Behavioral Finance and the Pricing Kernel Puzzle: Estimating Risk Aversion, Optimism, and Overconfidence^{*}

Giovanni Barone-Adesi[†] Swiss Finance Institute and University of Lugano Loriano Mancini[‡] Swiss Finance Institute and EPFL Hersh Shefrin[§] Leavey School of Business Santa Clara University

This version: May 2, 2012

^{*}For helpful comments we thank Malcolm Baker, Rob Engle, Paul Glasserman, Harrison Hong, Tom Howard, Danling Jiang, Fabio Maccheroni, and Wei Xiong. Barone-Adesi and Mancini acknowledge the financial support from the Swiss National Science Foundation NCCR-FinRisk. Shefrin acknowledges a course release grant from Santa Clara University.

[†]Giovanni Barone-Adesi, Institute of Finance, University of Lugano, Via G. Buffi 13, CH-6900 Lugano, Switzerland, E-mail: giovanni.baroneadesi@usi.ch

[‡]Loriano Mancini, Swiss Finance Institute at EPFL, Quartier UNIL-Dorigny, CH-1015 Lausanne, Switzerland. E-mail: loriano.mancini@epfl.ch

[§]Corresponding author: Hersh Shefrin, Department of Finance, Santa Clara University, 500 El Camino Real, Santa Clara, CA 95053, USA, E-mail: hshefrin@scu.edu

Behavioral Finance and the Pricing Kernel Puzzle: Estimating Risk Aversion, Optimism, and Overconfidence

Abstract

We combine two approaches to the pricing kernel, one empirical and one theoretical, which relax the restriction that the objective return distribution and risk neutral distribution share the same volatility and higher order moments. The empirical approach provides estimates for the evolution of the pricing kernel projection onto S&P 500 returns for the period 2002 through 2009. The theoretical approach provides a framework for extracting estimates of sentiment from the results of the empirical analysis, along with estimates of risk aversion and time preference. These estimates of sentiment turn out to be highly correlated with external measures such as the Baker–Wurgler sentiment index, the Yale/Shiller crash confidence index, and the Duke/CFO survey responses. We analyze the manner in which the three external measures reflect biases such as excessive optimism and overconfidence. Our analysis points out three significant issues related to overconfidence. The first issue is that the Baker–Wurgler sentiment index robustly reflects excessive optimism, but not the component of overconfidence that is uncorrelated with excessive optimism. The second issue is that overconfidence is strongly related to the presence of an upward sloping portion in the graph of the pricing kernel, a key feature of the "pricing kernel puzzle." The third issue is that the time series properties of excessive optimism and overconfidence appear to generate a negative relationship between perceived risk and return.

Keywords: Sentiment, Risk Aversion, Pricing Kernel, Optimism, Overconfidence JEL Codes: G02, G12

1. Introduction

Two pricing kernel frameworks relax the restriction that the variances of the objective return distribution and risk neutral distribution are equal, along with other higher moments such as skewness and kurtosis. One is the empirical approach of Barone-Adesi, Engle, and Mancini (2008) (BEM), and the other is the theoretical, behavioral approach of Shefrin (2008). In this paper we combine the two frameworks to estimate how the pricing kernel decomposes into two components, one relating to fundamentals and the second relating to sentiment.¹

Notably, the relaxation of the "equal volatility constraint" is critical to the estimation of overconfidence in market prices. Our analysis points out three significant issues related to overconfidence. The first issue is that the Baker–Wurgler sentiment index does not reflect the component of overconfidence that is uncorrelated with excessive optimism. The second issue is that overconfidence is strongly related to the presence of an upward sloping portion in the graph of the pricing kernel, a key feature of the "pricing kernel puzzle" (e.g., Jackwerth (2000)). The third issue is that the time series properties of excessive optimism and overconfidence appear to generate a negative relationship between perceived risk and return. We discuss each in turn.

The most prominent treatment of sentiment to date is Baker and Wurgler (2006), who develop a series for sentiment from a principal component analysis of six specific sentiment proxies.² Although Baker and Wurgler mention several psychological heuristics and biases that underlie sentiment, such as excessive optimism, overconfidence, and representativeness, they emphasize excessive optimism in their analysis.³

We define sentiment using a change of measure that transforms the objective return probability density function (pdf) into the representative investor's return pdf. In this respect, excessive optimism occurs when the representative investor overestimates mean returns, and overconfidence occurs when the representative investor underestimates return volatility. Notably, our estimation methodology provides the opportunity to test the degree to which the Baker–Wurgler series only reflects excessive optimism, or whether it reflects overconfidence as well. Our analysis suggests that

¹In Barone-Adesi, Mancini, and Shefrin (2012) we apply this decomposition to analyze the connection between sentiment and systemic risk during the financial crisis that began in 2008.

²The six specific series are: turnover on the New York Stock Exchange (NYSE); dividend premium; closed-end fund discount; number and first-day returns on IPOs; and the equity share in new issues.

³Baker and Wurgler (2007) state: "we view investor sentiment as simply optimism or pessimism about stocks in general \dots " p. 132.

the Baker–Wurgler series robustly reflects excessive optimism, but attaches much less weight to the component of overconfidence that is independent of optimism. Moreover, the weight attached to overconfidence is not robust in respect to statistical significance.

To estimate the "objective" pdf, we apply a GARCH methodology to historical data. We note that the traditional asset pricing literature often treats the historically-based estimate as the "subjective beliefs of the representative investor." To be sure, we do estimate the subjective beliefs of a representative investor. However, we do so using the risk neutral pdf, not the historically-based pdf.⁴ In our view, state prices, as expressed through the risk neutral pdf and term structure of interest rates, reflect the aggregate beliefs of investors more directly than the historically-based estimate.

This difference in perspective leads us to estimate risk aversion differently from the traditional literature. The traditional literature uses the risk neutral pdf to infer the risk aversion function from the historically-based pdf estimate. Notably, the empirical risk aversion function typically obtained in this way neither has a traditional neoclassical structure nor takes on values that are plausible from a neoclassical perspective. In contrast, we estimate risk aversion by imposing a neoclassical structure, and then assessing whether the estimated values are indeed plausible. For the most part, our estimates of risk aversion fall within a range that appears to be plausible and consistent with the economics literature.

We employ our estimate of the risk aversion function as an input to a neoclassical change of measure in which a pricing kernel transforms the risk neutral pdf into the representative investor's subjective pdf (beliefs). Of course, this approach contrasts with the traditional treatment of using the historically-based estimate as the representative investor's beliefs. Indeed, we use the difference between the historically-based estimate and the risk neutral-based estimate as the basis for computing sentiment. The degree to which our approach is reasonable seems to us as essentially an empirical issue. If our approach produces estimates of sentiment that correlate well with independent measures of sentiment based, for example, on closed-end fund discounts, IPO prices, dividend premiums, and investor surveys, then we would suggest that the approach we offer has merit. If the correlations are weak or nonexistent, then the approach is likely to have little merit. As it

⁴In the body of the paper, we describe two equivalent techniques for estimating the representative investor's pdf. One applies a change of measure to the objective pdf. A second applies a change of measure to the risk neutral pdf. The behavioral aspects are more clearly discerned using the first approach. However, the second approach is useful for the discussion in this section.

happens, our estimates of sentiment are highly correlated with other sentiment indicators such as the Baker–Wurgler series and survey data that we discuss below.

In addition to the findings about the Baker–Wurgler series and overconfidence, our analysis yields insights into two other issues about the impact of overconfidence on asset pricing. One issue pertains to the phenomenon known as the "pricing kernel puzzle," and the other to the sign of the relationship between risk and return. We discuss each in turn.

The term "pricing kernel puzzle" refers to the empirical finding that the pricing kernel features an upward sloping portion. Shefrin (2008) discusses how overconfidence can impart an inverted Ushape to the pricing kernel function, which of course features an upward sloping portion.⁵ Ziegler (2007) develops a heterogeneous beliefs continuous time model, with both diffusion and jump components. He allows the degree of optimism to vary across investors, but restricts overconfidence to the jump component.⁶ Ziegler argues that the degree of heterogeneity about expected returns necessary to resolve the pricing kernel puzzle is unrealistic.⁷ If so, this raises the question of whether the pricing kernel puzzle is mainly a reflection of overconfidence.⁸ Notably, we document that during particular time periods the pricing kernel does indeed feature an upward sloping portion, and that this coincides with overconfidence being high. To ensure that this finding is not just an artifact of our methodology, we employ two external measures related to overconfidence, the "crash confidence index," that was developed by Robert Shiller, and the Duke/CFO survey data. We conclude that overconfidence is an important feature of sentiment that impacts the shape of the pricing kernel, but which is not fully captured in the Baker–Wurgler series.

The third issue about overconfidence pertains to one of the main neoclassical tenets of pricing theory, namely that risk and return are positively related. In this regard, there is a behavioral

⁸In other words, do investors fail to learn the correct return standard deviation, even for just the diffusion component?

⁵The framework in Shefrin (2008) involves a discrete time heterogeneous investor model.

⁶Ziegler assumes that all investors know the correct time invariant standard deviation of the diffusion process because they are able to deduce it from the quadratic variation of the historical consumption stream. See p. 864 of his article. In contrast, Epstein and Ji (2012) develop a continuous time ambiguous beliefs model in which investors cannot be certain they have a correctly specified model of conditional returns. The ambiguous beliefs perspective is closer to our own. In our GARCH formulation, we allow for time variation of return standard deviation, which investors can possibly estimate with bias.

⁷Hens and Reichlin (2011) discuss three possible explanations for the pricing kernel puzzle, namely prospect theory preferences, heterogeneous beliefs, and incomplete markets. The explanation discussed in this paper focuses on heterogeneous beliefs, especially overconfidence. However, we do not contend that this is the only explanation, and acknowledge other possible explanations such as incomplete markets, or some mixture of neoclassical and behavioral elements. For example, Song and Xiu (2012) estimate nonparametric state price densities implicit in the S&P 500 and VIX options, together with their interactions, to provide insight into the empirical pricing kernel.

literature suggesting that some investors and corporate executives might perceive this relationship as being negative. See Ganzach (2000) who discusses investors' perceptions, Graham and Harvey (2001) who discuss corporate executives' perceptions for a one-year forecasting horizon,⁹ and Baker and Wurgler (2007) who analyze the cross-section of stocks. Shefrin (2008) provides additional evidence and suggests that the negative relationship is connected with excessive optimism and overconfidence being positively correlated.

We investigate the correlation issue in our framework in two ways. First, we update the Graham– Harvey negative risk-return correlation finding from 2001 and show that it has persisted through 2011. Notably, the correlations between Graham and Harvey's survey results and our representative investor estimates are quite strong, which we take as additional corroboration for our approach. Second, we examine the relationship between risk and expected return in the time series for both the representative investor's pdf series and the objective pdf series. Although the relationship appears to be positive for the objective pdf series, we find evidence of time periods in which it is negative for the representative investor's pdf series. Notably, the relationship between excessive optimism and overconfidence is strong and positive throughout our sample period. This feature, in combination with the biases being large, turns out to be key to driving the perception that risk and return are negatively related in our framework. We note that there are subperiods in our data during which the perceived relationship is indeed positive.

Overconfidence figures prominently in the recent behavioral asset pricing literature featuring heterogeneous beliefs. In this regard, see the overconfidence model in Scheinkman and Xiong (2003), the insightful explanation for empirical patterns in the term structure of interest rates in Xiong and Yan (2010), and the impact of investor disagreement on stock returns in Yu (2011).

The remainder of the paper is organized as follows. Section 2 presents the intuition underlying our approach. Section 3 describes our methodology for estimating the empirical pricing kernel. Section 4 reviews the main theoretical framework for analyzing the behavioral structure of the pricing kernel. Section 5 discusses the estimation procedure in our empirical approach. Section 6 presents our results. Section 7 relates our findings to the Baker–Wurgler sentiment series, the Yale/Shiller crash confidence index and the Duke/CFO survey responses. Section 8 discusses the

⁹Graham and Harvey study both a one-year forecast horizon and a ten-year forecast horizon. They report that the relationship between risk and return is positive for the ten-year horizon, which they emphasize in their subsequent updates to Graham and Harvey (2001).

representative investor's tendency to perceive that risk and return are negatively related. Section 9 concludes.

2. Intuition Underlying Our Approach

To develop the intuition underlying our approach to the connection between psychological biases and asset pricing, we provide a brief nontechnical introduction. Our starting point is the standard neoclassical framework in which equilibrium prices are set as if by a representative investor holding correct beliefs. The objective probability density function (pdf) associated with correct beliefs, is depicted in the top panel of Figure 1, and is labeled "Pobj."

In a behavioral framework, equilibrium prices are also set as if by a representative investor, but one whose beliefs possibly reflect biases in the investor population. Because of limits to arbitrage, investor biases are not necessarily eliminated in equilibrium. In the top panel of Figure 1, the function Prep denotes the pdf of the representative investor exhibiting two biases, excessive optimism and overconfidence. Relative to the objective pdf Pobj, excessive optimism means that the representative investor overestimates expected return. Overconfidence means that the representative investor underestimates return standard deviation. In Figure 1, notice that the mode of Prep is to the right of the mode of Pobj, and Prep attaches much less weight to tail events than Pobj.

Theoretically, excessive optimism is defined as expected return under Prep minus expected return under Pobj. Overconfidence is defined as return standard deviation under Pobj minus return standard deviation under Prep. Operationally, we estimate Pobj and Prep and then compute excessive optimism and overconfidence from their first and second moments. To estimate Pobj, we use a structural model based on historical returns for the S&P 500. To estimate Prep, we use S&P 500 index option prices (SPX) and the risk-free rate to infer the risk neutral pdf, and then apply a pricing kernel-based change of measure.

The pricing kernel lies at the heart of our process for inferring Prep. The pricing kernel is a function whose values are ratios of state prices to probabilities, which in this case we take to be objective probabilities Pobj. The bottom panel of Figure 1 displays three functions. The function "CRRAKernel" is the pricing kernel from a neoclassical representative investor model with CRRA preferences. As usual, the function is monotone decreasing, and measures intertemporal marginal

rate of substitution.

In contrast to "CRRAKernel," the function "BehavKernel" in Figure 1 depicts a pricing kernel associated with a representative investor whose beliefs exhibit excessive optimism and overconfidence. Notice how overconfidence manifests itself in tail events where the "BehavKernel" function lies below "CRRAKernel," as the behavioral representative investor underestimates tail event probabilities. Notably, in this example, the degree of overconfidence leads "BehavKernel" to feature an upward sloping portion in the left region of the figure. For the middle range, the combination of biases leads "BehavKernel" to lie above "CRRAKernel," so that "BehavKernel" has the shape of an inverted-U.

We use estimates of "BehavKernel," "CRRAKernel," and their difference to provide information which allows us to infer values for excessive optimism and overconfidence, and to disentangle their manifestation within prices. To estimate "BehavKernel" we use the ratio of the estimate of the risk neutral pdf to our estimate of the objective pdf. To estimate "CRRAKernel" we use a technique described later in the paper. To capture the differences between the two pricing kernels, we use the log of "BehavKernel" minus the log of "CRRAKernel," which is displayed as the function "LogDiff" in the bottom panel of Figure 1. We provide an exact interpretation of "LogDiff" later in the paper.

Our empirical measures of excessive optimism and overconfidence are computed relative to a process estimated from historical returns. We do not contend that historical returns are completely free of investor bias. Instead we investigate the extent to which market prices accurately reflect an econometrician's best estimate of future returns.¹⁰

The representative investor always holds the market portfolio, and therefore does not "lose money" because of biases. To the extent that the representative investor corresponds to a real investor, the biases cause the representative investor to be disappointed and surprised. Optimism leads to disappointment in the realized risk premium, and overconfidence leads to surprise about

¹⁰In the theoretical framework underlying our analysis, biases are defined relative to a market in which all investors hold objectively correct beliefs about the stochastic process governing aggregate consumption growth. Although biases impact the return distribution of the market portfolio, the magnitude of the impact is small, if not zero. See Theorem 17.2 in Shefrin (2008). In practice, the S&P 500 is often employed as a proxy for the market portfolio. This point is important because biases might have a greater impact on the S&P 500 than the market portfolio. Methodologically, we do not control for sentiment in estimating Pobj from historical returns. Instead we use external sentiment data to help us ascertain whether our working assumptions are reasonable or not.

the amount of volatility.¹¹

3. Formal Methodology for Estimating the Empirical SDF

By pricing kernel we mean a stochastic discount factor (SDF) defined as state price per unit objective probability. In order to estimate the empirical SDF, we use the same approach as in BEM. The SDF associated with returns between date t and date T, conditional on the information available at date $t \leq T$, is denoted by $M_{t,T}$. Throughout the paper, (T - t) is fixed and equal to one year. The SDF is estimated semiparametrically as

$$M_{t,T} = e^{-r_f (T-t)} \frac{q(S_T/S_t)}{p(S_T/S_t)}$$
(1)

where q is the risk-neutral density, p the objective (i.e., historical) density, S the S&P 500 index, and r_f is the instantaneous risk-free rate. The densities p and q are conditional on the information available at date t, but for ease of notation we omit such a dependence.¹² When no confusion arises, subscripts will be omitted.

Call and put options data are used to estimate the risk neutral density q and stock data are used to estimate the objective density p. We briefly recall the method for estimating p and q that was developed in BEM. For each Wednesday in our sample, we estimate two asymmetric Glosten, Jagannathan, and Runkle (1993) (GJR) GARCH models. A GJR GARCH model is fitted to historical daily log-returns of the S&P 500 to capture the index dynamic under the objective or historical distribution with pdf p. The model has the form

$$\log(S_t/S_{t-1}) = \mu_t + \epsilon_t \tag{2}$$

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha \epsilon_{t-1}^2 + \gamma I_{t-1} \epsilon_{t-1}^2$$
(3)

where $\epsilon_t = \sigma_t z_t$, z_t is the standardized historical innovation and $I_{t-1} = 1$ when $\epsilon_{t-1} < 0$, and $I_{t-1} = 0$ otherwise. BEM assumed that μ_t is just the historical average of 7%. In this paper, we

¹¹In heterogeneous investor models, there are complex issues about the degree to which the presence of informed investors leads biases to become smaller with time. Although important, this issue is peripheral to the scope of the paper.

paper. ¹²Specifically, the notation $q(S_T/S_t)$ does not reflect that our GARCH-based estimates of such a density, presented below, are conditional on the available history of stock prices, S_0, S_1, \ldots, S_t . The same remark applies to $p(S_T/S_t)$.

use a different specification for μ_t that is discussed below. The estimation is obtained via Gaussian Pseudo Maximum Likelihood; see, e.g., Gourieroux, Monfort, and Trognon (1984).

Another GJR GARCH model is calibrated to the cross section of out-of-the-money options on S&P 500 capturing the index dynamic under the risk neutral or pricing distribution with pdf q. This model has the same form as (2) and (3), but with different parameter values for ω , β , α , and γ , and the risk neutral drift term μ^* selected so that the daily expected return of the S&P 500 is equal to the risk-free return e^{r_f} . The calibration is achieved via non-linear least squares, i.e., minimizing the sum of squared pricing errors with respect to the risk neutral GARCH parameters. Pricing errors are defined as the difference between GARCH-based and market option prices. Then, for each Wednesday t, the conditional densities $p(S_T/S_t)$ and $q(S_T/S_t)$ of the S&P 500 are estimated by Monte Carlo Simulation. We use the Empirical Martingale Simulation method of Duan and Simonato (1998) and simulate 50,000 trajectories of daily S&P 500 returns from t to T, where (T-t) corresponds to one year. Transition densities p and q are obtained by nonparametric kernel density estimation, i.e., smoothing the corresponding simulated distribution. In this approach, first and second moments for annual returns under p and q respectively are based on GARCH models estimated at daily frequencies.

We consider two GJR GARCH models under the risk neutral density q, one we call Gauss and the other we call FHS. The difference between the two procedures is as follows: The Gaussian model uses randomly drawn Gaussian innovations for the simulation, whereas the FHS model uses the historical, nonparametric innovations obtained from the estimation of (2) and (3) for p; BEM point out that "using nonparametric innovations instead of Gaussian innovations induces an overall improvement in pricing options." p. 1247. We use both models in order to contrast the difference that FHS makes.

4. Theoretical Framework

Sentiment impacts the SDF, defined as the function state price per unit objective probability, by distorting state prices relative to a neoclassical counterpart. Therefore, a behavioral SDF effectively decomposes into a neoclassical component and a sentiment distortion. In a neoclassical framework featuring constant relative risk aversion (CRRA), the SDF has the following form:

$$M_{t,T}(\theta) = \theta_0 \left(S_T / S_t \right)^{-\theta_1} \tag{4}$$

In (4), M is the pricing kernel, t and T are indexes for time, S is a proxy for the value of the market portfolio, θ_0 is a discount factor measuring the degree of impatience and θ_1 is the coefficient of relative risk aversion, and $\theta = (\theta_0, \theta_1)$.

The logarithmic version of (4) is

$$\log(M_{t,T}(\theta)) = \log(\theta_0) - \theta_1 \log(S_T/S_t)$$
(5)

In Shefrin (2008), (5) generalizes to include an additional term Λ_t to reflect the impact of sentiment, so that the equation for the log-SDF becomes:¹³

$$\log(M_{t,T}) = \Lambda_t(S_T/S_t) + \log(\theta_{0,t}) - \theta_{1,t}\log(S_T/S_t)$$
(6)

Appendix A sketches a derivation of (6). In Shefrin's framework, Λ_t is a scaled log-change of measure, where the change of measure transforms the objective return pdf p into the representative investor's return pdf p_R . In other words, the function e^{Λ_t} is proportional to the change of measure p_R/p so that

$$p_R = p \, e^{\Lambda_t} \, \theta_{0,t,p} / \theta_{0,t} \tag{7}$$

where $\theta_{0,t,p}$ is a rescaling of $\theta_{0,t}$ whose purpose is to ensure that p_R integrates to unity. In the discussion below, we sometimes omit the *t*-subscript in Λ_t to simplify the exposition.

The log-change of measure $\log(p_R/p)$ specifies the percentage error in probability density which the representative investor assigns to the occurrence of a specific return. For example, suppose that the representative investor underestimates by 2% the probability that the market return will be 1%. In this case, the log-change of measure at 1% will be -2%.

In a Gaussian framework, a log-linear change of measure generates a variance preserving shift in mean (with the form $x\mu - \frac{1}{2}\mu^2$). If the mean shifts to the right by μ , the log-change of measure is a positively sloped linear function which, when applied to p, shifts probability mass from low

 $^{^{13}\}Lambda_t$ is a function of T as well as t and S_T/S_t , but we omit the T-subscript for ease of notation. In addition, $\theta_{1,t}$ is theoretically a function of T and S_T/S_t , and $\theta_{0,t}$ is a function of T. However, for simplicity we omit the explicit reference to T and treat $\theta_{1,t}$ as a constant for fixed t.

values to high values. If the mean shifts to the left, the log-change of measure is a negatively sloped linear function. To put it another way, a positively sloped log-linear change of measure gives rise to excessive optimism, while a negatively sloped log-linear change of measure gives rise to excessive pessimism.

If the log-change of measure is non-linear, then applying the change of measure impacts the second moment. A log-change of measure with a U-shape shifts probability mass from the center to the tails, thereby increasing the variance. A log-change of measure with an inverted U-shape shifts probability mass from the tails into the center, thereby lowering the variance. To put it another way, a U-shape gives rise to underconfidence, whereas an inverted U-shape gives rise to overconfidence. With respect to (6), if Λ is large enough, then the shape of the sentiment function will dominate the shape of the fundamental component. For example, if the log-change of measure has an inverted U-shape which is sufficiently strong, then Λ will overpower the other terms in (6), and dominate the shape of the log-SDF.

If the market reflects a mix of optimists and pessimists with optimism and overconfidence being positively correlated, then log-sentiment can feature an oscillating pattern which is sharply downward sloping in the left tail, upward sloping in the middle region, and downward sloping in the right tail. It is this shape which characterized the empirical findings for the shape of the pricing kernel in the work of Aït-Sahalia and Lo (2000) and Rosenberg and Engle (2002).

In neoclassical pricing theory, the risk neutral pdf q can be obtained from the objective pdf p by applying a change of measure using the normalized pricing kernel; see p. 51 of Cochrane (2005). Of course, this relationship can be inverted to express p as a function of q. In the behavioral framework, an analogous relationship holds between the representative investor's pdf p_R and q, rather than between p and q. In Shefrin (2008), the expression for p_R as a function of q is:

$$p_R(S_T/S_t) = q(S_T/S_t) \left(S_T/S_t\right)^{\theta_{1,t}} E_t^{p_R}[(S_T/S_t)^{-\theta_{1,t}}]$$
(8)

where $E_t^{p_R}$ is the time-t conditional expectation with respect to p_R .

5. Estimation Procedure

Options and stock data are from OptionMetrics, filtered as in BEM, and cover the period January 2002 to October 2009. On each Wednesday t, we estimate two versions of $M_{t,T}$ for the fixed horizon (T-t) = 1 year. Each version corresponds to a particular GJR GARCH model for the risk neutral density q. One GJR GARCH model driven by Gaussian innovations, and another GJR GARCH models and corresponding estimates of SDF simply as Gauss and FHS methods, respectively.

5.1. Drift Term

In estimating (2), we base μ_t on the earnings-to-price ratio (E/P), the inverse of the P/E ratio developed by Campbell and Shiller (1998) for U.S. stocks. This ratio features a stock price index P divided by an average of earnings over the prior ten years. Campbell and Shiller call their ratio the cyclically adjusted price-earnings ratio (CAPE). Here both price and earnings are adjusted for inflation. The key result of their analysis is that subsequent ten-year returns to stocks are negatively and statistically related to P/E.

Campbell and Shiller suggest that P/E reflects sentiment. When investors become "irrationally exuberant," prices rise relative to earnings in an unwarranted manner. That is, future returns are low because current prices are too high. Updated data series are available from Robert Shiller's website.¹⁴

For the market as a whole, E/P can be interpreted as expected steady state long-run return, as the present value of growth opportunities for the market is zero. A value of P/E equal to 25 corresponds to an expected long-term return of 4%. As a consistency check, we regress subsequent annualized ten-year returns for the Campbell–Shiller series on E/P. The regression equation for annualized return is 0.012 + 0.76 E/P. We also regressed returns just for the S&P 500 on E/Pand obtained the regression equation -0.022 + 1.21 E/P. In this regard, we ignore the issue of overlapping intervals in the estimation itself, as the bias strikes us as minor.

In our analysis, we used CAPE as the basis for expected return in estimating the objective pdf p. We report the results for the regression equation in which expected return is given by 14 http://www.econ.yale.edu/~shiller/

0.012 + 0.76 E/P. However, results based on the other specifications are similar in most respects.¹⁵

5.2. Decomposition of log-SDF

We use linear regressions to decompose the log-SDF into its constituent components, the sentiment function Λ and a fundamental component corresponding to a neoclassical SDF. Specifically, we fit a (constrained) CRRA-SDF to the unconstrained semiparametric SDF. The least square regression of the unconstrained or empirical pricing kernel $M_{t,T}$ on the CRRA pricing kernel $M_{t,T}(\theta)$ gives the closest (in least square sense) neoclassical CRRA pricing kernel to the empirical one. In using this procedure, we choose a decomposition that gives maximum weight to the neoclassical component. We then interpret the residuals from this regression as an estimate of the sentiment function Λ .

Given (5), the regression is run in log-log space because in this space the CRRA pricing kernel is linear. For each Wednesday t in our sample, we obtain a grid of 100 values of gross returns, $S_T^{(i)}/S_t$, i = 1, ..., 100, and regress the log of the unconstrained pricing kernel, $\log(M_{t,T}^{(i)})$, on the log gross return, $\log(S_T^{(i)}/S_t)$, using the 100 sample points.¹⁶ The time horizon (T - t) is fixed and equals one year. Then, we compute the pointwise difference, d_t , between the unconstrained and CRRA-constrained pricing kernel. For each gross return, this difference is defined as

$$d_t^{(i)} = \log(M_{t,T}^{(i)}) - \log(M_{t,T}^{(i)}(\theta))$$
(9)

where $M_{t,T}$ is the unconstrained SDF and $M_{t,T}(\theta)$ the CRRA-constrained SDF. The differences, $d_t^{(i)}$, i = 1, ..., 100, provide an estimate of the sentiment function Λ_t over the support of gross returns, $S_T^{(i)}/S_t$, i = 1, ..., 100.¹⁷ We repeat this procedure each Wednesday t between January 2002 and October 2009 and obtain the weekly time series $\theta_{0,t}$ and $\theta_{1,t}$ as well as a weekly series of the sentiment function Λ_t .

To measure the degree to which the unconstrained-SDF conforms to the CRRA-restriction, we

¹⁵The alternative specifications all have the form $\mu_t = \alpha + \beta (E/P)_t$, including the case $\mu_t = 0.07$ used in BEM. The latter specification sharply reduces the estimate of excessive optimism, but leaves it highly volatile. The estimates of overconfidence are robust to the alternative specifications.

¹⁶This procedure is different than Rosenberg and Engle (2002). They calibrated the constrained CRRA-SDF directly to option prices, whereas we fit the constrained CRRA-SDF to the unconstrained semiparametric-SDF. The two procedures give the same results when the true SDF conforms to CRRA.

¹⁷We note that $d_t^{(i)}$ is a function of T as well as t, as is also the case for Λ_t , for which it serves as an estimator.

use the following two distance measures for d_t :

$$\text{RMSE}_t = 100 \sqrt{\frac{1}{n} \sum_{i=1}^n (d_t^{(i)})^2}, \qquad \text{MAE}_t = 100 \frac{1}{n} \sum_{i=1}^n |d_t^{(i)}| \qquad (10)$$

which resemble traditional root mean square error and mean absolute error, expressed in percentage terms.

5.3. Range of Gross Returns

The pricing kernel is computed as the ratio of the risk neutral density, discounted by the riskfree rate, to its objective counterpart. In estimating the pricing kernel, we consider the range of gross returns approximately between 0.69 to 1.35. This range spans about six standard deviations around the mean for gross return. The range is large but, of course, is not the entire support of the objective and risk neutral densities, which comprise the inputs to computing the pricing kernel. Outside this range, the unconstrained SDF can be quite unstable and outlying observations in the unconstrained SDF can distort the least square fit of the CRRA-constrained SDF.

6. Empirical Results

Output from the estimation procedure described above consists of weekly estimates for the objective and risk neutral GJR GARCH parameters, the SDF $(M_{t,T})$, CRRA $(\theta_{1,t})$, time preference $(\theta_{0,t})$, the objective return pdf $(p(S_T/S_t))$, the risk neutral pdf $(q(S_T/S_t))$, the representative investor's pdf $(p_R(S_T/S_t))$, and sentiment (Λ_t) . Below we describe our main findings.

6.1. GARCH Estimation and Calibration

Table 1 shows estimation and calibration results for the GJR GARCH model in (2)–(3). The findings are in line with what reported in BEM. Although some GARCH parameter estimates exhibit a certain time-variation, the persistency of the GARCH volatility and its long-run mean are estimated quite precisely. Risk neutral GARCH volatilities appear to be larger and less persistent on average than objective GARCH volatilities.¹⁸ These findings are well in line with a recent

¹⁸We compared our risk neutral pdf estimates with Birru and Figlewski (2012), who use a shorter time to expiration than we do. Notably, the general patterns we find appear to be similar to those in Birru and Figlewski (2012).

literature on variance swap contracts; see, e.g., Aït-Sahalia, Karaman, and Mancini (2012) and references therein.

6.2. Shape of the Pricing Kernel Over Time

Figures 2 and 3 display the SDF estimated on each Wednesday from January 2002 to October 2009 using Gauss and FHS methods.

Consider Figure 2 which displays the evolution of the estimated pricing kernel for the Gaussian case over the course of the sample period. This figure provides a three-dimensional bird's eye view of how the shape of the Gaussian pricing kernel changed with time.

In Figure 2, at the beginning of the period, the (Gaussian) pricing kernel featured a declining pattern. By December of 2003, the pricing kernel featured a U-shape. During 2005, the shape of the pricing kernel had changed to an inverted-U. In 2009, the pricing kernel became steeper, similar to what it had been at the beginning of the sample period.

Figure 3 displays similar time series properties for the pricing kernel estimated using the FHS methodology. At the beginning of the sample period, the FHS-pricing kernel features a U-shape. Six months later, the pricing kernel has become flatter, and is no longer U-shaped. In early 2003, the pricing kernel has become quite flat at the left. By 2004, the left portion has declined, and the shape of the pricing kernel has become an inverted-U, a shape which essentially persists into 2008. In 2009, the shape has reverted to the pattern from 2003, flat at the left and then declining.

6.3. CRRA-Constrained Pricing Kernel

Figure 4 displays the time series estimates of θ_0 and θ_1 when the pricing kernel is constrained to conform to the case of a representative investor with correct beliefs and CRRA utility. In standard theory, θ_0 is the discount factor for time preference and θ_1 is the coefficient of relative risk aversion.

Meyer and Meyer (2005) survey some of the key studies by economists of how the coefficient of relative risk aversion varies across the population. Most of the survey data suggests values that lie between 0.23 and 8. Meyer and Meyer perform an adjustment to reconcile the scales used by the various studies and suggest an adjusted range of 0.8 to 4.72. We view this as a plausible range within which to evaluate our estimates.

The time series for the CRRA-coefficient, θ_1 , mostly varies between 0 and 3.1, with a mean of

1.14. During the middle of the sample period, θ_1 lies between 1 and 3.1. However, at the beginning and end of the sample period, which correspond to recessions, θ_1 falls between 0 and 1, and even dips below 0 in November 2007. For the most part, these values are quite realistic, and fall within the range described in Meyer and Meyer (2005).

One of the papers surveyed by Meyer and Meyer is Barsky, Juster, Kimball, and Shapiro (1997), hereafter BJKS. In addition to investigating risk aversion, BJKS also conduct surveys to identify time preference. They find considerable variation, but point out at zero interest rates, the modal household expresses a preference for flat consumption over time, and the mean household expresses a preference for increasing consumption over time. BJKS report a consumption growth range of 0.28% to 1.28% per year, which they interpret as negative time discounting. Their study provides the backdrop for discussing our estimates of time preference.

Shefrin (2008) discusses how in models with heterogeneous beliefs, there is an aggregation bias for θ_0 when $\theta_1 \neq 1$.¹⁹ To address this bias, we restrict attention to periods in which θ_1 lies between 0.9 and 1.1. For these periods, the mean value of θ_0 is 1.02, which is slightly above the range reported by BJKS.²⁰ That is, our estimates of time preference generally reflect negative discounting, but to a greater degree than the results reported by BJKS. There is also considerable time series variation, as the conditional minimum value for θ_0 is 1.01 and the conditional maximum value is 1.06.

The time series for the unconditional time preference variable θ_0 tends to lie between 0.99 and 1.04 during the early portion of the sample period, but in late 2004 gravitated to the range between 1.05 and 1.17, peaking in February 2005. The θ_0 -series then declined back to the range 1.05 to 1.11 until November 2007, when it declined sharply to the range 0.97 to 1.1. After the Lehman bankruptcy in September 2008, θ_0 rose sharply to 1.3 in October, and then declined back to the region around 1.0 from December on.

In typical asset pricing models, the risk-free rate of interest is negatively related to the rate of time discount, negatively related to return variance, and positively related to expected return.²¹

¹⁹ Jouini and Napp (2006) establish that the time preference parameter θ_0 is negatively related to θ_1 for $\theta_1 > 1$ and positively related to θ_1 for $\theta_1 < 1$. One surprise is that the signs in this relationship are opposite in our results. This feature might stem from the degree of belief dispersion, a variable for which we do not control. However, this is just a conjecture on our part.

²⁰The values for θ_0 here refer to $\theta_{0,t,p}$ associated with the normalization to ensure that the representative investor's pdf integrates to unity.

²¹Given normally distributed log-returns, Campbell, Lo, and MacKinlay (1997) establish that the risk-free rate r_f is given by $r_f = -\log(\theta_0) - \theta_1^2 \operatorname{Var}_t^p[\log(S_T/S_t)]/2 + \theta_1 E_t^p[\log(S_T/S_t)]$, where r_f is the log-gross risk-free rate, and $E_t^p[\log(S_T/S_t)]$ and $\operatorname{Var}_t^p[\log(S_T/S_t)]$ are the objective time-t conditional mean and variance log-return, respectively.

Inverting the relationship implies that the time preference parameter is negatively related to the risk-free rate, negatively related to return variance, and positively related to expected return. Therefore, we would expect the time preference parameter θ_0 to be positively related to both overconfidence and optimism, and negatively related to the risk-free rate. In this regard, consider an AR(2) regression of log(θ_0) on the interest rate, objective expected return, optimism, and overconfidence, all interacted with θ_1 , and objective return variance interacted with θ_1^2 . Although all the coefficients from this regression feature the expected signs, only the coefficients for objective return standard deviation and overconfidence turn out to be statistically significant.

6.4. RMSE and MAE for the Difference Measure d_t

Consider the time series for RMSE and MAE in (10), the two measures to summarize the function d_t in (9), representing the difference between the logarithms of the unconstrained pricing kernel estimate and its CRRA-constrained counterpart. Under the assumption that the constrained estimate proxies for the pricing kernel that would be in effect were all investors to hold correct beliefs, RMSE and MAE can be interpreted as measures of sentiment. Figure 5 displays the two series for the FHS-based estimation. The values of the two series are lowest in the early and late portions of the sample period, and highest in the middle portion. Notably, the magnitude of sentiment drops sharply during the second quarter of 2007 and again in the aftermath of the Lehman bankruptcy.

6.5. Estimates of Representative Investor's Beliefs p_R

Equations (7) and (8) provide the theoretical basis for estimating the beliefs p_R of the representative investor. Equation (7) implies that d_t can be interpreted as a scaled estimate for the sentiment function Λ_t . Therefore e^{d_t} can be interpreted as being proportional to a change of measure which transforms the objective density p into the representative investor's density p_R .²²

²²Figure 1 captures the general shape of the results during the middle portion of our sample period, for which a typical date is 21/12/2005. For this date, the d_t function is positive between 0.99 and 1.16, and negative outside this interval. This implies that a change of measure based on d_t , when applied to p, will shift probability mass to the region [0.99, 1.16] from the extremes. The modes of p and p_R are about the same, but the mass of p is more spread out than the mass of p_R .

6.5.1 Means and Standard Deviations

Figure 6 displays, for the FHS case, the time series for the objective mean and standard deviation of the market return, along with the difference between the means and standard deviations of the objective density and the representative investor's density, i.e., optimism and overconfidence.²³ Notice that in the middle of the sample, the representative investor is excessively optimistic and overconfident, judging the expected return as too high and the future volatility as too low. Notably, this pattern is reversed at the beginning and end of the sample period, when the representative investor is pessimistic and at times underconfident, especially in 2009.

For much of our sample period, the estimates of excessive optimism and overconfidence are economically large. The implications for valuation are also economically significant. Roughly speaking, excessive optimism will cause the S&P 500 to be misvalued by the ratio of excessive optimism to the net objective expected return.²⁴ As a fraction of the objective expected return, excessive optimism bias fluctuates in the range -62% at the bottom of the market in March 2009 to 60% in February 2007. Overconfidence reduces the required return, and correspondingly increases the value of the asset. Roughly speaking, the ratio of fundamental value to market value will be one minus the ratio of overconfidence to objective return standard deviation.²⁵ As a fraction of the objective return standard deviation, overconfidence bias fluctuates in the range -16% in December 2002 to 40% in February 2007.

With the exception of the period following the Lehman bankruptcy in September 2008, both optimism and overconfidence generally rose and fell with the market. The correlation coefficient for the two variables is 0.48. We note that the standard deviation of the risk neutral pdf features the

²³The respective means are $E_t^p[S_T/S_t - 1]$ which is based on the objective density p, and $E_t^{p_R}[S_T/S_t - 1]$ which is based on the representative investor's pdf p_R , where E_t denotes the time-t conditional expectation. The respective standard deviations are $\sqrt{\operatorname{Var}_t^p[S_T/S_t]}$ and $\sqrt{\operatorname{Var}_t^{p_R}[S_T/S_t]}$.

²⁴This statement is based on the equation P = E/r + PVGO, where P is the current asset price, E denotes expected earnings, PVGO is the present value of growth opportunities, and r is the expected (more precisely, required) return on equity. This equation derives from the Gordon discounted dividend stream formula P = D/(r - g), where D denotes dividends and g denotes growth rate. If we take r as the required return and associate excessive optimism to E and PVGO, then excessive optimism leads to a higher valuation for the asset. Define A = rPVGO as the annualized cash flow associated with PVGO, so that P = (E + A)/r. If we take E and A as given, and equate r to expected return, it follows that the extent of misvaluation due to optimism bias depends on the ratio of the representative investor's expected return to the objective expected return, with the former being equal to the latter plus excessive optimism. In this case, excessive optimism increases the discount rate and lowers the value of the asset. It seems more reasonable to associate excessive optimism with biases in the estimation of E and PVGO, and overconfidence with biases in r.

²⁵The argument is based on the Gordon formula in Footnote 24 for fixed E.

same general pattern as the representative investor's pdf. In particular, the risk neutral standard deviation tends to lie above the objective standard deviation during market declines, and fall during market increases.

During market increases, increased optimism leads the SDF to stay bounded away from zero. This occurs because in the right tail, there is more probability weight in the risk neutral pdf than the objective pdf, and so the ratio of the risk neutral to the objective will not fall to zero when computing the SDF. Notably, the corresponding rise in overconfidence dampens the effect, but does not eliminate it.

6.5.2 Skewness and Kurtosis

As Figure 7 shows, both the objective pdf and representative pdf feature negative skewness, in time series patterns which are essentially U-shaped. In respect to bias, the representative pdf is less skewed than the objective pdf. Skewness is least pronounced at the beginning and end of our sample period when returns were negative. Indeed, both objective and representative investor skewness turned positive during the final quarter of 2008, and then became negative again in 2009.

The time series patterns for kurtosis feature an inverted-U shape, and objective kurtosis values are higher than representative kurtosis values, with the latter values being much closer to 3.

As the correlation matrix in Table 2 indicates, there appears to be some structure to the way in which pdf moments evolve over time. For the objective pdf, an increase in objective return is associated with increased volatility, less negative skewness, and less weight in the tails (lower kurtosis). For the representative investor, an increase in expected return is associated with lower volatility, more negative skewness, and higher kurtosis. Hence, the signs for expected representative investor return are opposite to the signs for expected objective return. Later in the paper, we explore some of these differences in greater detail.

An increase in volatility, for both objective and representative investor, leads to less negativity in skewness and less kurtosis (positive correlation). Hence, on average, increased forecasted volatility moves in the direction of more evenness in both tails, and tails which are less heavy. Higher volatility is thus seen as less of a rare event.

6.6. Risk Premiums

The impact of excessive optimism and overconfidence on risk premiums constitutes one of the most fundamental findings in the paper. Risk free rates of interest are imbedded within the expected return under the risk neutral density q. The risk premium is the difference between the expected return and the risk free rate. There are two risk premiums, one objective and the other for the representative investor.

Consider the observations for end-of-month.²⁶ The objective risk premium is negatively correlated with both excessive optimism (-0.93) and overconfidence (-0.56). The signs are consistent with the intuition that increases in excessive optimism and overconfidence drive up prices, thereby reducing the risk premium. However, consider a regression of the objective risk premium on its two most recent lagged values, excessive optimism, and overconfidence, reported in Table 3. Although the coefficient on excessive optimism is negative (t-statistic = -3.7), the coefficient on overconfidence is positive (t-statistic = 2.8).

Interestingly, in a similar regression for the representative investor's risk premium, only the prior lagged value of the subjective risk premium is statistically significant, as shown in Table 3. Of course, it makes sense that investors would not receive compensation for risk they misperceive.

Both excessive optimism and overconfidence are positively related to past twelve month returns and negatively related to volatility. As shown in Table 4, controlling for own lagged values, the associated t-statistics for excessive optimism are 4.0 for past returns and -3.2 for past volatility.²⁷ The associated t-statistics for overconfidence are 0.4 for past returns and -3.1 for past volatility. Interestingly, excessive optimism is impacted by both past returns and past volatility, while overconfidence is only impacted by past volatility.

Chaining these relationships together, we have the following: High past returns and low volatility leads to high excessive optimism and high overconfidence. In turn, high optimism leads to a low objective equity premium, an effect which is mitigated by high overconfidence.

 $^{^{26}}$ As external measures of sentiment, introduced in the next section, are available only at a monthly (or quarterly) frequency, most regression results in this paper are based on end-of-month observations. However, regression results based on weekly observations are similar in most respects.

²⁷Volatility is measured using high and low values during the prior twelve months. If we measure volatility using monthly return standard deviation for daily returns, the associated t-statistic is -2.45.

7. External Measures of Sentiment

We compare the estimates of sentiment from our model with three independent measures, the Baker–Wurgler series, the Yale/Shiller confidence indexes, and the Duke/CFO survey responses.

7.1. Relationship of Biases to Baker–Wurgler Series

For our sample period, we analyze the relationship between the Baker–Wurgler series (BW) and the variables generated by estimation of sentiment.²⁸ An updated version of the Baker–Wurgler monthly series for sentiment is available from July 1965 through December 2010, at Jeff Wurgler's website.²⁹ Baker and Wurgler do not provide a precise interpretation of what their series exactly measures, although they do suggest thinking about the series as if it measures excessive optimism for stocks. Because we obtain estimates for both excessive optimism and overconfidence, we are able to assess the degree to which the Baker–Wurgler series reflects both biases.

We find that BW heavily reflects excessive optimism. Table 5 shows an AR(2) regression of BW on excessive optimism and overconfidence, the coefficient for excessive optimism has a coefficient of 6.0, with a t-statistic of 3.5. Interestingly, the coefficient of overconfidence, whose value is -0.9, is statistically insignificant with a t-statistic of -1.5.³⁰ The correlation coefficient between the Baker–Wurgler index and overconfidence is 0.27, with an associated t-statistic of 2.9. However, in regression analysis, the positive and statistically significant relationship between BW and overconfidence is overwhelmed by the impact of excessive optimism. Figure 8 (top panel) shows that BW and optimism comove significantly during our sample period.

Although the BW series weakly and negatively reflects overconfidence, we note that the estimated sentiment functions suggest significant overconfidence in much of our sample period. Recall that overconfidence is associated with a log-change of measure that has the shape of an inverted

²⁸Baker and Wurgler actually develop two series, one which reflects economic fundamentals and a second which removes the effect of economic fundamentals. Although we analyze both series, the results are quite similar for both, and for this reason we only report findings for the first series.

²⁹http://people.stern.nyu.edu/jwurgler/

³⁰A regression of BW on its own lagged value, excessive optimism, and overconfidence does yield a statistically significant coefficient for overconfidence, but the Durbin-Watson statistic for this regression is 1.6. The coefficient for excessive optimism is 5.3 with a t-statistic of 5.8. The coefficient for overconfidence is -1.2 with a t-statistic of -2.4. If we add a second own lag for BW, we can obtain a statistically significant coefficient for overconfidence (t-statistic = -2.1), with a Durbin Watson statistic equal to 2.0. However, the coefficient on the second lag of BW, the variable added to the previous regression, is not statistically significant. This leads us to conclude that statistical significance for overconfidence is not robust to alternative specifications. In contrast, the coefficient on excessive optimism is statistically significant in all of our specifications.

U. Figure 9 illustrates several log-change of measure functions for the first nine months of 2002. Notice the pronounced inverted U-shapes. We conclude that the BW series fails to capture an important aspect of sentiment, namely the overconfidence component that is independent of excessive optimism.

7.2. Yale/Shiller Confidence Indexes

In order to provide another external check on our overconfidence estimate, we use the Yale/Shiller surveys. The Yale/Shiller U.S. data is based on a survey of two samples of investors. The first is a sample of wealthy individual investors, and the second is a sample of institutional investors. For the time period studied in our paper, the first sample consists of a random sample of high-income Americans purchased from Survey Sampling, Inc. The second sample consists of investment managers listed in the *Money Market Directory of Pension Funds and Their Investment Managers*.

The Yale/Shiller confidence indexes consist of monthly six-month averages of monthly survey results. For example, an index value for January 2002 is an average of results from surveys between August 2001 and January 2002. Sample size has averaged a little over one hundred per six-month interval since the beginnings of the surveys. This means that standard errors are typically plus or minus five percentage points.

There are four confidence indexes. These measure confidence that there will be no crash in the next six months, confidence that the market will go up in the next twelve months, confidence that the market is fairly valued, and confidence that the market will reverse in the short-term (buying-on-dips). Given the two groups sampled, this leads to eight confidence series. Of these, the crash confidence index CP for institutional (professional) investors, and for individual investors CI, turn out to be the most informative about overconfidence. For this reason, we provide more detail about their construction below.

The crash confidence index measures the percent of the population who attach little probability to a stock market crash in the next six months. The survey question used to elicit the index is as follows:

What do you think is the probability of a catastrophic stock market crash in the U.S., like that of October 28, 1929 or October 19, 1987, in the next six months, including the

case that a crash occurred in the other countries and spreads to the U.S.? (An answer of 0% means that it cannot happen, an answer of 100% means it is sure to happen.)

The Crash Confidence Index is the percentage of respondents who think that the probability is less than 10%.

7.2.1 Weights in Tails

The crash confidence index pertains to the probability of a left tail event. To compare the crash confidence indexes CP and CI with our representative investor approach, we focus on left tail probabilities under the objective and representative investor's pdf associated with a return of -20%, i.e., $p\{S_T/S_t < 0.8\}$ and $p_R\{S_T/S_t < 0.8\}$, respectively, slightly abusing the notation.³¹

We find that the correlation coefficient between the representative investor's left tail probability $p_R\{S_T/S_t < 0.8\}$ and CP is -0.82, and for CI is -0.79. In and of itself, there is no prior stipulation that CP need reflect investor bias. However, we note that the correlation coefficient between CP and the objective left tail probability $p\{S_T/S_t < 0.8\}$ is -0.57. Notably, an AR(2) regression of CP on $p\{S_T/S_t < 0.8\}$ and the bias term $p\{S_T/S_t < 0.8\} - p_R\{S_T/S_t < 0.8\}$ has only the bias term being statistically significant with a t-statistic of 2.8. Figure 8 (bottom panel) shows CP and $p\{S_T/S_t > 0.8\}$, where the latter can be seen as the "probability of no crash" under the representative investors' pdf. The comovements between the two series are evident. For instance, both series reach lowest levels at the end of 2002 and 2008, during our sample period.

Although overconfidence certainly reflects beliefs about the left tail, by definition it also reflects beliefs about the right tail. We define the right tail probability as the probability associated with the event $\{S_T/S_t > 1.2\}$, in which the return exceeds 20%. The corresponding correlation coefficients for the representative investor's right tail probability $p_R\{S_T/S_t > 1.2\}$ is -0.79 for CP, and -0.76 for CI.

The somewhat weaker correlation coefficients for the right tail indicate that the left and right tails of p_R behave somewhat differently from each other. To study the different trajectories for the two tails, we compute the tail likelihood ratios (right over left) for the representative investor's pdf $L(p_R) = p_R \{S_T/S_t > 1.2\}/p_R \{S_T/S_t < 0.8\}$, and similarly for the objective pdf L(p). Then we

³¹For each Wednesday t in our sample, objective tail probabilities are computed numerically integrating the conditional density p of the gross return S_T/S_t , given the information available at date t. Similarly for the representative investor's tail probabilities.

take the ratio $L(p_R)/L(p)$ of the representative investor tail likelihood ratio over the objective tail likelihood ratio. Figure 10 shows the two tail probabilities $p_R\{S_T/S_t > 1.2\}$ and $p_R\{S_T/S_t < 0.8\}$, and their ratio, during our sample period.

We find that although the objective tail likelihood ratio L(p) is relatively stable around its mean of 2.5, the representative investor's tail likelihood ratio $L(p_R)$ is higher for much of the sample period, and far less stable. Although the mean of the representative investor's tail likelihood ratio $L(p_R)$ is 3.4, it lies near or below 2 at the beginning and end of the sample period, rises sharply to about 8 in the middle of the period, (range of 5 to 12), and peaks above 18 in late 2007. This suggests to us the presence of higher moment effects (beyond the first and second). In respect to sentiment, the ratio $L(p_R)/L(p)$ lies near or below 1 at the beginning and end of the sample period, rises sharply to about 6 in the middle of the period, (range of 5 to 7), and peaks above 11 in late 2007.

The patterns just described indicate that as excessive optimism and overconfidence increased in the middle of the sample period, the fear of a crash fell dramatically. In particular, the representative investor's left tail dropped relative to the right, much more than the corresponding pattern for the objective pdf. Interestingly, the correlation between the crash confidence index CP and the ratio $L(p_R)/L(p)$ is 0.71, whereas the correlation between CP and overconfidence is 0.61. The lower correlation for overconfidence is in line with what we would expect, given the different time series behavior of the two tail probabilities, along with the fact that overconfidence pertains to both tails, whereas CP only pertains to the left tail.

7.2.2 RMSE, MAE, and Yale/Shiller Crash Confidence Index

RMSE and MAE in (10), shown in Figure 5, are measures of the degree to which the SDF M deviates from conforming to the CRRA restriction. We find a strong relationship between these measures and CP. The correlation coefficients are 0.69 for both. Interestingly, RMSE and MAE are uncorrelated with the Baker–Wurgler series, suggesting that distortions to the shape of the SDF reflect overconfidence more than excessive optimism.

7.3. Duke/CFO survey responses

For our third external dataset, we use the Duke/CFO survey. Duke University and CFO magazine jointly conduct the CFO Outlook Survey in which they survey CFOs of companies and subscribers of CFO magazine every March, June, September and December. Between 1996 and 2003, survey respondents included members of the organization Financial Executives International, Duke University's survey partner at the time. An archive of past surveys is available under the "Past Results" tab at http://www.cfosurvey.org.

Some of the survey questions remain constant from survey to survey, in order to capture trend data on corporate optimism, overconfidence, expectations about economic conditions, and related business data categories. The Duke/CFO questions most relevant to this paper pertain to expected return, volatility, and skewness, both for a one-year horizon and a ten-year horizon. Consider the following example for the one-year horizon from March 2002, the beginning of our sample period.

On March 11th, the annual yield on 10-yr treasury bonds was 5.3%. Please complete the following:

- a. Over the next year, I expect the average annual S&P 500 return will be: There is a 1-in-10 chance it will be less than: ____
- b. Over the next year, I expect the average annual S&P 500 return will be: Expected return: ____
- c. Over the next year, I expect the average annual S&P 500 return will be: There is a 1-in-10 chance it will be greater than: ____

Graham and Harvey (2012) provide an overview of the survey results, in which they describe their procedure for computing expected returns, standard deviations, and kurtosis. Essentially, one-year forecasted volatility is the answer to question c minus the answer to question a, divided by 2.65, i.e., (c-a)/2.65. The estimate for skewness is $(c-2b+a)^3$ divided by standard deviation cubed.

The Duke/CFO survey derived estimates for expected return, volatility, and skewness provide an interesting contrast to our estimates from the objective pdf and representative investor pdf. Figure 11 (top panel) shows that the Duke/CFO expected return and representative investor's expected return are highly correlated after 2005 (correlation coefficient is 0.6).³² For the entire sample period, the correlation coefficient is 0.2.

The similarity in volatilities is striking. Although the Duke/CFO series exhibits very large overconfidence, as discussed in Ben-David, Graham, and Harvey (2010), the correlation with representative investor volatility is a very high 0.8; see Figure 11 (bottom panel).

As for skewness, the correlation between the Duke/CFO values and the representative investor values is distinctly negative. The former features an inverted-U shape over time, while the latter is U-shaped over time. The correlation coefficient between the two series is -0.39. The negative correlation points to the fact that when volatility increases, the respondents to the Duke/CFO series overfocus on volatility associated with negative returns. In contrast, the representative investor focuses on high positive returns, as well as negative returns, during periods of heightened volatility.

8. Risk and Return

The existence of a positive relationship between risk and return is a cornerstone concept of academic finance. For this reason, our finding of a negative perceived relationship between risk and return, discussed below, strikes us as being fundamentally important.

Our findings relate to a large empirical literature on predictive regressions (that is, regressions of ex-post returns on measures of risk, etc.). After two decades of empirical research, there is little consensus on the basic properties of the relation between the expected market return and volatility. Studies such as Glosten, Jagannathan, and Runkle (1993), Whitelaw (1994), and Brandt and Kang (2004) find a negative trade-off, while conversely Harvey (1989), Ghysels, Santa-Clara, and Valkanov (2005), Guo and Whitelaw (2006), and Ludvigson and Ng (2007) find a positive trade-off. Recently, Rossi and Timmermann (2010) provided some empirical evidence that the risk-return relation may be not linear.

A regression of expected return against return standard deviation under the objective pdf p has a slope coefficient of 0.12 and an intercept of 0.02, as shown in Table 6. Both estimates are statistically significant. These parameter values are generally consistent with neoclassical theory.

 $^{^{32}}$ Because Duke ran the survey with CFO magazine only after 2004, it appears that the data from 2005 on is more consistent than the data from the earlier period.

Since the frequency of the observation is weekly, to account for overlapping intervals, we use Newey and West (1987) robust standard errors with the number of lags optimally chosen as in Andrews $(1991).^{33}$

In the behavioral approach, prices reflect not the objective pdf p but the representative investor's pdf p_R . A regression of expected return against return standard deviation under the representative investor's pdf p_R has a slope coefficient of -0.13 and an intercept of 0.07. Both estimates are statistically significant.³⁴ The negative slope coefficient reflects the perspective that risk and return are negatively related, which is inconsistent with neoclassical theory.³⁵

Shefrin (2008) discusses several studies about the perception that risk and return are negatively related.³⁶ One key behavioral feature involves excessive optimism and overconfidence being positively correlated. In our data, the correlation between the two series for the entire sample period is 0.65. A positive correlation implies that whenever the representative investor is excessively optimistic and overestimates expected return, he tends to be overconfident and underestimates future volatility. Of course, associating high returns and low risk is precisely the hallmark of a negative perceived relationship.

We hasten to add that a positive correlation between excessive optimism and overconfidence does not necessarily imply a negative relationship between risk and return. This is because if sentiment is small, then it will not override the fundamental component. For example, during the period September 2008 through the end of our sample period, the representative investor's perceived risk and return were positively correlated (with a regression slope coefficient of 0.07). At the same time,

³³Recently, Bollerslev, Marrone, Xu, and Zhou (2012) showed that Newey–West estimators with sufficiently many lags, as implemented here, can correct estimation biases introduced by overlapping observations. Moreover, using end-of-month observations, regression estimates of slope and intercept are nearly the same as in Table 6 and are both statistically significant. The Durbin-Watson statistic for the regression is a very low 0.36. Adding an AR(1) error term alters the slope coefficient to 0.03 and the intercept to 0.04, with both estimates again statistically significant. ³⁴Using end-of-month observations, the slope coefficient is -0.11 and the intercept is 0.06, again both statistically significant. The Durbin-Watson statistic for the regression is a very low 0.60. Adding an AR(1) error term alters the slope coefficient to 0.01 and the intercept to 0.05, with only the intercept being statistically significant. The log-likelihood of a regression without the AR(1) term is 25, and with the AR(1) term is 127.

 $^{^{35}}$ By virtue of (8), the negative relationship between risk and return stems from the risk neutral pdf together with a change of measure that corresponds to a neoclassical CRRA-based SDF with values for CRRA that are plausible and in line with general estimates. Keep in mind that in a neoclassical CRRA-setting, (8) generates the objective pdf from the risk neutral pdf.

 $^{^{36}}$ These studies focus on behavior at the level of the individual, and suggest that excessive optimism and overconfidence are positively correlated across the population. From a dynamic perspective, wealth transfers resulting from trading will induce a time series correlation as well. This occurs as wealth shifts from, say, less optimistic, less confident investors to more optimistic, more confident investors, inducing an increase over time in both the representative investor's degree of excessive optimism and overconfidence.

excessive optimism and overconfidence were still positively related, with a correlation coefficient of 0.8. However, as can be seen in Figure 5, sentiment was small during this period.

Figure 12 shows the relationship between expected returns and return standard deviations under the objective pdf and the representative investor's pdf. The opposite relationships associated with objective pdf and representative investor pdf emerge clearly.

Graham and Harvey (2001) point out that for the one-year horizon, expected return and forecasted volatility are negatively correlated. We find that over time, this pattern has remained consistent in their survey data. Therefore, the negative relationship between risk and return is not an artifact of our estimation procedure, but is present in survey data. For our sample, period, the correlation coefficient in the Duke/CFO data is -0.19. For the period after 2005, the correlation is $-0.33.^{37}$

Table 6 also reports regression results of ex-post (i.e., realized) annual returns on ex-ante, expected returns. As the frequency of the observations is weekly, such predictive regression results need to be interpreted cautiously.³⁸ Over the course of our sample period, the relationship between the representative investor's expected return and subsequent realized return is negative and statistically insignificant with a t-statistic -1.84.³⁹ The relationship between objective expected return and subsequent realized return is positive and statistically significant, with a t-statistic of 4.8. Notice that the R-squared associated with the objective expected return is higher than the R-squared associated with the representative investor's expected return. In addition, maximizing the predictive power associated with the representative investor's expected return requires using the negative regression slope, in effect to control for the bias.

³⁷For the ten-year forecast horizon, expected return and forecasted volatility are positively related. This is consistent with the notion that sentiment is mean reverting, so that for longer horizons fundamentals dominate in belief formation.

 $^{^{38}}$ Long-horizon return regressions present various finite sample issues. Recently, Boudoukh, Whitelaw, and Richardson (2008) showed that even in the absence of any increase in the return predictability, the values of R^2 's in regressions involving highly persistent predictor variables and overlapping returns will by construction increase roughly proportionally to the return horizon and the length of the overlap. Baker, Taliaferro, and Wurgler (2006) and Goyal and Welch (2008) provide related studies. However, our focus is not on the best model to predict future market returns, but rather on the opposite risk-return relationships under the objective and representative investor's distributions. As discussed above, such relationships are robust to a number of alternative specifications.

³⁹Using end-of-month observations, the t-statistic drops to -0.2.

9. Conclusion

This paper applies behavioral asset pricing theory to estimate the components of the pricing kernel. In this respect, the pricing kernel decomposes into a fundamental component and sentiment. Our estimates for the fundamental component are plausible in the sense of being consistent with independent estimates of risk aversion, and to a lesser extent time preference. Our estimates for the sentiment component are consistent with independent measures of investor sentiment such as the Baker–Wurgler series, the Yale/Shiller crash confidence index, and the Duke/CFO survey data. We find that for much of our sample period, the pricing kernel features an upward sloping portion, which is consistent with overconfidence bias.

Our analysis suggests that the Baker–Wurgler series robustly reflects excessive optimism, but not the component of overconfidence that is independent of excessive optimism. We also find that the Yale/Shiller crash confidence index is effectively a sentiment measure that is highly correlated with overconfidence, and very strongly correlated with the tail probability associated with the representative investor's pdf.

Excessive optimism and overconfidence are positively correlated throughout our sample period. During periods in which these biases are strong, the representative investor perceives risk and return to be negatively related, whereas objectively they are positively related. Notably, the negative correlation is also a pronounced feature in the one-year forecasts from the Duke/CFO survey data.

A. Derivation of the Sentiment Function Λ

The sentiment function encapsulates the representative investor's biases. In this section, we briefly describe the structure of the sentiment function, and its manifestation within the SDF.

Let ξ denote state price. Then the SDF is given by $M = \xi/p$, which in a representative investor CRRA-framework has the form $\xi = p_R \theta_0 (S_T/S_t)^{-\theta_1}$. This last relationship follows from the optimizing condition in which marginal rate of substitution (for expected utility) is set equal to relative state prices, with consumption at t = 0 serving as numeraire.⁴⁰

To simplify notation, we drop the *t*-subscripts and the argument of the pdf. Divide both sides of the previous equation for ξ by $p \theta_{0,e}$, where $\theta_{0,e}$ corresponds to the value of θ_0 that would prevail if all investors held correct beliefs. Here, the subscript *e* denotes efficiency. This last operation leads to the expression $\xi/p = (\theta_0/\theta_{0,e}) (p_R/p) \theta_{0,e} (S_T/S_t)^{-\theta_1}$. Define $e^{\Lambda} = (\theta_0/\theta_{0,e}) (p_R/p)$, which is a scaled change of measure and corresponds to (7).

The change of measure (p_R/p) associated with Λ exactly specifies the transformation of the objective pdf p into the representative investor's pdf p_R . Therefore, Λ encapsulates the representative investor's biases.

Shefrin (2008) establishes that θ_1 does not vary as investors' beliefs change. Then, in the preceding expression for the SDF, e^{Λ} multiplies the term $\theta_{0,e} (S_T/S_t)^{-\theta_1}$, and the latter is the SDF M_e that would prevail if all investors held correct beliefs. Therefore $M = e^{\Lambda}M_e$. Taking logs, obtain $\log(M) = \Lambda + \log(M_e)$. This expression stipulates that the log-SDF can be decomposed into two components, one being the sentiment function and the other being the neoclassical log-SDF that would prevail if all investors held correct beliefs.

Rearranging the decomposition of the log-SDF yields $\Lambda = \log(M) - \log(M_e)$. Notably, the last relationship corresponds to (9) and explains why d serves as our estimate of the sentiment function Λ .

 $^{^{40}}$ For the purpose of this discussion, we assume a representative investor with CRRA-preferences. Shefrin (2008) develops an aggregation theorem for a model involving heterogeneous investors in which the representative investor's preferences are approximately CRRA, and whose beliefs are given by a Hölder average. To reduce complexity, we mostly abstract from aggregation issues in this paper.

References

- Aït-Sahalia, Y., M. Karaman, and L. Mancini, 2012, "The Term Structure of Variance Swaps, Risk Premia, and the Expectation Hypothesis," working paper, EPFL.
- Aït-Sahalia, Y., and A. Lo, 2000, "Nonparametric Risk Management and Implied Risk Aversion," Journal of Econometrics, 94, 9–51.
- Andrews, D., 1991, "Heteroskedasticity and Autocorrelation Consistent Covariance Matrix Estimation," *Econometrica*, 59, 817–858.
- Baker, M., R. Taliaferro, and J. Wurgler, 2006, "Predicting Returns with Managerial Decision Variables: Is There a Small-Sample Bias?," *Journal of Finance*, 61, 1711–1730.
- Baker, M., and J. Wurgler, 2006, "Investor Sentiment and the Cross-section of Stock Returns," Journal of Finance, 61, 1645–1680.
- , 2007, "Investor Sentiment in the Stock Market," *Journal of Economic Perspectives*, 21, 129–151.
- Barone-Adesi, G., R. Engle, and L. Mancini, 2008, "A GARCH Option Pricing Model in Incomplete Markets," *Review of Financial Studies*, 21, 1223–1258.
- Barone-Adesi, G., L. Mancini, and H. Shefrin, 2012, "Sentiment, Asset Prices, and Systemic Risk," in *Handbook of Systemic Risk*, ed. by J.-P. Fouque, and J. Langsam. Cambridge University Press, Cambridge.
- Barsky, R., F. Juster, M. Kimball, and M. Shapiro, 1997, "Preference Parameters and Behavioral Heterogeneity: An Experimental Approach in the Health and Retirement Survey," *Quarterly Journal of Economics*, 107, 537–579.
- Ben-David, I., J. Graham, and C. Harvey, 2010, "Managerial Miscalibration," working paper, Duke University.
- Birru, J., and S. Figlewski, 2012, "Anatomy of a Meltdown: The Risk Neutral Density for the S&P 500 in the Fall of 2008," *Journal of Financial Markets*, forthcoming.
- Bollerslev, T., J. Marrone, L. Xu, and H. Zhou, 2012, "Stock Return Predictability and Variance Risk Premia: Statistical Inference and International Evidence," working paper, Duke University.
- Boudoukh, J., R. Whitelaw, and M. Richardson, 2008, "The Myth of Long-Horizon Predictability," *Review of Financial Studies*, 21, 1577–1605.
- Brandt, M., and Q. Kang, 2004, "On the Relationship Between the Conditional Mean and Volatility of Stock Returns: A Latent VAR Approach," *Journal of Financial Economics*, 72, 217–257.
- Campbell, J., A. Lo, and A. C. MacKinlay, 1997, *The Econometrics of Financial Markets*. Princeton University Press, Princeton.
- Campbell, J., and R. Shiller, 1998, "Valuation Ratios and the Long Run Market Outlook," Journal of Portfolio Management, 24, 11–26.

Cochrane, J., 2005, Asset Pricing. Princeton University Press, Princeton, Second edition.

- Duan, J.-C., and J. Simonato, 1998, "Empirical Martingale Simulation for Asset Prices," Management Science, 44, 1218–1233.
- Epstein, L. G., and S. Ji, 2012, "Ambiguous Volatility, Possibility and Utility in Continuous Time," working paper, Boston University.
- Ganzach, Y., 2000, "Judging Risk and Return of Financial Assets," Organizational Behavior and Human Decision Processes, 8, 353–370.
- Ghysels, E., P. Santa-Clara, and R. Valkanov, 2005, "There is a Risk-Return Trade-off After All," Journal of Financial Economics, 76, 509–548.
- Glosten, L., R. Jagannathan, and D. Runkle, 1993, "On the Relation Between the Expected Value and the Volatility of the Nominal Excess Return on Stocks," *Journal of Finance*, 48, 1779–1801.
- Gourieroux, C., A. Monfort, and A. Trognon, 1984, "Pseudo Maximum Likelihood Methods: Theory," *Econometrica*, 52, 681–700.
- Goyal, A., and I. Welch, 2008, "A Comprehensive Look at the Empirical Performance of Equity Premium Prediction," *Review of Financial Studies*, 21, 1455–1508.
- Graham, J., and C. Harvey, 2001, "Expectations of Equity Risk Premia, Volatility, and Asymmetry from a Corporate Finance Perspective," working paper, Duke University.
- ———, 2012, "The Equity Risk Premium in 2012," working paper, Duke University.
- Guo, H., and R. Whitelaw, 2006, "Uncovering the Risk-Return Relation in the Stock Market," Journal of Finance, 61, 1433–1463.
- Harvey, C., 1989, "Time-Varying Conditional Covariances in Tests of Asset Pricing Models," Journal of Financial Economics, 24, 289–317.
- Hens, T., and C. Reichlin, 2011, "Three Solutions to the Pricing Kernel Puzzle," working paper, University of Zurich.
- Jackwerth, J., 2000, "Recovering Risk Aversion from Options Prices and Realized Returns," *Review of Financial Studies*, 13, 433–451.
- Jouini, E., and C. Napp, 2006, "Heterogeneous Beliefs and Asset Pricing in Discrete Time," Journal of Economic Dynamics and Control, 30, 1233–1260.
- Ludvigson, S., and S. Ng, 2007, "The Empirical Risk-Return Relation: A Factor Analysis Approach," Journal of Financial Economics, 83, 171–222.
- Meyer, D., and J. Meyer, 2005, "Relative Risk Aversion: What Do We Know?," Journal of Risk and Uncertainty, 31, 243–262.
- Newey, W. K., and K. D. West, 1987, "A Simple, Positive Semi-definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix," *Econometrica*, 55, 703–708.

- Rosenberg, J., and R. Engle, 2002, "Empirical Pricing Kernels," *Journal of Financial Economics*, 64, 341–372.
- Rossi, A., and A. Timmermann, 2010, "What is the Shape of the Risk-Return Relation?," working paper, University of California San Diego.
- Scheinkman, J., and W. Xiong, 2003, "Overconfidence and Speculative Bubbles," Journal of Political Economy, 111, 1183–1219.
- Shefrin, H., 2008, A Behavioral Approach to Asset Pricing. Elsevier Academic Press, Boston, Second edition.
- Song, Z., and D. Xiu, 2012, "A Tale of Two Option Markets: State-Price Densities Implied from S&P 500 and VIX Option Prices," working paper, University of Chicago.
- Whitelaw, R., 1994, "Time Variations and Covariations in the Expectation and Volatility of Stock Market Returns," *Journal of Finance*, 49, 515–541.
- Xiong, W., and H. Yan, 2010, "Heterogeneous Expectations and Bond Markets," Review of Financial Studies, 23, 1433–1466.
- Yu, J., 2011, "Disagreement and Return Predictability of Stock Portfolios," Journal of Financial Economics, 99, 162–183.
- Ziegler, A., 2007, "Why Does Implied Risk Aversion Smile?," *Review of Financial Studies*, 20, 859–904.

	$\omega \times 10^6$	β	$\alpha \times 10^3$	γ	Persist.	Ann. vol.		
	Objective GARCH parameters							
Mean	1.215	0.926	3.473	0.117	0.989	0.198		
Stdv.	0.207	0.005	4.141	0.013	0.002	0.016		
Risk Neutral FHS GARCH parameters								
Mean	4.153	0.789	2.169	0.358	0.970	0.212		
Stdv.	5.600	0.208	9.366	0.360	0.033	0.074		
Risk Neutral GAUSS GARCH parameters								
Mean	3.987	0.756	3.479	0.448	0.983	0.252		
Stdv.	5.575	0.201	12.280	0.371	0.021	0.112		

Table 1. GARCH estimation and calibration. The GJR GARCH model is $\log(S_t/S_{t-1}) = \mu_t + \epsilon_t$, where S_t is the S&P 500 index at day t, μ_t is the drift, and the conditional variance $\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha \epsilon_{t-1}^2 + \gamma I_{t-1} \epsilon_{t-1}^2$, where $\epsilon_t = \sigma_t z_t$, z_t is a standardized innovation and $I_{t-1} = 1$ when $\epsilon_{t-1} < 0$, and $I_{t-1} = 0$ otherwise. For each Wednesday from January 2002 to October 2009, a GJR GARCH model is estimated using historical daily S&P 500 returns and applying Gaussian Pseudo Maximum Likelihood; a GJR GARCH model driven by Gaussian innovations is calibrated to out-of-the-money options on the S&P 500 index minimizing the sum of squared pricing errors; a GJR GARCH model driven by filtered historical innovations is similarly calibrated to options on the S&P 500 index. Persist. is persistency of the GARCH volatility and given by $\beta + \alpha + \gamma/2$. Ann. vol. is unconditional annualized volatility.

	Objective				Representative Investor			
	Exp	Std	Skew	Kurt	Exp	Std	Skew	Kurt
ExpObj	1							
StdObj	0.82	1						
SkewObj	0.58	0.82	1					
KurtObj	-0.74	-0.89	-0.93	1				
ExpRepInv	-0.56	-0.45	-0.56	0.55	1			
$\operatorname{StdRepInv}$	0.86	0.87	0.77	-0.88	-0.68	1		
SkewRepInv	0.56	0.59	0.68	-0.64	-0.81	0.66	1	
$\operatorname{KurtRepInv}$	-0.56	-0.62	-0.64	0.64	0.71	-0.69	-0.93	1

Table 2. Correlations between objective and representative investor's expected moments of returns. Ex-ante, expected returns (Exp) under the objective (Obj) and representative investor's (RepInv) distributions are given by $E_t^p[S_T/S_t-1]$ and $E_t^{p_R}[S_T/S_t-1]$, respectively, where E_t^p is the conditional expectation at date t under the objective pdf p, similarly $E_t^{p_R}$ under the representative investor's pdf p_R , S_t is the S&P 500 index at date t, and (T - t) is one year. Ex-ante, conditional standard deviation (Std), skewness (Skew), kurtosis (Kurt) of one-year returns are similarly computed for each Wednesday t in our sample, from January 2002 to October 2009.

Intercept	Lag1	Lag2	Optimism	Overconfidence	R^2			
Objective Risk Premium								
0.003	1.08	-0.23	-0.20	0.04	0.97			
(2.36)	(9.13)	(-2.34)	(-3.71)	(2.81)				
Representative Investor's Risk Premium								
0.01	0.36	0.09	-0.01	0.04	0.15			
(4.17)	(3.25)	(0.86)	(-0.11)	(1.30)				

Table 3. Risk premium and sentiment. Upper panel: Time series regression of objective risk premium on a constant (Intercept), its two most recent lagged values (Lag1, Lag2), optimism and overconfidence; t-statistics in parentheses. Objective risk premium is $E_t^p[S_T/S_t] - E_t^q[S_T/S_t]$, where E_t^p is the conditional expectation at date t under the objective pdf p, E_t^q is the conditional expectation at date t under the sevent lagged values (Lag1, Lag2), optimism one year; optimism is $E_t^{p_R}[S_T/S_t] - E_t^p[S_T/S_t]$, where $E_t^{p_R}$ is the conditional expectation at date t under the representative investor's pdf p_R ; overconfidence is $\sqrt{\operatorname{Var}_t^p[S_T/S_t]} - \sqrt{\operatorname{Var}_t^{p_R}[S_T/S_t]}$. Lower panel: Same time series regression for the risk premium perceived by the representative investor, defined as $E_t^{p_R}[S_T/S_t] - E_t^q[S_T/S_t]$. R^2 is the adjusted R-squared. Observations are end-of-month from January 2002 to October 2009.
Intercept	Lag1	Lag Ret	Lag Vol	R^2			
Optimism							
0.01	0.61	0.02	-0.03	0.89			
(3.22)	(8.06)	(3.99)	(-3.20)				
Overconfidence							
0.03	0.60	0.005	-0.05	0.60			
(4.37)	(7.85)	(0.40)	(-3.12)				

Table 4. Optimism and overconfidence. Upper panel: Time series regression of optimism on a constant (Intercept), its most recent lagged value (Lag1), past monthly S&P 500 return (Lag Ret) and past volatility measured using high and low values during the prior twelve months (Lag Vol); t-statistics in parentheses. Optimism is $E_t^{p_R}[S_T/S_t] - E_t^p[S_T/S_t]$, where $E_t^{p_R}$ is the conditional expectation at date t under the representative investor's pdf p_R , E_t^p is the conditional expectation at date t under the objective pdf p, S_t is the S&P 500 index at date t, and (T-t) is one year. Lower panel: Same time series regression for overconfidence, defined as $\sqrt{\operatorname{Var}_t^p[S_T/S_t]} - \sqrt{\operatorname{Var}_t^{p_R}[S_T/S_t]}$. R^2 is the adjusted R-squared. Observations are end-of-month from January 2002 to October 2009.

Baker–Wurgler Series								
Intercept	Lag1	Lag2	Optimism	Overconfidence	R^2			
-0.17	1.13	-0.20	6.02	-0.88	0.92			
(-1.15)	(11.27)	(-2.18)	(3.54)	(-1.48)				

Table 5. Baker–Wurgler series and sentiment. Time series regression of Baker–Wurgler series on a constant (Intercept), its two most recent lagged values (Lag1, Lag2), optimism and overconfidence; t-statistics in parentheses. Optimism is $E_t^{p_R}[S_T/S_t] - E_t^p[S_T/S_t]$, where $E_t^{p_R}$ is the conditional expectation at date t under the representative investor's pdf p_R , E_t^p is the conditional expectation at date t under the representative investor's pdf p_R , E_t^p is the conditional expectation at date t under the objective pdf p, S_t is the S&P 500 index at date t, and (T - t) is one year. Overconfidence is $\sqrt{\operatorname{Var}_t^p[S_T/S_t]} - \sqrt{\operatorname{Var}_t^{p_R}[S_T/S_t]}$. R^2 is the adjusted R-squared. Observations are end-of-month from January 2002 to October 2009.

	Intercept	Slope	R^2
1. Objective return vs. risk	0.02	0.12	0.75
	(7.99)	(5.94)	
2. Rep. Inv. return vs. risk	0.07	-0.13	0.50
	(15.20)	(-5.03)	
3. Actual vs. objective return	-0.48	11.26	0.19
	(-3.56)	(4.80)	
4. Actual vs. Rep. Inv. return	0.34	-6.36	0.12
	(2.26)	(-1.84)	

Table 6. Risk and return. Regression 1: Time series regression of objective expected return on a constant (Intercept) and expected objective volatility (Slope). Objective expected return is $E_t^p[S_T/S_t-1]$, where E_t^p is the conditional expectation at date t under the objective pdf p, S_t is the S&P 500 index at date t, and (T-t) is one year; expected objective volatility is $\sqrt{\operatorname{Var}_t^p[S_T/S_t-1]}$. Regression 2: Same regression as Regression 1 for expected return and volatility of the representative investor. Regression 3: Time series regression of actual, ex-post annual return on a constant (Intercept) and objective expected return (Slope). Regression 4: Same regression as Regression 3 for representative investor's expected return. Observations are weekly from January 2002 to October 2009. Robust t-statistics in parentheses based on Newey and West (1987) robust standard errors and optimal number lags as in Andrews (1991). R^2 is the adjusted R-squared.



Figure 1. Upper graph: Objective (Pobj) and representative investor's (Prep) probability density functions for the date 21/12/2005. Lower graph: Behavioral unconstrained SDF (BehavKernel), CRRA-constrained SDF (CRRAKernel), and the LogDiff function, i.e., log BehavKernel minus log CRRAKernel. The latter difference is the function d_t in (9), for the date 21/12/2005.



Figure 2. SDF estimated semiparametrically using Gauss method for each Wednesday between January 2002 and October 2009 and for the horizon of one year. For each Wednesday t in our sample, the stochastic discount factor (SDF), $M_{t,T}$, is estimated semiparametrically as $M_{t,T} = e^{-r_f (T-t)} q(S_T/S_t)/p(S_T/S_t)$, where q is the risk-neutral density of S_T/S_t , p the objective (i.e., historical) density of S_T/S_t , r_f is the instantaneous risk-free rate, S_t the S&P 500 index at date t, and (T-t) is one year. The densities p and q are conditional on the information available at date t and based on GJR GARCH models with Gaussian innovations estimated using historical S&P 500 returns and SPX options, respectively.



Figure 3. SDF estimated semiparametrically using FHS method for each Wednesday between January 2002 and October 2009 and for the horizon of one year. For each Wednesday t in our sample, the stochastic discount factor (SDF), $M_{t,T}$, is estimated semiparametrically as $M_{t,T} = e^{-r_f (T-t)} q(S_T/S_t)/p(S_T/S_t)$, where q is the risk-neutral density of S_T/S_t , p the objective (i.e., historical) density of S_T/S_t , r_f is the instantaneous risk-free rate, S_t the S&P 500 index at date t, and (T-t) is one year. The densities p and q are conditional on the information available at date t and based on GJR GARCH models with FHS innovations estimated using historical S&P 500 returns and SPX options, respectively.



Figure 4. Time series estimates of $\theta_{1,t}$ and $\theta_{0,t}$ in the CRRA SDF. $\theta_{1,t}$ is the coefficient of relative risk aversion and $\theta_{0,t}$ is the discount factor measuring the degree of impatience at date t. The constat relative risk aversion (CRRA) SDF is $M_{t,T}(\theta) = \theta_{0,t} (S_T/S_t)^{-\theta_{1,t}}$, where S_t is the S&P 500 index at date t, and (T - t) is one year. For each Wednesday t in our sample, $\theta_{0,t}$ and $\theta_{1,t}$ are estimated fitting the CRRA-constrained SDF, $M_{t,T}(\theta)$, to the unconstrained SDF, $M_{t,T} = e^{-r_f (T-t)} q(S_T/S_t)/p(S_T/S_t)$, where q is the conditional risk-neutral density, p is the conditional objective (i.e., historical) density, S_t is the S&P 500 index at date t, and r_f is the instantaneous risk-free rate. The fitting is in the log-log space.



Figure 5. Time series of distance measures between unconstrained SDF and CRRA-constrained SDF, RMSE and MAE, using FHS method. Under the assumption that the CRRA-constrained SDF proxies for the pricing kernel that would be in effect were all investors to hold correct beliefs, RMSE and MAE can be interpreted as measures of sentiment. For each Wednesday t in our sample and for each gross return $S_T^{(i)}/S_t$, $i = 1, \ldots, 100$, the pointwise distance between the unconstrained SDF, $M_{t,T}^{(i)}$, and the CRRA-constrained SDF, $M_{t,T}^{(i)}(\theta) = \theta_{0,t}(S_T^{(i)}/S_t)^{-\theta_{1,t}}$, is $d_t^{(i)} = \log(M_{t,T}^{(i)}) - \log(M_{t,T}^{(i)}(\theta))$. The two distance measures are $\text{RMSE}_t = 100 (\sum_{i=1}^n |d_t^{(i)}|/n$.



Figure 6. Upper graph: Time series for objective mean and standard deviation of the market return. Objective mean of market return is $E_t^p[S_T/S_t - 1]$, where E_t^p is the time-t conditional expectation under the objective pdf p, S_t is the S&P 500 index at date t, and (T - t) is one year; objective standard deviation is $\sqrt{\operatorname{Var}_t^p[S_T/S_t]}$. Lower graph: Time series of optimism and overconfidence. Optimism is $E_t^{p_R}[S_T/S_t] - E_t^p[S_T/S_t]$, where $E_t^{p_R}$ is the time-t conditional expectation under the representative investor's pdf p_R . Overconfidence is $\sqrt{\operatorname{Var}_t^p[S_T/S_t]} - \sqrt{\operatorname{Var}_t^{p_R}[S_T/S_t]}$. Values are in percentage. Density estimates are obtained using the FHS method.



Figure 7. Upper graph: Time series for conditional skewness of the market return under objective and representative investor's distributions. Conditional skewness under the objective pdf (Obj) is computed as $E_t^p[(S_T/S_t - E_t^p[S_T/S_t])^3]/(\operatorname{Var}_t^p[S_T/S_t])^{3/2}$, where E_t^p and Var_t^p are the conditional mean and variance for each Wednesday t in our sample, S_t is the S&P 500 index at date t, and (T - t) is one year. Conditional skewness under the representative investor's pdf (RepInv) is similarly computed. Lower graph: Time series for conditional kurtosis of the market return under objective and representative investor's distributions. Conditional kurtosis is computed as $E_t^p[(S_T/S_t - E_t^p[S_T/S_t])^4]/(\operatorname{Var}_t^p[S_T/S_t])^2$ under the objective pdf, and similarly under the representative investor's pdf.



Figure 8. Upper graph: Baker–Wurgler sentiment series and optimism. Baker and Wurgler (2006) monthly series of sentiment extracted using principal component analysis of six specific sentiment proxies, i.e., turnover on the New York Stock Exchange (NYSE), dividend premium, closed-end fund discount, number and first-day returns on IPOs, and the equity share in new issues. Optimism is $E_t^{p_R}[S_T/S_t] - E_t^p[S_T/S_t]$, where $E_t^{p_R}$ is the conditional expectation at date t under the representative investor's pdf p_R , S_t is the S&P 500 index at date t, (T - t) is one year, and similarly E_t^p is the conditional expectation under the objective pdf p. Lower graph: Yale/Shiller crash confidence index (CP) and "probability of no crash" under the representative investor's pdf. The latter is $Prob\{S_T/S_t > 0.8\}$ under the representative investor's pdf p_R . For each Wednesday t, from January 2002 to October 2009, the conditional probability $Prob\{S_T/S_t > 0.8\}$ is computed numerically integrating the conditional density p_R of the gross return S_T/S_t , given the information available at date t.



Figure 9. Sentiment functions plotted for several days in 2002. The sentiment function at date t is $\Lambda_t(S_T/S_t) = \log(M_{t,T}) - \log(M_{t,T}(\theta))$, where $M_{t,T} = e^{-r_f (T-t)} q(S_T/S_t)/p(S_T/S_t)$ is the unconstrained SDF and $M_{t,T}(\theta) = \theta_{0,t} (S_T/S_t)^{-\theta_{1,t}}$ is the CRRA-constrained SDF. q is the conditional risk-neutral density of S_T/S_t , p is the conditional objective (i.e., historical) density of S_T/S_t , r_f is the instantaneous risk-free rate, $\theta_{0,t}$ is the time discount factor, $\theta_{1,t}$ is the coefficient of relative risk aversion, S_t is the S&P 500 index at date t, and (T-t) is one year. On the x-axis, gross return is S_T/S_t .



Figure 10. Time series estimates of right and left conditional tail probabilities under the representative investor's pdf p_R , i.e., $Prob\{S_T/S_t > 1.2\}$ and $Prob\{S_T/S_t < 0.8\}$, respectively, and their ratio, i.e., $Prob\{S_T/S_t > 1.2\}/Prob\{S_T/S_t < 0.8\}$. For each Wednesday t from January 2002 to October 2009, conditional tails probabilities are obtained numerically integrating the conditional density p_R of the gross return S_T/S_t , given the information available at date t. (T-t) is one year.



Figure 11. Upper graph: Time series of one-year S&P 500 expected return based on Duke/CFO survey responses and the representative investor's distribution. Duke/CFO survey responses are elicited as described in Section 7.3, quarterly frequency. One-year S&P 500 expected return under the representative investor's pdf is given by $E_t^{p_R}[S_T/S_t - 1]$, where $E_t^{p_R}$ is the conditional expectation at each Wednesday t in our sample under the representative investor's pdf p_R , S_t is the S&P 500 return standard deviation based on Duke/CFO survey responses and the representative investor's distribution. For each Wednesday t in our sample, return standard deviation under the representative investor's distribution. For each Wednesday t in our sample, return standard deviation under the representative investor's pdf is $\sqrt{\operatorname{Var}_t^{p_R}[S_T/S_t]}$. Values are in percentage.



Figure 12. Risk and return. For each Wednesday t from January 2002 to October 2009, "Expected Return, Objective" is the time-t conditional expected market return under the objective pdf p, i.e., $E_t^p[S_T/S_t - 1]$, where S_t is the S&P 500 index at date t, and (T - t) is one year; "Stdv. Return, Objective" is the time-t conditional expected volatility of market return under the objective pdf p, i.e., $\sqrt{\operatorname{Var}_t^p[S_T/S_t]}$. "Expected Return, Rep. Inv." and "Stdv. Return, Rep. Inv." are representative investor's expected return and volatility, respectively, computed using time-t conditional representative investor's pdf, p_R . In each graph, superimposed is the regression line.