma}$. For each fixed SDF mean, the Cressie Read bounds are obtained via dual utility maximization problems based on HARA functions parameterized by γ. The HJ bound is obtained by linearly projecting a generic admissible SDF on the subspace of observed returns. For each estimator, each line is obtained by the ratio of the bound values obtained including all basis assets (three FF factors and ten industry portfolios) over the bound values obtained including only FF factors.

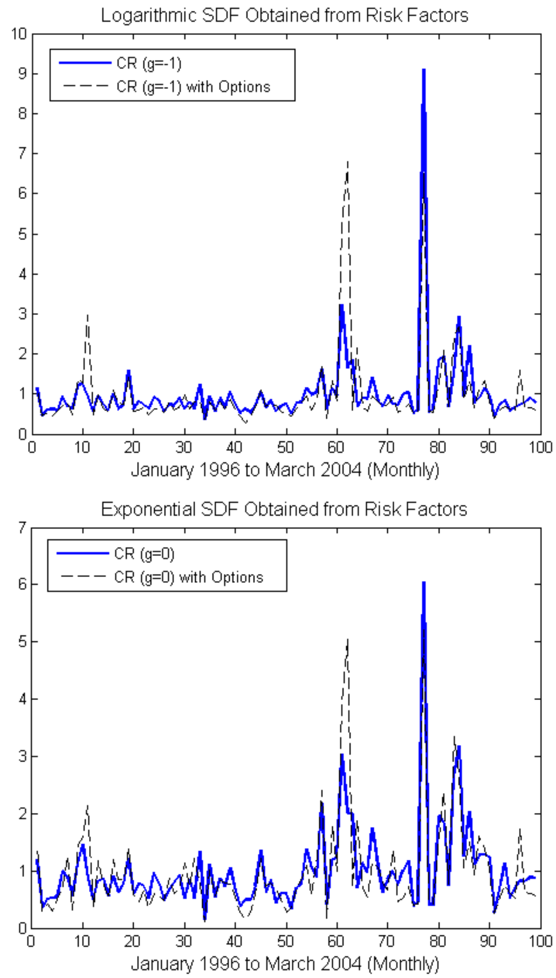

Figure 7: CR Exponential and Logarithmic Stochastic Discount Factors Extracted from Market Risk Factors
This picture presents CR stochastic discount factors based on monthly returns on a set of 9 risk factors over the period 1996:1 to 2004:3. The CR exponential SDF minimizes the discrepancy $\phi(\pi) = \pi log \pi$ and is achieved via the first-order conditions of the dual utility maximization problem based on an exponential function. The CR logarithmic SDF minimizes the discrepancy $\phi(\pi) = -log\pi$ and is achieved via the first-order conditions of the dual utility maximization problem based on a logarithimic function. The SDFs have a fixed mean at 0.996. The Bond and Credit risk factors are respectively the Lehman U.S. Corporate AA and BAA Intermediate Bond Indexes. The stock market factor is the CRSP value-weighted NYSE, AMEX, and NASDAQ combined index. The FX factor is an equally-weighted portfolio of British, German and Japanese one-month eurocurrency deposits. The commodity factor is the Goldman Sachs Commodity Index. The four option factors come from Agarwal and Naik (2004) and represent at-the-money and out-of-the-money call and put options. The solid SDFs include the first 5 risk factors while the dashed ones include also the 4 additional option factors.

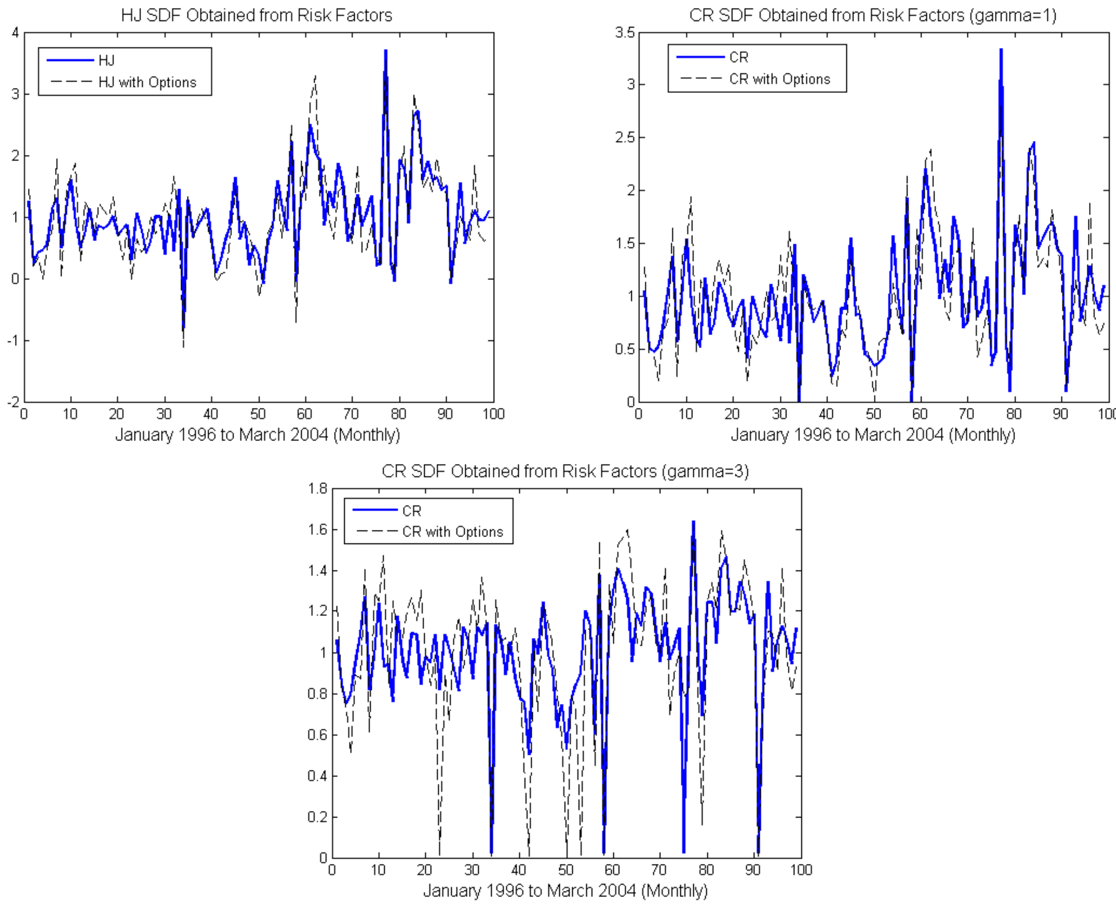

Figure 8: HJ and Other CR Stochastic Discount Factors Extracted from Market Risk Factors

This picture presents HJ and CR stochastic discount factors based on monthly returns on a set of 9 risk factors over the period 1996:1 to 2004:3. Each γ indicates a different discrepancy function on the Cressie Read family $\phi(\pi) = \frac{\pi^{\gamma+1}-1}{(\gamma+1)\gamma}$. The Cressie Read SDFs are obtained via the first-order conditions of dual utility maximization problems based on HARA functions parameterized by γ. The HJ SDF is obtained by linearly projecting a generic admissible SDF on the subspace of observed returns. The SDFs have a fixed mean at 0.996. The Bond and Credit risk factors are respectively the Lehman U.S. Corporate AA and BAA Intermediate Bond Indexes. The stock market factor is the CRSP value-weighted NYSE, AMEX, and NASDAQ combined index. The FX factor is an equally weighted portfolio of British, German and Japanese one-month eurocurrency deposits. The commodity factor is the Goldman Sachs Commodity Index. The four option factors come from Agarwal and Naik (2004) and represent at-the-money and out-of-the-money call and put options. The solid SDFs include the first 5 risk factors while the dashed ones include also the 4 additional option factors.

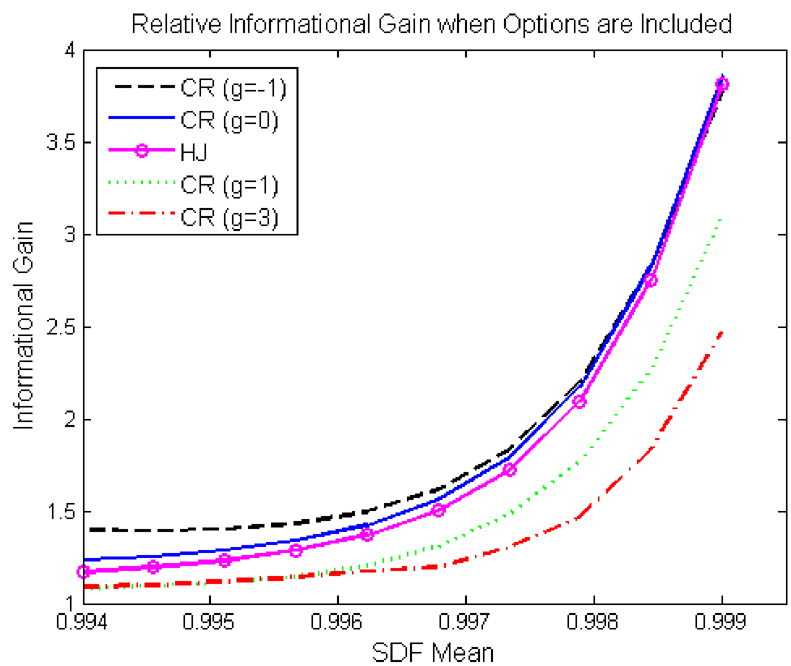

Figure 9: Informational Gain When Options Are Traded

This picture presents ratios of HJ and CR bounds based on monthly returns on 9 risk factors over the period 1996:1 to 2004:3. Each γ indicates a different discrepancy function on the Cressie Read family $\phi(\pi) = \frac{\pi^{\gamma+1}-1}{(\gamma+1)\gamma}$. The Cressie Read bounds are obtained via dual utility maximization problems based on HARA functions parameterized by γ. The HJ bound is obtained by linearly projecting a generic admissible SDF on the subspace of observed returns. The Bond and Credit risk factors are respectively the Lehman U.S. Corporate AA and BAA Intermediate Bond Indexes. The stock market factor is the CRSP value-weighted NYSE, AMEX, and NASDAQ combined index. The FX factor is an equally-weighted portfolio of British, German and Japanese one-month eurocurrency deposits. The commodity factor is the Goldman Sachs Commodity Index. The four option factors come from Agarwal and Naik (2004) and represent at-the-money and out-of-the-money call and put options. For each CR estimator, each line is obtained by the ratio of the bound value estimated with all 9 risk factors including options over the bound value estimated with only the first 5 risk factors (bond, credit, stock, currency and commodities).

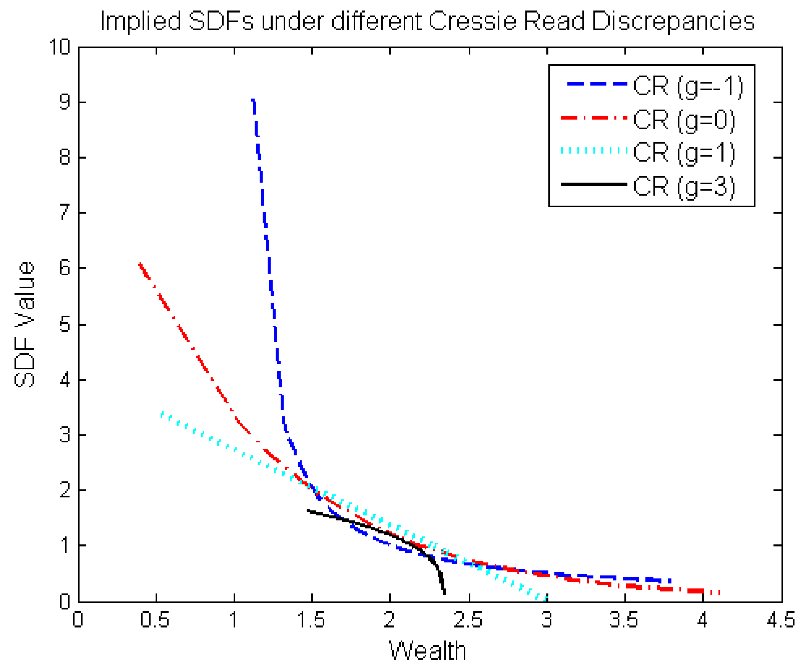

Figure 10: CR Stochastic Discount Factors as a Function of Aggregate Wealth

This picture presents CR stochastic discount factors based on monthly returns on a set of 5 risk factors over the period 1996:1 to 2004:3. Each γ indicates a different discrepancy function on the Cressie Read family $\phi(\pi) = \frac{\pi^{\gamma+1}-1}{(\gamma+1)\gamma}$. The Cressie Read SDFs are obtained via the first-order conditions of dual utility maximization problems based on HARA functions parameterized by γ. The SDFs have a fixed mean at 0.996. The Bond and Credit risk factors are respectively the Lehman U.S. Corporate AA and BAA Intermediate Bond Indexes. The stock market factor is the CRSP value-weighted NYSE, AMEX, and NASDAQ combined index. The FX factor is an equally weighted portfolio of British, German and Japanese one-month eurocurrency deposits. The commodity factor is the Goldman Sachs Commodity Index. The SDFs are plotted as a function of aggregate wealth defined by $W = W_0 * r_f + \lambda_{CR}^{opt}(R - r_f)$, where $(R - r_f)$ represents a vector with excess returns of the 5 risk factors, and W_0 is the initial wealth, arbitrarily chosen to be 2.

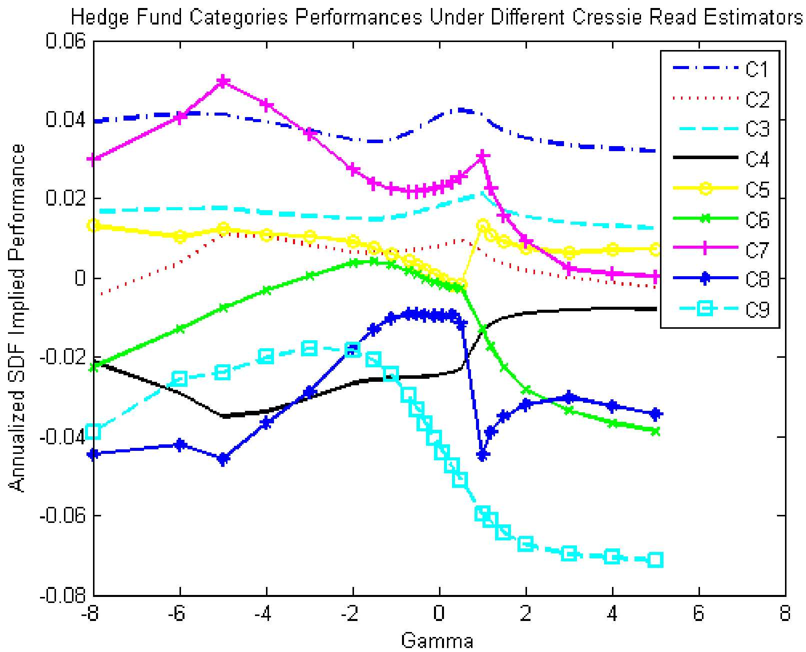

Figure 11: Hedge Fund Alphas for various Discrepancy Measures
This picture contains the alphas of different Hedge Fund categories (listed in Appendix A) obtained with implied SDFs from different Cressie Read estimators. Risk factors are composed by monthly returns over the period 1996:1 to 2004:3. The Bond and Credit risk factors are represented respectively by the Lehman U.S. Corporate AA and BAA Intermediate Bond Indexes. The stock market factor is represented by the CRSP value-weighted NYSE, AMEX, and NASDAQ combined index. The FX factor is represented by a equally-weighted portfolio of British, German and Japanese one-month eurocurrency deposits. The commodity factor is captured by the Goldman Sachs Commodity Index. Cressie Read estimators solve HARA utility maximization problems whose portfolios are linear combinations of the listed risk factors. A fixed SDF mean equal to 0.9962 is adopted.

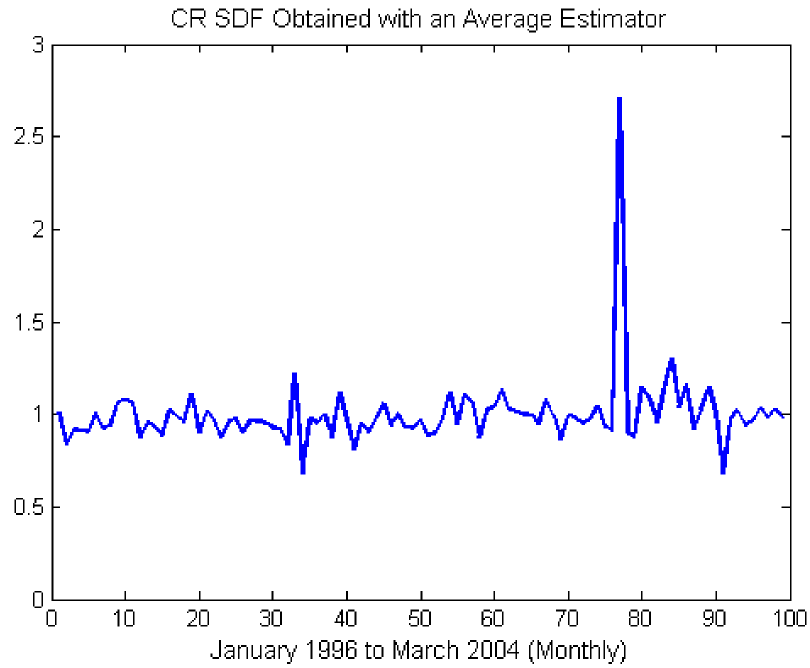

Figure 12: A SDF implied by a weighted HARA-utility maximization Problem
This picture presents a CR stochastic discount factor based on monthly returns on a set of 5 risk factors over the period 1996:1 to 2004:3. The Cressie Read SDF is obtained via the first-order conditions of a utility maximization problem based on the average of a set of HARA functions parameterized by $\gamma = \{-6, -2, -1.1, 0.1, 2, 4\}$. The SDF has a fixed mean at 0.996. The Bond and Credit risk factors are respectively the Lehman U.S. Corporate AA and BAA Intermediate Bond Indexes. The stock market factor is the CRSP value-weighted NYSE, AMEX, and NASDAQ combined index. The FX factor is an equally-weighted portfolio of British, German and Japanese one-month eurocurrency deposits. The commodity factor is the Goldman Sachs Commodity Index.