Not Only What But also When: A Theory of Dynamic Voluntary Disclosure

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Abstract

The extant theoretical literature on voluntary disclosure focuses on settings with a single period and a single piece of private information. We extend this literature by studying a dynamic model of voluntary disclosure of multiple pieces of private information. Such situations are prevalent in real life, e.g., in corporate disclosure environments that are characterized by information asymmetry between the firm and the capital market with respect to whether, when, and what kind of private information the firm has learned. We show (perhaps surprisingly) that, due to dynamic strategic interaction between the firm and the capital market, later disclosures are interpreted more favorably. We provide sufficient conditions for the equilibrium to be in threshold strategies. Our model also predicts higher likelihood of early voluntary disclosure by managers who, for various reasons (such as higher short term incentives, prior to issuing new debt or equity, higher probability of the firm being taken over) care more about short-term price.

1 Introduction

In this paper, we study a dynamic model of voluntary disclosure of information by a potentially informed agent. The extant theoretical literature on voluntary disclosure focuses on static models in which an interested party (e.g., a manager of a firm) may privately observe a single piece of private information (e.g., Grossman 1981, Milgrom 1981, Dye 1985, and Jung and Kwon 1988) or dynamic models in which the disclosure timing does not play a role (e.g., Shin 2003, 2006) as the manager's decision is what to disclose but not when to disclose it. Corporate disclosure environments, however, are characterized by multi-period and multi-dimensional flows of information from the firm to the market, where the information asymmetry between the firm and the capital market can be with respect to whether, when, and what relevant information the firm might have learned. For example, firms with ongoing R&D projects can obtain new information about the state of their projects, where the time of information arrival and its content is unobservable to the market. This is common, for example, in pharmaceutical companies that get results of a drug clinical trial (prior to FDA approval). Such results are not required to be publicly disclosed in a timely manner and investors' beliefs about the result of a drug's clinical trial may have a great effect on the firm's price. The multidimensional nature of the disclosure game (multi-period and multi-signal) plays a critical role in shaping the equilibrium; e.g., when deciding whether to disclose one piece of information the agent must also consider the possibility of learning and potentially disclosing a new piece of information in the future.

In order to study a dynamic model of voluntary disclosure, we extend Dye's (1985) and Jung and Kwon's (1988) voluntary disclosure model with uncertainty about information endowment to a two-period, two-signal setting. We describe the potentially informed agent as a manager of a publicly traded firm. In our two-period setting, a manager who cares about both periods' stock price may receive up to two private signals about the value of the firm. In each period, the manager may voluntarily disclose any subset of the signals he has received but not yet disclosed. Our model demonstrates how dynamic considerations shape the disclosure strategy of a privately informed agent and the market reactions to what he releases (or doesn't release) and when. Absent information asymmetry, the firm's price at the end of the second period is independent of the arrival and disclosure times of the firm's private information. Nevertheless, we show that in equilibrium, the market price depends not only on what information has been disclosed so far, but also on when it was disclosed. In particular, we show that the price at the end of the second period given disclosure of one signal is *higher if the signal is disclosed later* in the game. This result might be counter intuitive, as one might expect the market to reward the manager for early disclosure of information, since then he seems less likely to be "hiding something."

To see the intuition, let time be $t \in \{1, 2\}$ and suppose it is now $t = 2$. Consider the following two histories on the equilibrium path in which the manager discloses a single signal, x. In history 1, the manager disclosed x at $t = 1$ while in history 2 he discloses x at $t = 2$. The difference between prices under the two histories stems from the scenario where the agent has learned the second signal, y, but it was not disclosed at either $t = 1$ or $t = 2$. If x was disclosed at $t = 1$ then the market can infer that y is less than x and that y is sufficiently low so that revealing it at $t = 2$ would not increase the price at this time. If x is disclosed at $t = 2$, then the market also considers the possibility that the manager knew y already at $t = 1$ and has learned x only at $t = 2$. Conditional on this scenario the market learns that y is not only lower than x but also lower than the threshold for disclosure of a single signal at $t = 1$.[1] One might expect that this should lead to a more negative belief about y. But the opposite is true! There are two effects that impact investors' beliefs in opposite directions. On one hand, the lower threshold implies that the expected value of y, conditional on this event, is lower. On the other hand, the lower threshold also implies that it is less likely that the manager learned y at $t = 1$. In a threshold equilibrium the expectation of y conditional on the manager not knowing it is higher; hence, this increases the expected value of y. We argue that the second effect always dominates! The reason is that the manager still discloses at $t = 2$ all signals y that exceed the market beliefs about y at this time. Hence, in the case of the history with late disclosure, investors additionally rule out any y that is above the disclosure threshold at $t = 1$ but below the threshold for disclosure at $t = 2$. Since the disclosure threshold at $t = 2$ equals the average y for all managers' types who do not disclose y at $t = 2$ (including types who did and did not learn y), ruling out these types, which are lower than the expected value of y according to investors' beliefs, increases the expectation of y and hence increases the market price.

In Section 3, we formalize and extend this intuition to establish the main result of the paper. We argue that later disclosure receives a better interpretation provided that the equilibrium is monotone and symmetric. To further characterize the strategic behavior and market inferences in our model, we characterize threshold equilibria in Section 4. We show that a threshold equilibrium

[1] To simplify the demonstration of the intuition, we assume that the agent follows a threshold strategy at $t = 1$. However, our proof does not make this assumption.

exists under suitable conditions.[2] We then characterize the equilibrium disclosure strategy and the properties of the corresponding equilibrium prices. We find that managers that assign higher weight to the first period's stock price compared to the second period's stock price tend to issue voluntary disclosure more frequently in the first period; i.e., their first period's disclosure threshold is lower. Assigning higher weight to the short term price compared to the long-term price may be due to many reasons that are beyond the scope of our model. Several such examples are: managers who face higher short-term incentives, managers of firms that are about to issue new debt or equity, managers of firms with a higher probability of being taken over, managers with a shorter expected horizon with the firm. The manager's disclosure threshold is also affected by the probability of obtaining private information. In particular, similar to the single-period models, the disclosure threshold is decreasing in the likelihood of the manger to obtain private information.

1.1 Related Literature

The voluntary disclosure literature goes back to Grossman and Hart (1980), Grossman (1981), and Milgrom (1981), who established the "unraveling result," which states that under certain assumptions (including: common knowledge that the agent is privately informed, disclosing is costless, and information is verifiable) all types disclose their information in equilibrium. In light of companies' propensity to withhold some private information, the literature on voluntary disclosure evolved around settings in which the unraveling result does not prevail. The two major streams of this literature are: (*i*) assuming that disclosure is costly (pioneered by Jovanovic 1982 and Verrecchia 1983) and (*ii*) investors' uncertainty about information endowment (pioneered by Dye 1985 and Jung and Kwon 1988). Our model follows Dye (1985) and Jung and Kwon (1988) and extends it to a multi-signal and a multi-period setting.

As mentioned in the introduction, in spite of the vast literature on voluntary disclosure, very little has been done on multi-period settings and on multi-signal settings.[3]

To the best of our knowledge the only papers that study multi-period voluntary disclosures are Shin (2003, 2006), Einhorn and Ziv (2008), and Beyer and Dye (2011). The settings studied in these papers as well as the dynamic considerations of the agents are very different from ours. Shin

[2] In most of the existing voluntary disclosure literature (e.g., Verrecchia 1983, Dye 1985, Acharya et al. 2011), the equilibrium always exists, is unique, and is characterized by a threshold strategy. In our model, due to multiple periods and signals, existence of a threshold equilibrium is not guaranteed, and therefore we provide sufficient conditions for existence (similar to Pae 2005).

[3] For example, this gap in the literature is pointed out in a survey by Hirst, Koonce, and Venkataraman (2008), who write "much of the prior research ignores the iterative nature of management earnings forecasts."

(2003, 2006) studies a setting in which a firm may learn a binary signal for each of its independent projects, where each project may either fail or succeed. In this binary setting, Shin (2003, 2006) studies the "sanitization" strategy, under which the agent discloses only the good (success) news. The timing of disclosure does not play a role in such a setup. Einhorn and Ziv (2008) study a setting in which in each period the manager may obtain a single signal about the period's cash flows, where at the end of each period the realized cash flows are publicly revealed. If the agent chooses to disclose his private signal, he incurs some disclosure costs. Acharya, DeMarzo, and Kremer (2011) examine a dynamic model in which a manager learns one piece of information at some random time and his decision to disclose it is affected by the release of some external news. They show that a more negative external signal is more likely to trigger the release of information by the firm. Perhaps surprisingly this clustering effect is present only in a dynamic model and not in a static one. Given that the firm may learn only one piece of information the effect that we study in our paper cannot be examined in their model. Finally, Beyer and Dye (2011) study a reputation model in which the manager may learn a single private signal in each of the two periods. The manager can be either "forthcoming" and disclose any information he learns or he may be "strategic." At the end of each period, the firm's signal/cash flow for the period becomes public and the market updates beliefs about the value of the firm and the type of the agent. Importantly, the option to "wait for a better signal" that is behind our main result is not present in any of these papers.

Our paper also adds to the understanding of management's decision to selectively disclose information. Most voluntary disclosure models assume a single signal setting, in which the manager can either disclose all of his information or not disclose at all. In practice, managers sometimes voluntarily disclose part of their private information while concealing another part of their private information. To the best of our knowledge, the only exceptions in the voluntary disclosure literature in which agents may learn multiple signals are Shin (2003, 2006), which we discussed above, and Pae (2005). The latter considers a single-period setting in which the agent can learn up to two signals. We add to Pae (2005) dynamic considerations, which are again crucial for creating the option value of waiting for a better signal.

2 The Model

We study a dynamic voluntary disclosure game. There is an agent, who we refer to as a manager of a publicly traded company, and a competitive market of risk neutral investors. The value of the company is a realization of a random variable V and is not known to the market or the manager, but the agents share a common prior over the distribution of V. There are two signals, X and Y, that are random variables, distributed conditional on realization of V symmetrically over $\mathbb{R}^2$ according to some atomless distribution that is symmetric across the two signals. The expected value of V given the realizations of the two signals, (x, y), which equals the stock price P, given the disclosure of the two signals, (x, y), is

$$E[V|X = x, Y = y] = P(x, y) = P(y, x).$$

We assume that the distributions are such that P is continuous and strictly increasing in both arguments. We also assume that the conditional distributions of the signals, $\Phi(x|y)$, have full support.

The game has two periods, $t \in \{1, 2\}$. At the start of the game nobody knows the signal realizations. At the beginning of period 1 the manager privately learns each of the signals with probability p. Learning is independent across the two signals, so that the probability of learning both signals at $t = 1$ is p^2. The probability of learning a signal is also independent of the value of any of the signals or the value of the company. In the beginning of period 2 the agent learns with probability p any signal that he has not yet learned in period 1.[4]

Each period, after potentially learning some signals, the agent decides whether to reveal some or all of the signals he has learned and not yet disclosed (so disclosure is voluntary and can be selective). We follow Dye (1985) and assume that: (*i*) the agent cannot credibly convey the fact that he did not obtain a signal, and (*ii*) any disclosure is truthful (or verifiable at no cost) and does not impose a direct cost on the manager or the firm. Upon disclosure, the market can recognize if it is signal X or Y - for example, they correspond to information about revenues and costs, or domestic and foreign markets.

A public history at time t contains the set of signals that the agent has revealed and the times at which he has revealed them. We denote it by $h_t^P \in H_t^P = \{\emptyset, (x, t_x), (y, t_y), (x, y, t_x, t_y)\}$. The

[4] All the model's analysis and results are robust to the introduction of a third period in which the private signals learned by the manager are publicly revealed.

possible histories are that no signal has been revealed, that one signal has been revealed with value x, or value y and that two signals have been revealed with values (x, y). The times $t_x, t_y \in \{1, 2\}$ are the time signals are revealed.

We assume that when the agent reveals a signal the market does not know when he learned it. For example, if the agent learns signal X at time 1 and reveals it in period 2, the market cannot directly observe whether he has learned that signal in period 1 or 2.

A private history of the agent at time $t = 1$ is the set of signals that he has learned so far, $h_1^A \in H_1^A = \{\emptyset, x, y, (x, y)\}$. At $t = 2$ the private history is the set of signals that he has learned by then and the times he has learned them: $h_2^A \in H_2^A = \{\emptyset, (x, \tau_x), (y, \tau_y), (x, y, \tau_x, \tau_y)\}$, where (τ_x, τ_y) are the times the agent has learned the signals x and y respectively. At $t = 2$ the agent also knows the public history, i.e., whether he has revealed any signals at $t = 1$ and if so, which one. A (behavioral) strategy of the agent at $t = 1$ is a mapping from private history h_1^A into a disclosure policy; at $t = 2$ the strategy is a mapping from $H_2^A \times H_1^P$ to a disclosure policy.

We model the investors in a reduced form: given the public history, they form beliefs about the value of the firm and set the market price at time t equal to that expectation:

$$P_t\left(H_t^P\right) = E\left[V | H_t^P\right] = E\left[P(x, y) | H_t^P\right].$$

If the agent reveals two signals, the market price is $P_t\left(H_t^P\right) = P(x, y)$ no matter in which period the agent revealed those signals since upon revealing both signals there is no more information asymmetry about V (recall that we have assumed that the probability of learning a signal is independent of the value of the company).[5] However, if the agent does not reveal any signal the market prices $P_1(\emptyset)$ and $P_2(\emptyset)$ will be different from the prior expectation of $P(x, y)$ because investors form beliefs based on the equilibrium strategy of the agent and will infer that the agent might have learned some signals and decided not to reveal them (since with positive probability the agent does not learn any signals, both these histories are on the equilibrium path). Finally, when only one signal, e.g., x, has been revealed the price will be

$$P_t(x, t_x) = E_y\left[P(x, y) | H_t^P\right],$$

where the beliefs over the second signal, y, are formed consistently with Bayes rule and the equilibrium strategy of the agent, whenever possible. In fact, some histories in which only x has been

[5] Prices are set this way even after an off-equilibrium disclosure of (x, y).

revealed may be out-of-equilibrium and in this case we do not put restrictions on the market's beliefs about the second signal.

We restrict attention to symmetric equilibria in which if only signal Y is revealed then the market price $P_t(y, t_y)$ is the same as if only signal X is revealed at the same time and happens to have the same value. This allows us to save on some notation. We also restrict attention to problems and equilibria in which $P_t(x, t_x)$ is increasing in x (so that if the agent knows that he plans to reveal only one signal, he will prefer to reveal the higher one).[6] We describe the properties of $P_t(x, t_x)$ in greater detail in the next section.

The manager's objective is to maximize a weighted average of the firm's price over the two periods. For simplicity and without loss of generality we assume throughout most of the analysis that the manager weighs the prices equally across the two periods.

A (perfect Bayesian) equilibrium is a profile of disclosure policies of the agent and a set of price functions $\{P_t(\emptyset), P_t(x, t_x), P(x, y)\}$ (both on and off the equilibrium path) such that the agent optimizes given the price functions and the prices are consistent with the strategy of the agent by applying Bayes rule whenever applicable. The equilibrium is monotone (which is the class we analyze) if the price function $P_t(x, t_x)$ is increasing in x for all t, t_x.

Figure 1 summarizes the sequence of events in the model.

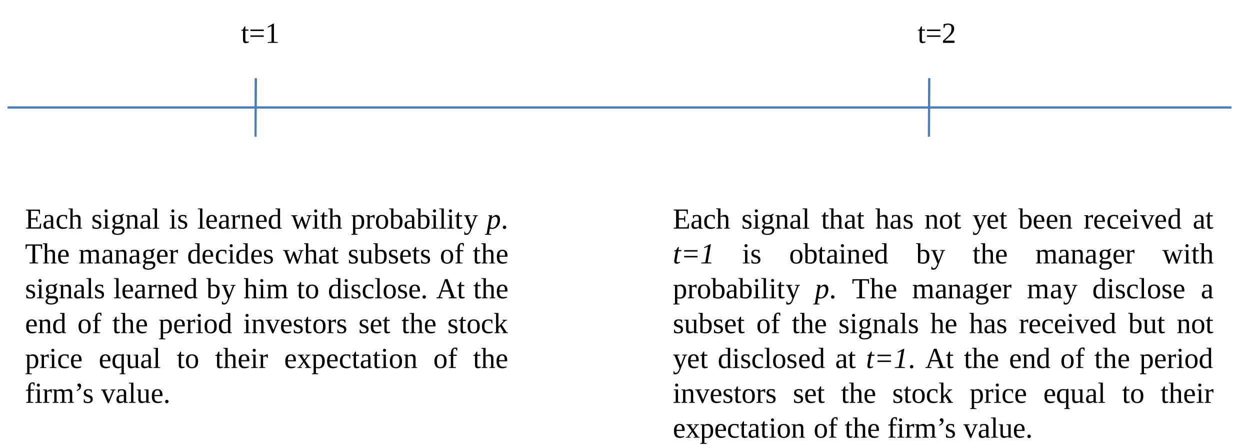

Figure 1: Timeline

[6]For example, if X and Y are positively correlated and the agent follows a threshold strategy for revealing y conditional on revealing x, then the equilibrium will satisfy this constraint.

3 Later Disclosures Receive Better Responses

As mentioned above, we focus on symmetric equilibria. We also restrict attention to monotone pure strategy equilibria, so that $P_t(x, t_x)$ is increasing in x for all t and t_x. Finally, without loss of generality we focus on histories in which x is disclosed before y or both signals are not disclosed so, $t_x \leq t_y$ if $t_y \leq 2$.

In this section we present our main result: if we compare two public histories in which one signal is revealed but at different times, the market price is higher in the history with later disclosure.[7] In other words, the market forms its beliefs based on what is revealed *and also when it is revealed.* It is so even though the time of learning a signal is not informative about the value of the company. Formally:

Theorem 1 *Consider any symmetric monotone PBE in which public histories* $h_2^P = (x, 1)$ *and* $\hat{h}_2^P = (x, 2)$ *are on the equilibrium path.*[8] *Then:*

$$P_2(x, 2) \geq P_2(x, 1),$$

i.e., in period 2 *the price upon revelation of only one signal is higher if that signal was revealed later.*

Theorem 1 characterizes a property of any equilibrium that satisfies the monotonicity assumption. In Section 4 we demonstrate the existence of a particular equilibrium, a threshold equilibrium which satisfies the monotonicity (and symmetry) assumption. Moreover, we show in Section 4 that there exists an x' such that $P_2(x, 2) > P_2(x, 1)$ for all $x > x'$ such that x is revealed on the equilibrium path.

We establish the result of Theorem 1 via a series of claims and lemmas. Some of the details are in the appendix but we try to present the basic intuition in this section. We assume existence of equilibrium with these two public histories $h_2^P = (x, 1)$ and $\hat{h}_2^P = (x, 2)$ on the equilibrium path and keep these two histories fixed. We start with the following observation:

[7] Since in our model there are only 2 signals that can be revealed, the effect of the time of disclosure is relevant only for public histories where one signal is revealed.

[8] I.e., for some x, both at $t = 1$ and $t = 2$ there are some private histories in which the agent learns that $X = x$ and his disclosure policy is to disclose only that signal.

Claim 1 *At $t = 2$ a manager who knows y follows a threshold strategy: conditional on revealing x at $t_x = 1$ or $t_x = 2$, the manager reveals y if and only if $P(x, y) \geq P_2(x, t_x)$.*[9]

This claim follows immediately from the objective function of the manager and the assumption that $P(x, y)$ is increasing in y.

We refer to managers who disclose x and by $t = 2$ know the second signal y as *informed* and those who have not learned y as *uninformed.* Since we look at histories in which x has been disclosed, the only uncertainty of the investors is about y. Formally, the set of *informed* managers is given by $\{(y, \tau_y) | \tau_y \leq 2\}$ and the set of *uninformed* managers is given by $\{(y, \tau_y) | \tau_y > 2\}$, where $\tau_y > 2$ indicates that the manager did not learn y in either of the periods.

A key concept we use is that of *potential disclosers.* This is the set of agents for which Claim 1 applies. These are informed agents at $t = 2$ ($\tau_y \leq 2$) who have not disclosed y at $t = 1$ and their disclosure decision at $t = 2$ can be described as myopic given that x is disclosed (either at $t = 1$ or $t = 2$). In order to get to the set of *potential disclosers* from the set of informed agents, we need to eliminate the following agents: agents who have disclosed y earlier, agents who would have disclosed y but not x, and agents who would have preferred to disclose nothing given x and y. Alternatively, the set of potential disclosures can be describes as the set of all (y, τ_y) that are consistent with the history $\left(h_2^P \text{ or } \hat{h}_2^P\right)$ *plus* all the types (y, τ_y) who reveal x (at either $t = 1$ or $t = 2$) and reveal y at $t = 2$ and it is consistent with the history $\left(h_2^P \text{ or } \hat{h}_2^P\right)$. In the next two subsections we characterize the set of "*potential disclosers*" for the two scenarios/histories h_2^P (when x is disclosed at $t = 1$) and $\hat{h}_2^P$ (when x is disclosed at $t = 2$).

Let A denote the set of uninformed agents and B_1 denote the set of potential disclosers when x is disclosed at $t = 1$. Since the price equals the expectation over y of the firm's value conditional on non-disclosure of y, Claim 1 implies that

$$P_2(x, 1) = E[P(x, y) | y \in S_{A,B_1}], \tag{1}$$

where

$$S_{A,B_1} = A \cup B_1 \backslash \{B_1 \cap \{y : P(x, y) \geq E[(P(x, y) | y \in S_{A,B_1})]\}\}.$$

The set S_{A,B_1} is constructed by looking at the union of A and B_1 when we remove "above average" types from B_1 (above average in the sense that the price when disclosing y at $t = 2$

[9] To simplify the exposition, throughout this section we assume that an agent who is indifferent will disclose his information.

$(P(x,y))$ is higher than the the non-disclosure price $(P_2(x,1))$, as described by Claim 1). Similarly, denoting the set of potential disclosers when x is disclosed at $t=2$ by B_2, we have

$$P_2(x,2) = E[P(x,y)\,|y \in S_{A,B_2}], \tag{2}$$

where

$$S_{A,B_2} = A \cup B_2 \backslash \{B_2 \cap \{\{y : P(x,y) \geq E[(P(x,y)\,|y \in S_{A,B_2})]\}.$$

Note that the above definitions of S_{A,B_1} and S_{A,B_2} rely on the existence and uniqueness of a certain fixed point. We prove the existence and uniqueness of such fixed point below. S_{A,B_1} and S_{A,B_2} play a central role in our proof of Theorem 1.

The roadmap for the proof is as follows. We show that a sufficient condition for $P_2(x,2) \geq P_2(x,1)$ is that the set of potential disclosers when x is disclosed at $t=2$ is a subset of the set of potential disclosers when x is disclosed at $t=1$, that is, $B_2 \subseteq B_1$. Rather than proving this directly we prove it by way of contradiction. In particular, we show that if $P_2(x,1) > P_2(x,2)$, then $B_2 \subseteq B_1$, but we also show that if $B_2 \subseteq B_1$ then $P_2(x,2) > P_2(x,1)$, which leads to a contradiction.

3.1 Generalized Minimum Principle

We now establish a general statistical lemma (it is an extension of the minimum principle that appeared first in Acharya, DeMarzo, and Kremer (2011)).[10] Given sets A and B, let $S_{A,B}$ be defined as

$$S_{A,B} = A \cup B \backslash \{B \cap \{y \geq E[y|y \in S_{A,B}]\}\}.$$

Lemma 1 *Generalized Minimum Principle*

(0) $S_{A,B}$ *exists and is unique.*

(*i*) $E[y|y \in A \cup B] \geq E[y|y \in S_{A,B}]$*, with equality if and only if any* $y \in B$ *satisfies* $y < E[y|y \in S_{A,B}]$.

(*ii*) *Suppose that* $B' \supseteq B$"*. Then* $E[y|y \in S_{A,B''}] \geq E[y|y \in S_{A,B'}]$.

(*iii*) *Suppose that* $B' \supset B$"*. Then* $S_{A,B''} = S_{A,B'}$ *if and only if* $y > E[y|y \in S_{A,B''}]$ *for all* $y \in B' \backslash B$"*.*[11]

[10] Acharya et al. (2011) established a claim that is similar to (0) and (*i*) of the lemma below.

[11] Note that (*ii*) and (*iii*) imply that if there are elements $y \in B' \backslash B$" such that $y < E[y|y \in S_{A,B''}]$ then $E[y|y \in S_{A,B''}] > E[y|y \in S_{A,B'}]$.

To see the intuition behind the existence of $S_{A,B}$ consider an iterative procedure in constructing $S_{A,B}$. In each step the average goes down since we remove some types higher than the previous average. This procedure converges since if we remove all types in B we are left with the set A. The intuition behind (ii) is that having a smaller domain in set B can have two effects: if we reduce the domain of B by removing realizations of y that are higher than the overall average, then the expectation does not change (as these elements would have been removed anyhow from B). If we remove some elements that are lower than the average (even if these are above-average elements of B, but are smaller than the overall average that includes A) then the average goes up. The proof of (ii) and (iii) and the formalization of (0) and (i) are in the appendix.

3.2 The Set of Potential Disclosers when x is Disclosed at t=1

We first note that once x is revealed, prices drop over time if no further disclosure is made, which is formalized in the following Lemma.

Lemma 2 $P_1(x,1) \geq P_2(x,1)$.

We present the proof in the appendix. The intuition is that once the agent reveals x, the market grows more and more worried about adverse selection over y because the probability that the agent learned y grows over time. Since at $t=2$ an agent that disclosed X but did not disclose Y behaves myopically (see Claim 1), Lemma 2 implies the following corollary:

Corollary 1 *A manager that has disclosed x at $t=1$ is myopic with respect to the decision to disclose y. That is, conditional on disclosing x at $t=1$ an informed manager reveals also y at $t=1$ if and only if $P(x,y) \geq P_1(x,1)$.*

We decompose the set B_1 into two disjoint subsets, $B_1 = B_1^1 \cup B_1^2$ where $B_1^1 \cap B_1^2 = \varnothing$. The subsets B_1^1 and B_1^2 are given by

$$B_1^1 = \{(y,\tau_y) | \tau_y = 1,\ y \text{ is consistent with } x \text{ being revealed at } t=1 \text{ and } y \text{ not being revealed}\},$$
$$B_1^2 = \{(y,\tau_y) | \tau_y = 2,\ y \text{ is consistent with } x \text{ being revealed at } t=1\}.$$

If x is disclosed at $t=1$ and y was known only at $t=2$ then no realization of y can be ruled out. Hence, B_1^2 consists of all the agents who became informed at $t=2$, i.e., $B_1^2 = \{(y,\tau_y) | \tau_y = 2\}$. What can we infer about B_1^1 from x being revealed at $t_x = 1$? Under the contradictory assumption that $P_2(x,2) < P_2(x,1)$ we can infer the following.

Claim 2 *Assume that $P_2(x,2) < P_2(x,1)$ and define the set $B_1^{\prime 1} \equiv \{(y,\tau_y)\,|\tau_y = 1, y \leq \min\{x, y^*(x)\}\}$, where $y^*(x)$ satisfies $P_2(x,2) \leq P(x, y^*(x))$. Then using either the set $B_1^{\prime 1}$ or B_1^1 when computing $P_2(x,1)$ using equation (1) yields the same price $P_2(x,1)$.*

See the appendix for the proof.

3.3 The Set of Potential Disclosers when x is Disclosed at t=2

Consider the set B_2. As in the case of B_1, we decompose B_2 into two disjoint subsets, $B_2 = B_2^1 \cup B_2^2$ where $B_2^1 \cap B_2^2 = \varnothing$. The subsets B_2^1 and B_2^2 are given by

$$B_2^1 = \{y|\tau_y = 1,\ y \text{ is consistent with } x \text{ being revealed at } t = 2\}$$

$$B_2^2 = \{y|\tau_y = 2,\ y \text{ is consistent with } x \text{ being revealed at } t = 2\}$$

What do we learn about B_2^1 under the contradictory assumption that $P_2(x,2) < P_2(x,1)$? We first argue that

Lemma 3 *Suppose that $P_2(x,2) < P_2(x,1)$. If only x is disclosed at time $t = 2$ then the agent could not have known both signals at $t = 1$.*

The proof is provided in the appendix.

Based on this lemma and using monotonicity of equilibrium, we can infer that if y was learned at $t = 2$ it must be that $y \leq x$. We can also rule out that y is such that it would have been disclosed at $t = 1$ if it were the only signal known to the agent. Therefore, an immediate corollary is

Corollary 2 *Suppose that $P_2(x,2) < P_2(x,1)$. Then $B_2^1 = \{(y,\tau_y)|\tau_y = 1, y \leq x, y \in ND\}$ where ND is the set of values of y that are not disclosed at $t = 1$ when the agent only knows y at $t = 1$.*

Considering B_2^2 we learn that the agent would not have preferred to disclose just y to disclosing just x, and therefore $B_2^2 = \{(y,\tau_y)|\tau_y = 2, y \leq x\}$.[12]

[12] In principle we also know that y is such that the agent prefers to reveal x than to keep both x and y hidden. This can be ignored as it does not include additional information about y (it implies only that $P_2(\emptyset) \leq P_2(x,2)$, which is independent of y).

3.4 Proof of the Main Theorem

Based on the definition of S_{A,B_2} we have $E[P(x,y)|y \in S_{A,B_2}] = E[P(x,y)|y \in S_{A,B_2'}]$ where $S_{A,B_2'} = S_{A,B_2} \cap \{y < P_2(x,2)\}$. Hence we can replace B_2 with B_2'. Note that we have

$$B_2'^1 = \{(y,\tau_y)|\tau_y = 1, y \leq x, y \in ND, y < P_2(x,2)\}.$$

Given Claim 2 we have that $B_1^1 \supseteq B_2'^1$. Similarly

$$B_2'^2 = \{(y,\tau_y)|\tau_y = 2, y \leq x, y < P_2(x,2)\}$$

and we have $B_1^2 \supseteq B_2'^2$ which implies that $B_1 \supseteq B_2'$. However, Lemma 1(ii) shows that $B_1^2 \supseteq B_2'^2$ implies $E[y|y \in S_{A,B_2'}] \geq E[y|y \in S_{A,B_1}]$ and therefore that $P_2(x,2) \geq P_2(x,1)$. This contradicts the assumption that $P_2(x,2) < P_2(x,1)$ and implies that Theorem 1 holds.

In section 4 we analyze a threshold equilibrium and show that the inequality in Theorem 1 is strict for sufficiently high values of x. This may hold either for all values of x that are disclosed on the equilibrium path, or else for values of x that are higher than some finite x'. In the general case of Theorem 1 a sufficient condition for the inequality to be strict is that the set B_2' is strictly subsumed by the set B_1' (where $B_1' = B_1'^1 \cup B_1^2$). More intuitively, a sufficient condition for a strict inequality is that there are values $y < P_2(x,2)$ which are not precluded under the assumption that the agent learned only y at $t = 1$ and concealed it, i.e., there are values $y \in ND$ that are lower than $P_2(x,2)$.

4 A Threshold Equilibrium

In this section we demonstrate that a symmetric threshold equilibrium exists under suitable conditions. In this way, we establish that the assumptions in the previous section are non-vacuous. We also show that for large x the inequality in Theorem 1 is strict.

In static voluntary disclosure models with one signal, it is easy to prove that any equilibrium is in threshold strategies. It follows from the facts that (i) the payoff upon non-disclosure is fixed; and (ii) the payoff upon disclosure equals the expected type given the signal, which is increasing in the signal. Hence, the incentives to disclose are increasing in type and equilibrium is characterized by a threshold strategy. In contrast, in our dynamic setting (i) the agent's expected payoff upon non-disclosure is not constant but rather increasing in type; and (ii) the payoff upon disclosure of one signal is difficult to compute since it depends on the equilibrium beliefs (regarding the second signal). Therefore, unlike the static case, we are not able to prove that all equilibria are in threshold strategies. Instead we show existence of such equilibria by showing that if market prices are set using beliefs that the manager follows a threshold strategy then indeed it is optimal for the manager to follow a threshold strategy (for some thresholds consistent with the beliefs).

To demonstrate the existence and properties of a threshold equilibrium, we make the following additional assumptions. The value of the firm, V, is normally distributed. Without loss of generality, V has zero mean, i.e., $V \sim N(0, \sigma^2)$. The private signals that the manager may learn are given by $X = V + \tilde{\varepsilon}_x$ and $Y = V + \tilde{\varepsilon}_y$ where $\tilde{\varepsilon}_x, \tilde{\varepsilon}_y \sim N(0, \sigma_\varepsilon^2)$ and $\tilde{\varepsilon}_x, \tilde{\varepsilon}_y$ are independent of V and of each other.

We denote investors' equilibrium expectation at time t of the signal y, conditional on the manager disclosing $X = x$ at time $t_x \leq t$ by

$$h_t(x, t_x).$$

Properties of the joint normal distribution of the signals imply the following conditional expectations:

$$\begin{aligned}
E[V|X = x] &= \beta_1 x, \\
E[Y|X = x] &= \beta_1 x \\
E[V|X = x, Y = y] &= \beta_2(x + y)
\end{aligned}$$

where $\beta_1 = \frac{\sigma^2}{\sigma^2+\sigma_\varepsilon^2}$ and $\beta_2 = \frac{\sigma^2}{2\sigma^2+\sigma_\varepsilon^2}$. Note that $\beta_2 < \beta_1 < 2\beta_2 < 1$ and $\beta_2(1 + \beta_1) = \beta_1$.

Of course, equilibrium prices are more complicated because when the manager discloses only one signal investors form beliefs about the other signal. In particular, the expectation of the firm's value given disclosure of a single signal, $X = x$, at $t = t_x$ as calculated at the end of period t is:

$$\begin{aligned}
P_t(x, t_x) &= E[V|x \text{ was disclosed at } t_x] \\
&= \beta_2(x + h_t(x, t_x)).
\end{aligned}$$

We define a threshold strategy in our dynamic setting with two signals as follows:

Definition 1 *Denote the information set of an agent by $\{x, y\} \in (\mathbb{R} \cup \emptyset)^2$ where $x = \emptyset$ means that the agent has not learned the signal X yet. We say that the equilibrium is a threshold equilibrium if* *an agent with information set* $\{(x, \tau_x), (y, \tau_y)\}$ *who discloses* x *at time* $t_x \in \{1, 2\}$ *also discloses any* $x' > x$ *by time* t_x *when his information set is* $\{(x', \tau_x), (y, \tau_y)\}$*, and symmetrically for the signal* Y*.*[13]

As shown in Section 3 the agent is myopic at $t = 2$, so he follows a threshold strategy then. Hence, we focus our attention on the manager's disclosure decision at $t = 1$. The main result of this section is:

Proposition 1 *For* $p < 0.95$ *there exists a threshold equilibrium in which*[14]

(i) an agent who at $t = 1$ *learns only one signal discloses it at* $t = 1$ *if and only if it is greater than* x^**. If the agent learns both signals at* $t = 1$ *and one of them is greater than* x^* *then he discloses at least the highest signal at* $t = 1$*. Disclosing a single signal* $x < x^*$ *at* $t = 1$ *is not part of the equilibrium disclosure strategy.*

(ii) there exists $x' \geq x^*$ *such that* $P_2(x, 2) > P_2(x, 1)$ *for any* $x \geq x'$*.*

Note that an agent who learns both signals at $t = 1$ and one of them is greater than x^* may choose to disclose at $t = 1$ both signals or just the higher signal.

The proof of Proposition 1 is quite complex and therefore we provide few interim steps. We start by describing the expected payoff the various manager types obtain from different disclosure decisions. Next, in Section 4.1, we discuss a variant of a static disclosure model that provides some of the insights we later use in the proof for our dynamic setting. We should note that this variant of the static model may be of independent interest. In Section 4.2, we characterize the equilibrium prices if the manager is believed to follow a threshold strategy. For these prices, we show that the agent's expected payoff upon disclosure at $t = 1$ increases in his type faster than his expected payoff upon non-disclosure. Therefore, given these prices, the agent's best response would indeed be to follow a threshold strategy in the first period, which allows us to complete the proof of Proposition 1. In Section 4.3 we discuss some empirical predictions that our model offers.

We first assume (and later confirm) that there exists a threshold equilibrium in which a manager who learns a single signal, x, at $t = 1$ discloses it at $t = 1$ if and only if $x \geq x^*$. We later show that

[13] While we do not know whether a non-threshold equilibrium exists, one can show that it is always the case that the equilibrium-reporting strategy of the second period is a threshold strategy.

[14] We believe that the threshold equilibrium exists also for values of p greater than 0.95; however, for tractability reasons we restrict the values of p.

a manager with $x \geq x^*$ who deviates and does not disclose x at $t = 1$ still prefers to disclose x at $t = 2$ over not making any disclosure at $t = 2$.

It is useful to partition the set of managers who learn a signal $x \geq x^*$ at $t = 1$ into the following three subsets: (I) managers who learn only x at $t = 1$; (II) managers who learn both signals at $t = 1$ but the signal y is sufficiently high, such that if the manager doesn't disclose any signal at $t = 1$ he will disclose both x and y at $t = 2$; and (III) managers who learn both signals at $t = 1$ but the signal y is sufficiently low, such that if the manager doesn't disclose any signal at $t = 1$ he will not disclose y at $t = 2$ (and will disclose x at $t = 2$). In cases (II) and (III) we assume without loss of generality that $y \leq x$ (if $y > x$ then our reasoning is symmetric when replacing x with y).

Consider case (I).

If the manager discloses x at $t = 1$ he will disclose Y at $t = 2$ (if learns it) only if $P(x, y) \geq P_2(x, 1)$, i.e., only if $y \geq h_2(x, 1)$. On the other hand, if the manager does not disclose x at $t = 1$ he may benefit at $t = 2$ from one of two "real options." The first option value will be realized if the manager learns at $t = 2$ a sufficiently high y, so that at $t = 2$ his optimal strategy will be to disclose y and conceal x. This will happen for sufficiently high realizations of y, such that $y > y^H(x)$.[15] For such realizations of y, not disclosing x at $t = 1$ increases the manager's payoff at $t = 2$ relative to his payoff at $t = 2$ if he were to disclose x at $t = 1$. The second option value will be realized if either the manager does not learn y at $t = 2$ or if he learns a sufficiently low y such that he does not disclose y and only discloses x at $t = 2$. When he learns y at $t = 2$ the manager will not disclose it (and will disclose only x) if $P(x, y) < P_2(x, 2)$, i.e., if $y < h_2(x, 2)$. In such a case, since $P_2(x, 2) \geq P_2(x, 1)$ (see Theorem 1), the manager's payoff at $t = 2$ is higher than his payoff would have been had he disclosed x at $t = 1$.

In order for the agent who learned a single signal at $t = 1$ to disclose it at $t = 1$, his expected benefit from these two real options should be (weakly) lower than the difference in equilibrium prices at $t = 1$ conditional on disclosing or not disclosing x at $t = 1$. This implies that in equilibrium, each agent who discloses x at $t = 1$ benefits from an immediate gain upon disclosure relative to concealing his signal at $t = 1$, that is,

$$P_1(x, 1) > P_1(\emptyset) \text{ for any } x \geq x^*. \tag{3}$$

This inequality means that following the disclosure of any single signal at $t = 1$ (on the equi-

[15] The value $y^H(x)$ is the value of y for which $P_2(y, 2) = P(x, y)$, or equivalently $h_2(y, 2) = x$.

librium path) the price goes up by some uniformly bounded amount compared to the price when no information is disclosed. This is different from the standard Dye (1985) and Jung and Kwon (1988) model.

More explicitly, the optimal disclosure policy of a manager who learns only x at $t=1$ depends on the comparison of the following two payoffs. The expected payoff at $t=1$ of an agent that learned only x and chooses to disclose it is:

$$\begin{aligned}
&E_{t=1}\left(U|\tau_x=1,\tau_y\neq 1,t_x=1,x\right)=\beta_2\left(x+h_1\left(x,1\right)\right)+E_y\left[\max\left\{\beta_2\left(x+h_2\left(x,1\right)\right),\beta_2\left(x+y\right)\right\}|x\right]\\
&=\beta_2\left(x+h_1\left(x,1\right)\right)+\left(1-p\right)\beta_2\left(x+h_2\left(x,1\right)\right)\\
&+p\beta_2\left[\left(x+\int_{-\infty}^{h_2(x,1)}h_2\left(x,1\right)f\left(y|x\right)dy\right)+\int_{h_2(x,1)}^{\infty}\left(x+y\right)f\left(y|x\right)dy\right].
\end{aligned}$$

where $f\left(y|x\right)$ is the pdf of the distribution of Y conditional on $X=x$. The first term is the period 1 price, the second term is the period 2 price if the agent does not learn Y and the last term is the average second period price if the agent learns Y and follows the optimal strategy of revealing y if and only if $y\geq h_2\left(x,1\right).$

The expected payoff at $t=1$ of an agent that learned only x at $t=1$ and withholds his information at $t=1$ is:[16]

$$\begin{aligned}
&E_{t=1}\left(U|\tau_x=1,\tau_y\neq 1,t_x\neq 1,x\right)=P_1\left(\emptyset\right)+E_y\left[\max\left\{\beta_2\left(x+h_2\left(x,2\right)\right),\beta_2\left(x+y\right),\beta_2\left(y+h_2\left(y,2\right)\right)\right\}|x\right]\\
&=P_1\left(\emptyset\right)+\left(1-p\right)\beta_2\left(x+h_2\left(x,2\right)\right)\\
&+p\beta_2\left[\int_{-\infty}^{h_2(x,2)}\left(x+h_2\left(x,2\right)f\left(y|x\right)dy\right)+\int_{h_2(x,2)}^{y^H(x)}\left(x+y\right)f\left(y|x\right)dy+\int_{y^H(x)}^{\infty}\left(y+h_2\left(y,2\right)f\left(y|x\right)dy\right)\right].
\end{aligned}$$

The first two terms are analogous to the first two terms in the previous case. The last term is the average price from an optimal strategy in period 2 which is: if $y<h_2\left(x,2\right)$ then disclose only x, if $y\in\left[h_2\left(x,2\right),y^H\left(x\right)\right]$ disclose both signals, and if $y>y^H\left(x\right)$ disclose only y.

A manager that learns a single signal, x, at $t=1$ prefers to disclose x at $t=1$ to not disclosing it if

$$E_{t=1}\left(U|\tau_x=1,\tau_y\neq 1,t_x=1,x\right)\geq E_{t=1}\left(U|\tau_x=1,\tau_y\neq 1,t_x\neq 1,x\right). \tag{4}$$

Next, consider subset (II).

[16] The agent considers $E_y\left[\max\left\{\beta_2\left(x+h_2\left(x,2\right)\right),\beta_2\left(x+y\right),\beta_2\left(y+h_2\left(y,2\right)\right),P_2\left(\emptyset\right)\right\}|x\right]$. However, for any $x>x^*$ an agent who did not disclose at $t=1$ is better off disclosing x at $t=2$ than not disclosing at all. Therefore, we omit the price at $t=2$ given no disclosure, $P_2\left(\emptyset\right)$, from the agent's expected utility.

Subset (II) consists of managers who learn both signals at $t = 1$ and y is such that $P(x, y) \geq P_2(x, 2) \geq P_2(x, 1)$, i.e., $y \geq h_2(x, 2)$. Such types disclose both signals at $t = 2$ no matter if they disclose x at $t = 1$ or not. Therefore, such types do not benefit from any of the real options that managers in subset (I) do. Such a manager will disclose x at $t = 1$ if

$$\max\{P_1(x, 1), P(x, y)\} \geq P_1(\emptyset).$$

So types in subset (II) will clearly follow a threshold strategy.

Finally, consider subset (III).

Subset (III) consists of managers who learn both signals at $t = 1$ and y is such that $P(x, y) < P_2(x, 2)$, i.e., $y < h_2(x, 2)$. If such a type discloses nothing at $t = 1$, he will benefit from the delay since $P_2(x, 2) \geq P_2(x, 1)$. So, they trade off a higher price at $t = 1$ against a lower price at $t = 2$. Such a manager will disclose x at $t = 1$ if

$$P_1(x, 1) + P_2(x, 1) \geq P_1(\emptyset) + P_2(x, 2), \tag{5}$$

or, equivalently, if

$$\beta_2(x + h_1(x, 1)) + \beta_2(x + h_2(x, 1)) \geq P_1(\emptyset) + \beta_2(x + h_2(x, 2)).$$

To show that given the equilibrium prices the manager's best strategy is a threshold strategy at $t = 1$ it is sufficient to show that $LHS - RHS$ of both inequalities (4) and (5) increase in x. We will show below that if $LHS - RHS$ of (5) is increasing in x so does $LHS - RHS$ of (4). Hence, it will be sufficient to prove that:

$$\frac{\partial}{\partial x}P_1(x, 1) + \frac{\partial}{\partial x}P_2(x, 1) \geq \frac{\partial}{\partial x}P_2(x, 2),$$

or equivalently,

$$\frac{\partial}{\partial x}h_1(x, 1) + \frac{\partial}{\partial x}h_2(x, 1) \geq \frac{\partial}{\partial x}h_2(x, 2) - 1. \tag{6}$$

Next, in Section 4.1 we study an extension of Dye's (1985) static model, in which we assume that the disclosure threshold of the agent is determined exogenously and is stochastic. This will be instrumental in Section 4.2 which is the last stage of proving Proposition 1.

4.1 A Variant of a Static Model

Consider the following static disclosure setting, similar to Dye (1985) and Jung and Kwon (1988). The agent's potential signal is now the firm's value and it is a realization of $S \sim N(\mu, \sigma^2)$.[17] With

[17] The reason we are considering general μ is that in our dynamic setting investors will update their beliefs about the undisclosed signal, y, based on the value of the disclosed signal, x.

probability p the agent learns this value. If the agent learns the realization of S he may choose to disclose it. We are interested in investors' beliefs about the firm's value given no disclosure for an arbitrary threshold disclosure policy. That is, what is the expectation of S given that the agent discloses s if and only if $s \geq z$, for exogenously determined z. Unlike Dye (1985) and Jung and Kwon (1988), we are not constraining z to be consistent with optimal disclosure strategy by the agent's, i.e., z is not part of an equilibrium. We will refer to this setting as the *"Dye setting with an exogenous disclosure threshold."*

Denote by $h^{stat}(\mu, z)$ investors' expectation of S given that no disclosure was made and given that the disclosure threshold is z. Figure 2 plots $h^{stat}(\mu, z)$ for $S \sim N(0,1)$ with $p = 0.5$.

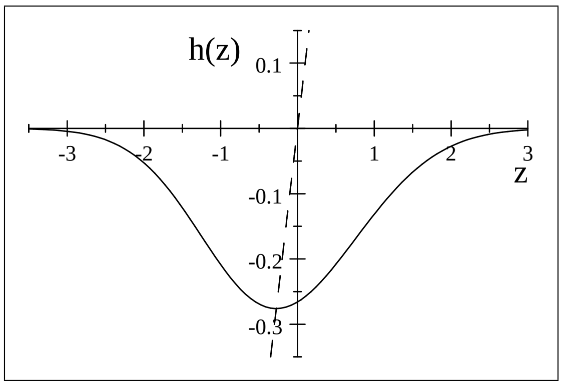

Figure 2: Price Given No-Disclosure in a Dye Setting with Exogenous Disclosure Threshold z

For $z \to \infty$ none of the agents discloses, and hence, following no disclosure investors do not revise their beliefs relative to the prior. For $z \to (-\infty)$ all agents who obtain a signal disclose it, and therefore, following no disclosure investors infer that the agent is uninformed, and therefore (as for $z \to \infty$) investors posterior beliefs equal the prior distribution. As the exogenous disclosure threshold, z, increases from $-\infty$, upon observing no disclosure investors know that the agent is either uninformed or that the agent is informed and his type is lower than z. Therefore, for any finite disclosure threshold, z, investors' expectation of s following no disclosure is lower than the prior mean (zero). The following lemma provides a further characterization of investors' expectation about s given no disclosure, $h^{stat}(\mu, z)$.

Lemma 4 *Consider the Dye setting with an exogenous disclosure threshold. Then:*

1. $h^{stat}(\mu + \Delta, z + \Delta) = h^{stat}(\mu, z) + \Delta$ *for any constant* Δ*; this implies that* $\frac{\partial}{\partial \mu} h^{stat}(\mu, z) + \frac{\partial}{\partial z} h^{stat}(\mu, z) = 1$.

2. $z^* = \arg\min_z h^{stat}(\mu, z)$ *if and only if* $z^* = h^{stat}(\mu, z^*)$*. This implies that the equilibrium* *disclosure threshold in the standard Dye (1985) and Jung and Kwon (1988) equilibrium minimizes* $h^{stat}(\mu, z)$.

The second point follows from Lemma 1 (the Generalized Minimum Principle). Note that for all $z < h^{stat}(\mu, z)$ the price given no disclosure, $h^{stat}(\mu, z)$, is decreasing in z (and for $z > h^{stat}(\mu, z)$ it is increasing in z).

Direct analysis of the $h^{stat}(\mu, z)$ shows that for $p < 0.95$ the absolute value of the slope of $h^{stat}(\mu, z)$ with respect to z is uniformly bounded by 1. This bound allows to bound how future prices change with x and that allows us to establish the existence of a threshold strategy equilibrium; this is where we use the assumption $p < 0.95$ in Proposition 1.

For the analysis of our dynamic model it will prove useful to consider a richer variant of this model, allowing a random threshold policy. In particular, with probability λ_i, $i \in \{1, .., K\}$, where $\sum_{i=1}^{K} \lambda_i = p$, the agent discloses only if his type is above $z_i(\mu)$. The reason we are considering a random disclosure policy is as follows. In our dynamic setting, when by $t = 2$ the agent disclosed a single signal investors do not know whether the agent learned a second signal and if so, whether he learned it at $t = 1$ or at $t = 2$. Since the agent follows different disclosure thresholds at the two possible dates, investors' beliefs about the agent's disclosure threshold for the signal y are stochastic. Moreover, the disclosure thresholds for y change with x, but x affects also the investor's belief about the unconditional mean of the second signal Y, so we write z as a function of the mean.

Let us denote by $h^{stat}(\mu, \{z_i(\mu)\})$ the conditional expectation of the type given no disclosure and given that the disclosure thresholds are $z_i(\mu)$.

Lemma 5 *For* $p \leq 0.95$ *suppose that* $z_i(\mu) < h^{stat}(\mu, \{z_i(\mu)\})$ *and* $z_i'(\mu) \in [0, c]$ *for all* i. *Then* $\frac{d}{d\mu} h^{stat}(\mu, \{z_i(\mu)\}) \in (1, 2 - c)$.

The intuition for the random case, in which $K > 1$, is somewhat complicated, and, therefore, we defer it to the appendix where we formally prove Lemma 5. In order to provide the basic intuition for the result, we analyze the particular case in which the disclosure strategy is nonrandom, i.e., $K = 1$. We start by providing the two simplest examples, for the cases where $z'(\mu) = 1$ and $z_i'(\mu) = 0$. These examples are useful in demonstrating the basic logic and how it can be analyzed using Figure 2. These two examples also provide most of the intuition for the case with no restriction on $z_i'(\mu)$, which is presented in Example 3. Note that example 3 also provides the upper and lower bounds for the more general case in Lemma 5.

Examples (all the examples assume $K = 1$):

1. If $z'(\mu) = 1$ then $\frac{d}{d\mu} h^{stat}(\mu, z(\mu)) = 1$.

   Using point 1 in Lemma 4 we have $\frac{d}{d\mu} h^{stat}(\mu, z(\mu)) = \frac{\partial}{\partial\mu} h^{stat}(\mu, z) + z'(\mu) * \frac{\partial}{\partial z} h^{stat}(\mu, z) = 1$. The intuition can be demonstrated using Figure 2. A unit increase in μ (keeping z constant) shifts the entire graph both upwards and to the right by one unit. However, since also z increases by one unit, the overall effect is an increase in $h^{stat}(\mu, z(\mu))$ by one unit.

2. If $z'(\mu) = 0$ and $z(\mu) = z^*$, then $\frac{d}{d\mu} h^{stat}(\mu, z(\mu)) \in (1, 2)$.

   From Lemma 4 we know that $\frac{\partial}{\partial\mu} h^{stat}(\mu, z^*) + \frac{\partial}{\partial z^*} h^{stat}(\mu, z^*) = 1$ and therefore $\frac{\partial}{\partial\mu} h^{stat}(\mu, z^*) = 1 - \frac{\partial}{\partial z^*} h^{stat}(\mu, z^*)$. We also know that $\frac{\partial}{\partial z^*} h^{stat}(\mu, z^*) \in [(-1, 0)$ since $z^* \leq h^{stat}(\mu, z^*)$. Therefore, $\frac{\partial}{\partial\mu} h^{stat}(\mu, z^*) \in (1, 2)$. The intuition can be demonstrated using Figure 2. The effect of a unit increase in μ can be presented as a sum of two effects: (i) a unit increase in the disclosure threshold z as well as a shift of the entire graph both to the right and upwards by one unit and (ii) a unit decrease in the disclosure threshold z (as $z'(\mu) = 0$). The first effect is similar to Example 1 above and therefore increases $h^{stat}(\mu, z(\mu))$ by one. The second effect increases $h^{stat}(\mu, z(\mu))$ by the absolute value of the slope of $h^{stat}(\mu, z)$, which is between zero and one.

3. In the general case $z'(\mu) = c$, we have $\frac{d}{d\mu} h^{stat}(\mu, z(\mu)) \in (1, 2 - c)$.

   This covers previous examples as special cases. Following a similar logic, we conclude that $\frac{d}{d\mu} h^{stat}(\mu, z(\mu)) = \frac{\partial}{\partial\mu} h^{stat}(\mu, z) + c * \frac{\partial}{\partial z} h^{stat}(\mu, z) = 1 + (c - 1)\frac{\partial}{\partial z} h^{stat}(\mu, z(\mu))$. Recall that $\frac{\partial}{\partial z} h^{stat}(\mu, z(\mu)) \in (-1, 0)$ for $p < 0.95$.

4.2 Existence of a Threshold Equilibrium and Characterization of Prices

In this section, we first assume the existence of a threshold equilibrium in which a manager who learns only x at $t = 1$ discloses it if and only if $x \geq x^*$. We derives some characteristics of prices that always hold for such a disclosure strategy. Then, using these characteristics of prices, we finalize the proof of Proposition 1 by showing that it is indeed optimal for the manager to follow a threshold strategy.

Recall three observations that we discussed earlier. First, at $t = 2$ the manager behaves myopically and follows a threshold disclosure strategy. Second, the price at $t = 1$ given no disclosure, $P_1(\emptyset)$, is lower than the price at $t = 1$ given disclosure of any $x \geq x^*$. That is, $P_1(\emptyset) < P_1(x, 1)$ (see Equation 3). Third, an agent who learns both signals at $t = 1$ such that $y < x$ and discloses x at $t = 1$, behaves myopically with respect to the disclosure of his signal y in both $t = 1$ and $t = 2$ (see Corollary 1).

Next, we characterize the slopes of the various prices given disclosure of a single signal; i.e., we characterize: $P_1(x, 1) = \beta_2 (x + h_1(x, 1))$, $P_2(x, 1) = \beta_2 (x + h_2(x, 1))$, and $P_2(x, 2) = \beta_2 (x + h_2(x, 2))$. As we discussed earlier, a sufficient condition for a threshold strategy to indeed be optimal for the manager is to show that inequality (6), which uses the derivatives of the above prices, holds.

Claim 3 *Suppose that investors believe that the manager follows a threshold reporting strategy similar to the one in Proposition 1. Then, for* $p \leq 0.95$*:*

$$\frac{\partial}{\partial x} h_1(x,1) \begin{cases} = \beta_1 & if \quad h_1(x,1) < x \\ \in (2\beta_1 - 1, \beta_1) & if \quad h_1(x,1) > x \end{cases}$$

$$\frac{\partial}{\partial x} h_2(x,2) \begin{cases} = \beta_1 & if \quad h_2(x,2) < x^* \\ \in (2\beta_1 - 1, 2\beta_1) & if \quad h_2(x,2) > x^* \end{cases} \tag{7}$$

$$\frac{\partial}{\partial x} h_2(x,1) \begin{cases} = \beta_1 & if \quad h_2(x,1) < x \\ \in (2\beta_1 - 1, \beta_1) & if \quad h_2(x,1) > x \end{cases}$$

Proof. We provide here the reasoning for $h_1(x, 1)$ and delegate the other two cases to the appendix.

As we showed in Section 3, for any x that is disclosed at $t = 1$ such that $h_1(x, 1) < x$ (the non binding case),[18] if $\tau_y = 1$ the agent is myopic with respect to the disclosure of y and discloses it whenever $y \geq h_1(x, 1)$. This makes the analysis of the effect of an increase in x on $h_1(x, 1)$ qualitatively similar to the analysis of an increase in the mean of the distribution in a standard Dye (1985) and Jung and Kwon (1988) equilibrium. In this case, an increase in the mean of the distribution results in an identical increase in both the equilibrium beliefs and the equilibrium disclosure threshold. This case is captured by Example 1 in Section 4.1. The quantitative difference in our dynamic setting is that a unit increase in x increases investors' beliefs about y by β_1 (rather

[18] We use the term non-binding to indicate that the constraint/investors' inference $y < x$ is not binding. The reason is that in this case $h_1(x, 1) < x$ also implies that $y < x$.

than by 1) and therefore also increases both the beliefs about y and the threshold for disclosure of y by β_1. As a result, in our dynamic setting, for $h_1(x,1) < x$ we have $\frac{\partial}{\partial x} h_1(x,1) = \beta_1$.[19]

In the binding case, i.e., for all x such that $h_1(x,1) > x$ (if such $x > x^*$ exists) we know that if $\tau_y = 1$ then $y < x$ (otherwise, the manager would have disclosed y). An increase in x increases the beliefs about y at a rate of β_1 while the increase in the constraint/disclosure threshold ($y < x$) is at a rate of 1. Therefore, this is a special case of Example 3 in Section 4.1, where we increase the mean by β_1 and $z'(\mu) \equiv c = \frac{1}{\beta_1}$. From Example 3 we know that an increase in the beliefs about y given a unit increase in x (which is equivalent to an increase of β_1 in the value of μ in Example 3) is given by $\beta_1 \left(1 + (c-1) \frac{\partial}{\partial z} h^{stat}(\mu, z)\right)$. Substituting $c = \frac{1}{\beta_1}$ and rearranging terms yields

$$\frac{\partial}{\partial x} h_1(x,1) = \beta_1 + (1-\beta_1) \frac{\partial}{\partial z} h^{stat}(\mu, z).$$

Since $\frac{\partial}{\partial z} h^{stat}(\mu, z) \in (-1, 0)$ we have $\frac{\partial}{\partial x} h_1(x,1) \in (2\beta_1 - 1, \beta_1)$.

Analyzing the effect of x on $h_2(x,2)$ and $h_2(x,1)$ is more involved and more technical. Therefore, we defer it to the appendix (See Proof of Claim 3). The reason these cases are more complicated is that when pricing the firm at $t = 2$ investors do not know whether the manager learned y at $t = 1$ or at $t = 2$ (in the case where the agent did in fact learn y). Investors' inferences about y depend on when the agent learned it, and therefore the analysis of $h_2(x,2)$ and $h_2(x,1)$ requires stochastic disclosure thresholds. This is where we use Lemma 5. ■

Equipped with this characterization of how prices change with x, under the assumption of a threshold disclosure strategy, we are now ready to establish the existence of a threshold equilibrium. The existence proof is complicated and technical, and therefore we provide only a sketch of the proof below, while the formal proof is deferred to the appendix.

Sketch of Proof for Proposition 1. We show that if the manager learns a sufficiently high signal at $t = 1$ he will disclose it at $t = 1$ (and if he learns a second signal at $t = 1$ he sometimes discloses it as well). If the highest signal the manager learns at $t = 1$ is sufficiently low he will not make a disclosure at $t = 1$. Finally, using the properties of the slopes of the various prices that we derived in Claim (3), we show that the difference between the agent's expected payoff at $t = 1$ from disclosing a signal x at $t = 1$ and his expected payoff at $t = 1$ from not disclosing it is increasing in the signal x. This guarantees the existence of a threshold equilibrium.

[19] Since both the beliefs about y and the disclosure threshold increase at the same rate, the probability that an agent who discloses only x at $t = 1$ learned y at $t = 1$ but did not disclose it is independent of x.

Finally, we show that there exists an x' such that for $x > x'$ later disclosure receives a strictly better interpretation, i.e., $P_2(x, 2) > P_2(x, 1)$. ■

While Proposition 1 establishes the existence of a threshold equilibrium, it does not describe how x^*, the disclosure threshold at $t = 1$, is determined. We complete this analysis below.

At the beginning of Section 4, when discussing the manager's trade-offs we partitioned the set of managers who make a disclosure at $t = 1$ into subsets $(I) - (III)$. We will use the same partition in order to describe how the threshold for disclosure of a single signal at $t = 1$ is determined.

For a given x, if a manager in subset (III) prefers to disclose x at $t = 1$ then it is easy to see that every manager in subset (II) strictly prefers to disclose x at $t = 1$. It is not easy, however, to compare the incentives to disclose in subsets (I) and (III). The reason is that a type in subset (I) that does not disclose x at $t = 1$ may benefit from either one of the real options, or none of them, while a type in subset (III) benefits for sure from just one of the real options (the increased price at $t = 2$ when disclosing a single signal at $t = 2$ relative to the price at $t = 2$ when disclosing x at $t = 1$).

To obtain an equilibrium with a threshold for disclosure of a single signal at $t = 1$, we set x^* to equal the lowest value of x for which all agents with $x = x^*$ from all subsets $(I) - (III)$ weakly prefer to disclose x^* at $t = 1$ to not disclosing at $t = 1$. The binding constraint might be either Equation (4) or Equation (5). Since in equilibrium, there are agents who learn a signal x^* at $t = 1$ and strictly prefer to disclose it at $t = 1$ to not disclosing at $t = 1$ (these are agents in subset (II) and possibly some agents from one of the other subsets), the price given disclosure of $x < x^*$ at $t = 1$, which is off the equilibrium path, must be sufficiently low to prevent the above types from deviating from the equilibrium strategy and disclosing $x < x^*$ at $t = 1$. This implies that a necessary condition for our equilibrium is that prices exhibit a discontinuity at x^*. Note that on the equilibrium path, prices are continuous.

4.3 Empirical Implications

Our model demonstrates how strategic considerations affect an agent's voluntary disclosure decisions in a dynamic setting. An immediate empirical implication of our model is that, given all else equal, a later occurrence of voluntary disclosure receives a better interpretation. There are many real life circumstances in which, as our model assumes, investors are uncertain about the time in which a firm observes value-relevant information and the disclosure of such information is voluntary. Firms that have ongoing R&D projects can obtain new information about the state of their projects, where the time of information arrival and its content is unobservable to the market. Moreover, such results are not required to be publicly disclosed. One such example is pharmaceutical companies that get results of drug clinical trials. Investors' beliefs about a drug's clinical trial may have a great effect on the firm's price and may also affect beliefs about the prospect of other projects of the firm. In such a setting, our model predicts that when the firm discloses the results of only part of its projects, a later disclosure gets a more positive market reaction. Another related example is firms that apply for patents. After the initial application, the firm first waits to receive a notice of allowance (NOA) from the US PTO (U.S. Patent and Trademark Office) for each of the applications, which indicates that the patent is near approval. Typically, patent applications may include many claims to be covered under the patent and the NOA informs the firms which of the claims have been approved and which have not been approved. Following the NOA, the firm waits for the formal issuance, indicating that the PTO has formally bestowed patent protection.[20] As Lansford (2006, page 5) indicates: "It is important to note that firms enjoy wide discretion as to when to announce a patent event." Lansford (2006) documents that firms indeed time the disclosure of NOA strategically.

In light of the focus of the current literature on static models, an interesting question is how the dynamic/multi period nature of our setting, which we believe is very prevalent in practice, affects the agent's disclosure decisions. One way to demonstrate this effect is to compare the disclosure strategy in a single-period setting (as in Pae 2005) with the disclosure strategy in our setting. In particular, consider a single-period benchmark setting, in which the probability of obtaining each signal equals the probability of obtaining each signal in the first period of our two-period setting. That is, the first period of our two-period setting differs from the single-period benchmark setting only in that there is a future in which the agent may learn additional information and may voluntarily disclose information. Our model indicates that the existence of a continuation to the first period decreases the amount of disclosure in the first period. The reason is that in the two-period setting, withholding information in the first period generates a real option for the agent, and, therefore, increases the disclosure threshold relative to a setting where such real options for

[20]It typically takes a few months between the NOA and the time at which the patent is published in the US PTO website.

non-disclosure do not exist.

The disclosure strategy and investors' beliefs in the single-period benchmark are identical to the ones in the first period of the limit case of our two-period setting in which the manager assigns an infinite weight to the first period's price relative to the second period's price, or, equivalently, when the discount rate used by the manager goes to infinity. This result can be generalized and it demonstrates that the higher the weight the manager assigns to the first period's price (short term) relative to the second period's price (long term), the higher the expected probability of disclosure in the first period. In other words, increasing the weight assigned to the first period's price decreases the first period's disclosure threshold. Higher weight assigned by the manager to the first period's price can reflect, for example, managers who face higher short-term incentives, managers of firms that are about to issue new debt or equity, a higher probability of the firm being taken over, a shorter expected horizon for the manager with the firm, etc. This gives rise to the testable empirical prediction of a higher likelihood of early voluntary disclosure by managers of firms who care more about short-term price (for various reasons, including the ones mentioned above).

Our model also provides some predictions regarding the extent to which managers' voluntary disclosures tend to cluster. For example, conditional on disclosure of two signals the disclosed values are on average closer to each other than in the original distribution of the signals; i.e., disclosed values are clustered. Another type of clustering for which our model can generate predictions is the time-clustering of the disclosures.

5 Conclusion

The vast literature on voluntary disclosure models focuses on static models in which an interested party (e.g., a firm's manager) may privately observe a single piece of private information (e.g., Dye 1985 and Jung and Kwon 1988). Many real-life voluntary disclosure environments, for example corporate disclosure environments, are multi-period in their nature and the informed party often obtains more than a single piece of private information. In such settings, the decisions whether to disclose one piece of information must take into account the possibility of learning and potentially disclosing a new piece of information in the future. To the best of our knowledge, such dynamic considerations of voluntary disclosure have not been studied in the literature.

In this paper, we show that the interaction between these two dimensions affects disclosure decisions and equilibrium prices in a qualitatively new way. In our model, absent information asymmetry, the firm's price at the end of the second period would be independent of the disclosure time of the firm's private information. Nevertheless, we show that when managers strategically choose what and when to disclose the market price does depends on the timing of disclosure. In particular, we show that the price at the end of the second period given disclosure of one signal is higher if the signal is disclosed later in the game. That result illustrates the importance of considering dynamic aspects of voluntary disclosure.

The model generates several empirical predictions. For example: market reaction to later voluntary disclosure is more positive and the amount of voluntary disclosure in the short term increases if the manager assigns higher relative weight to short-term prices.

Appendix

Proof of Claim . 1

Using the notation in Lemma 1, let the set A correspond to the set of y if the managers is uniformed even at $t = 2$ (and who disclose x at $t = 1$). Let B correspond to the set of y that is informed either at $t = 1$ and does not disclose then or learns y at $t = 2$. The claim follows from Lemma 1 as $P_1(x, 1) = E[y|y \in |A \cup B]$ and $P_2(x, 1) = E[g(z)|S_{A,B}]$ (note that the distribution over A is $\Phi(y|x)$ but the distribution over B is different since the investors have to form beliefs on the relative likelihood of the agent learning y in the two periods, which is why it is useful to prove the previous lemma for general distributions). ■

Prof of Lemma . 1

(0) For a constant c let $S^c_{A,B} \quad = A \cup \{B \cap \{(y, \tau_y) : y \leq c\}$. For $c \to -\infty$ we have that $E_y(S^c_{A,B}) = E_y(A) > c$ and for $c \to \infty$ we have that $E_y(S^c_{A,B}) = E_y(A \cup B) < c$. From continuity we can find c^* for which $E_y(S^{c^*}_{A,B}) = c^*$. This establishes existence.

Now suppose by way of contradiction that there are multiple solutions. Specifically, assume there are $c' < c$" so that $E_y(S^{c'}_{A,B}) = c'$ and $E_y(S^{c"}_{A,B}) = c$". When we compare $S^{c'}_{A,B}$ to $S^{c"}_{A,B}$ we note that $S^{c"}_{A,B} \supset S^{c'}_{A,B}$ and that for $(y, \tau_y) \in S^{c"}_{A,B} \backslash S^{c'}_{A,B}$ we have $y < E_y(S^{c"}_{A,B})$. This implies that $S^{c"}_{A,B}$ can be represented as a union of $S^{c'}_{A,B}$ where the average c' ($< c$") and a set of types that are lower than c". This however, implies that $E_y(S^{c"}_{A,B}) < c$" and we get a contradiction.

(i) When comparing $S_{A,B}$ to $A \cup B$ we note that we have excluded above average types for which $y > E_y(S_{A,B})$. This results in lower average type.

(ii) Suppose first that there exists $(y, \tau_y) \in S_{A,B"} \backslash S_{A,B'}$. Since $B' \supseteq B''$ it must be that these $(y, \tau_y) \in B' \cap B''$. From the definition of $S_{A,B}$ since $(y, \tau_y) \in S_{A,B''}$ we conclude that $E_y(S_{A,B''}) > y$. Since $(y, \tau_y) \notin S_{A,B'}$, we conclude that $E_y(S_{A,B'}) < y$ which implies the claim. Hence, we will assume that $S_{A,B'} \supseteq S_{A,B"}$ and we consider $(y, \tau_y) \in S_{A,B'} \backslash S_{A,B"}$; this implies $y < E_y(S_{A,B'})$. Hence, all the elements $(y, \tau_y) \in S_{A,B'} \backslash S_{A,B"}$ have y that is below the average in $S_{A,B'}$ which implies that $E_y(S_{A,B"}) \geq E_y(S_{A,B'})$.

(iii) Consider the set $S_{A,B"}$, and note that it satisfy the definition for $S_{A,B'}$. Hence, the claim follows from uniqueness that was proven in (0) ■

Proof of Claim 2.

B_1^1 can be described as the intersection of three conditions, $B_1^1 = C_1(x) \cap C_2(x) \cap C_3(x)$ where:

- $C_1(x)$: At $t = 1$, the agent prefers to reveal x instead of both x and y. By Corollary 1, y is such that $P(x, y) \leq P_1(x, 1)$.

- $C_2(x)$: At $t = 1$, the agent prefers to reveal x rather than y. Monotonicity of the equilibrium then implies $y \leq x$.

- $C_3(x)$: At $t = 1$, the agent prefers to reveal x rather than to hide both x and y.

The constraint $C_3(x)$ can be described using the inequality $\Pi_x \geq \Pi_\emptyset$ where:

$$\Pi_\emptyset = P_1(\emptyset) + \max\{P(x, y), P_2(x, 2), P_2(y, 2), P_2(\emptyset)\}$$

$$\Pi_x = P_1(x, 1) + \max\{P(x, y), P_2(x, 1)\}$$

where the first expression is the expected payoff of a type that knows x and y at time 1 and decides not to reveal anything; and Π_x is the payoff of the same type that decides to reveal x only. Since x is revealed alone on the equilibrium path at time $t = 1$, the inequality $\Pi_x \geq \Pi_\emptyset$ needs to hold. We also know that on the equilibrium path x is being disclosed at $t = 2$ which implies that $P_2(x, 2) > P_2(\emptyset)$. Condition $C_2(x)$ implies already that $y \leq x$ and by monotonicity of equilibrium $P_2(x, 2) > P_2(y, 2)$. So, without changing the intersection of $C_1(x) \cap C_2(x) \cap C_3(x)$ we can define $C_3(x)$ by replacing $\Pi_\emptyset$ with

$$\Pi'_\emptyset = P_1(\emptyset) + \max\{P(x, y), P_2(x, 2)\}.$$

If $\Pi_x \geq \Pi'_\emptyset$ for all y then the constraint $C_3(x)$ can be ignored by defining $y^*(x) = \infty$. If this condition does not hold for any y then the agent does not disclose x at $t = 1$ if he knows both signals. This can be ruled out as an agent who only knows x decides to disclose it at $t = 1$. If for each realization of y he would have preferred to keep quiet then this would be the case also when he does not know y. So we focus on the case where $\Pi_x < \Pi'_\emptyset$ holds for some y. As a function of y, for low y, both Π_x and $\Pi'_\emptyset$ are constants that depend only on x;for high y we have that Π_x and $\Pi'_\emptyset$ have the same slope with respect to y. Let y^* be defined as the unique y that equates Π_x and $\Pi'_\emptyset$. Suppose first that $\Pi'_\emptyset$ is larger than Π_x for small y but starts increasing later than Π_x. That case would require $P_2(x, 1) < P(x, y^*) < P_2(x, 2)$, but that contradicts the supposition. Hence, we are left with $\Pi'_\emptyset$ is smaller than Π_x for small y, but starts increasing sooner than Π_x. This implies that $C_3(x)$ is given by $y \le y^*$ where y^* satisfies $P_2(x,2) < P(x,y^*) < P_2(x,1)$. Finally, since $P_2(x,1) \le P_2(x,1)$, $C_1(x)$ can be ignored which implies the claim. QED ■

Proof of Lemma 3. We know that x is being disclosed with positive probability if it is the only signal known at $t=1$. Let Π_D denote the payoff for such an agent, who learned only x at $t=1$, from disclosing x at $t=1$ and Π_N his payoff from not disclosing at $t=1$. We have:

$$\Pi_D = P_1(x,1) + E_y[\max\{P_2(x,1), P(x,y)\}],$$
$$\Pi_N = P_1(\emptyset) + E_y[\max\{P_2(x,2), P_2(y,2), P(x,y), P_2(\emptyset)\}].$$

We know that for some x we have that $\Pi_D - \Pi_N \ge 0$. Consider an agent who knows both signals at $t=1$ and prefers to disclose just x at $t=2$ (an agent with $x \in (P_2(\emptyset), P_1(\emptyset))$ and a sufficiently low y). Such an agent knows at time $t=1$ that he will disclose x and not disclose y at $t=2$. So, for this to happen it must be that $\Pi'_N - \Pi'_D \ge 0$ where:

$$\Pi'_D = P_1(x,1) + P_2(x,1),$$
$$\Pi'_N = P_1(\emptyset) + P_2(x,2).$$

This leads to contradiction as $P_2(x,1) > P_2(x,2) \Rightarrow \Pi'_D - \Pi'_N > \Pi_D - \Pi_N \ge 0$. ■

Proof of Lemma 5

By applying Bayes role, $h^{stat}(\mu, \{z_i(\mu)\})$ is given by:

$$h^{stat}(\mu, \{z_i(\mu)\}) = \frac{(1-p)\mu + \sum_{i=1}^K \lambda_i \int_{-\infty}^{z_i(\mu)} y\phi(y|\mu)\,dy}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(\mu)|\mu)}.$$

Taking the derivative of $h^{stat}(\mu, \{z_i(\mu)\})$ with respect to μ and applying some algebraic manipulation yields:

$$\frac{d}{d\mu} h^{stat}(\mu, \{z_i(\mu)\}) = 1 + \frac{\sum_{i=1}^K \lambda_i (z_i'(\mu) - 1)\phi(z_i(\mu)|\mu)\left(z_i(\mu) - h^{stat}(\mu, \{z_i(\mu)\})\right)}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(\mu)|\mu)}.$$

We start by proving the supremum of this derivative

Given that $z_i'(\mu) \ge 0$ and $z_i(\mu) \le h^{stat}(\mu, \{z_i(\mu)\})$ for all $i \in \{1,..,K\}$ we have

$$\begin{aligned}\frac{d}{d\mu} h^{stat}(\mu, \{z_i(\mu)\}) &\le 1 + \frac{\sum_{i=1}^K \lambda_i \phi(z_i(\mu)|\mu)\left(z_i(\mu) - h^{stat}(\mu, \{z_i(\mu)\})\right)}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(\mu)|\mu)} \\ &\le 1 + \max_{\substack{z_i \le h(x) \\ i \in \{1,...K\}}} \frac{\sum_{i=1}^K \lambda_i \phi(z_i(\mu)|\mu)\left(z_i(\mu) - h^{stat}(\mu, \{z_i(\mu)\})\right)}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(\mu)|\mu)}\end{aligned}$$

Due to symmetry, for all $i \in \{1,..,K\}$ the maximum is achieved at $z_i(\mu) = z^*(\mu)$. To see this, note that the FOC of the maximization with respect to $z_i(\mu)$ is

$$0 = \left(\phi'\left(z_i(\mu)|\mu\right)\left(h^{stat}\left(\mu,\{z_i(\mu)\}\right) - z_i(\mu)\right) - \phi\left(z_i(\mu)|\mu\right)\right)\left((1-p) + \sum_{i=1}^{K}\lambda_i\Phi\left(z_i(\mu)|\mu\right)\right)$$
$$-\left(\sum_{i=1}^{K}\lambda_i\phi\left(z_i(\mu)|\mu\right)\left(h^{stat}\left(\mu,\{z_i(\mu)\}\right) - z_i(\mu)\right)\right)\phi\left(z_i(\mu)|\mu\right).$$

Since $\phi'\left(z_i(\mu)|\mu\right) = -\alpha\left(z_i(\mu) - \mu\right)\phi\left(z_i(\mu)|\mu\right)$ (for some constant $\alpha > 0$), this simplifies to

$$-\alpha\left(z_i(\mu) - \mu\right)\left(h^{stat}\left(\mu,\{z_i(\mu)\}\right) - z_i(\mu)\right) = \frac{\sum_{i=1}^{K}\lambda_i\phi\left(z_i(\mu)|\mu\right)\left(z_i(\mu) - h^{stat}\left(\mu,\{z_i(\mu)\}\right)\right)}{(1-p) + \sum_{i=1}^{K}\lambda_i\Phi\left(z_i(\mu)|\mu\right)} + 1.$$

In the range $z_i(\mu) \leq h^{stat}\left(\mu,\{z_i(\mu)\}\right) \leq \mu$, the LHS is decreasing in $z_i(\mu)$.[21] The RHS is the same for all i. Therefore, the unique solution to this system of FOC is for all $z_i(\mu)$ to be equal (and note that the maximum is achieved at an interior point since at $z_i(\mu) = h^{stat}\left(\mu,\{z_i(\mu)\}\right)$ the LHS is zero and the RHS is positive; and as $z_i(\mu)$ goes to $-\infty$ the LHS goes to $+\infty$ while the RHS is bounded). This implies that example 3 which we discussed following the statement of the Lemma also provides an upper bound. Recall that this lower bound was $\frac{d}{d\mu}h^{stat}\left(\mu,\{z_i(\mu)\}\right) \geq 1$. The lower bound can be concluded in a similar way by observing that if we want to minimize the slope we will again choose the same $z_i(\mu)$ for all i, and therefore our example provides also a lower bound which is $\frac{d}{d\mu}h^{stat}\left(\mu,\{z_i(\mu)\}\right) \leq 2 - z_i'(\mu)$.

QED

Proof of Claim 3

The case of $h_1(x,1)$ has been proved right bellow Claim 3 in the main text. We analyze the cases of $h_2(x,2)$ and $h_2(x,1)$ bellow.

We first analyze $h_2(x,2)$.

When an agent discloses $x > x^*$ at $t = 2$ investors know that $\tau_x = 2$ (otherwise the agent would have disclosed x at $t = 1$). Investors' beliefs about the manager's other signal at $t = 2$ is set as a weighted average of three scenarios: $\tau_y = 1$, $\tau_y = 2$ and $\tau_y > 2$. We start by describing the disclosure thresholds conditional on each of the three scenarios.

(i) If $\tau_y > 2$ the agent cannot disclose y and therefore the disclosure threshold is not relevant. In the pricing of the firm conditional on $\tau_y > 2$ investors use $E(y|x)$ which equals $\beta_1 x$.

[21] Since $z_i(x) \leq h\left(x,\{z_i(\cdot)\}\right)$ also $h\left(x,\{z_i(\cdot)\}\right) \leq E[x|y] = \beta_1 x$.

(ii) If $\tau_y = 2$ investors know that $y < h_2(x, 2)$ and also that $y < x$. We need to distinguish between the binding case and the non-binding case. In the non-binding case, where $h_2(x, 2) \leq x$, investors know that $y < h_2(x, 2)$, so conditional on $\tau_y = 2$ investors set their beliefs as if the manager follows a disclosure threshold of $h_2(x, 2)$. In the binding case, where $h_2(x, 2) > x$, investors know that $y < x$, so it is equivalent to a disclosure threshold of x.

(iii) If $\tau_y = 1$ investors know that $y < x^*$ (where $x^* \leq x$) and also $y < h_2(x, 2)$. Here again we should distinguish between a non-binding case, in which $h_2(x, 2) < x^*$ (if such case exists), and a binding case in which $h_2(x, 2) > x^*$. In the non-binding case the disclosure threshold is $h_2(x, 2)$. In the binding case the disclosure threshold is x^*, which is independent of x.

The next Lemma provides an upper and lower bound for $\frac{\partial}{\partial x} h_2(x, 2)$. The proof of the Lemma, uses the disclosure thresholds for each of the three scenarios above. This Lemma holds not only for $h_2(x, 2)$ but also for $h_2(x, 1)$.

Lemma 6 *For $p < 0.95$*

$$\frac{\partial}{\partial x} h_2(x, 2), \frac{\partial}{\partial x} h_2(x, 1) \in (2\beta_1 - 1, 2\beta_1).$$

Proof of Lemma 6

We first prove the case of $h_2(x, 2) > x^*$.

In this proof we use a slightly different notation, as part of the proof is more general than our setting. Note that the first part of this proof is quite similar to the proof of Lemma 5.

Suppose that x and y have joint normal distribution and the agent is informed about y with probability p and uninformed with probability $1 - p$. Conditional on being informed the agent's disclosure strategy is assumed to be as follows: with probability λ_i, $i \in \{1, .., K\}$, he discloses if his type is above $z_i(x)$, where the various $z_i(x)$ are determined exogenously such that $z_i(x) \leq h(x, \{z_i(x)\})$ for all i (which always holds in our setting). Note that $\sum_{i=1}^{K} \lambda_i = p$. Let's denote the conditional expectation of y given x and given the disclosure thresholds, $z_i(x)$, by $h(x, \{z_i(x)\})$.

By applying Bayes role, $h(x, \{z_i(x)\})$ is given by:

$$h(x, \{z_i(x)\}) = \frac{(1-p) E[y|x] + \sum_{i=1}^{K} \lambda_i \int_{-\infty}^{z_i(x)} y \phi(y|x) \, dy}{(1-p) + \sum_{i=1}^{K} \lambda_i \Phi(z_i(x)|x)}.$$

Taking the derivative of $h(x, \{z_i(x)\})$ with respect to x and applying some algebraic manipulation (recall that $\frac{\partial E[y|x]}{\partial x} = \beta_1$) yields:

$$h'(x, \{z_i(x)\}) = \beta_1 + \frac{\sum_{i=1}^K \lambda_i (z_i'(x) - \beta_1) \phi(z_i(x)|x)(z_i(x) - h(x, \{z_i(x)\}))}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(x)|x)}. \tag{8}$$

We start by proving the supremum of $h'(x, \{z_i(x)\})$.

Given that $z_i'(x) \geq 0$ and $(z_i(x) - h(x, \{z_i(x)\})) \leq 0$ for all $i \in \{1, .., K\}$ we have

$$\begin{aligned} h'(x, \{z_i(x)\}) &\leq \beta_1 + \frac{\beta_1 \sum_{i=1}^K \lambda_i \phi(z_i(x)|x)(h(x, \{z_i(x)\}) - z_i(x))}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(x)|x)} \\ &\leq \beta_1 + \max_{\substack{z_i \leq h(x) \\ i \in \{1,\ldots,K\}}} \frac{\beta_1 \sum_{i=1}^K \lambda_i \phi(z_i|x)(h(x, \{z_i(x)\}) - z_i)}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i|x)} \end{aligned}$$

Due to symmetry, for all $i \in \{1, .., K\}$ the maximum is achieved at $z_i(x) = z^*(x)$. To see this, note that the FOC of the maximization with respect to $z_i(x)$ is

$$\begin{aligned} 0 = &\left(\phi'(z_i(x)|x)(h(x, \{z_i(x)\}) - z_i(x)) - \phi(z_i(x)|x)\right)\left((1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(x)|x)\right) \\ &- \left(\sum_{i=1}^K \lambda_i \phi(z_i(x)|x)(h(x, \{z_i(x)\}) - z_i(x))\right) \phi(z_i(x)|x). \end{aligned}$$

Since $\phi'(z_i(x)|x) = -\alpha(z_i(x) - \beta_1 x)\phi(z_i(x)|x)$ (for some constant $\alpha > 0$), this simplifies to

$$-\alpha(z_i(x) - \beta_1 x)(h(x, \{z_i(x)\}) - z_i(x)) = \frac{\sum_{i=1}^K \lambda_i \phi(z_i(x)|x)(h(x, \{z_i(x)\}) - z_i(x))}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(x)|x)} + 1.$$

In the range $z_i(x) \leq h(x, \{z_i(x)\}) \leq \beta_1 x$, the LHS is decreasing in $z_i(x)$.[22] The RHS is the same for all i. Therefore, the unique solution to this system of FOC is for all $z_i(x)$ to be equal (and note that the maximum is achieved at an interior point since at $z_i(x) = h(x, \{z_i(x)\})$ the LHS is zero and the RHS is positive; and as $z_i(x)$ goes to $-\infty$ the LHS goes to $+\infty$ while the RHS is bounded).

Let $z^*(x)$ be the maximizing value. Then

$$\begin{aligned} h'(x, \{z_i(x)\}) &\leq \beta_1 + \frac{\beta_1 \sum_{i=1}^K \lambda_i \phi(z^*(x)|x)(h(x, \{z_i(x)\}) - z^*(x))}{(1-p) + p\Phi(z^*(x)|x)} \\ &= \beta_1 + \frac{p\beta_1 \phi(z^*(x)|x)(h(x, \{z_i(x)\}) - z^*(x))}{(1-p) + p\Phi(z^*(x)|x)}. \end{aligned}$$

The right hand side of the above inequality is identical to the slope in a Dye setting with exogenous disclosure threshold with probability of being uninformed $(1-p)$ and an exogenously determined disclosure threshold of $z^*(x)$, where the disclosure threshold does not change in x. In

[22] Since $z_i(x) \leq h(x, \{z_i(\cdot)\})$ also $h(x, \{z_i(\cdot)\}) \leq E[x|y] = \beta_1 x$.

such a setting, we can think of the effect of a marginal increase in x as the sum of two effects. The first effect is a shift by β_1 in both the distribution and the disclosure threshold. This will increase $h(x)$ by β_1. The second effect is a decrease in the disclosure threshold by β_1 (as the disclosure threshold does not change in x). Since $z^*(x) < \beta_1 x$ we are in the decreasing part of the beliefs about y given no disclosure (to the left of the minimum beliefs). Therefore, the decrease in the disclosure threshold increases the beliefs about y by the change in the disclosure threshold times the slope of the beliefs about y given no disclosure. Since for $p < 0.95$ the slope of the beliefs about y given no disclosure is greater than -1, the latter effect increases the beliefs about y by less than β_1. The overall effect is therefore smaller than $2\beta_1$.

Next we prove the infimum of $h'(x, \{z_i(x)\})$.

Equation (8) capture a general case with any number of potential disclosure strategies. In our particular case $K = 1$ where $i = 1$ represents the case of $\tau_y = 1$ and $i = 2$ represents the case of $\tau_y = 2$. So, in our setting equation (8) can be written as

$$\begin{aligned} h'(x, \{z_i(x)\}) = \beta_1 &+ \frac{\lambda_1 (z_1'(x) - \beta_1)\, \phi(z_1(x)|x)(z_1(x) - h(x, \{z_i(x)\}))}{(1-p) + \sum_{i=1}^{2} \lambda_i \Phi(z_i(x)|x)} \\ &+ \frac{\lambda_2 (z_2'(x) - \beta_1)\, \phi(z_2(x)|x)(z_2(x) - h(x, \{z_i(x)\}))}{(1-p) + \sum_{i=1}^{2} \lambda_i \Phi(z_i(x)|x)} \end{aligned}$$

When calculating $h_2(x, 2)$ and $h_1(x, 1)$ in our setting, the disclosure threshold, $z_i(x)$, in any possible scenario (the binding and non-binding case for both $\tau_y = 1$ and $\tau_y = 2$) takes one of the following three values: $h_i(x, \cdot)$, x or x^*. Note that whenever $z_i(x) = h(x, \{z_i(x)\})$ we have $\frac{(z_i'(x) - \beta_1)\phi(z_i(x)|x)(z_i(x) - h(x, \{z_i(x)\}))}{(1-p) + \sum_{i=1}^{K} \lambda_i \Phi(z_i(x)|x)} = 0$.

For the remaining two cases ($z_i(x) = x$ and $z_i(x) = x^*$), for all $i \in \{1, 2\}$ we have $z_i'(x) \leq 1$ and $(z_i(x) - h(x, \{z_i(x)\})) \leq 0$. This implies

$$h'(x) \geq \beta_1 - \frac{(1 - \beta_1) \sum_{i=1}^{K} \lambda_i \phi(z_i(x)|x)(h(x, \{z_i(x)\}) - z_i(x))}{(1-p) + \sum_{i=1}^{K} \lambda_i \Phi(z_i(x)|x)}.$$

Using the same symmetry argument for the first order condition as before, $h'(x, \{z_i(x)\})$ is minimized for some $z^{\min}(x)$ and hence

$$h'(x, \{z_i(x)\}) \geq \beta_1 + \frac{p(1 - \beta_1)\, \phi\left(z^{\min}(x)|x\right)\left(h(x, \{z_i(x)\}) - z^{\min}(x)\right)}{(1-p) + p\Phi(z^{\min}(x)|x)}.$$

The right hand side of the above inequality is identical to the slope in a Dye setting with exogenous disclosure threshold in which: the probability of being uninformed is $(1 - p)$, the exogenously determined disclosure threshold is $z^{\min}(x)$ and $\frac{\partial}{\partial x} z^{\min}(x) = 1$. In such a setting, we can think of the effect of a marginal increase in x as the sum of two effects. The first is a shift by β_1 in both the distribution and the disclosure threshold. This will increase $h(x)$ by β_1. The second effect is an increase in the disclosure threshold by $(1-\beta_1)$ (as the disclosure threshold increases by 1). Since $z^{\min}(x) < \beta_1 x$ we are in the decreasing part of the beliefs about y given no disclosure (to the left of the minimum beliefs). Therefore, the increase in the disclosure threshold decreases the beliefs about y by the change in the disclosure threshold, $(1-\beta_1)$, times the slope of the beliefs about y given no disclosure. Since for $p < 0.95$ the slope of the beliefs about y given no disclosure is greater than -1 the latter effect decreases the beliefs about y by less than $(1-\beta_1)$. The overall effect is therefore greater than $\beta_1 - (1-\beta_1) = 2\beta_1 - 1$.

QED Lemma 6

Finally, we analyze $h_2(x,1)$

Recall that Lemma 6 applies also to $h_2(x,1)$. However, for $h_2(x,1)$ we can show tighter bounds.

We first show that for the case where $h_2(x,1) < x$ we have $h_2'(x,1) = \beta_1$.

If $h_2(x,1) < x$ (the non-binding case) then when pricing the firm at $t=2$ investors know that if the agent learned y (at either $t=1$ or $t=2$) then $y < h_2(x,1)$. If the agent did not learn y then investors use in their pricing $E(y|x) = \beta_1 x$. So, the beliefs about y are a weighted average of $E(y|y < h_2(x,1))$ and $E(y|x) = \beta_1 x$. This is similar to a Dye (1985) and Jung and Kwon (1988) setting, and therefore, in equilibrium we have $h_2'(x,1) = \beta_1$.

Next we show that for x such that $h_2(x,1) > x$ (if such case exists) $h_2'(x,1) \in (2\beta_1 - 1, \beta_1)$.

The argument is similar to the one we made in the proof that $h_1'(x,1) \in (2\beta_1 - 1, \beta_1)$ for x such that $h_1(x,1) > x$. First note that for $h_2(x,1) > x$ investors' beliefs about y conditional on that the agent has learned y are independent of whether he learned y at $t=1$ or at $t=2$. Moreover, given that $\tau_y \leq 2$ investors know that $y < x$. So, from investors' perspective, it doesn't matter if the agent learned y at $t=1$ or at $t=2$. Their pricing, $h_2(x,1)$, will reflect a weighted average between $E(y|y < x)$ and $E(y|\tau_y > 2, x) = \beta_1 x$. From here on the proof is qualitatively the same as in the proof for $h_1'(x,1) \in (2\beta_1 - 1, \beta_1)$, where the only quantitative difference is the probability that the agent learned y.

Next, we show that for the particular case in which $h_2(x,2) < x^*$ (if such case exists) $h_2'(x,2) = \beta_1$.

$h_2(x,2)$ is a weighted average of the beliefs about y over the three scenarios $\tau_y = 1$, $\tau_y = 2$ and $\tau_y > 2$. That is, we can write

$$h_2(x,2) = \lambda_1 h_1 + \lambda_2 h_3 + (1 - \lambda_1 - \lambda_2) h_3,$$

where $\lambda_i = \Pr(\tau_y = i | ND_y)$ and $h_i = E(y|\tau_y = i, ND_y)$ for $i = 1, 2, 3$ where $i = 3$ represents the case of $\tau_y > 2$. ND_y stands for No-Disclosure of y (where x was disclosed at $t = 2$). Since $h_2(x,2) < x^*$ the disclosure threshold for both $\tau_y = 2$ and $\tau_y = 1$ is $h_2(x,2)$. Assume by contradiction that $\frac{\partial}{\partial x} h_2(x,2) > \beta_1$. Then, an increase in x increases $h_2(x,2)$ by more than the increase in the expectation of y (which is β_1) and therefore, the probability of obtaining a signal below the disclosure threshold increases for both the first and the second period. This implies that both λ_1 and λ_2 increase. In addition, note that the increase in h_1 and in h_2 is lower than $\frac{\partial}{\partial x} h_2(x,2)$ and the increase in h_3 is β_1 - which is also lower than $\frac{\partial}{\partial x} h_2(x,2)$. The fact that both h_1 and h_2 are lower than h_3 leads to a contradiction, since an increase in x put more weight on the lower values (λ_1 and λ_2 increase) and in addition all the values h_1, h_2, h_3 increase at a rate weakly lower than the assumed increase in $h_2(x,2)$. A symmetric argument can be made when assuming by contradiction that $\frac{\partial}{\partial x} h_2(x,2) < \beta_1$. The case of $\frac{\partial}{\partial x} h_2(x,2) = \beta_1$ does not lead to a contradiction, as an increase in x does not affect the probabilities λ_1, λ_2 and the derivatives of h_1 and h_2 and h_3 are all equal to β_1.

QED Claim 3

Proof of Proposition 1

We first prove the existence of a threshold equilibrium and then show that there exists an x' such that $P_2(x,2) > P_2(x,1)$ for any $x \geq x'$.

In proving the existence of a threshold equilibrium, we first consider partially informed agents that learn a single signal, x, at $t = 1$ ($\tau_x = 1, \tau_y \neq 1$) and then we consider fully informed agents that learn both signals at $t = 1$. For each of these cases we show that: (i) for sufficiently high (low) realizations of x the agent discloses (does not disclose) x at $t = 1$; and (ii) On the equilibrium path, the difference between the agent's expected payoff if he discloses only x at $t = 1$ and if he does not disclose at $t = 1$ is increasing in x.

Partially Informed Agents ($\tau_x = 1, \tau_y \neq 1$)

For sufficiently low realizations of x the agent is always better off not disclosing it at $t = 1$, as he can "hide" behind uninformed agents. We next establish that an agent that learns a single signal, x, at $t = 1$ and this signal is sufficiently high will disclose it at $t = 1$.

Lemma 7 *Consider an agent that learns a single signal, x, at $t = 1$. In a threshold equilibrium, the difference between the agent's expected payoff (as calculated at $t = 1$) from disclosing his signal at $t = 1$ and from not disclosing it at $t = 1$ is increasing in x. That is,*

$$\frac{\partial}{\partial x}\left(E\left(U|\tau_x = 1, \tau_y \neq 1, t_x = 1\right) - E\left(U|\tau_x = 1, \tau_y \neq 1, t_x \neq 1\right)\right) > 0$$

Proof. For simplicity of exposition, we partition the support of x into two cases: realizations of x for which $\beta_2\left(x + h_2\left(x, 2\right)\right) \geq P_2\left(\emptyset\right)$ and for which $\beta_2\left(x + h_2\left(x, 2\right)\right) < P_2\left(\emptyset\right)$.[23]

Case I - $\beta_2\left(x + h_2\left(x, 2\right)\right) \geq P_2\left(\emptyset\right)$

Rewriting $E\left(U|\tau_x = 1, \tau_y \neq 1, t_x = 1, x\right) - E\left(U|\tau_x = 1, \tau_y \neq 1, t_x \neq 1\right)$ yields:

$$\begin{aligned}&\beta_2\left[x + h_1\left(x, 1\right) + h_2\left(x, 1\right) - h_2\left(x, 2\right)\right] - P_1\left(\emptyset\right)\\&+ p\beta_2\left[\int_{h_2(x,1)}^{\infty}\left(y - h_2\left(x, 1\right)\right) f\left(y|x\right) dy - \int_{h_2(x,2)}^{\infty}\left(y - h_2\left(x, 2\right)\right) f\left(y|x\right) dy - \int_{y^H(x)}^{\infty}\left(h_2\left(y, 2\right) - x\right) f\left(y|x\right) dy\right].\end{aligned}$$

The derivative of this expression with respect to x has the same sign as

$$D = 1 + \frac{\partial}{\partial x}\left(h_1\left(x, 1\right) + h_2\left(x, 1\right) - h_2\left(x, 2\right)\right) + p\left[A + B + C\right], \tag{9}$$

where

$$\begin{aligned}A &= \frac{\partial}{\partial x}\int_{h_2(x,1)}^{\infty}\left(y - h_2\left(x, 1\right)\right) f\left(y|x\right) dy\\B &= -\frac{\partial}{\partial x}\int_{h_2(x,2)}^{\infty}\left(y - h_2\left(x, 2\right)\right) f\left(y|x\right) dy\\C &= -\frac{\partial}{\partial x}\int_{y^H(x)}^{\infty}\left(h_2\left(y, 2\right) - x\right) f\left(y|x\right) dy.\end{aligned}$$

To evaluate this derivative we use the following, easy to obtain, equations:

$$\begin{aligned}\frac{\partial}{\partial x} f\left(y|x\right) &= -\beta_1\frac{\partial}{\partial y} f\left(y|x\right),\\\frac{\partial}{\partial x}\left(F\left(y\left(x\right)|x\right)\right) &= f\left(y\left(x\right)|x\right)\left(\frac{\partial}{\partial x} y\left(x\right) - \beta_1\right).\end{aligned}$$

[23] Note that on the equilibrium path we are always in case I, i.e., $\beta_2\left(x + h_2\left(x, 2\right)\right) \geq P_2\left(\emptyset\right)$.

Next, we analyze the three terms A, B, and C. Note that the derivative with respect to the limits of integrals for A, B and C is zero.

$$A = -\frac{\partial h_2(x,1)}{\partial x}(1 - F(h_2(x,1)|x)) - \beta_1 \int_{h_2(x,1)}^{\infty} (y - h_2(x,1)) \frac{\partial}{\partial y} f(y|x)\, dy.$$

Integrating by parts (w.r.t. y) the term $\int_{h_2(x,1)}^{\infty} (y - h_2(x,1)) \frac{\partial}{\partial y} f(y|x)\, dy$ yields:

$$\begin{aligned}
&\int_{h_2(x,1)}^{\infty} (y - h_2(x,1)) \frac{\partial}{\partial y} f(y|x)\, dy \\
&= -(h_2(x,1) - h_2(x,1)) f(h_2(x,1)|x) - \int_{h_2(x,1)}^{\infty} f(y|x)\, dy = -(1 - F(h_2(x,1)|x)).
\end{aligned}$$

Plugging it back to A we get

$$A = -\left(\frac{\partial h_2(x,1)}{\partial x} - \beta_1\right)(1 - F(h_2(x,1)|x)).$$

Next, we calculate B:

$$\begin{aligned}
B &= \int_{h_2(x,2)}^{\infty} \frac{h_2(x,2)}{\partial x} f(y|x)\, dy + \beta_1 \int_{h_2(x,2)}^{\infty} (y - h_2(x,2)) \frac{\partial}{\partial y} f(y|x)\, dy \\
&= \left(\frac{\partial h_2(x,2)}{\partial x} - \beta_1\right)(1 - F(h_2(x,2)|x)).
\end{aligned}$$

Finally,

$$\begin{aligned}
C &= \left(1 - F\left(y^H(x)|x\right)\right) + \beta_1 \int_{y^H(x)}^{\infty} (h_2(y,2) - x) \frac{\partial}{\partial y} f(y|x)\, dy \\
&= \left(1 - F\left(y^H(x)|x\right)\right) - \beta_1 \int_{y^H(x)}^{\infty} \frac{\partial h_2(y,2)}{\partial y} f(y|x)\, dy.
\end{aligned}$$

Substituting A, B and C back to (9) and re-arranging terms yields:

$$\begin{aligned}
D &= 1 + \frac{\partial}{\partial x}(h_1(x,1) + h_2(x,1) - h_2(x,2)) \\
&\quad - p \left[\begin{array}{c} \left(\frac{\partial h_2(x,1)}{\partial x} - \beta_1\right)(1 - F(h_2(x,1)|x)) + \left(\frac{\partial h_2(x,2)}{\partial x} - \beta_1\right)(1 - F(h_2(x,2)|x)) + \\ \left(1 - F\left(y^H(x)|x\right)\right) - \beta_1 \int_{y^H(x)}^{\infty} \frac{\partial h_2(y,2)}{\partial y} f(y|x)\, dy \end{array} \right] \\
&= (1-p)\left(1 + \frac{\partial}{\partial x}(h_1(x,1) + h_2(x,1) - h_2(x,2))\right) \\
&\quad + p \left[\begin{array}{c} 1 + \frac{\partial h_1(x,1)}{\partial x} + \frac{\partial h_2(x,1)}{\partial x} F(h_2(x,1)|x) + \beta_1 (1 - F(h_2(x,1)|x)) - \frac{\partial h_2(x,2)}{\partial x} F(h_2(x,2)|x) \\ -\beta_1 (1 - F(h_2(x,2)|x)) + 1 - F\left(y^H(x)|x\right) - \beta_1 \int_{y^H(x)}^{\infty} \frac{\partial h_2(y,2)}{\partial y} f(y|x)\, dy \end{array} \right] \\
&= (1-p)\left(1 + \frac{\partial}{\partial x}(h_1(x,1) + h_2(x,1) - h_2(x,2))\right) \\
&\quad + p \left[\begin{array}{c} 1 + \frac{\partial h_1(x,1)}{\partial x} + \frac{\partial h_2(x,1)}{\partial x} F(h_2(x,1)|x) - F(h_2(x,1)|x)\beta_1 - \frac{\partial h_2(x,2)}{\partial x} F(h_2(x,2)|x) \\ +F(h_2(x,2)|x)\beta_1 + \left(1 - F\left(y^H(x)|x\right)\right) - \beta_1 \int_{y^H(x)}^{\infty} \frac{\partial h_2(y,2)}{\partial y} f(y|x)\, dy \end{array} \right]
\end{aligned}$$

Additional rearranging yields:

$$
\begin{aligned}
D = {} & (1-p)\left(1+\frac{\partial}{\partial x}\left(h_1(x,1)+h_2(x,1)-h_2(x,2)\right)\right) \\
& + p\left[1+\frac{\partial h_1(x,1)}{\partial x}+\left(\frac{\partial h_2(x,1)}{\partial x}-\beta_1\right)F\left(h_2(x,1)|x\right)-\left(\frac{\partial h_2(x,2)}{\partial x}-\beta_1\right)F\left(h_2(x,2)|x\right)\right] \\
& + p\beta_1\int_{y^H(x)}^{\infty}\frac{1}{\beta_1}-\frac{\partial h_2(y,2)}{\partial y}f(y|x)\,dy.
\end{aligned}
$$

Since $\frac{\partial h_2(x,1)}{\partial x}\le\beta_1$ (see Claim 3) and $F\left(h_2(x,2)|x\right)\ge F\left(h_2(x,1)|x\right)$ we have

$$
\begin{aligned}
D \ge {} & (1-p)\left(1+\frac{\partial}{\partial x}\left(h_1(x,1)+h_2(x,1)-h_2(x,2)\right)\right) \\
& + p\left[1+\frac{\partial h_1(x,1)}{\partial x}+\left(\frac{\partial h_2(x,1)}{\partial x}-\frac{\partial h_2(x,2)}{\partial x}\right)F\left(h_2(x,2)|x\right)\right] \\
& + p\beta_1\int_{y^H(x)}^{\infty}\frac{1}{\beta_1}-\frac{\partial h_2(y,2)}{\partial y}f(y|x)\,dy \\
= {} & \left(1-p\left(1-F\left(h_2(x,2)|x\right)\right)\right)\left(1+\frac{\partial}{\partial x}\left(h_1(x,1)+h_2(x,1)-h_2(x,2)\right)\right)+ \\
& + p\left(1-F\left(h_2(x,2)|x\right)\right)\left(1+\frac{\partial h_1(x,1)}{\partial x}\right)+p\beta_1\int_{y^H(x)}^{\infty}\frac{1}{\beta_1}-\frac{\partial h_2(y,2)}{\partial y}f(y|x)\,dy \\
= {} & \left(1-p\left(1-F\left(h_2(x,2)|x\right)\right)\right)\left(1+\frac{\partial}{\partial x}\left(h_1(x,1)+h_2(x,1)-h_2(x,2)\right)\right) \\
& + p\int_{h_2(x,2)}^{\infty}\left(1+\frac{\partial h_1(x,1)}{\partial x}\right)f(y|x)\,dy+p\int_{y^H(x)}^{\infty}1-\beta_1\frac{\partial h_2(y,2)}{\partial y}f(y|x)\,dy \\
= {} & \left(1-p\left(1-F\left(h_2(x,2)|x\right)\right)\right)\left(1+\frac{\partial}{\partial x}\left(h_1(x,1)+h_2(x,1)-h_2(x,2)\right)\right)+ \\
& + p\int_{h_2(x,2)}^{y^H(x)}\left(1+\frac{\partial h_1(x,1)}{\partial x}\right)f(y|x)\,dy+p\int_{y^H(x)}^{\infty}2+\frac{\partial h_1(x,1)}{\partial x}-\beta_1\frac{\partial h_2(y,2)}{\partial y}f(y|x)\,dy \\
\ge {} & \left(1-p\left(1-F\left(h_2(x,2)|x\right)\right)\right)\left(1+\frac{\partial}{\partial x}\left(h_1(x,1)+h_2(x,1)-h_2(x,2)\right)\right) \\
& + p\int_{y^H(x)}^{\infty}2+\frac{\partial h_1(x,1)}{\partial x}-\beta_1\frac{\partial h_2(y,2)}{\partial y}f(y|x)\,dy
\end{aligned}
$$

So, the following two conditions are sufficient for the proof of Case I.

For all x:

1. $\frac{\partial}{\partial x}h_1(x,1)+\frac{\partial}{\partial x}h_2(x,1)\ge\frac{\partial}{\partial x}h_2(x,2)-1$
2. $\frac{\partial h_2(y,2)}{\partial y}\le\left(2+\frac{\partial h_1(x,1)}{\partial x}\right)\frac{1}{\beta_1}$ for any $y>x$

Case II -$\beta_2\left(x+h_2(x,2)\right)<P_2(\emptyset)$

The analysis of Case I was for generic bounds of the integrals $h_2(x,1)$ and $y^H(x)$. The difference between Case I and Case II is that the price at $t=2$ given no disclosure of y (which occurs when the agent does not obtain a signal y or obtains a low realization of y) is $P_2(\emptyset)$ in Case II and $\beta_2(x+h_2(x,2))$ in Case I. Therefore, the expected payoff of the agent in Case II is less sensitive to x than in Case I. As a result, the fact that for values of x for which $\beta_2(x+h_2(x,2)) \geq P_2(\emptyset)$ (in Case I) $\frac{\partial}{\partial x}(E(U|\tau_x=1,\tau_y\neq 1,t_x=1)-E(U|\tau_x=1,\tau_y\neq 1,t_x\neq 1))>0$ implies that it also holds for $\beta_2(x+h_2(x,2))<P_2(\emptyset)$.

To summarize the analysis of Partially Informed Agents, conditions 1 and 2 above are sufficient for both Case I and Case II. Claim 3 established that condition 2 above holds.

So, it is only left to show that also condition 1 holds. For any $\beta_1>\frac{1}{2}$, it is immediate to see that condition 1 holds since the LHS of condition 1 is greater than $2(2\beta_1-1)>0$ and the RHS is less than $2\beta_1-1$. We defer the case of $\beta_1<\frac{1}{2}$ to later in the proof. ■

Fully Informed Agent ($\tau_x=\tau_y=1$)

We next discuss the case of an agent that learns both signals at $t=1$ (such that $x>y$). If the signal y is sufficiently high, such that it will be disclosed at $t=2$ if it was not disclosed at $t=1$, then it is straightforward that $\frac{\partial}{\partial x}(E(U|\tau_x=1,\tau_y=1,t_x=1)-E(U|\tau_x=1,\tau_y=1,t_x\neq 1))>0$. The reason is that the price at $t=2$ will be $P(x,y)$ regardless of the disclosure decision at $t=1$.

The only case we still haven't analyzed is a fully informed agent ($\tau_x=\tau_y=1$) whose signal y is sufficiently low such that it will not be disclosed at $t=2$ if it was not disclosed at $t=1$. The Lemma bellow shows that such an agent with a sufficiently high signal x will disclose at least one signal at $t=1$ and that for such an agent the difference between disclosing and not disclosing x at $t=1$ is increasing in x.

Lemma 8 *Assume an agent that learned both signals at $t=1$ and the realization of y is sufficiently low such that it will not be disclosed. Then*

(i) For sufficiently high realizations of x the agent prefers to disclose x at $t=1$ over not disclosing x at $t=1$.

(ii) $\frac{\partial}{\partial x}(E(U|\tau_x=1,\tau_y=1,t_x=1)-E(U|\tau_x=1,\tau_y=1,t_x\neq 1))>0$.

Proof.

(i) We need to show that

$$\beta_2[x+h_1(x,1)]+\beta_2[x+h_2(x,1)]>P_1(\emptyset)+\beta_2[x+h_2(x,2)].$$

Rearranging yields

$$\beta_2 [x + h_2 (x, 1)] - P_1 (\emptyset) > \beta_2 [h_2 (x, 2) - h_1 (x, 1)] .$$

Since $h_2 (x, 1)$ is not decreasing for sufficiently high x the LHS of the above inequality, $\beta_2 [x + h_2 (x, 1)] - P_1 (\emptyset)$, goes to infinity as x goes to infinity. Therefore, it is sufficient to show that $h_2 (x, 2) - h_1 (x, 1)$ is bounded from above. Both $h_2 (x, 2)$ and $h_1 (x, 1)$ are lower than $\beta_2 x$. From the Generalized Minimum Principle (Lemma 1) we know that $h_1 (x, 1)$ is higher than the price given no disclosure in a Dye (1985), Jung and Kwon (1988) setting where $y \sim N (\beta_1 x, Var (y|x))$. The price given no disclosure in such a setting is $\beta_1 x - Cons$, so $h_1 (x, 1) > \beta_1 x - Cons$. Hence, given that $h_2 (x, 2) < \beta_1 x$ we have $h_2 (x, 2) - h_1 (x, 1) < Const$.

(ii) We need to show that

$$\frac{\partial}{\partial x} (\beta_2 [x + h_1 (x, 1)] + \beta_2 [x + h_2 (x, 1)] - P_1 (\emptyset) - \beta_2 [x + h_2 (x, 2)]) > 0,$$

which is identical to condition 2 in the proof of Lemma 7.

QED ■

The following Lemma establishes the last part of Proposition 1.

Lemma 9 *There exists an $x' \geq x^*$ such that $P_2 (x, 2) > P_2 (x, 1)$ for any $x \geq x'$.*

Proof. In Theorem 1 we have shown that $P_2 (x, 2) \geq P_2 (x, 1)$ for any x, which implies in the setting of section 4 that $h_2 (x, 2) \geq h_2 (x, 1)$.

As established in section 3, given disclosure of the signal x the manager behaves myopically in the sense that he discloses the signal y (when he learned y) if and only if it increases the price relative to the price when y is not disclosed. This holds for both $t = 1$ and $t = 2$. We can now introduce the equilibrium inference on the sets B_1^1, B_1^2, B_2^1 and B_2^2 that were defined in section 3. In particular, we adjust the set B_i^j by taking into account also the equilibrium disclosure strategy when defining the potential disclosures and denote it by b_i^j. The sets b_i^j for $i, j = 1, 2$ are given by:

$$b_1^1 = \{(y, \tau_y) | \tau_y = 1,\ t_x = 1 \text{ and } y \leq \min \{x, h_1 (x, 1), h_2 (x, 1)\}\}$$
$$b_1^2 = \{(y, \tau_y) | \tau_y = 2,\ t_x = 1 \text{ and } y \leq h_2 (x, 2)\}$$

$$b_2^1 = \{(y, \tau_y) | \tau_y = 1,\ t_x = 2 \text{ and } y \leq \min \{x^*, h_2 (x, 2)\}\}$$
$$b_2^2 = \{(y, \tau_y) | \tau_y = 2,\ t_x = 2 \text{ and } y \leq \min \{x, h_2 (x, 2)\}\}$$

Note that $h_1(x,1) > h_2(x,1)$ so b_1^1 can be written as $b_1^1 = \{(y,\tau_y) | \tau_y = 1,\ t_x = 1 \text{ and } y \leq \min\{x, h_2(x,1)\}\}$.

We next show that $h_2(x,2) > h_2(x,1)$ for all x such that $h_2(x,2) > x^*$. From section 3 we know that $h_2(x,2) \geq h_2(x,1)$ so we only need to preclude $h_2(x,2) = h_2(x,1)$. Assume by contradiction that $h_2(x,2) = h_2(x,1)$. Since $x > x^*$ we have $b_2^1 \subset b_1^1$ and $b_2^2 \subseteq b_1^2$. Moreover, any $y \in (x^*, h_2(x,2))$ is strictly lower than $h_2(x,2)$ which equals $E[y|y \in S_{A,b_2}]$. From part (i) of the Generalized Minimum Principle (Lemma 1) we have $h_2(x,2) > h_2(x,1)$ which leads to a contradiction. Therefore, for all values of x such that $h_2(x,2) > x^*$ we have $h_2(x,2) > h_2(x,1)$.

The last thing to be shown is that there exists an x' such that $h_2(x,2) > x^*$ for any $x \geq x'$. This is immediate given that $\frac{\partial}{\partial x} h_2(x,2) = \beta_1 (> 0)$ for value of x such that $h_2(x,2) < x^*$ (see Claim 3). Note that x' can be, but is not necessarily, grater than x^*.

QED ■

The last thing we still need show in order to establish the existence of a threshold equilibrium is that Condition 1 in the proof of Lemma 7, i.e. that $\frac{\partial}{\partial x} h_1(x,1) + \frac{\partial}{\partial x} h_2(x,1) \geq \frac{\partial}{\partial x} h_2(x,2) - 1$, holds also for the case of $\beta_1 < \frac{1}{2}$. In the upper and lower bounds for the slopes of the various prices derived in Claim 3, there is a lot of slack that can still be used. However, the simplest way to prove the last part of Proposition 1 is not to use this slack, but rather to show the existence of a threshold equilibrium in which x^* is such that $h_2(x^*,1) \leq x^*$. For such x^* we know from Claim 3 that $\frac{\partial}{\partial x} h_2(x,1) = \beta_1$. Substituting this into condition 1 above yields $\frac{\partial}{\partial x} h_1(x,1) + \beta_1 \geq \frac{\partial}{\partial x} h_2(x,2) - 1$ which given Claim 3 is always satisfied.[24]

QED Proposition 1.

[24] Since $\frac{\partial}{\partial x} h_2(x,1) \leq \beta_1$ there exists an x such that $h_2(x,1) \leq x$ for all higher values of x. Using adequate off-equilibrium beliefs, as discussed in section 4.2, we can always set x^* to satisfy $h_2(x^*,1) \leq x^*$.

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