A Monetary Theory with Non-Degenerate Distributions*

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Abstract

We construct and analyze a tractable search model of money with a non-degenerate distribution of money holdings. We assume search to be directed in the sense that buyers know the terms of trade before visiting particular sellers. Directed search makes the monetary steady state block recursive in the sense that individuals' policy functions, value functions and the market tightness function are all independent of the distribution of individuals over money balances, although the distribution affects the aggregate activity by itself. Block recursivity enables us to characterize the equilibrium analytically. By adapting lattice-theoretic techniques, we characterize individuals' policy and value functions, and show that these functions satisfy the standard conditions of optimization. We prove that a unique monetary steady state exists. Moreover, we provide conditions under which the steady-state distribution of buyers over money balances is non-degenerate and analyze the properties of this distribution.

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1. Introduction

In this paper, we construct a tractable model with a microfoundation of money where a non-degenerate distribution of money holdings among individuals can be persistent. We analytically characterize the steady state of the equilibrium in this economy. In particular, we prove the existence of a unique monetary steady state, provide the conditions under which the distribution of money is non-degenerate, and examine the properties of the distribution.

A persistent, non-degenerate distribution of money is critical for policy analysis. As documented by Christiano et al. (1999) using vector autoregression (VAR), even an impulse monetary shock can have a persistent effect on aggregate real activity. This persistent effect of monetary shocks seems difficult to be captured by models where the distribution of money is not persistent, as illustrated by Lucas (1990). In Lucas' model, a shock to the money stock can generate a temporary liquidity effect by affecting the money constraint on the individuals in the financial market. However, this effect lasts for only one period because the members in a household pool the resource at the end of a period, which results in a degenerate distribution of money holdings across households. In contrast, Rotemberg (1984) follows Baumol (1952) and Tobin (1956) to model an individual's money holdings as the solution to a problem of optimal inventory management. Because cash withdrawals are staggered across individuals, the distribution of money is non-degenerate in the steady state. In that model, a monetary shock affects aggregate real activity persistently. These papers, and the large literature spawned from them, are useful indications of the importance of money dispersion in analyzing policy, but they assume an exogenous role of money rather than deriving this role from the fundamentals of the model.

We use monetary search theory originated in Kiyotaki and Wright (1989) as a natural framework to provide a microfoundation of money and to generate a non-degenerate distribution of money. On the microfoundation, monetary search theory models exchange as a decentralized process in which each trade involves only a small group of anonymous individuals who do not have a double coincidence of wants. In this environment, an object with no intrinsic value, such as fiat money, can have positive value by facilitating exchange. On the non-degenerate distribution of money, decentralized exchange naturally induces dispersion in money balances among individuals. Because matching is stochastic, different individuals may end up trading away different amounts of money. Even if all individuals hold the same amount of money initially, the distribution of buyers over money balances can fan out as decentralized exchange continues.

Despite these potentially desirable features, monetary search theory has mostly imposed tractability assumptions to make money distribution degenerate. The difficulty lies in the endogeneity and the potentially large dimensionality of the distribution. The distribution is an aggregate state variable that can affect individuals' decisions in general. In turn, the decisions of all individuals together affect the evolution of the distribution. An equilibrium typically needs to determine individuals' decisions and the aggregate distribution simultaneously. This is a difficult task because the distribution can potentially have a large dimension. To avoid this difficulty, Kiyotaki and Wright (1989) assume indivisible goods and money; Shi (1995) and Trejos and Wright (1995) allow goods to be divisible but retain the assumption of indivisible money. When allowing both money and goods to be divisible, models impose other assumptions to make money distribution degenerate. Notably, Shi (1997) assumes that each household consists of a large number of members who share consumption and utility, and so all households hold the same amount of money in the equilibrium. Lagos and Wright (2005) assume that each decentralized market is followed by a centralized market in which individuals have quasi-linear preferences over a good, and so trading in the centralized market eliminates all dispersion in money holdings. Green and Zhou (1998, *GZ* henceforth) are the first to formally analyze a non-degenerate money distribution, but their model is tractable only under the assumptions that goods come as a fixed endowment and money can only be accumulated in discrete units. Without these restrictive assumptions, an analytically tractable microfoundation of money with a non-degenerate distribution has eluded monetary theory. Very little is known about even the fundamental properties of an equilibrium. For example, does such a steady state exist, is it unique, and when is the distribution is non-degenerate?

Given this state of the literature, we focus on the analytical characterization of the steady state with a non-degenerate distribution, despite that the broad research project is motivated by the dynamic effect of monetary policy. The focus on the steady state is a useful first step because it addresses the fundamental questions raised above on the equilibrium and provides a long-run anchor for a dynamic analysis. As a standalone contribution, the steady-state analysis provides a set of theoretical tools that help overcome the main difficulties in building a monetary theory with a non-degenerate money distribution. A dynamic analysis is left for future work.

The main deviation of our model from the monetary search literature lies in the way we model search. The literature assumes search to be undirected in the sense that individuals do not know the terms of trade before they are matched. In contrast, we assume search to be directed in the sense that individuals know the terms of trade before a match, as in Peters (1991), Moen (1997), Acemoglu and Shimer (1999), Burdett et al. (2001) and Julien et al. (2000). For each type of good, there can be a continuum of submarkets, each of which specifies the terms of trade and a tightness (i.e., the ratio of trading posts to buyers). Buyers choose which submarket to visit and firms choose how many trading posts to create in each submarket. There is a cost of creating a trading post for a period, and the number of trading posts in each submarket is determined by free entry. Once inside a submarket, buyers and trading posts are brought into bilateral meetings through a frictional matching function that has constant returns to scale. The matching probability for a buyer and a trading post is a function of the tightness of the submarket. In equilibrium, the tightness in each submarket is consistent with buyers' choices on which submarket to enter and firms' choices on the creation of trading posts.

Directed search is a realistic feature of an actual economy. It reflects the fact that individuals have information about the location and the price range of the goods they want to buy. They go directly to the sellers who sell the goods they want, rather than randomly search among all the sellers. Moreover, buyers with different money holdings optimally sort into submarkets that differ in the terms of trade. Specifically, because the marginal value of money is lower to a buyer who has a relatively high money balance, such a buyer has a strong desire to spend a relatively large amount of money on consumption and to spend it sooner rather than later. With this desire, the buyer chooses to enter a submarket where he has a relatively high matching probability to trade a relatively large amount of money for a large quantity of goods. Firms cater to this desire by creating a relatively large number of trading posts per buyer in this submarket. Because buyers with different money holdings choose not to mix with each other, a buyer's optimal choices depend on the buyer's own money balance and the tightness of the particular submarket he visits, but not on the distribution of individuals over money balances. Moreover, because each submarket is tailored to only one group of buyers with a particular money balance, the tightness of each submarket that ensures zero profit for a trading post does not depend on the distribution of money holdings. Precisely, individuals' policy functions, value functions and the market tightness function are all independent of the distribution in the steady state. We refer to this feature of the steady state as *block recursivity.*

Block recursivity makes the analytical characterization of the steady state tractable. Although the distribution affects the aggregate activity, it is not part of the state space in individuals' decision problems. As a result, we can characterize an individual's policy and value functions solely as functions of the individual's own balance. Having done so for each balance separately, we can compute the flows of individuals across money balances to obtain the distribution. In the steady state, the support of the distribution consists of a finite number of values of money balance, each of which is associated with one active submarket. Moreover, an individual goes through purchasing cycles. When the individual has no money, he works to obtain money and then becomes a buyer. Starting with a high balance, a buyer enters a submarket where he has a high matching probability, spends a large amount of money and obtains a large quantity of goods. For the next trade, the buyer will go into a submarket where his matching probability, the required spending and the quantity of goods obtained in a trade are all lower. The buyer will continue this pattern until he depletes his balance, at which point he will work again.

The analytical characterization of the equilibrium enables us to prove that a unique monetary steady state exists, to determine when the steady-state distribution of buyers over money balances is non-degenerate, and to analyze the properties of this distribution. The unique monetary steady state unifies the literature by nesting some well-known models as special cases. In one case, the distribution is degenerate in the steady state, which occurs when individuals are sufficiently impatient. In this case, all buyers hold the same amount of money and spend the entire amount in one trade. A purchasing cycle consists of only one purchase. This endogenous pattern resembles the one assumed in the models with indivisible money (e.g., Shi, 1995, and Trejos and Wright, 1995). However, whether the pattern is endogenous or exogenous is important for policy. A one-time change in the money stock does not affect the real activity in the steady state in our model, but it does when money is indivisible by assumption.

The other case of the unique steady state features a non-degenerate distribution of buyers over money balances. As we will explain in subsection 4.3, this case arises if individuals are sufficiently patient, if the utility function of consumption is sufficiently concave, if the disutility function of labor supply is not very convex, and if the cost of creating a trading post is low. In this case, each buyer runs down money balance in a purchasing cycle through consecutive purchases, and the purchasing pattern is staggered across the buyers. Moreover, because the buyers who hold a high balance trade relatively quickly and exit from that balance, there are more buyers with low balances than with high balances, and so the density of the distribution in a purchasing cycle is a decreasing function of money balance. The purchasing cycle resembles the one in the inventory model of money by Baumol (1952) and Tobin (1956), but the latter authors model the role of money in a reduced form. The decreasing density of money distribution resembles that in GZ, but GZ restrict that goods come as a fixed endowment and money can only be accumulated in discrete units. Moreover, in contrast to both Baumol-Tobin and GZ, we allow individuals to choose among submarkets that differ in the terms of trade and the matching probability. As a realistic feature, this endogeneity should be important for how monetary policy affects the aggregate activity. We will contrast our model further to the Baumol-Tobin model in subsection 4.1 and to GZ in subsection 4.3.

A large part of this paper is devoted to the analysis of a buyer's decision problem, which establishes the properties of the policy and value functions that are needed for block recursivity.

We provide a set of analytical tools to overcome some difficulties in the use of dynamic programming. The difficulties arise from the features that a buyer's objective function is not concave and that a buyer's value function cannot be assumed to be differentiable a priori. These difficulties prevent us from using the standard approach in dynamic programming (e.g., Stokey et al., 1989) to analyze the policy and value functions. In subsection 3.2.1, we will give an overview of these difficulties and the way in which we resolve them. A short description is that we adapt lattice-theoretic techniques (see Topkis, 1998) to prove that a buyer's policy functions are monotone functions of the real balance. Using this result, we prove further that optimal choices obey the first-order conditions, the value functions are differentiable and the envelope conditions hold. By establishing these standard conditions formally, we hope to make the model easy to use. This procedure of analyzing a dynamic programming problem is of independent interest because it is applicable in a variety of dynamic models that involve both discrete and continuous choices.[1]

It is important to clarify that our analysis and the main results do not follow simply from the labor search literature, despite the fact that this literature has explored directed search and block recursivity (e.g., Shi, 2009, Gonzalez and Shi, 2010, and Menzio and Shi, 2011). Several elements in our model are important for monetary theory, but not necessarily so for labor theory. First, an individual's gain from a monetary trade depends not only on how the match surplus is split, but also on how all individuals in the economy value money. A monetary equilibrium must determine this value of money. This is not necessary in a labor search model because the value of a match is determined by preferences there. Second, money balance is a stock variable that can be accumulated or decumulated over time through trade, and there is a market clearing condition on the aggregate stock of money. Such a stock variable has been absent in a labor search model. Third, a buyer in a monetary model optimally chooses the length of a purchasing cycle and the amount of money to be spent in each period within a cycle. Most parts of this paper (e.g., sections 3 and 4) are devoted to resolving these monetary issues. Even at the technical level of using lattice-theoretic techniques, our analysis differs from that in Gonzalez and Shi (2010). While Gonzalez and Shi explore convexity of the value function to apply lattice-theoretic techniques, a buyer's value function in our model is neither convex nor concave. To apply lattice-theoretic techniques, we analyze a buyer's decision problem in steps (see subsection 3.2.2).

In the monetary search literature, Corbae et al. (2003) seem the first to incorporate directed search. They focus on the formation of trading coalitions and assume that money and goods are

[1]There are a large number of applications of lattice-theoretic techniques in dynamic programming. But as explained by Gonzalez and Shi (2010, see the references therein), most of them assume that the objective function in the maximization problem is concave. This assumption is not satisfied in our model in general, and the restrictions to ensure a concave objective function are too restrictive to be interesting.

indivisible. Rocheteau and Wright (2005) examine directed search as a robustness check, and Galenianos and Kircher (2008) and Julien et al. (2008) examine directed search with auctions. These papers do not formulate a block recursive equilibrium. Moreover, the latter three papers impose the tractability assumption as in Lagos and Wright (2005), which makes either money distribution degenerate or its real effect temporary. This assumption is shared by a large literature inspired by Lagos and Wright (2005), which we do not survey here.[2]

Under the assumption of undirected search, some papers have studied a non-degenerate money distribution. As mentioned earlier, GZ are the first along this line, but they restrict that goods come as a fixed endowment and money can only be accumulated in discrete units.[3] Eliminating these restrictions, Molico (2006) and Chiu and Molico (2008) numerically compute the equilibrium. In particular, Chiu and Molico (2008) generate a non-degenerate money distribution by extending the Lagos-Wright model to allow the cost function of the good traded in the centralized market to be strictly convex. These numerical exercises are useful, but they are not able to address the fundamental issues about an equilibrium with a non-degenerate money distribution that have eluded the literature, such as existence and uniqueness of the steady state. These models are analytically intractable precisely because the assumption of undirected search makes the equilibrium not block recursive. Even for a quantitative analysis, our model is easier to compute than these models of undirected search, as we will discuss in section 5. Moreover, undirected search models, including Molico (2006) and Chiu and Molico (2008), have several implications in contrast with our model, which we will discuss at the end of subsection 2.1.

2. A Monetary Economy with Directed Search

2.1. The model environment

There are I types of individuals and I types of perishable goods indexed by $i \in \{1, 2, ..., I\}$, where $I \geq 3$. Each type i consists of a continuum of individuals with measure one who are specialized in the consumption of good i and the production of good $i+1$ (modulo I). The preferences of a type i individual are represented by the utility function $\sum_{t=0}^{\infty} \beta^t [U(q_t) - h(\ell_t)]$, where $\beta \in (0,1)$ is the discount factor, $U : \mathbb{R}_+ \to \mathbb{R}$ is the utility of consumption of good i, and $h : [0,1] \to \mathbb{R}$

[2] Berentsen, Camera and Waller (2005) extend the Lagos-Wright (2005) model to allow for two rounds of trading in the decentralized market before a centralized market opens for a homogeneous good over which individuals have quasi-linear preferences. The non-degenerate distribution of money is not persistent, because it is re-set frequently by the trading in the centralized market. An extension of the Lagos-Wright model to allow for many consecutive rounds of decentralized trading is not tractable. Faig (2008) introduces lotteries into the Lagos-Wright model to partially avoid the use of quasi-linear preferences. Again, the money distribution is degenerate in his model.

[3] Zhou (1999) extends the GZ model by introducing production cost. Berentsen et al. (2004) introduce lotteries into GZ. Zhu (2005) extends the GZ model by making goods divisible. He studies a sequence of economies with discrete money holdings and characterizes the limit where the size of the discreteness goes to zero.

is the disutility of labor. We assume that U is strictly increasing, strictly concave and twice continuously differentiable, with the boundary properties: $U(0) = 0$, $U'(\infty) = 0$, and $U'(0)$ is sufficiently large. Similarly, we assume that h is strictly increasing, strictly convex and twice continuously differentiable, with the boundary properties: $h(0) = 0$ and $h'(1) = \infty$. In addition to consumption goods, there is an object called fiat money which is intrinsically worthless, perfectly divisible and costlessly storable. In this paper, we focus on the case in which the supply of fiat money per capita, M, is constant over time.

The economy is also populated by I types of firms. Each type i consists of a large number of firms that are specialized in the production and distribution of good i. A type i firm operates a technology of constant returns to scale that transforms any amount of labor supplied by individuals of type $i-1$ (modulo I) into the same amount of good i.[4] Moreover, a type i firm can open a trading post in the market for good i using $k > 0$ units of labor supplied by individuals of type $i-1$ (modulo I). Firms are owned by the individuals through a balanced mutual fund.

In every period, a labor market and a product market open. Firms can participate in both markets in the same period. In contrast, individuals can participate in either the labor market or the product market. That is, in a given period, individuals must choose whether to become workers or buyers. Before making this choice, individuals can play a fair lottery. Even though individuals are risk averse, a lottery can be desirable because the value function without the lottery can be non-concave at particular money balances. One cause of non-concavity is the discrete nature of the decision on which market to enter. Another cause is the tradeoff between the matching probability and the surplus of trade in the product market, to be described later.

The labor market is centralized and frictionless. Taking the nominal wage rate as given, each firm chooses how much labor to demand and each worker chooses how much labor to supply. In equilibrium, the nominal wage rate equates the demand for and the supply of labor of each type. Workers are paid in money instead of goods because they do not want to consume the good produced by the firm in which they work and because goods are perishable between periods. Moreover, a firm cannot pay its employees with an IOU because firms are better off exiting the market than honoring their IOUs. All labor is paid from the proceeds of sales.

To simplify the notation, we choose labor, instead of goods or money, as the numeraire in this model. Let ωM be the nominal wage rate, and so one unit of money is worth $1/(\omega M)$ units of labor. We refer to the quantity of money expressed in terms of labor units as the *real balance*. Thus, the real balance per capita in the economy is equal to $1/\omega$. We will also express the price

[4] The linear production technology is assumed without loss of generality. Note that the disutility of labor supply, $h(\ell)$, is strictly convex and the utility of consumption is strictly concave.

of goods in labor units later. Although ω is the nominal wage rate normalized by the money stock, we simply refer to ω as the nominal wage rate whenever there is no confusion.

The product market is decentralized and has search frictions. Buyers and trading posts meet in pairs and there is no record keeping of their actions once they exit a trade. The market for each type i goods is organized in a continuum of submarkets indexed by the terms of trade $(x, q) \in \mathbb{R}_+ \times \mathbb{R}_+$, where x is the real balance paid by the buyer and q is the quantity of goods obtained by the buyer in a trade. A firm chooses how many trading posts to create in each submarket, and a buyer chooses which submarket to visit. As is standard in search models, the length of a period is such that a buyer can visit at most one submarket in a period. The matching process is frictional in each submarket. Let θ denote the *tightness*, i.e., the ratio of trading posts to buyers, in a submarket. In a submarket with tightness θ, a buyer is matched with probability $b = \lambda(\theta)$, and a trading post is matched with probability $s = \rho(\theta)$. The function $\lambda : \mathbb{R}_+ \to [0, 1]$ is a strictly increasing function with boundary conditions $\lambda(0) = 0$ and $\lambda(\infty) = 1$. The function $\rho : \mathbb{R}_+ \to [0, 1]$ is a strictly decreasing function such that $\rho(\theta) = \lambda(\theta)/\theta$, $\rho(0) = 1$ and $\rho(\infty) = 0$. Since b and s are both functions of θ, we can express a trading post's matching probability as a function of a buyer's matching probability: $s = \mu(b) \equiv \rho(\lambda^{-1}(b))$. Clearly, $\mu(b)$ is a decreasing function. We assume that $1/\mu(b)$ is strictly convex in b.

Because firms and buyers choose which submarket to enter, a type i buyer will choose to participate only in the submarkets where type i goods are produced, i.e., where trading posts are created by type i firms. Moreover, across the submarkets that sell the same good, search is directed as in Moen (1997), Acemoglu and Shimer (1999), Burdett et al. (2001) and Shi (2001). That is, firms and buyers take into account the fact that market tightness varies with the terms of trade across the submarkets according to a function $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$. The function θ is endogenously determined in the equilibrium by the requirement that $\theta(x, q)$ be equal to the ratio of trading posts to buyers in submarket (x, q) for all (x, q). As a result, matching probabilities, b and s, are endogenous functions of (x, q).[5]

When a buyer meets a trading post in submarket (x, q), the buyer pays the real balance x for q units of the consumption good. The buyer must pay the seller with money because neither barter nor credit is feasible. The buyer cannot pay the seller with goods because goods are perishable and there is no double coincidence of wants in goods between the buyer and the seller. Moreover, the buyer cannot pay the seller with an IOU because individuals are anonymous; once they exit

[5]Note that the price of goods in a submarket alone is not adequate for describing a submarket because a buyer may not spend all the money in a trade. In subsection 2.4, we will briefly contrast directed search with a perfectly competitive goods market and a trading-post model with no search frictions.

a trade, they can renege on their IOUs without fear of retribution. Thus, the amount of money that a buyer can spend in a trade is bounded above by the balance he carries into the trade.

The environment above retains many standard features of a search model. In particular, matching is stochastic in each submarket, an individual can perform only one role in a period, and a buyer has at most one match in a period. Directed search is the main difference of our environment from the monetary search literature. In order to direct search, some commitment is needed for individuals to make a meaningful tradeoff between the surplus of a trade and the matching probability. In our paper, the commitment is to the posted terms of trade. Let us clarify this assumption with three remarks. First, directed search is a realistic feature of many markets in an actual economy. The trading posts in a submarket may be interpreted as a collection of stores in a particular location or sellers who offer similar terms of trade. Buyers often choose the location and the price range of goods at which they shop, rather than randomly search among all the sellers. Second, modeling a market as a collection of trading posts can be traced at least back to Shapley and Shubik (1977), who argue that the setup is useful for retaining the force of competition while allowing for coordination frictions. Directed search differs from the Shapley-Shubik setup in that matching is frictional and stochastic inside each submarket, while Shapely and Shubik assume that the shorter side of the market at a trading post always has a trade. Third, commitment to the terms of trade is a simple way to model directed search, but it is not the only one. Search can be directed by commitment to a variety of trading arrangements, from posted prices (e.g., Burdett et al., 2001) to auctions (Julien et al., 2000).

Directed search has other desirable implications for modeling monetary exchange, which contrast with undirected search models (e.g., Shi, 1997, Lagos and Wright, 2005, Molico, 2006, and Chiu and Molico, 2008). First, undirected search models have often (although not always) assumed that the individuals in a match bargain over the terms of trade. To characterize the bargaining outcome simply, the models require a strong assumption that individuals' money holdings are public information. If money holdings are private information, instead, the bargaining game can have multiple equilibria, which complicate the analysis significantly and weaken the predictive power. Our model does not suffer from this problem. Because search is directed, whether an individual's money balance is public or private information does not matter to the analysis. In any submarket, the expected payoff to an individual is determined by the terms of trade and the individual's own money balance. An individual can calculate this expected payoff without the need to know how much money the trading partner holds. Second, with undirected search and bargaining (under public information), a buyer with a higher balance pays a higher price in the equilibrium not because the buyer optimally chooses so, ex ante, but rather because the buyer is held up by the seller, ex post. In our model, as will be proven later, a buyer with a higher balance optimally chooses to pay a higher price because the submarket with a higher price compensates the buyer with a higher matching probability.

2.2. An individual's decisions

Let $V(m)$ denote the lifetime utility of an individual who starts a period with the real balance m (expressed in units of labor). We refer to V as the *ex-ante value function*, since it is measured before the individual makes any decision in a period. Let $B(m)$ denote a *buyer's value function*, i.e., the lifetime utility of an individual who enters the product market as a buyer with the real balance m. Similarly, let $W(m)$ denote a *worker's value function*, i.e., the lifetime utility of an individual who enters the labor market as a worker with the real balance m.

A worker chooses labor supply, ℓ, which generates ℓ units of the real balance as wage income. The individual also owns a diversified portfolio of the firms. However, the return to this portfolio is zero since all firms earn zero profit in the equilibrium. Thus, a worker who enters the labor market with a real balance m will have a real balance $m + \ell$ at the end of the period. The discounted value of this balance is $\beta V(m + \ell)$. The worker's value function, W, obeys:

$$W(m) = \max_{\ell \in [0,1]} \left[\beta V(m + \ell) - h(\ell) \right]. \tag{2.1}$$

Denote the optimal choice of ℓ as $\ell^*(m)$ and the implied real balance at the end of the period as $y^*(m) = m + \ell^*(m)$. We refer to $\ell^*(.)$ and $y^*(.)$ as a worker's policy functions.

A buyer chooses which submarket (x, q) to enter, taking the tightness function $\theta(x, q)$ as given. In submarket (x, q), the buyer will meet a trading post with probability $\lambda(\theta(x, q))$, in which case he will trade a real balance x for q units of goods. Current consumption yields utility $U(q)$, and the residual balance $(m - x)$ yields the discounted value $\beta V(m - x)$. With probability $1 - \lambda(\theta(x, q))$, the buyer will not have a match, in which case he will retain the balance m whose discounted value will be $\beta V(m)$. Thus, the buyer's value function, B, obeys:

$$B(m) = \max_{x \in [0,m],\ q \geq 0} \left\{ \lambda(\theta(x, q)) \left[U(q) + \beta V(m - x) \right] + \left[1 - \lambda(\theta(x, q)) \right] \beta V(m) \right\}. \tag{2.2}$$

The buyer's optimal choices are represented by the policy functions $(x^*(m), q^*(m))$.

An individual chooses whether to be a worker or a buyer in the period. This choice induces:

$$\tilde{V}(m) = \max\{W(m), B(m)\}. \tag{2.3}$$

Notice that $\tilde{V}$ may be non-concave for some real balances, even when W and B are concave functions. Thus, there is a potential gain to the individual from playing fair lotteries before making the above choice on whether to be a worker or a buyer. Denote a lottery as $(z_j, \pi_j)_{j=1,2}$, with $z_2 \geq z_1$, where π_j is the probability that the prize z_j is realized. The lifetime utility of the prize z_j is $\tilde{V}(z_j)$. Thus, the ex ante value function induced by the lottery choice is:

$$V(m) = \max_{(z_1,z_2,\pi_1,\pi_2)} \left[\pi_1 \tilde{V}(z_1) + \pi_2 \tilde{V}(z_2)\right] \tag{2.4}$$

$$\text{s.t.} \quad \pi_1 z_1 + \pi_2 z_2 = m, \quad \pi_1 + \pi_2 = 1, \quad z_2 \geq z_1,$$
$$\pi_j \in [0,1] \text{ and } z_j \geq 0 \text{ for } j = 1, 2.$$

Let $(z_j^*(m), \pi_j^*(m))_{j=1,2}$ denote the individual's optimal choice of a lottery.[6]

2.3. A firm's decisions

A firm chooses how many trading posts to create in each submarket and how much labor to employ. The firm hires labor to create trading posts and produce goods, and pays labor with money received from selling goods. Consider submarket (x, q). The cost of creating a trading post is k units of labor. A trading post in submarket (x, q) will have a match with probability $\rho(\theta(x, q))$, in which case the firm uses q units of labor to produce q units of goods and exchanges for a real balance x. Thus, the expected benefit of creating a trading post in submarket (x, q) is $\rho(\theta(x, q))(x - q)$. If $\rho(\theta(x, q))(x - q) < k$, it is optimal for the firm not to create any trading post in submarket (x, q). If $\rho(\theta(x, q))(x - q) > k$, it is optimal for the firm to create infinitely many trading posts in submarket (x, q). If $\rho(\theta(x, q))(x - q) = k$, the firm is indifferent between creating different numbers of trading posts in submarket (x, q). The case $\rho(\theta(x, q))(x - q) > k$ never occurs, because it implies $\theta(x, q) = \infty$ and, hence, $\rho(\theta(x, q)) = 0$, which contradicts the condition for the case. Thus, in any submarket (x, q) visited by a positive number of buyers, the tightness is consistent with the firm's incentive to create trading posts if and only if

$$\rho(\theta(x, q))(x - q) \leq k \quad \text{and} \quad \theta(x, q) \geq 0, \tag{2.5}$$

where the two inequalities hold with complementary slackness. In any submarket (x, q) that is not visited by buyers, the tightness can be arbitrary if k is greater than $\rho(\theta(x, q))(x - q)$. However, following Shi (2009), Menzio and Shi (2010, 2011) and Gonzalez and Shi (2010), we restrict attention to equilibria in which (2.5) also holds for such submarkets.[7] Note that (2.5)

[6] Because a lottery is defined for any given m, it is used by individuals who hold the *same* balance. Thus, a lottery is not introduced here for individuals with heterogeneous holdings to exchange their balances among each other between trading rounds. Moreover, a lottery may or may not be played in the equilibrium and, when it is played in the equilibrium, it is played at only one level of money balance (see Lemma 4.1).

[7] This restriction on the beliefs out of the equilibrium "completes" the market in the following sense: A submarket is inactive only if, given that some buyers are present in the submarket, the expected benefit to a lone trading post in the submarket is still lower than the cost of the trading post. This restriction can be justified by a "trembling-hand" argument that a small measure of buyers appear in every submarket exogenously. Similar restrictions are common in the literature on directed search, e.g., Moen (1997) and Acemoglu and Shimer (1999).

implies that the firm earns zero profit. If $N(x,q)$ is the distribution of trading posts across the submarkets, then the sum of money received from sales, $\int \rho(\theta(x,q))x dN(x,q)$, exactly covers the sum of money paid to the workers, $\int [k + \rho(\theta(x,q))q] dN(x,q)$. That is, the cash constraint on a firm's wage payment is always satisfied with equality.

2.4. Equilibrium definition and block recursivity

We define a monetary steady state as follows:

Definition 2.1. *A* monetary steady state *consists of value functions, (V,W,B), policy functions, $(\ell^*, x^*, q^*, z^*, \pi^*)$, market tightness function θ, a wage rate ω, and a distribution of individuals over real balances, G, that satisfy the following requirements:*
(i) W satisfies (2.1) with ℓ^* as the associated policy function;
(ii) B satisfies (2.2) with (x^*, q^*) as the associated policy functions;
(iii) V satisfies (2.4) with (z^*, π^*) as the associated policy functions;
(iv) θ satisfies (2.5) for all $(x,q) \in \mathbb{R}_+^2$;
(v) G is the ergodic distribution generated by $(\ell^*, x^*, q^*, z^*, \pi^*, \theta)$;[8]
(vi) ω is such that $\omega < \infty$ and $\int m \, dG(m) = 1/\omega$.

Requirements (i)-(iv) are explained in previous subsections. Requirement (v) asks the distribution of individuals over real balances to be stationary and consistent with the flows of individuals induced by optimal choices. Requirement (vi) asks that money should have a positive value and that all money should be held by the individuals. Specifically, the sum of real balances according to the distribution G must be equal to the total real balance in the economy, $1/\omega$. We did not specify the labor market clearing condition in the above definition, because such a condition is implied by requirement (vi) in a closed economy.

As we will show in section 4, the support of the equilibrium distribution is a discrete set. However, for individuals to optimally choose to hold only the balances in this set, they need to know the optimal choice and the payoff of holding any balance outside the equilibrium set. This information is provided by the policy and value functions in (i)-(iii) above. Similarly, although the number of submarkets participated by a positive measure of individuals is finite in the equilibrium, individuals need to know the tightness in all submarkets, given by the function θ in (iv) above, in order to choose optimally the submarkets to participate in (see footnote 7).

[8]The general specification of the law of motion of G is cumbersome at this point and not necessary for the equilibrium analysis. In section 4 we will characterize the law of motion of G implied by optimal choices.

Equilibrium objects and requirements in Definition 2.1 can be grouped into two blocks. The first block consists of the value functions, the policy functions and the market tightness function, which are determined by requirements (i) - (iv). The second block consists of the distribution of individuals over real balances and the wage rate, which are determined by requirements (v) and (vi). The second block depends on the objects in the first block, but the first block is self-enclosed and not affected by the second block. That is, the value functions, the policy functions and the market tightness function are independent of the distribution and the wage rate. We refer to this property of the equilibrium as *block recursivity*, following the usage in recent literature on labor search (Shi, 2009, Menzio and Shi, 2010, 2011, and Gonzalez and Shi, 2010). Clearly, even when an equilibrium is block recursive, the distribution still affects the aggregate activity.

Block recursivity is an attractive property of our model because it enables us to solve for equilibrium value functions, policy functions and the market tightness function without solving for the distribution of individuals over real balances. After obtaining the objects in the first block, we can compute the equilibrium distribution by simply equating the flows of individuals into and out of each level of the real balance. Thus, the steady state is tractable even when the distribution of real balances is non-degenerate. In contrast, when the steady state is not block recursive, the distribution is an aggregate state variable that appears in the policy and value functions. In this case, one must compute the objects in the two blocks simultaneously and, since the distribution is endogenous and potentially has a large dimension, the computation of an equilibrium is complicated. In fact, it is to circumvent this complexity that monetary models have imposed assumptions on the model environment to make the distribution degenerate (e.g., Shi, 1997, Lagos and Wright, 2005).

Directed search and free-entry of trading posts are responsible for the steady state to be block recursive. With directed search, individuals choose to enter only the submarkets with the best tradeoff between the terms of trade and the matching probability, as formulated in subsection 2.2. The tightness function provides all information on the market that is relevant for this tradeoff. Given the tightness in each submarket, a buyer's optimal decision on which submarket to visit depends only on the buyer's own real balance, and not on how real balances are distributed among other buyers. Similarly, given the tightness in each submarket, the expected profit of a trading post in a submarket depends only on the particular real balance of the buyers who are expected to enter that submarket, and not on how real balances are distributed in other submarkets. In turn, the tightness in each submarket is determined by free-entry of trading posts, which drives the expected profit of a trading post down to zero wherever $x - q \geq k$, and the tightness to zero wherever $x - q < k$. Because the expected profit of a trading post in each submarket depends only on the real balance of the buyers who will enter that submarket, so does the resulting tightness. Thus, value functions, policy functions, and the market tightness function are all independent of money distribution in the steady state.

To appreciate the role of directed search, consider an environment with undirected search in which all buyers and trading posts go through the same random matching process first and then decide whether to trade. The terms of trade can be either posted before the meeting (without serving the function of directing search) or bargained after the meeting. If the terms of trade are posted before a meeting takes place, whether a particular match generates a non-negative surplus and, hence, a trade depends on the real balance of the buyer in the match. Because the buyer is randomly drawn, the trading probability depends on the distribution of buyers over real balances. If the terms of trade are instead bargained after a meeting takes place, they depend on real balances of both individuals in the match which are randomly drawn from the distribution. In both cases, the distribution of individuals over real balances affects the value function and the expected benefit of a trading post. Because the tightness of the market is such that the expected benefit of a trading post is equal to the cost, the tightness is also a function of the distribution. That is, when search is undirected, the equilibrium is not block recursive.

With directed search, individuals self-select into submarkets by making the trade-off between the terms of trade and the matching probability. This trade-off contrasts directed search with two other environments. One is an economy where individuals can only trade in one uniform market, which may or may not be perfectly competitive. In this environment, because all buyers with different balances are forced to pay the same price, equilibrium price depends on the distribution of real balances, and so the equilibrium is not block recursive. Another environment is the trading-post model of Shapley and Shubik (1977), where search frictions do not exist. Individuals supply money and goods to trading posts, and the price level at each trading post is equal to the ratio between the total amount of money and goods supplied to the post. Although it is generally possible that many trading posts can be supported by various beliefs of the market participants, a similar restriction on beliefs as in footnote 7 implies that the equilibrium will have only one active trading post. Again, the distribution of real balances affects the price at this active trading post, and so the equilibrium is not block recursive.

Finally, let us remark on the assumption of a perfectly competitive labor market. Although this assumption is standard in macro, most money-search papers treat a worker synonymously to a producer whose income depends on the matching outcome and, hence, is random (e.g., Shi, 1995, and Trejos and Wright, 1995). In our model, each firm hires workers to produce goods and to maintain a large number of trading posts. Although each trading post may or may not have a trade, the law of large numbers implies that a firm's total revenue from all trading posts together is deterministic, which enables the firm to pay a deterministic competitive wage rate. Moreover, we will show in section 4 that all workers go to work with zero balance. However, neither the deterministic wage income nor the degenerate real balance among workers in the equilibrium is important for block recursivity. This should be clear from the fact that block recursivity is a statement about the independence of the entire value functions, policy functions and the tightness function on the distribution, not just the independence at particular real balances. For example, if there are idiosyncratic shocks to the disutility of labor or a worker's matching outcome, then the distribution of wage income among workers will be non-degenerate in the equilibrium. But the equilibrium will still be defined in the same way as above and will remain block recursive. We abstract from such heterogeneity among workers in order to focus on a buyer's decision and money distribution among the buyers.

3. Equilibrium Policy and Value Functions

In this section we establish existence, uniqueness and other features of value and policy functions. A center piece of this analysis is subsection 3.2 on a buyer's value and policy functions. In particular, we prove that a buyer's policy functions $(x^*(m), q^*(m))$ are monotonically increasing, which implies that buyers choose to sort themselves out according to the real balance. A buyer with a higher balance chooses to search in a submarket where he can spend a larger balance and get a higher quantity of goods. In such a submarket the buyer also has a higher matching probability. Sorting leads to a stylized pattern of purchases over time by a buyer and a clear characterization of the equilibrium in section 4.

Monotonicity of policy functions is also critical for us to prove that the standard conditions of optimization, such as the first-order conditions and the envelope conditions, hold in our model. The characterization of a buyer's problem is technically challenging because the problem is not well-behaved. In fact, a buyer's objective function is not concave in the choice and state variables jointly. We cannot use standard arguments in dynamic programming (e.g., Stokey et al., 1989) to establish monotonicity of the policy functions and differentiability of the value function and, in turn, to establish the validity of the envelope and first-order conditions. Instead, we develop an alternative set of arguments that first prove monotonicity of the policy functions, then differentiability of the value function and finally the validity of the first-order and envelope conditions. These arguments are of independent interest because they are likely to apply to a variety of dynamic models that involve both discrete and continuous choices.

A map of the analysis in this section is as follows. First, we assume that an individual's real balance is bounded above by $\bar{m} < \infty$, which we will validate later in Theorem 3.5. Let $\mathcal{C}[0, \bar{m}]$ denote the set of continuous and increasing functions on $[0, \bar{m}]$, and let $\mathcal{V}[0, \bar{m}]$ denote the subset of $\mathcal{C}[0, \bar{m}]$ that contains all concave functions. Taking an arbitrary ex ante value function $V \in \mathcal{V}[0, \bar{m}]$, we use subsection 3.1 to characterize a worker's problem. Second, with the same function V, we use subsection 3.2 to characterize a buyer's problem. Third, in subsection 3.3, we characterize an individual's lottery choice and obtain an update of the ex ante value function, denoted as TV. We prove that T is a monotone contraction mapping on $\mathcal{V}[0, \bar{m}]$, and so there is a unique fixed point for the ex ante value function. Finally, we verify in Theorem 3.5 that an individual's real balance is indeed bounded above by $\bar{m} < \infty$.

3.1. A worker's value and policy functions

Let $\bar{m}$ be a sufficiently large upper bound on individuals' real balances and V any arbitrary function in $\mathcal{V}[0, \bar{m}]$. Given V, the worker's problem, (2.1), generates the worker's value function $W(m)$, the policy function of labor supply $\ell^*(m)$, and the policy function of the end-of-period balance $y^*(m) = m + \ell^*(m)$. We have the following lemma (see Appendix A for a proof):

Lemma 3.1. *For any $m \in [0, \bar{m}]$ and $V \in \mathcal{V}[0, \bar{m}]$, the following properties hold:*
(i) $W \in \mathcal{V}[0, \bar{m}]$; i.e., W is continuous, increasing and concave on $[0, \bar{m}]$;
(ii) $\ell^*(m)$ is unique, continuous and decreasing in m, and $y^*(m)$ is unique, continuous and strictly increasing in m;
(iii) For all m such that $\ell^*(m) > 0$, $W'(m)$ and $V'(y^*(m))$ exist and satisfy:

$$W'(m) = \beta V'(m + \ell^*(m)) = h'(\ell^*(m)). \tag{3.1}$$

The first equality is the envelope condition and the second equality the first-order condition.

The value function of a worker is continuous and increasing in the worker's real balance because the ex ante value function has these properties. A worker's value function is also concave, because concavity of the ex ante value function and convexity of the disutility of labor supply imply that the worker's objective function is concave jointly in the choice ℓ and the state variable m. Part (ii) states existence, uniqueness and monotonicity of a worker's policy functions. These properties are intuitive. By supplying higher labor, a worker obtains a higher balance which increases the ex ante value function next period. Since the ex ante value function is concave, the marginal value of money obtained by working is decreasing. In contrast, the marginal disutility of labor supply is strictly increasing. Thus, for any given balance, a worker's optimal labor supply is unique.

Such uniqueness implies that the policy function of labor supply is continuous in the worker's real balance. Moreover, since the marginal gain from working is smaller when a worker already has a relatively high balance, the policy function of labor supply is decreasing in the worker's real balance. Similarly, a worker's policy function of the end-of-period real balance is unique and continuous. This function is strictly increasing in m because a higher balance has a strictly positive marginal benefit to a worker.

Part (iii) states that if a worker's optimal labor supply is strictly positive, then the worker's value and policy functions satisfy the envelope condition and the first-order condition. Notice that the choice $\ell = 1$ is never optimal, because the marginal disutility of labor supply at this choice is infinite. Hence, a worker's optimal labor supply is interior if it is strictly positive. An interior choice is a common requirement for the first-order and the envelope conditions to hold, and the requirement is not binding in the equilibrium.[9] When the optimal choice $\ell^*(m)$ is interior, the derivative $W'(m)$ is given by $W'(m) = h'(\ell^*(m))$ and, hence, exists. This condition expresses the fact that the marginal value of having a higher balance prior to work is equal to the marginal reduction in the disutility of labor needed to obtain any given end-of-period balance.

For part (iii), we transform a worker's problem into one where the choice is the end-of-period balance y instead of labor supply:

$$W(m) = \max_{y \geq m} \left[\beta V(y) - h(y - m) \right]. \tag{3.2}$$

We apply the following standard approach in dynamic programming (see Stokey et al., 1989, p85). First, with any concave V, the objective function in (3.2) is concave in (y, m) jointly. This feature implies that the optimal choice $y^*(m)$ is unique for each m and that W is concave. Second, since W and the objective function are concave, Benveniste and Scheinkman (1979) have shown that $W'(m)$ exists and satisfies the envelope condition, provided that the optimal choice is interior. Third, using concavity of W and convexity of h, we can deduce from the envelope condition, $W'(m) = h'(\ell^*(m))$, that the policy function $\ell^*(m)$ is decreasing.

Note that the derivative $V'(y^*(m))$ exists, despite that we have not assumed that V is differentiable everywhere. To prove this result, we first note that concavity of V implies that V is differentiable almost everywhere and that the one-sided derivatives of V exist (see Royden, 1988, pp113-114). Then, we prove that optimal labor supply always generates an end-of-period balance at which the future value function is differentiable. Specifically, we use a generalized version of the envelope theorem to show that both of the one-sided derivatives of $\beta V(y^*(m))$ are equal to

[9] If an individual's balance is so high that optimal labor supply is zero at such a balance, then it is optimal for the individual to choose to enter the goods market as a buyer rather than the labor market as a worker.

those of $W(m)$ (See Appendix A). Because $W'(m)$ exists, so does $\beta V'(y^*(m))$.

Lemma 3.1 holds for all $m \geq 0$. Of particular interest is the case $m = 0$. For a worker with $m = 0$, denote the optimal end-of-period balance as $\hat{m} = y^*(0) = \ell^*(0)$. This worker's value function is $W(0) = \beta V(\hat{m}) - h(\hat{m})$. Lemma 3.1 implies that

$$V'(\hat{m}) = \frac{1}{\beta} h'(\hat{m}) = \frac{1}{\beta} W'(0). \tag{3.3}$$

3.2. A buyer's value and policy functions

We now analyze a buyer's problem (2.2), given any arbitrary ex ante value function $V \in \mathcal{V}[0, \bar{m}]$. In subsection 3.2.1, we reformulate the buyer's problem, describe the difficulty in analyzing the problem, outline our approach, and present the main results in Theorem 3.2. In subsections 3.2.2 and 3.2.3, we establish two lemmas which together constitute a proof of Theorem 3.2.

3.2.1. The difficulty, our approach and main results

For convenience, we use (x, b) instead of (x, q) to represent a buyer's choices and express q as a function of (x, b), where b is the buyer's matching probability in a submarket. Recall that $b = \lambda(\theta(x, q))$, that a trading post's matching probability is $s = \rho(\theta(x, q))$, and that $s = \mu(b) \equiv \rho(\lambda^{-1}(b))$. Thus, the market tightness condition (2.5) can be written as

$$s = \mu(b) = \begin{cases} \frac{k}{x-q}, & \text{if } k \leq x - q \\ 1, & \text{otherwise.} \end{cases} \tag{3.4}$$

In any submarket with $x - q \leq k$, the tightness is 0, and a buyer's matching probability is $b = \mu^{-1}(1) = 0$. In any submarket with $x - q > k$, the tightness is strictly positive, and a buyer's matching probability is $b = \mu^{-1}(\frac{k}{x-q}) > 0$. Thus, in any submarket (x, q) with positive tightness, we can express the quantity of goods traded in a match as

$$q = Q(x, b) \equiv x - \frac{k}{\mu(b)}. \tag{3.5}$$

Note that if a buyer has a balance $m \leq k$, the only submarkets that the buyer can afford to visit have $x - q \leq m \leq k$ and, hence, have zero tightness. For such a buyer, the optimal choice is $b^*(m) = 0$, and the value function is $B(m) = \beta V(m)$.

Let us focus on the non-trivial case $m > k$. In this case, the buyer's problem (2.2) can be transformed into the following one in which the choices are (x, b):

$$\begin{aligned} B(m) \quad &= \max_{(x,b)} \{\beta V(m) + b\left[u(x, b) + \beta V(m - x) - \beta V(m)\right]\} \\ \text{s.t.} \quad & x \in [0, m], \quad b \in [0, 1], \end{aligned} \tag{3.6}$$

where $u(x, b) = U(Q(x, b))$.[10] Let $(x^*(m), b^*(m))$ denote the buyer's policy functions for (x, b) and let $\phi(m)$ denote the policy function for the residual balance $(x - m)$. Then,

$$q^*(m) \equiv Q\left(x^*(m), b^*(m)\right), \quad \phi(m) \equiv m - x^*(m). \tag{3.7}$$

The objective function in (3.6) is not concave jointly in (x, b) and m. The objective function involves the product of the buyer's trading probability, b, and the buyer's surplus of trade. Even if these terms are concave separately, the product of the two may not be concave in (x, b, m) jointly. The lack of concavity presents a major difficulty in using the standard approach in dynamic programming to analyze policy and value functions, because the approach starts with the requirement that the objective function be concave jointly in the choice and state variables (see Stokey et al., 1989, and the analysis of a worker's problem in subsection 3.1). Attempts to make a buyer's objective function concave entail additional restrictions on the endogenous function V, which are difficult to verify as the outcome of (2.4).

To analyze a buyer's problem, we use lattice-theoretic techniques (see Topkis, 1998). The procedure almost reverses the steps of the standard approach. First, we establish monotonicity of the policy functions using lattice-theoretic techniques. Second, using monotonicity of the policy functions, we prove that the value functions $B(m)$ and $V(m)$ are differentiable at real balances induced by optimal choices from any initial balance. This result allows us to characterize the policy functions with the first-order conditions and envelope conditions. Finally, we prove that the ex ante value function is differentiable at all real balances. This procedure is natural in the sense that monotonicity of the policy functions is a basic property that does not necessarily require differentiability of the value functions.[11] Recall that $\mathcal{C}[0, \bar{m}]$ denotes the set of continuous and increasing functions on $[0, \bar{m}]$, and $\mathcal{V}[0, \bar{m}]$ denotes the subset of $\mathcal{C}[0, \bar{m}]$ that contains all concave functions. The following theorem states the main result of our procedure:

Theorem 3.2. *Take any arbitrary $V \in \mathcal{V}[0, \bar{m}]$. Then, $B \in \mathcal{C}[0, \bar{m}]$. If $m \leq k$, then $b^*(m) = 0$ and $B(m) = \beta V(m)$; if $m > k$, then $B(m)$ satisfies (3.6). Consider any $m \in [k, \bar{m}]$ such that*

[10] Note that for $q \geq 0$, the buyer's choices must satisfy $x \geq k/\mu(b)$. However, there is no need to add this constraint to the problem (3.6) because it is not binding in any realized trade. For any choices (x, b) such that $x < k/\mu(b)$ and $x > 0$, the quantity of goods is $q < 0$ and the utility of consumption is $u(x, b) < U(0) = 0$. In this case, the buyer's surplus from trade is $u(x, b) + \beta V(m - x) - \beta V(m) < 0$. The buyer can avoid this loss by choosing $b = 0$.

[11] There are other approaches that establish differentiability of the value function in the presence of a non-concave objective function. However, these approaches do not prove monotonicity of the policy functions. Moreover, they are not applicable in our model. Specifically, these approaches assume the objective function to be equi-differentiable (Milgrom and Segal, 2002) or differentiable with respect to the state variable (Clausen and Strub, 2010). In our model, the objective function in (2.2) contains both $V(m)$ and $V(m - x)$, where x is a choice and m a state variable. For this objective function to satisfy either of the aforementioned assumptions, the value function V must be differentiable, which is a result to be proven.

$b^*(m) > 0$. The results (i)-(iii) below hold:
(i) For each m, the optimal choices $(x^*(m), b^*(m))$ and the implied quantities $(q^*(m), \phi(m))$ are unique. The policy functions $x^*(m)$, $b^*(m)$, $q^*(m)$ and $\phi(m)$ are continuous and increasing.
(ii) The optimal choice $b^*(m)$ satisfies the first-order condition:

$$u(x,b) + b\,u_2(x,b) = \beta\left[V(m) - V\left(m - x\right)\right]. \tag{3.8}$$

For all m such that $\phi(m) > 0$, $\phi(m)$ satisfies the first-order condition:[12]

$$V'\left(\phi(m)\right) = \frac{1}{\beta}u_1\left(x^*(m), b^*(m)\right). \tag{3.9}$$

(iii) $B'(m)$ exists if and only if $V'(m)$ exists, and B is strictly increasing.
Consider any $m < \bar{m}$ such that $b^*(m) > 0$. If $B(m) = V(m)$ and if there exists a neighborhood O of m such that $B(m') \leq V(m')$ for all $m' \in O$, then (iv) and (v) below hold. These two parts also hold at $m = \bar{m}$ if $B'(\bar{m}) = V'(\bar{m})$:
(iv) The derivatives $B'(m)$ and $V'(m)$ exist and satisfy:

$$V'(m) = \frac{b^*(m)}{1 - \beta\left[1 - b^*(m)\right]}u_1\left(x^*(m), b^*(m)\right) = B'(m). \tag{3.10}$$

(v) If $\phi(m) > 0$, then b^* and ϕ are strictly increasing at m, and V is strictly concave at $\phi(m)$, with $V'(\phi(m)) > V'(m)$.

Parts (ii)-(iv) of this theorem assure that one can use the standard apparatus of optimization to analyze a buyer's optimal decisions and value function. We will establish Lemmas 3.3 and 3.4 below, which together prove Theorem 3.2. A reader who is eager to see the implications of the above theorem may want to go directly to subsection 3.3.

3.2.2. A buyer's policy functions and monotonicity

To apply lattice-theoretic techniques (Topkis, 1998) to (3.6), we investigate whether the objective function in (3.6) is supermodular in the choice variables (x, b) and the state variable m, i.e., whether the objective function has increasing differences in (x, b), (x, m) and (b, m). Supermodularity implies that the variables are complementary with each other in the objective function, which intuitively leads to increasing policy functions. As a preliminary step that also develops the intuition, we examine the properties of the functions $Q(x, b)$ and $u(x, b)$. The function $Q(x, b)$, defined in (3.5), determines the quantity of goods sold to a buyer who has a matching probability

[12] If $\phi(m) = 0$, then (3.9) is replaced with $V'\left(0\right) \leq \frac{1}{\beta}u_1\left(m, b^*\left(m\right)\right)$.

b and spends a real balance x in the trade in a submarket with positive tightness. For all (x, b) such that $Q(x, b) > 0$, it is easy to verify that the function Q has the following properties:

$$Q_1(x, b) > 0,\ Q_2(x, b) < 0,\ Q(x, b) \text{ is (weakly) concave, and } Q_{12} = 0. \tag{3.11}$$

It is intuitive that Q strictly increases in x and strictly decreases in b. For any given matching probability, the more a buyer is willing to pay, the higher the quantity of goods he can obtain. For any given payment, a buyer who wants to have a higher matching probability must accept a lower quantity of goods. Equivalently, a firm must be compensated with the reduction in the quantity of goods for creating more trading posts to increase a buyer's matching probability.

The second-order properties of Q are also intuitive. The function Q is strictly concave in b because increasing the number of trading posts has a diminishing marginal effect on increasing a buyer's matching probability. In order for a firm to create more trading posts to increase a buyer's matching probability further, the firm must be compensated with an increasingly larger reduction in the quantity of goods traded for a given x. The function Q is linear in x because the amount of labor needed to produce any quantity of goods is assumed to be a linear function of the quantity. Moreover, $Q_{12} = 0$ because Q is separable in (x, b).[13]

The function $Q(x, b)$ is used in the objective function in (3.6) to express $u(x, b) = U\left(Q(x, b)\right)$. Because the utility function U is strictly increasing and strictly concave, the properties of Q above imply similar properties of u for all (x, b) such that $Q(x, b) > 0$:

$$u_1(x, b) > 0,\ u_2(x, b) < 0,\ u(x, b) \text{ is strictly concave, and } u_{12} > 0. \tag{3.12}$$

The second-order properties of u are stronger than those of Q. In particular, $u(x, b)$ is strictly supermodular since $u_{12} > 0$. This property is intuitive. Consider $u_1(x, b)$, the marginal utility of spending. In a submarket where the buyer's matching probability is relatively high, the quantity of goods that the buyer obtains in a trade with any given spending is relatively low, because a firm must be compensated for creating a large number of trading posts to deliver the high matching probability for the buyer. At such low consumption, an increase in spending can increase the utility of consumption by a relatively large amount. Thus, $u_1(x, b)$ is strictly higher in a submarket with a higher b than with a lower b.

In addition to $u(x, b)$, a buyer's objective function in (3.6) contains other functions. Let us write this objective function as $\beta V(m) + b\, R(x, b, m)$, where R is the buyer's surplus:

$$R\left(x, b, m\right) \equiv u(x, b) + \beta V\left(m - x\right) - \beta V(m). \tag{3.13}$$

[13]If the amount of labor needed to produce any quantity of goods is assumed to be a strictly convex function of the quantity, then Q is strictly concave in (x, b) and $Q_{12} > 0$. These features of Q will strengthen our results.

Even if b and $R(x,b,m)$ are supermodular, their product is not necessarily supermodular. Thus, we cannot apply Topkis' (1998) theorems directly to (3.6). To resolve this problem, we decompose the buyer's problem into two steps. In the first step, we fix b and characterize the optimal choice of x. For any given (b,m), the optimal choice of x maximizes $R(x,b,m)$. Denote

$$\tilde{x}\left(b,m\right) = \arg\max_{x\in[0,m]} R(x,b,m), \quad \tilde{R}\left(b,m\right) = R\left(\tilde{x}\left(b,m\right), b, m\right). \tag{3.14}$$

In the second step, we characterize the optimal choice of b as

$$b^{*}(m) = \arg\max_{b\in[0,1]} \; b \; \tilde{R}(b,m). \tag{3.15}$$

Much weaker properties are required to apply Topkis' theorems in the two steps than in the direct approach. In the first step, we prove that a buyer's surplus $R(x,b,m)$, rather than bR, is supermodular in (x,b,m). Because a higher m enlarges the feasibility set in (3.14), supermodularity of R implies that $\tilde{x}(b,m)$ is increasing in (b,m) and that $\tilde{R}(b,m)$ is supermodular. In the second step, we prove that $b\,\tilde{R}(b,m)$ is supermodular in (b,m). That is, $b\,R(x,b,m)$ is supermodular in (b,m) at the *particular spending level* $x=\tilde{x}(b,m)$, which is weaker than supermodularity of $bR(x,b,m)$ for all (x,b,m). Since the feasibility set in (3.15) is independent of m, supermodularity of $b\,\tilde{R}(b,m)$ implies that the policy function $b^{*}(m)$ is increasing. This implies that the policy function $x^{*}(m)=\tilde{x}(b^{*}(m),m)$ is increasing. By changing the choices from (x,b) to (x,q) and to $(m-x,b)$, in turn, we use the same procedure to prove that $q^{*}(m)$ and $\phi(m)$ are increasing. The following lemma summarizes the results (see Appendix B for a proof):

Lemma 3.3. *For any $V\in\mathcal{V}[0,\bar{m}]$, $B\left(m\right)\in\mathcal{C}[0,\bar{m}]$. If $m\le k$, then $b^{*}(m)=0$ and $B(m)=\beta V(m)$; if $m>k$, $B(m)$ solves (3.6). Moreover, for all $m\in[k,\bar{m}]$ such that $b^{*}(m)>0$, the policy functions are monotone as stated in part (i) of Theorem 3.2.*

As stated in part (i) of Theorem 3.2, optimal choices $(x^{*}(m),b^{*}(m))$ are unique for any given balance m, and so the policy functions are continuous. We prove this result by establishing the feature that the logarithmic transformation of the part to be maximized in the buyer's problem, $bR(x,b,m)$, is strictly concave in (x,b). Uniqueness is intuitive. Because the matching function has diminishing marginal returns to the number of trading posts, a buyer's matching probability is strictly concave in the tightness of the submarket. This implies that the marginal cost of increasing a buyer's matching probability is increasing, in the sense that the buyer must either spend an increasingly larger real balance to purchase a given quantity of goods or obtain an increasingly smaller quantity of goods for any given spending. Thus, for any given balance, a buyer finds a unique pair of (x,b) that offers the best trade-off between the quantity of goods traded and the probability of the trade. That is, given the balance, a buyer chooses a unique submarket to enter, rather than being indifferent between different submarkets.

The policy functions $x^*(m)$, $b^*(m)$, $q^*(m)$ and $\phi(m)$ are all increasing, provided $b^*(m) > 0$. This monotonicity arises from the assumption that V is concave. A buyer's residual balance after trade is valued with V next period. Because V is concave, the marginal value of the residual balance is diminishing. For a buyer with a higher balance, it is then optimal to increase current utility by entering a submarket where he has a higher matching probability and higher spending. In addition, because consumption is "normal" in both the current and the next period, it is optimal for a buyer with a higher balance to increase consumption in both periods. This requires the residual balance $\phi(m)$ to be increasing in the buyer's current balance. Not surprisingly, the proof of supermodularity of a buyer's surplus function, $R(x, b, m)$, relies heavily on concavity of V and the properties of $u(x, b)$ listed in (3.12).

In summary, buyers sort themselves into different submarkets according to the real balance. A buyer with more money chooses to enter a submarket where he will have a higher matching probability and once he is matched in the submarket, he will spend a larger amount of money, buy a larger quantity of goods, and exit the trade with a higher balance.

3.2.3. First-order conditions, envelope theorems and value functions

The remaining parts of Theorem 3.2, (ii)-(v), describe the first-order conditions, the envelope condition and additional properties of the value functions. They are restated in the following lemma and proven in Appendix C:

Lemma 3.4. *Consider any $m \in [k, \bar{m}]$ such that $b^*(m) > 0$. For any $V \in \mathcal{V}[0, \bar{m}]$, parts (ii) and (iii) of Theorem 3.2 hold. For any $m < \bar{m}$ such that $b^*(m) > 0$, if $B(m) = V(m)$ and if there exists a neighborhood O of m such that $B(m') \leq V(m')$ for all $m' \in O$, then parts (iv) and (v) of Theorem 3.2 hold. These two parts also hold at $m = \bar{m}$ if $B'(\bar{m}) = V'(\bar{m})$.*

Part (ii) of Theorem 3.2 states that optimal choices b^* and x^* satisfy the first-order conditions. In the first-order condition of b^*, (3.8), the left-hand side is the marginal benefit of increasing b, represented by the increase in expected utility of consumption resulting from a higher matching probability. The right-hand side of (3.8) is the buyer's opportunity cost of a trade represented by the reduction in the future value function resulting from a lower future balance. The optimal choice b^* equates the marginal benefit and the marginal cost of changing b. Similarly, the first-order condition of x^*, (3.9), requires that the marginal cost of increasing spending, represented by the marginal value of the residual balance ϕ, should be equal to the marginal utility of consumption brought about by higher spending.

The first-order condition of b^* holds regardless of whether or not V is differentiable. This is because b does not appear in V, which implies that the buyer's objective function in (3.6) is differentiable with respect to b for any given (x, m). In contrast, the choice x appears in V through the residual balance. Thus, the first-order condition of the optimal choice x^*, (3.9), implies that the derivative $V'(\phi(m))$ exists. That is, it is optimal for a buyer to choose spending in such a way to steer away from residual balances at which V is not differentiable. This result is similar to the existence of $V'(y^*(m))$ in Lemma 3.1. Its proof relies on the features that the utility of consumption is differentiable and optimal spending is a continuous function of the real balance. Once $V'(\phi(m))$ exists, $V(m)$ is the only function on the right-hand side of the Bellman equation for $B(m)$ whose differentiability has not be proven. It is then not surprising that $B'(m)$ exists if and only if $V'(m)$ does, as stated in part (iii) of Theorem 3.2. Moreover, B is strictly increasing because utility is strictly increasing in consumption.

Part (iv) of Theorem 3.2 is the envelope condition of a buyer's problem. It is valid for any $m < \bar{m}$ that satisfies $B(m) = V(m)$ and that is surrounded by a neighborhood O such that $B(m') \leq V(m')$ for all $m' \in O$. In this neighborhood, the one-sided derivatives of B and V satisfy $B'(m^+) \leq V'(m^+)$ and $B'(m^-) \geq V'(m^-)$. Notice that the marginal value of money to a buyer is a weighted average of the marginal value of consumption in the case of making a purchase and the marginal value of retaining the balance in the case of not trading. For $B'(m^+) \leq V'(m^+)$ to hold, the marginal value of retaining the balance, $V'(m^+)$, must be greater than or equal to the marginal value of money related to a purchase, which is the expression in the middle of (3.10). Similarly, this expression must be less than or equal to $V'(m^-)$ in order for $B'(m^-) \geq V'(m^-)$ to hold. These two requirements imply $V'(m^+) \geq V'(m^-)$. On the other hand, $V'(m^+) \leq V'(m^-)$, because V is concave. Thus, $V'(m^+) = V'(m^-)$; that is, $V'(m)$ exists.

Note that the neighborhood O above always exists if V is the *equilibrium* function. The neighborhood is required in Theorem 3.2 because the theorem takes V as an arbitrary function in $\mathcal{V}[0, \bar{m}]$. Also, the neighborhood O may not exist around the arbitrary upper bound $\bar{m}$, because we have not characterized B and V on the right-hand side of $\bar{m}$. This is why the additional condition $B'(\bar{m}) = V'(\bar{m})$ is needed for part (iv) of Theorem 3.2 to hold at $m = \bar{m}$.

Part (v) of Theorem 3.2 describes additional properties. First, for any m with $B(m) = V(m)$, a buyer with the balance m does not have the need to use a lottery. This implies that the marginal value of the real balance is strictly decreasing at m and so, for any given trading probability, the buyer's surplus of trade is strictly increasing in m locally. If such a buyer has additional money, he prefers to enter a submarket with a strictly higher trading probability and spend more in order to capture the higher surplus of trade. That is, $b^*(m)$ and $x^*(m)$ are strictly increasing at such m. Second, since future consumption is a normal good, it is optimal for the buyer to keep part of this additional money as the residual balance. That is, $\phi(m)$ is also strictly increasing at such m. Third, the ex ante value function must be strictly concave at $\phi(m) > 0$: if V is linear at $\phi(m)$, the buyer should have spent more because the marginal cost of doing so is locally constant.

3.3. Lotteries and the equilibrium ex ante value function

We have characterized a worker's and a buyer's optimal decisions, taking the ex ante value function as any arbitrary $V \in \mathcal{V}[0, \bar{m}]$, i.e., any continuous, increasing and concave function on $[0, \bar{m}]$. Part (i) of Lemma 3.1 states that a worker's problem (2.1) defines a mapping $T_W : \mathcal{V}[0, \bar{m}] \to \mathcal{V}[0, \bar{m}]$ that maps $V \in \mathcal{V}[0, \bar{m}]$ into a worker's value function $W \in \mathcal{V}[0, \bar{m}]$. Theorem 3.2 implies that a buyer's problem (2.2) defines a mapping $T_B : \mathcal{V}[0, \bar{m}] \to \mathcal{C}[0, \bar{m}]$ that maps $V \in \mathcal{V}[0, \bar{m}]$ into a buyer's value function $B \in \mathcal{C}[0, \bar{m}]$. In the equilibrium, the ex ante value function must satisfy (2.4). If we substitute $W = T_W V$ and $B = T_B V$ into (2.3) to obtain $\tilde{V}$, then the right-hand side of (2.4) defines a mapping T on V, and we can write (2.4) as $V(m) = TV(m)$. That is, the equilibrium ex ante value function is a fixed point of T. In this subsection, we show that T maps $\mathcal{V}[0, \bar{m}]$ into $\mathcal{V}[0, \bar{m}]$ and that it has a unique fixed point. Moreover, we verify that there indeed exists a finite upper bound $\bar{m}$ on individuals' real balances in the equilibrium.

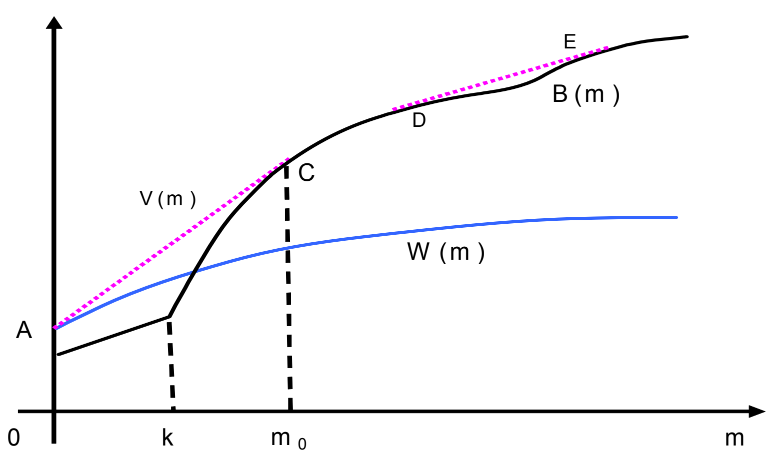

Figure 1. Lotteries and the ex ante value function

The functional equation (2.4) involves maximization over the choice of lotteries. This choice is necessary for T to preserve concavity of the ex ante value function V. In the preceding analysis of a worker's and a buyer's optimal decisions, we required V to be concave to prove optimal choices to be unique and policy functions to be monotone. However, in the absence of lotteries, the ex ante value function would be $\tilde{V}$ defined by (2.3), instead of V. The function $\tilde{V}$ can fail to be concave for two reasons. One is that a buyer's value function, $B(m)$, may be non-concave. If this happens when $B(m) > W(m)$, then $\tilde{V}(m)$ is equal to $B(m)$ and, hence, is non-concave. In this case, there is a gain to individuals to play a fair lottery to convexify the feasibility set of values. Depicted in Figure 1, this *lottery for high balances* makes V the dashed line connecting points D and E. The second cause of non-concavity of $\tilde{V}$ is the maximum operator in (2.3), which represents the choice between being a buyer and a worker. It is optimal for an individual to be a worker if the real balance is sufficiently small, and a buyer if the real balance is sufficiently large. When the real balance is close to the level at which an individual is indifferent between the two roles in the market, the maximum of the two functions, $B(m)$ and $W(m)$, is convex, and so there is a gain to the individual from playing a fair lottery. In fact, individuals whose real balances are below this level of indifference can also gain from the lottery. Depicted in Figure 1, this *lottery for low balances* makes V the dashed line connecting points A and C.

In the lottery for low balances, the low prize is $z_1^* = 0$, the high prize is $z_2^* = m_0$, and the probability of winning the high prize is $\pi_2^*(m) = m/m_0$. The high prize is determined as

$$m_0 = \arg\max_{z \geq m} \left[\frac{m}{z} \tilde{V}(z) + \left(1 - \frac{m}{z}\right) \tilde{V}(0) \right]. \tag{3.16}$$

It is clear that m_0 is independent of the individual's balance m, provided $m \leq m_0$.

The following theorem states existence, uniqueness and other properties of the equilibrium value functions as well as the properties of the upper bound $\bar{m}$ (see Appendix D for a proof):

Theorem 3.5. *(i) T is a self-map on $\mathcal{V}[0, \bar{m}]$ and has a unique fixed point V.*
(ii) $V(m) > W(m) > 0$ for all $m > 0$; $V(0) = W(0) > 0$, and $W(m) \geq B(m)$ for all $m \in [0, k]$.
(iii) There exists $m_0 \in (k, \bar{m}]$ such that an individual with $m < m_0$ will play the lottery with the prize m_0, which satisfies $V(m_0) = B(m_0)$, $b^*(m_0) > 0$ and $\phi(m_0) = 0$. Moreover, if $m_0 < \bar{m}$, then (3.10) holds for $m = m_0$, and $V'(m_0) = B'(m_0) > 0$.
(iv) $V'(m) > 0$ exists for all $m \in [0, \bar{m})$; $B'(m)$ exists for all $m \in [k, \bar{m})$ such that $b^*(m) > 0$.
(v) There exists $\bar{m} < \infty$ such that individuals' balances satisfy $m \leq \bar{m}$ in equilibrium. Moreover, $\bar{m}$ satisfies $\bar{m} = \hat{z}_2 = z_2^*(\hat{m})$, $B(\bar{m}) = V(\bar{m})$ and $B'(\bar{m}) = V'(\bar{m})$.

For part (i), we verify that the mappings on V defined by a worker's problem, (2.1), and a buyer's problem, (2.2), are monotone and feature discounting with the factor β. As a result, T is a monotone contraction mapping that maps continuous, increasing and concave functions into continuous and increasing functions. In addition, since TV is generated by the optimal choice of a two-point lottery, it is a concave function (see Lemma F.1, Menzio and Shi, 2010). Thus, T is a monotone contraction mapping on $\mathcal{V}[0, \bar{m}]$ and has a unique fixed point.

Part (ii) compares V, W and B, as depicted in Figure 1. Part (iii) formally characterizes the lottery for low balances. In particular, after winning the prize m_0, an individual strictly prefers to be a buyer and to spend the entire balance m_0 in one trade. Note that this part implies that all workers hold zero balance prior to work in the equilibrium. Part (iv) states that the ex ante value function is differentiable and strictly increasing for all $m < \bar{m}$, and a buyer's value function is differentiable at all balances which yield positive matching probability. Differentiability of V is more general than that in Theorem 3.2, and it can be explained as follows. If V were not differentiable at some $m \in (m_0, \bar{m})$, then V would be strictly concave at m and hence $V(m) = B(m)$, in which case part (iv) of Theorem 3.2 would imply the contradiction that V is differentiable at m. Thus, V is differentiable for all $m \in (m_0, \bar{m})$. Also, V is differentiable for all $m \in [0, m_0)$ because V is a linear function in this interval. Furthermore, if $m_0 < \bar{m}$, then V is differentiable at m_0 as a result of part (iv) of Theorem 3.2 and the fact $B(m_0) = V(m_0)$.

Finally, part (v) states that individuals' real balances are endogenously bounded above $\bar{m} < \infty$, which is equal to the high prize of the lottery at $\hat{m}$. Recall that $\hat{m}$ is the wage income of a worker who has zero balance prior to work. Moreover, the upper bound satisfies $B(\bar{m}) = V(\bar{m})$ and $B'(\bar{m}) = V'(\bar{m})$, and so we can eliminate the qualifications "if $m < \bar{m}$" and "if $m_0 < \bar{m}$" in various parts of Theorems 3.2 and 3.5. It is intuitive that individuals' real balances are finite in the equilibrium. Because the marginal utility of consumption is diminishing, the marginal value of the real balance is diminishing. In contrast, the marginal cost of labor needed to obtain money is strictly increasing. Thus, if an individual has a sufficiently large balance, he strictly prefers spending some money rather to working and accumulating even more money. This force endogenously puts an upper bound on the real balance in the equilibrium.

Although it is intuitive that equilibrium balances are bounded above, proving the result is not simple. In order to demonstrate the optimality of the bounded balance, we need to compare the value function over all possible balances including, in principle, an infinite balance. However, allowing for an infinite balance makes the choice set non-compact which, in turn, makes it difficult to characterize optimal decisions. Instead, we determine the value functions under any arbitrarily fixed and finite upper bound $\bar{m}$, and then vary $\bar{m}$ to prove that individuals never chooses to hold an infinite balance. As the first step, we recognize that a worker's wage income in a period is a finite number $\hat{m} = \ell^*(0) \leq 1$. If the value function V is strictly concave at $\hat{m}$, then an individual with $\hat{m}$ will go to the market as a buyer to spend money rather than work to obtain more money or play a lottery in the next period. In this case, $\hat{m}$ is the endogenous upper bound on equilibrium balances. Similarly, if V is linear at $\hat{m}$ as a result of a lottery and if the high prize of the lottery, $z_2^*(\hat{m})$, is finite, then we can define $\bar{m}$ as the least upper bound above which $z_2^*(\hat{m}) < \bar{m}$. This least upper bound is finite. Thus, the only potential case in which equilibrium balances are unbounded is when $z_2^*(\hat{m})$ is infinite, in which case the lottery at $\hat{m}$ is not well-defined for endogenously determined $\bar{m}$. For this unbounded case to occur, a buyer's value function, $B(m)$, must be strictly increasing and (weakly) convex for all large enough m. In addition, the marginal value of the balance near $\bar{m}$ must be increasing. But these two conditions are inconsistent with the diminishing marginal utility of consumption, because the value of money is derived ultimately from the utility of consumption that money buys. In Appendix D, we formalize these two conditions and the proof.[14]

4. Monetary Equilibrium

In this section we characterize the spending pattern, prove existence and uniqueness of the monetary steady state, and examine the steady-state distribution of real balances.

4.1. Equilibrium pattern of spending

Let us begin with some features of optimal choices established in section 3. First, all workers go to work with zero balance and obtain wage income $\hat{m}$. Second, an individual with $m < m_0$ plays a lottery before going to the market, with the low prize $z_1^*(m) = 0$ and the high prize $z_2^*(m) = m_0$. An individual with $m > m_0$ may or may not play a lottery. Third, buyers sort into different submarkets according to real balances. A buyer with m chooses to enter the submarket where he has a matching probability $b^*(m)$ and, after being matched, he spends a balance $x^*(m)$, buys $q^*(m)$ units of goods, and exits the trade with a residual balance $\phi(m) = m - x^*(m)$. The functions $b^*(m)$, $x^*(m)$, $q^*(m)$ and $\phi(m)$ are all increasing in m.

Denote $\phi^0(m) = m$ and $\phi^{i+1}(m) = \phi(\phi^i(m))$ for $i = 0, 1, 2, ...$ For any arbitrary $m \geq m_0$, let $n(m)$ be the number of purchases that a buyer with m can make before his balance falls below m_0, i.e., $\phi^{n(m)-1}(m) \geq m_0 > \phi^{n(m)}(m)$. Also, denote $\hat{n} = n(\hat{m})$, $\hat{z}_j = z_j^*(\hat{m})$ and $\hat{n}_j = n(\hat{z}_j)$, where $j \in \{1, 2\}$. We prove the following lemma in Appendix E:

[14] The issue on the finite upper bound resolved here is related to, but different from, that in Zhou (1999). In Zhou's paper, money can be accumulated only in discrete units, search is undirected, and there is no lottery. The issue on the upper bound arises there from the feature that an individual may encounter a match in which he is a seller rather than a buyer, even if he already holds a high balance. In contrast, directed search in our model ensures that an individual who intends to buy never chooses to enter a match in which he could end up being a seller. The issue on the upper bound arises from the lottery instead.

Lemma 4.1. *(i) If $\hat{m} < m_0$, then $\hat{z}_1 = 0$, $\hat{z}_2 = m_0$, and $\hat{n}_2 = 1$;*
(ii) The only lottery that is possibly played in the steady state is the lottery at $\hat{m}$, with $\hat{z}_1$ and $\hat{z}_2$ as the prizes, and this lottery is indeed played iff $B(\hat{m}) < V(\hat{m})$;
(iii) If $\hat{m} \geq m_0$, then the following properties hold for $j \in \{1, 2\}$: (a) $b^*(\phi^{i-1}(\hat{z}_j)) > 0$ for all $i = 1, 2, ..., \hat{n}_j$; (b) $\phi^i(\hat{z}_j) \geq m_0$, $V(\phi^i(\hat{z}_j)) = B(\phi^i(\hat{z}_j))$ and V is strictly concave at $\phi^i(\hat{z}_j)$ for all $i = 1, 2, ..., \hat{n}_j - 1$; (c) $\phi^{\hat{n}_j}(\hat{z}_j) = 0$.

Part (i) describes the case where wage income is $\hat{m} < m_0$. In this case, an individual with $\hat{m}$ plays a lottery and, if he wins the prize m_0, he will spend it in one trade. This part is implied by part (iii) of Theorem 3.5. Note that the case $\hat{m} < m_0$ can occur if the disutility of labor, $h(\ell)$, is sufficiently convex. That is, when the marginal disutility increases rapidly with labor supply, it is optimal for a worker to work only a small amount of time in a period, which leads to $\hat{m} < m_0$. However, regardless of how convex h is, a worker does not work for consecutive periods to build up the balance. Instead, after working for one period, an individual tries to get the prize m_0 of the lottery and then go to buy goods. The use of the lottery with the prize m_0 is a better way to smooth the cost of labor than working for consecutive periods.

Part (ii) of Lemma 4.1 states that the only possible lottery played in the steady state is the one at $\hat{m}$. If $\hat{m} < m_0$, the statement is true and a lottery will be played, as explained above. If $\hat{m} \geq m_0$, the statement is implied by part (iii) of Lemma 4.1, which we explain below, and a lottery will be played only when $B(\hat{m}) < V(\hat{m})$.

Part (iii) of Lemma 4.1 describes a buyer's stylized purchasing cycle, starting with the prize $\hat{z}_j$ of the lottery at $\hat{m}$.[15] The buyer will trade with positive probability every period until running out of money (part (a)), the value function will be strictly concave at the residual balance of each trade if this balance is strictly positive (part (b)), and in the last trade in the cycle, he will spend all of his money (part (c)). Because of (b) and (c), the buyer has no need for a lottery at any residual balance resulted from trade. Moreover, as the balance is reduced by each trade, the buyer chooses to go into the next submarket where the trading probability, the required spending and the quantity of goods are all lower than in the previous trade.

This stylized pattern can be derived from repeatedly applying parts (iv) and (v) of Theorem 3.2. Starting at the prize $\hat{z}_j$ of the lottery at $\hat{m} \geq m_0$, we have $B(\hat{z}_j) = V(\hat{z}_j)$, $B'(\hat{z}_j) = V'(\hat{z}_j)$ and $b^*(\hat{z}_j) \geq b^*(m_0) > 0$. If the buyer spends the entire amount $\hat{z}_j$, he will become a worker next period. If the residual balance is $\phi(\hat{z}_j) > 0$, then all the hypotheses in part (v) of Theorem 3.2 are satisfied with $m = \hat{z}_j$. In this case, the ex ante value function is strictly concave at $\phi(\hat{z}_j)$,

[15] We will treat the case where a lottery is not played at $\hat{m} > m_0$ as a degenerate lottery at $\hat{m}$.

at which $B = V$ and no lottery is needed. Then, the hypotheses in part (iv) of Theorem 3.2 are satisfied at $m = \phi(\hat{z}_j)$, which imply that B and V are differentiable at $\phi(\hat{z}_j)$ and their derivatives are equal. Moreover, because V is linear for all $m < m_0$, strict concavity of V at $\phi(\hat{z}_j)$ implies $\phi(\hat{z}_j) \geq m_0$. With the balance $\phi(\hat{z}_j)$, the buyer will have the next trade with strictly positive probability. If the residual balance is $\phi^2(\hat{z}_j) = 0$, the round of purchases ends. If the residual balance is strictly positive, we can repeat the above argument to conclude that, at $m = \phi^2(\hat{z}_j)$, the function V is strictly concave, $B = V$, and the two functions' derivatives are given by (3.10). Moreover, $\phi^2(\hat{z}_j) \geq m_0$, and the buyer has no need for a lottery at $\phi^2(\hat{z}_j)$. This pattern continues until the $\hat{n}_j$-th trade, in which the buyer spends all of the money.

The purchasing cycle above has some similarity to that in Baumol (1952) and Tobin (1956). These authors model the role of money in a reduced form by assuming that money is necessary for purchasing goods. There is a fixed cost of converting other assets, such as nominal bonds, into money, in addition to foregoing the higher return on those assets. Because of the fixed cost, an individual manages cash holdings as a problem of optimal inventory management. That is, after each withdrawal of cash, an individual makes several consecutive purchases until cash holdings fall to a critical level, at which the individual makes another withdrawal. In our model, the value of money is derived endogenously from the fundamental features of preferences (no double coincidence of wants) and the information on individuals trading history (anonymity). Instead of the fixed cost, search frictions behave as an implicit trading cost and make it possibly optimal for an individual to make several consecutive purchases before working to re-stock money balance. Another main feature of our model that is missing from the Baumol-Tobin model and its variations is that a buyer optimally chooses the trading probability and the price at which to spend money in each stage of a purchasing cycle. Intuitively, these choices are important for how monetary policy affects the real activity and the distribution of money.[16]

4.2. Equilibrium distribution of real balances

Let $G(m)$ be the equilibrium measure of individuals holding balances less than or equal to m immediately after the lotteries are played in a period. The support of this distribution is a discrete set. Start with a worker who has just earned wage income $\hat{m}$. If $\hat{m} \geq m_0$, the individual may or may not play a lottery whose prize is either $\hat{z}_1$ or $\hat{z}_2$. After receiving the prize $\hat{z}_j$, an individual will go through a purchasing cycle which results in a sequence of real balances, $\{\phi^i(\hat{z}_j)\}_{i=0}^{\hat{n}_j-1}$. In this case, the support of G is $\{\phi^i(\hat{z}_1)\}_{i=0}^{\hat{n}_1-1} \cup \{\phi^i(\hat{z}_2)\}_{i=0}^{\hat{n}_2-1} \cup \{0\}$. If $\hat{m} < m_0$, the

[16]Although Baumol (1952) and Tobin (1956) assume an exogenous flow of income and an equal amount of spending in each period of a purchasing cycle, these assumptions are eliminated by Jovanovic (1982).

individual with $\hat{m}$ plays a lottery whose prize is either 0 or m_0. In this case, all buyers in the goods market hold the same balance m_0, which will be spent in one trade, and so the support of G is $\{m_0, 0\}$. Denote the corresponding frequency function as g.

It is straightforward to calculate the steady-state distribution of real balances. In the steady state, the measure of individuals who hold each balance in the support of G should be constant over time. If $\hat{m} \geq m_0$ (i.e., $\hat{n}_2 \geq 1$), we can express this requirement as follows:

$$0 = g(0)\,\hat{\pi}_j - b^*(\hat{z}_j)g(\hat{z}_j), \qquad j = 1, 2; \tag{4.1}$$

$$\begin{gathered} 0 = b^*(\phi^{i-1}(\hat{z}_j))g(\phi^{i-1}(\hat{z}_j)) - b^*(\phi^{i}(\hat{z}_j))g(\phi^{i}(\hat{z}_j)) \\ \text{for } 1 \leq i \leq \hat{n}_j - 1 \text{ and } j = 1, 2; \end{gathered} \tag{4.2}$$

$$g(0) = \sum_{j=1,2} b^*(\phi^{\hat{n}_j - 1}(\hat{z}_j))g(\phi^{\hat{n}_j - 1}(\hat{z}_j)), \tag{4.3}$$

where $\hat{\pi}_j = \pi_j^*(\hat{m})$ for $j \in \{1, 2\}$. Equation (4.1) sets the change in the measure of individuals who hold the balance $\hat{z}_j$ to zero. The flow of individuals into the balance $\hat{z}_j$ consists of the workers in the current period who will win the prize $\hat{z}_j$ of the lottery next period. The size of this inflow is $g(0)\hat{\pi}_j$. The outflow of individuals from the balance $\hat{z}_j$ consists of the buyers with the balance $\hat{z}_j$ who successfully trade in the current period. The size of this outflow is $b^*(\hat{z}_j)g(\hat{z}_j)$. Similarly, (4.2) sets the change in the measure of individuals who hold the balance $\phi^i(\hat{z}_j)$ to zero, where $i \in \{1, 2, ..., \hat{n}_j - 1\}$. The inflow of individuals into the balance $\phi^i(\hat{z}_j)$ consists of the buyers with the balance $\phi^{i-1}(\hat{z}_j)$ who successfully trade in the current period, and the outflow consists of the buyers with the balance $\phi^i(\hat{z}_j)$ who successfully trade in the current period. Finally, (4.3) sets the change in the measure of individuals who hold no money to zero. In any period, the individuals who have no money are the workers. Since every worker obtains a balance $\hat{m}$ by working for one period, the size of the outflow from the group is $g(0)$. The inflow comes from the buyers who are in the last period of their purchasing cycle and who successfully trade in the current period, as given by the right-hand side of (4.3).

In the case $\hat{m} > m_0$, (4.1) – (4.3) solve for the steady-state distribution as

$$\left.\begin{aligned} & g(\phi^i(\hat{z}_j)) = \tfrac{g(0)\hat{\pi}_j}{b^*(\phi^i(\hat{z}_j))} \text{ for } j = 1, 2, \text{ and } 0 \leq i \leq \hat{n}_j - 1; \\ & g(0) = \left[1 + \textstyle\sum_{j=1,2} \sum_{i=0}^{\hat{n}_j - 1} \tfrac{\hat{\pi}_j}{b^*(\phi^i(\hat{z}_j))}\right]^{-1}. \end{aligned}\right\} \tag{4.4}$$

The formula (4.4) is also valid in the case $\hat{m} < m_0$. In this case, $\hat{z}_1 = 0$ and $\hat{n}_2 = 1$ in (4.4), and so the steady-state distribution is $g(m_0) = 1 - g(0)$ and $g(0) = b^*(m_0)/[b^*(m_0) + \pi_2^*(m_0)]$.

4.3. Existence and uniqueness of a monetary steady state

In section 3, we have characterized individuals' policy and value functions, which are independent of the nominal wage rate ω. The market tightness function θ is solved by (2.5), which is independent of ω. Moreover, given the policy functions, (4.4) solves the steady-state distribution of real balances independently of ω. Thus, for a monetary steady state to exist, it suffices to solve for ω by requirement (vi) of Definition 2.1. This requirement yields:

$$\omega = \left[\sum_{j=1,2} \sum_{i=0}^{\hat{n}_j - 1} \phi^i(\hat{z}_j) g(\phi^i(\hat{z}_j)) \right]^{-1}. \tag{4.5}$$

Because all of the elements on the right-hand side of (4.5) have been solved independently of ω, the formula determines a unique, finite value of ω in the steady state. We summarize this result and other properties of the steady state below (see Appendix F for a proof):

Theorem 4.2. *A unique monetary steady state exists and is block recursive. Money is neutral in the steady state. The distribution of buyers over real balances is degenerate if $\beta \leq \beta_0$, where $\beta_0 > 0$ is defined in Appendix F. On the other hand, if β is sufficiently close to one, the distribution of buyers over real balances is non-degenerate if and only if*

$$m_c > m_0(m_c) = q_0(m_c) + \frac{k}{\mu(b_0(m_c))}, \tag{4.6}$$

where m_c, $q_0(m)$, and $b_0(m)$ are defined in Appendix F. Moreover, for each $j = 1, 2$, the distribution satisfies $g(\phi^i(\hat{z}_j)) > g(\phi^{i-1}(\hat{z}_j))$ for all $i = 1, 2, ..., \hat{n}_j - 1$.

Money is neutral in the long run; i.e., a one-time change in the nominal stock of money has no effect on real variables in the steady state. This is intuitive in our model. The real balance, m, the quantity of money traded in a match, x^*, and the residual balance after a trade, ϕ, are all measured in units of labor. They are given by the policy functions that are independent of the nominal stock. Similarly, a one-time change in the nominal stock of money does not affect the quantity of goods traded, labor supply and the distribution of individuals. However, money is not neutral in the short run in our model, as we will remark in section 5.

The distribution of buyers over real balances may or may not be degenerate in the steady state. The distribution is degenerate if $\hat{m} \leq m_0$, which occurs when individuals are sufficiently impatient in the sense $\beta \leq \beta_0$. In this case, an individual without money works for one period to obtain the balance $\hat{m}$ and then plays the lottery whose prize is m_0. Thus, all buyers hold the same balance, m_0, and spend all the money whenever they have a match. An individual alternates stochastically between being a buyer with a balance m_0 and a worker with no money, as determined by the lottery and the matching outcome. Thus, when $\beta \leq \beta_0$, our model endogenously generates the patterns of money holdings that were assumed in earlier models with indivisible money (e.g., Shi, 1995, and Trejos and Wright, 1995). Even in this case, however, our model does not share the result of those models that a one-time change in the money stock affects real activities. Instead, money is neutral in the steady state here.

It is intuitive that money distribution among buyers is degenerate when individuals are sufficiently impatient.[17] Consider a buyer with the highest equilibrium balance, $\hat{z}_2$. The buyer can spend this balance in one trade or spread it over several periods in a sequence of purchases. If the buyer spends the entire balance in one trade, he consumes a large amount of goods in the period. The upside of doing so is that the utility of current consumption is not discounted. The downside is that the marginal utility of large consumption is low, relative to spreading consumption over several periods. When the buyer is sufficiently impatient, the upside of spending all the money at once outweighs the downside. In fact, if $\beta \leq \beta_0$, the highest equilibrium balance is $\hat{z}_2 = m_0$. In this case, all buyers in the market hold the same balance m_0.

For money distribution to be non-degenerate among buyers, a necessary condition is that individuals are patient. However, high patience is not sufficient. Even in the limit $\beta \to 1$, it is still possible that optimal labor supply is so low that wage income is $\hat{m} \leq m_0$, in which case all buyers hold the same balance m_0. The additional condition, (4.6), is needed for $\hat{m} > m_0$. In the limit $\beta \to 1$, this condition is also sufficient for the distribution to be non-degenerate. Unfortunately, the condition (4.6) is too complicated to be expressed in terms of model parameters, because m_c, $q_0(m)$ and $b_0(m)$ are defined implicitly through some equations (see Appendix F). To illustrate the elements involved, consider:

Example: $U(q) = \frac{(q+0.1)^{1-\sigma}-(0.1)^{1-\sigma}}{1-\sigma}$, $h(\ell) = 10[1-(1-\ell)^{\eta}]$, and $\mu(b) = 1-b$. A higher value of σ corresponds to a more concave U, and a higher value of η (< 1) to a less convex h. For any given (σ, η), we denote $K(\sigma, \eta)$ as the critical level of k such that (4.6) is satisfied if and only if $k < K(\sigma, \eta)$. Figure 2.1 depicts $K(\sigma, 0.5)$ for $\sigma \in [1.1, 3]$ and Figure 2.2 depicts $K(2, \eta)$ for $\eta \in [0.1, 0.9]$. The function $K(\sigma, 0.5)$ is increasing in σ. This means that (4.6) is more easily satisfied for any given (k, η) when the utility function of consumption is more concave. Also, $K(2, \eta)$ is increasing in η. This means that (4.6) is more easily satisfied for any given (k, η) when the disutility function of labor supply is less convex.

[17] The critical level β_0 depends on other parameters of the model. In particular, β_0 increases in the degree of convexity of the disutility function of labor supply. Thus, consistent with an earlier explanation, the case $\hat{m} < m_0$ is more likely to occur if the disutility function of labor supply is more convex.

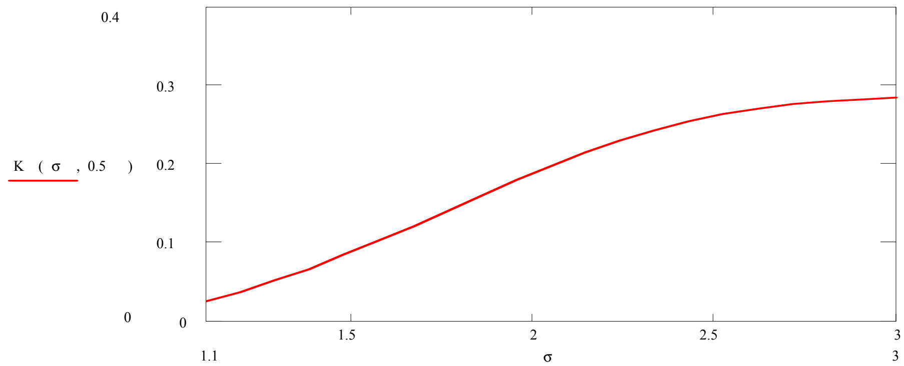

Figure 2.1

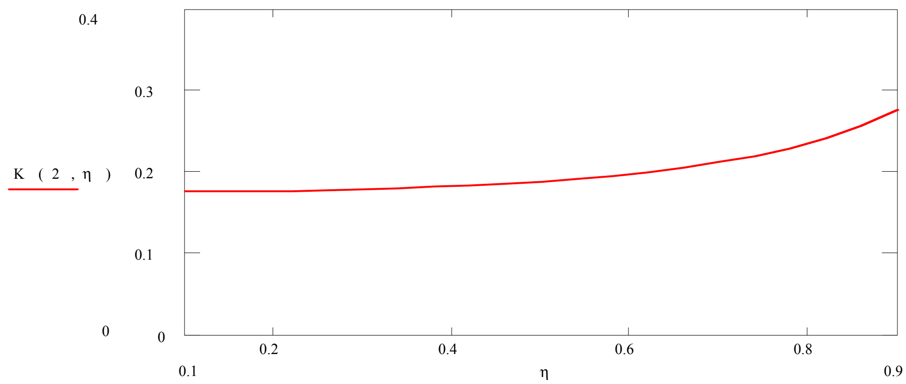

Figure 2.2

The above example illustrates the elements, in addition to high patience, that make it optimal for a buyer to spread purchases in several periods. First, the cost of creating a trading post cannot be too high. If a trading post is very costly to create, the number of trading posts in each submarket is small in the equilibrium and, hence, the matching probability is low for a buyer. Then it is optimal to spend all the money in one trade because, if the buyer keeps a positive residual balance, he will find it difficult to get a match in the future to spend the money. Second, the utility function of consumption needs to be sufficiently concave. Intuitively, a more concave utility function increases a buyer's incentive to smooth consumption over time by making a sequence of relatively small purchases rather than one large purchase. Third, the disutility of labor supply cannot be very convex. If the marginal cost of labor supply increases very quickly, the optimal choice is to work for a relatively small balance in a period, play the lottery, spend all the prize in one trade, and work again.[18]

[18]Another element for a non-degenerate distribution is that $\mu(b)$ should not increase very quickly with b. If $\mu(b)$ increases very quickly with b, the amount of money required for obtaining any given quantity of goods increases quickly with b. In this case, the benefit of acquiring a large balance and going through a sequence of purchases is small relative to the cost of labor supply, and so a buyer will make only one purchase before working again.

Theorem 4.2 also describes the shape of the steady-state distribution. First, consider the case where an individual with the balance $\hat{m}$ has no need for a lottery. In this case, the frequency function of the distribution, g, is strictly decreasing in real balances among buyers. This is an intuitive implication of buyers' optimal choices described in Theorem 3.2. Because buyers with more money choose to trade with a relatively high probability, they exit quickly from the high balance into a lower balance and, hence, a relatively small number of buyers are left holding a high balance in the steady state. The measure of buyers increases as their real balances strictly decrease in the purchasing cycle. Next, consider the case where an individual with the balance $\hat{m}$ has the need for a lottery. From each prize of the lottery, $\hat{z}_j$ $(j = 1, 2)$, a buyer's balance in a purchasing cycle follows the sequence $\{\phi^i(\hat{z}_j)\}_{i=0}^{\hat{n}_j-1}$. The above feature of the distribution holds true for each of these two sequences. That is, for each $j \in \{1, 2\}$, the measure of buyers holding $\phi^i(\hat{z}_j)$ increases with i and, hence, decreases with $\phi^i(\hat{z}_j)$ for all $i \in \{0, 1, ..., \hat{n}_j - 1\}$. However, with a non-degenerate lottery at $\hat{m}$, the overall frequency function of real balances is not necessarily monotone. For example, $g(\phi^i(\hat{z}_1))$ may be greater than, less than or equal to $g(\phi^i(\hat{z}_2))$ for a particular i, and the comparison between the two may vary over i.

A non-degenerate distribution of real balances has a wealth effect in the sense that a transfer of money between two sets of buyers with different balances affects the sum of the values of these buyers. Recall that a buyer's marginal value of the real balance increases strictly as the balance decreases with each purchase. That is, $V'(\phi^i(\hat{z}_j)) > V'(\phi^{i-1}(\hat{z}_j))$ for all $i = 0, 1, ..., \hat{n}_j$ and $j = 1, 2$ (see part (v) of Theorem 3.2). A transfer of money from a buyer with a relatively high balance to a buyer with a relatively low balance reduces the gap between the two buyers' marginal values of money. This transfer increases the sum of the values of the two buyers.

Let us relate our work to GZ. In a model of undirected search, GZ allow money to be accumulated in discrete units and assume that goods come as a fixed endowment to a seller. They prove that there exist monetary steady states where the support of money distribution is discrete and the density is decreasing. In our model, the support of money distribution is also discrete in the equilibrium and, when no lottery is played in the equilibrium, the density of the distribution is decreasing. The intuition for the decreasing density is similar in the two models, but the mechanisms that generate the feature differ. In our model, the density is decreasing because a buyer with more money optimally chooses to visit a submarket where he has a higher probability of trade. As buyers with more money trade more quickly, fewer buyers are left holding high balances than low balances in the steady state. In GZ, search is undirected, which does not allow individuals to choose which market to visit. Nevertheless, a buyer with more money is more likely to accept a seller's offer and, hence, to trade more quickly. There are other differences between our model and GZ. First, there is a continuum of monetary steady states in GZ, called single-price equilibria, which differ from each other in the price level and real allocation. This multiplicity depends on a combination of assumptions in GZ, including the assumption that a seller cannot supply more than the unit endowment of the good. Our analysis shows that the steady state is unique when individuals can trade whatever quantities of money and goods that are optimal for them. Second, an individual in GZ does not go through a purchasing cycle. Instead, in any period, an individual can be either a buyer or a seller with positive probability, depending on the outcome of random matching. Finally, because GZ assume undirected search, an extension of their model to eliminate the assumptions of indivisibility and fixed endowment of goods will not be block recursive and, hence, will be intractable analytically.

5. Concluding Remarks

In this paper, we have constructed and analyzed a tractable search model of money where the distribution of real money balances can be non-degenerate. Search is directed in the sense that buyers know the terms of trade before visiting particular sellers. We showed that the monetary steady state is block recursive in the sense that individuals' policy functions, value functions and the market tightness function are all independent of the distribution of individuals over real balances, although the distribution affects the aggregate activity by itself. Using lattice-theoretic techniques, we characterized individuals' policy and value functions, and showed that these functions satisfy the standard conditions of optimization. We proved that a unique monetary steady state exists, provided conditions under which the steady-state distribution of buyers over real balances is non-degenerate, and analyzed the properties of this distribution.

We hope that our model provides a new starting point for studying monetary policy. Although the monetary steady state is block recursive, the non-degenerate distribution of money does matter for the aggregate real activity and welfare. This distribution serves as a channel through which monetary policy affects the real activity. In particular, a one-time change in the money stock can have persistent real effects in the short run, despite that this policy is neutral in the long run. To see the short-run effects, consider a one-time increase in the lump-sum transfer of money to the buyers. This transfer represents a larger proportional increase in the balance to a buyer with a low balance than to a buyer with a high balance. As a result, the transfer changes the support of the distribution of real balances in the short run, which changes aggregate real consumption and output. Moreover, because of this response of the distribution of real balances, the nominal wage rate (normalized by the money stock), ω, deviates immediately from the steady state. As ω adjusts back to the steady state during the transition, the rate of return to money, which can be expressed as ω_t/ω_{t+1}, is non-constant over time. It can be verified that individuals' policy and value functions depend on this rate of return to money, and so the one-time monetary transfer also affects individuals' optimal choices in the short run. These real effects are persistent if the distribution of real balances or ω_t/ω_{t+1} returns to the steady state slowly. Thus, our model has the potential to explain why temporary monetary shock have persistent real effects as alluded to in the introduction. Furthermore, these real effects depend on how and where money is injected. For example, if the one-time injection increases all buyers' balances in the same proportion, rather than in a lump-sum, then it has no effect on ω or the distribution.[19]

The steady-state analysis in this paper provides a long-run anchor for studying the dynamic effects of monetary policy described above. As alluded to above, the equilibrium is not block recursive in the transition to the steady state, because individuals' policy and value functions depend on the distribution of money through the rate of return to money, ω_t/ω_{t+1}. Despite this non-block recursivity in the short run, it is much simpler to compute the dynamics in our model than in models of undirected search. The reason is that the distribution of money in our model does not affect individuals' policy and value functions directly, but rather exclusively through the rate of return to money, ω_t/ω_{t+1}. Since ω is a one-dimensional object whose reciprocal is equal to the mean of real balances, the dynamic equilibrium can be solved using the approximation technique of Krusell and Smith (1998), i.e., by approximating the distribution with a small number of moments. In contrast, in undirected search models including Molico (2006) and Chiu and Molico (2008), the entire distribution of money affects individuals' policy and value functions directly. In that case, there is no intuitive reason why the Krusell-Smith approximation can work well, and the computation suffers from the well-known problem of large dimensionality. We will undertake the dynamic analysis in future work, which will also incorporate other elements relevant for policy analysis, such as nominal bonds and aggregate shocks.

[19] The model can also be used to study a permanent change in the money growth rate, which is not neutral in the long run. Similar to Molico (2006) and Chiu and Molico (2008), part of the non-neutrality in the long run comes from the effect that a permanent change in money growth changes the distribution of real balances in the steady state by redistributing the purchasing power between individuals with different balances.

Appendix

A. Proof of Lemma 3.1

Take any $V \in \mathcal{V}[0, \bar{m}]$ as the ex ante value function appearing in a worker's maximization problem, (2.1). The objective function in (2.1) is continuous, bounded on $[0, \bar{m}]$ and increasing in m. Then, the Theorem of the Maximum implies $W \in \mathcal{C}[0, \bar{m}]$, i.e., a continuous and increasing function on $[0, \bar{m}]$. Because the objective function $[\beta V(m + \ell) - h(\ell)]$ is strictly concave in (ℓ, m) jointly, its maximized value, $W(m)$, is concave in m, and the optimal choice ℓ^* is unique. With uniqueness, the Theorem of the Maximum implies that the policy function $\ell^*(m)$ is continuous (see Stokey et al., 1989, p62). The choice $\ell = 1$ can never be optimal under the assumption $h'(1) = \infty$. It may be possible that the optimal choice is $\ell^*(m) = 0$ when m is sufficiently high. In this case, it is evident that $\ell^*(m) = 0$ is (weakly) increasing in m and $y^*(m) = m$ is strictly increasing in m.

The remainder of this proof focuses on the case where $\ell^*(m) > 0$. In this case, $y^*(m) = m + \ell^*(m) > m$. Reformulate a worker's problem as (3.2), where the choice is the end-of-period balance $y = m+\ell$. The objective function in (3.2) is strictly concave in (y, m) jointly and $h(y-m)$ is continuously differentiable in (y, m). Thus, the result in Benveniste and Scheinkman (1979) applies (see also Stokey et al., 1989, p85). That is, for all m such that the optimal choice $y^*(m)$ is interior, $W(m)$ is differentiable and the derivative satisfies:

$$W'(m) = h'(y^*(m) - m) = h'(\ell^*(m)).$$

In addition, using concavity of W and strict convexity of h, we can deduce from the equation $W'(m) = h'(\ell^*(m))$ that $\ell^*(m)$ is decreasing in m.

Return to the original maximization problem of a worker, (2.1). Consider any $m \in [0, \bar{m}]$ such that $\ell^*(m) > 0$. Because $\ell^*(m)$ is continuous, there exists $\varepsilon_0 > 0$ such that $\ell^*(m \pm \varepsilon) > 0$ for all $\varepsilon \in [0, \varepsilon_0]$. Moreover, we can choose sufficiently small ε_0 so that for any $\varepsilon \in [0, \varepsilon_0]$, the choice $\ell^*(m - \varepsilon)$ is feasible to a worker who holds a balance m and the choice $\ell^*(m)$ is feasible to a worker who holds a balance $m - \varepsilon$. Then, for any $\varepsilon \in [0, \varepsilon_0]$, the optimality of ℓ^* implies:

$$\begin{aligned} W(m) &= F(\ell^*(m), m) \geq F(\ell^*(m - \varepsilon), m), \\ W(m - \varepsilon) &= F(\ell^*(m - \varepsilon), m - \varepsilon) \geq F(\ell^*(m), m - \varepsilon), \end{aligned}$$

where $F(\ell, m)$ temporarily denotes the objective function in (2.1). Hence,

$$\frac{F(\ell^*(m - \varepsilon), m) - F(\ell^*(m - \varepsilon), m - \varepsilon)}{\varepsilon} \leq \frac{W(m) - W(m - \varepsilon)}{\varepsilon} \leq \frac{F(\ell^*(m), m) - F(\ell^*(m), m - \varepsilon)}{\varepsilon}.$$

Since $W'(m)$ exists, taking the limit $\varepsilon \searrow 0$ on the above relations yields $\beta V'(y^{*-}(m)) = W'(m)$, where $y^{*-}(m) = m^- + \ell^*(m)$. Note that the one-sided derivatives $V'(y^{*-})$ and $V'(y^{*+})$ exist because V is a concave function (see Royden, 1988, pp113-114). Similarly, we can prove that $\beta V'(y^{*+}(m)) = W'(m)$, where $y^{*+}(m) = m^+ + \ell^*(m)$. Therefore, V is differentiable at $y^*(m)$ and the derivative satisfies $\beta V'(y^*(m)) = W'(m)$, which is the first equality in (3.1). Substituting $W'(m) = h'(\ell^*(m))$ yields the second equality in (3.1).

Finally, since V' is decreasing and h' is strictly increasing, the first-order condition $\beta V'(y^*(m)) = h'(y^*(m) - m)$ implies that $y^*(m)$ is strictly increasing. **QED**

B. Proof of Lemma 3.3

Take any $V \in \mathcal{V}[0, \bar{m}]$ as the ex ante value function appearing in a buyer's maximization problem, (2.2). Applying the Theorem of the Maximum to the problem, we conclude that B is continuous on $[0, \bar{m}]$ and that a solution to the buyer's maximization problem exists (see Stokey et al., 1989, p62). Since V is increasing, the objective function in (2.2) is increasing in m. Since the feasibility set in the maximization problem is also increasing in m, then B is increasing, i.e., $B \in \mathcal{C}[0, \bar{m}]$. As explained earlier in subsection 3.2.1, B has two segments. If $m \leq k$, then $b^*(m) = 0$ and $B(m) = \beta V(m)$; if $m > k$, $B(m)$ solves (3.6).

If $b^* = 0$, the choice of x is irrelevant for the buyer because a trade does not take place. For the remainder of the proof, we focus on the case where $b^*(m) > 0$. Temporarily denote $F(x, b, m) = b\, R(x, b, m)$, where the surplus function R is defined in (3.13). Optimal choices (x^*, b^*) maximize $F(x, b, m)$.

(1) A buyer's optimal choices are unique and the policy functions are continuous.

If $R(x, b^*, m) < 0$, the optimal choice is $b^* = 0$; if $R(x, b^*, m) = 0$, then the choice $b^* = 0$ is not dominated by other choices of b. Because we focus on $b^* > 0$, it suffices to examine a buyer's optimal choices when $R(x, b, m) > 0$. With $b > 0$ and $R(x, b, m) > 0$, we can transform a buyer's maximization problem as

$$B(m) = \beta V(m) + \exp\left\{ \max_{x,b} \left[\ln b + \ln R(x, b, m) \right] \right\}.$$

The function $(\ln b)$ is concave. Recall that $u(x, b)$ is strictly concave in (x, b) jointly. Since V is concave, then $V(m - x)$ is concave in x. Thus, $R(x, b, m)$ defined in (3.13) is strictly concave in (x, b) jointly. Since the logarithmic function is strictly increasing and strictly concave, the function $[\ln b + \ln R(x, b, m)]$ is strictly concave in (x, b) jointly. The Theorem of the Maximum implies that a buyer's optimal choices (x^*, b^*) are unique for each m and the policy functions $(x^*(m), b^*(m))$ are continuous. So are the policy functions $q^*(m)$ and $\phi(m)$.

(2) Monotonicity of the policy functions $x^*(m)$ and $b^*(m)$.

Consider any $m \in [k, \bar{m}]$ such that $b^*(m) > 0$. As discussed in the main text, we solve a buyer's maximization problem in two steps: first, the optimal choice of x solves the problem in (3.14) for any given (b, m); second, the optimal choice of b solves the problem in (3.15).

Take the first step. For any given (b, m), the optimal choice of x maximizes $R(x, b, m)$ and is denoted $\tilde{x}(b, m)$ as in (3.14). Because $u(x, b)$ is strictly concave in x and V is concave, $R(x, b, m)$ is strictly concave in x, which implies that a unique $\tilde{x}$ exists for any given (b, m). We prove that $R(x, b, m)$ is supermodular. Since the choice set of x, $[0, m]$, is increasing in m and independent of b, supermodularity of $R(x, b, m)$ and uniqueness of $\tilde{x}$ imply that the maximizer $\tilde{x}(b, m)$ is an increasing function of (b, m) (see Topkis, 1998, p76) and that the maximized value of R is supermodular in (b, m) (see Topkis, 1998, p70).

To prove that $R(x, b, m)$ is supermodular, note that the feasibility set of (x, b, m) is $\{(x, b, m) : 0 \le x \le m,\ 0 \le b \le 1,\ k \le m \le \bar{m}\}$. This set is a sublattice in $\mathbb{R}^3_+$ with the usual relation "$\ge$". It suffices to prove that R has increasing differences in the three pairs, (b, m), (x, b) and (x, m) (see Topkis, 1998, p45). Take arbitrary m_1, m_2, x_1, x_2, b_1 and b_2 from the feasibility set, with $m_2 > m_1$, $x_2 > x_1$, and $b_2 > b_1$. Because R is separable in b and m, it is clear that R has (weakly) increasing differences in (b, m). For the differences in (x, b), compute:

$$R(x_2, b, m) - R(x_1, b, m) = [u(x_2, b) - u(x_1, b)] + \beta [V(m - x_2) - V(m - x_1)].$$

Since $u(x, b)$ is strictly supermodular in (x, b), we have:

$$\begin{aligned} & [R(x_2, b_2, m) - R(x_1, b_2, m)] - [R(x_2, b_1, m) - R(x_1, b_1, m)] \\ = \ & [u(x_2, b_2) - u(x_1, b_2)] - [u(x_2, b_1) - u(x_1, b_1)] > 0. \end{aligned}$$

That is, R has strictly increasing differences in (x, b). For the differences in (x, m), we have:

$$\begin{aligned} & [R(x_2, b, m_2) - R(x_1, b, m_2)] - [R(x_2, b, m_1) - R(x_1, b, m_1)] \\ = \ & \beta [V(m_1 - x_1) - V(m_1 - x_2)] - \beta [V(m_2 - x_1) - V(m_2 - x_2)] \ge 0. \end{aligned}$$

The inequality follows from concavity of V (see Royden, 1988, p113) and the facts that $m_1 - x_1 < m_2 - x_1$, $m_1 - x_2 < m_2 - x_2$, and $(m_1 - x_1) - (m_1 - x_2) = (m_2 - x_1) - (m_2 - x_2) = x_2 - x_1 > 0$. Thus, $R(x, b, m)$ has increasing differences in (x, m).

Denote $\tilde{R}(b, m) = R(\tilde{x}(b, m), b, m)$ as in (3.14). From the above proof, $\tilde{R}(b, m)$ is supermodular in (b, m). Because $R(x, b, m)$ strictly decreases in b for any given (x, m), then $\tilde{R}(b, m)$ is strictly decreasing in b. To examine the dependence of $\tilde{R}(b, m)$ on m, take arbitrary m_1 and m_2 in $[k, \bar{m}]$, with $m_2 \ge m_1$. We have:

$$R(x, b, m_2) - R(x, b, m_1) = \beta [V(m_1) - V(m_1 - x)] - \beta [V(m_2) - V(m_2 - x)] \ge 0,$$

where the inequality follows from concavity of V. Since the above result holds for all (x, b), then

$$\tilde{R}(b, m_1) = R(\tilde{x}(b, m_1), b, m_1) \le R(\tilde{x}(b, m_1), b, m_2) \le R(\tilde{x}(b, m_2), b, m_2) = \tilde{R}(b, m_2).$$

Note that for the second inequality we have used the fact that $\tilde{x}(b, m_1)$ is feasible in the problem $\max_{x \leq m_2} R(x, b, m_2)$. Thus, $\tilde{R}(b, m)$ increases in m.

Now let us take the second step, i.e., to characterize the optimal choice of b. Denote the optimal choice of b as $b^*(m) = \arg\max_{b \in [0,1]} f(b, m)$, where

$$f(b, m) = F(\tilde{x}(b, m), b, m) = b\tilde{R}(b, m).$$

We show that f is supermodular in (b, m). Take arbitrary $b_1, b_2 \in [0, 1]$, with $b_2 > b_1$, and arbitrary $m_1, m_2 \in [k, \bar{m}]$, with $m_2 > m_1$. Compute:

$$\begin{aligned} & [f(b_2, m_2) - f(b_1, m_2)] - [f(b_2, m_1) - f(b_1, m_1)] \\ = \;& b_2 \left[\tilde{R}(b_2, m_2) - \tilde{R}(b_1, m_2) + \tilde{R}(b_1, m_1) - \tilde{R}(b_2, m_1)\right] \\ & + (b_2 - b_1)\left[\tilde{R}(b_1, m_2) - \tilde{R}(b_1, m_1)\right]. \end{aligned}$$

Because $\tilde{R}(b, m)$ is supermodular in (b, m), the first difference on the right-hand side is positive. Because $\tilde{R}(b, m)$ is increasing in m, the second difference on the right-hand side is also positive. Thus, $f(b, m)$ is supermodular in (b, m) on $[0, 1] \times [k, \bar{m}]$. Note also that the choice set for b, $[0, 1]$, is independent of m and that the optimal choice b^* is unique. Thus, $b^*(m)$ is increasing in m (see Topkis, 1998, p76). Since $\tilde{x}(b, m)$ is increasing in (b, m), the optimal choice of x, given by $x^*(m) = \tilde{x}(b^*(m), m)$, is increasing in m.

(3) $q^*(m)$ is an increasing function.

Denote $a = m - x + q$ and use (q, a) as a buyer's choices. Using (3.4), we can express:

$$m - x = a - q, \quad b = \mu^{-1}\left(\frac{k}{m - a}\right).$$

Because $b \geq 0$, the relevant domain of a is $[0, m - k]$. The relevant domain of q is $[0, a]$. A buyer chooses $(q, a) \in [0, a] \times [0, m - k]$ to solve:

$$\max_{(q,a)} \mu^{-1}\left(\frac{k}{m - a}\right)[U(q) + \beta V(a - q) - \beta V(m)].$$

We can divide this problem into two steps: first solve q for any given (a, m) and then solve a.

For any given (a, m), the optimal choice of q, denoted as $\tilde{q}(a)$, solves:

$$J(a) \equiv \max_{0 \leq q \leq a} [U(q) + \beta V(a - q)].$$

Note that q and J do not depend on m for any given a. It is easy to see that the objective function above is supermodular in (q, a). Since the choice set, $[0, a]$, is increasing in a and $\tilde{q}$ is unique, then $\tilde{q}(a)$ and $J(a)$ increase in a (see Topkis, 1998, p76 and p70).

The optimal choice of a is $a^*(m) = \arg\max_{0\le a\le m-k} \Delta(a,m)$, where

$$\Delta(a,m) = \mu^{-1}\left(\frac{k}{m-a}\right)[J(a) - \beta V(m)].$$

Note that if $J(a) < \beta V(m)$, the buyer can choose $a = m - k$ to obtain $\Delta = 0$. Thus, focus on the case where $J(a) \ge \beta V(m)$. Since $\mu(b)$ is strictly decreasing in b and $1/\mu(b)$ is strictly convex in b, the function $\mu^{-1}(\frac{k}{m-a})$ strictly increases in m, strictly decreases in a, and is strictly supermodular in (a,m). Thus, for arbitrary $a_2 > a_1$ and $m_2 > m_1 \ge k$, we have:

$$\begin{aligned}
& \Delta(a_2,m_2) - \Delta(a_1,m_2) - \Delta(a_2,m_1) + \Delta(a_1,m_1) \\
= & \left[\mu^{-1}\left(\tfrac{k}{m_2-a_2}\right) - \mu^{-1}\left(\tfrac{k}{m_1-a_2}\right)\right][J(a_2) - J(a_1)] \\
& + \left[\mu^{-1}\left(\tfrac{k}{m_2-a_1}\right) - \mu^{-1}\left(\tfrac{k}{m_2-a_2}\right)\right][\beta V(m_2) - \beta V(m_1)] \\
& + \left[\mu^{-1}\left(\tfrac{k}{m_2-a_2}\right) - \mu^{-1}\left(\tfrac{k}{m_2-a_1}\right) - \mu^{-1}\left(\tfrac{k}{m_1-a_2}\right) + \mu^{-1}\left(\tfrac{k}{m_1-a_1}\right)\right][J(a_1) - \beta V(m_1)].
\end{aligned}$$

The first term on the right-hand side is positive because $J(a)$ increases in a and $\mu^{-1}(\frac{k}{m-a})$ increases in m. The second term on the right-hand side is positive because $\mu^{-1}(\frac{k}{m-a})$ decreases in a and $V(m)$ increases in m. The third term on the right-hand side is strictly positive because $\mu^{-1}(\frac{k}{m-a})$ is strictly supermodular in (a,m). Therefore, $\Delta(a,m)$ is strictly supermodular. Since the choice set $[0, m-k]$ is also increasing in m, the solution $a^*(m)$ increases in m (see Topkis, 1998, p76). Since $\tilde{q}(a)$ increases in a, then $q^*(m) = \tilde{q}(a^*(m))$ increases in m.

(4) $\phi(m)$ is an increasing function.

We reformulate a buyer's problem by letting the choices be (ϕ, a), where a is defined as $a = \phi + q$. From the definition of a and (3.4), we can express

$$q = a - \phi, \quad b = \mu^{-1}\left(\frac{k}{m-a}\right).$$

The relevant domain of ϕ is $[0, \min\{m, a\}]$, and of a is $[0, m-k]$. A buyer solves:

$$\max_{(\phi,a)} \mu^{-1}\left(\frac{k}{m-a}\right)[U(a-\phi) + \beta V(\phi) - \beta V(m)]. \tag{B.1}$$

As in the above formulation where the choices are (q, a), we can divide the maximization problem into two steps. First, for any given a, the optimal choice of ϕ solves:

$$J(a) = \max_{\phi\ge 0}[U(a-\phi) + \beta V(\phi)]. \tag{B.2}$$

Note that we have written the constraint on ϕ as $\phi \ge 0$, instead of $\phi \in [0, \min\{m, a\}]$. The optimal choice satisfies $\phi < m$, because $\phi = m$ implies $x = 0$ which is not optimal (in the case with $b > 0$). Also, $\phi < a$ under the assumption that $U'(0)$ is sufficiently large. Denote the solution for ϕ as $\tilde{\phi}(a)$. Second, the optimal choice of a solves

$$B(m) - \beta V(m) = \max_{0 \le a \le m-k} \mu^{-1}\left(\frac{k}{m-a}\right)[J(a) - \beta V(m)]. \tag{B.3}$$

Similar to the procedure used in the above formulation of the problem where the choices are (q, a), we can prove that $a^*(m)$ increases in m and, hence, $\phi^*(m)$ increases in m. **QED**

C. Proof of Lemma 3.4

Take any $V \in \mathcal{V}[0, \bar{m}]$ as the ex ante value function appearing in a buyer's problem and consider any arbitrary $m \in [k, \bar{m}]$ such that $b^*(m) > 0$. Parts (1) - (4) below establish Lemma 3.4.

(1) The one-sided derivatives of B satisfy:

$$B'\left(m^+\right) = b^*(m)u_1\left(x^*(m), b^*(m)\right) + \beta\left(1 - b^*(m)\right)V'\left(m^+\right) \tag{C.1}$$

$$B'\left(m^-\right) = b^*(m)u_1\left(x^*(m), b^*(m)\right) + \beta\left(1 - b^*(m)\right)V'\left(m^-\right). \tag{C.2}$$

$B'(m)$ exists if and only if $V'(m)$ exists. Moreover, $B(m)$ is strictly increasing.

Consider the formulation of a buyer's problem, (B.1), where the choices are ϕ and $a = \phi + q$. Let a and a' be arbitrary levels in $[0, m-k]$. Note that the constraint on the choice ϕ is $\phi \ge 0$, which does not depend on a. Thus, the choice $\tilde{\phi}(a)$ is feasible in the maximization problem with a' and the choice $\tilde{\phi}(a')$ is feasible in the maximization problem with a. Using a proof similar to the one in Appendix A that established the existence of $V'(y^*(m))$, we can prove that $J'(a^-)$ and $J'(a^+)$ both exist and are equal to

$$J'(a) = U'(\tilde{q}(a)) > 0, \tag{C.3}$$

where $\tilde{\phi}$ and $\tilde{q}(a) \equiv a - \tilde{\phi}(a)$ are given in part (4) of the above proof of Lemma 3.3.

Next, we prove that the objective function in (B.3) is strictly concave in a and derive the first-order condition of a. Recall that $\tilde{q}(a)$ is an increasing function, as shown in the above proof of Lemma 3.3. This result and (C.3) together imply that $J'(a)$ is decreasing, i.e., that $J(a)$ is concave. Because $J(a)$ is increasing and concave, and $\mu^{-1}(\frac{k}{m-a})$ is strictly decreasing and strictly concave in a, it can be verified that the objective function in (B.3) is strictly concave in a. Strict concavity of the objective function implies that the optimal choice of a is unique. Also, because the objective function is differentiable in a, the optimal choice of a is given by the first-order condition. Deriving the first-order condition, substituting $J'(a)$ from (C.3), and substituting $\mu^{-1}(\frac{k}{m-a^*}) = b^*(m)$, we obtain:

$$J(a^*) - \beta V(m) + U'(\tilde{q}(a^*)) \frac{k\mu' b^*(m)}{\mu^2} \leq 0 \quad \text{and} \quad a^* \leq m - k, \tag{C.4}$$

where the two inequalities hold with complementary slackness.

Now we derive (C.1) and (C.2), which clearly imply that $B'(m)$ exists if and only if $V'(m)$ exists. Note that $b^*(m) > 0$ implies $a^*(m) < m - k$. Because $a^*(m) < m - k$ and $a^*(m)$ is continuous, there exists $\varepsilon > 0$ such that $a^*(m + \varepsilon) < m - k$ and $a^*(m) < m - \varepsilon - k$. Consider the neighborhood $O(m) = (m - \varepsilon, m + \varepsilon)$. For any $m' \in O(m)$, the choice $a^*(m')$ is feasible in the problem where the balance is m, and the choice $a^*(m)$ is feasible in the problem where the balance is m'. Applying to (B.3) a proof similar to Appendix A that established the existence of $V'(y^*(m))$, we can derive the formulas of $B'(m^+)$ and $B'(m^-)$ for any m such that $b^*(m) > 0$. These formulas and the first-order condition of a^*, (C.4), together yield:

$$\begin{aligned} B'(m^+) &= b^*(m) \left[J'(a^*) - \beta V'(m^+) \right] + \beta V'(m^+), \\ B'(m^-) &= b^*(m) \left[J'(a^*) - \beta V'(m^-) \right] + \beta V'(m^-). \end{aligned}$$

Again, we have used the fact that a concave function has one-sided derivatives. Substituting $J'(a^*)$ from (C.3) and $U'(q^*) = u_1(x^*, b^*)$ into the above equations, we obtain (C.1) and (C.2).

Finally, we prove that B is strictly increasing. Since V is concave and increasing, $V'(m^-) \geq V'(m^+) \geq 0$. Since $b^* \leq 1$ and $J'(a^*(m)) > 0$, the above equations for $B'(m^+)$ and $B'(m^-)$ imply that $B'(m^-) \geq B'(m^+) \geq b^*(m) J'(a^*) > 0$, where we have used the hypothesis $b^*(m) > 0$. Therefore, $B(m)$ is strictly increasing if $b^*(m) > 0$.

(2) The optimal choice b^* satisfies the first-order condition (3.8). If $\phi(m) > 0$, then $V'(\phi(m))$ exists, and the optimal choice x^*satisfies the first-order condition (3.9).

For any given (x, m), the objective function in a buyer's problem (3.6) is differentiable with respect to b. Thus, if the optimal choice b^* is interior, it satisfies the first-order condition (3.8). Now consider the optimal choice x^* and assume $\phi(m) > 0$ (i.e., $x^*(m) < m$). Since $x^*(m) < m$, a procedure similar to the derivation of $J'(a)$ in part (1) above but applied to (3.6) yields:

$$\begin{aligned} B'(m^+) &= \beta \left[b^*(m) V'(\phi^+(m)) + (1 - b^*(m)) V'(m^+) \right] \\ B'(m^-) &= \beta \left[b^*(m) V'(\phi^-(m)) + (1 - b^*(m)) V'(m^-) \right], \end{aligned}$$

where $\phi^+(m) = m^+ - x^*(m)$ and $\phi^-(m) = m^- - x^*(m)$. Comparing these equations with (C.1) and (C.2) yields that $V'(\phi^+(m)) = V'(\phi^-(m)) = V'(\phi)$ which is given by (3.9).

(3) For any $m \in [k, \bar{m})$ such that $b^*(m) > 0$, if $B(m) = V(m)$ and if there exists a neighborhood $O \ni m$ such that $B(m') \leq V(m')$ for all $m' \in O$, then $B'(m)$ and $V'(m)$ exist and satisfy (3.10) in part (iv) of Theorem 3.2. Also, (3.10) holds for $m = \bar{m}$ if $B'(\bar{m}) = V'(\bar{m})$.

Take any $m \in [k, \bar{m})$ that satisfies the hypotheses described in this part. Because $B(m') \leq V(m')$ for all $m' \in O(m)$ and $B(m) = V(m)$, continuity of B and V implies that $B'(m^-) \geq V'(m^-)$ and $B'(m^+) \leq V'(m^+)$. Substituting $V'(m^-) \leq B'(m^-)$ into the right-hand side of (C.2), we get:

$$V'\left(m^-\right) \leq B'(m^-) \leq \frac{b^*(m)}{1 - \beta\left[1 - b^*(m)\right]} u_1\left(x^*(m), b^*(m)\right). \tag{C.5}$$

Similarly, substituting $V'(m^+) \geq B'(m^+)$ into the right-hand side of (C.1), we get:

$$V'\left(m^+\right) \geq B'(m^+) \geq \frac{b^*(m)}{1 - \beta\left[1 - b^*(m)\right]} u_1\left(x^*(m), b^*(m)\right). \tag{C.6}$$

On the other hand, concavity of V implies $V'(m^-) \geq V'(m^+)$. Thus, $V'(m^-) = B'(m^-) = B'(m^+) = V'(m^+)$. Moreover, $V'(m)$ and $B'(m)$ satisfy (3.10).

If $m = \bar{m}$, it is still true that $B'(m^-) \geq V'(m^-)$, and so (C.5) also holds at $m = \bar{m}$. However, since we cannot presume $V'(\bar{m}^+) \geq B'(\bar{m}^+)$, we cannot conclude that (C.6) holds at this point. If $B'(\bar{m}) = V'(\bar{m})$, however, (C.1) and (C.2) imply that (3.10) holds at $m = \bar{m}$.

(4) Consider any $m \in [k, \bar{m})$ such that $b^*(m) > 0$ and $\phi(m) > 0$. If $B(m) = V(m)$ and if there exists a neighborhood O surrounding m such that $B(m') \leq V(m')$ for all $m' \in O$, then b^* and ϕ are strictly increasing at m and V is strictly concave at $\phi(m)$, with $V(\phi(m)) = B(\phi(m))$ and $V'(\phi(m)) > V'(m)$. These properties also hold for $m = \bar{m}$ if $B'(\bar{m}) = V'(\bar{m})$.

Take any arbitrary $m_1 \in [k, \bar{m}]$ that satisfies the hypotheses for m described in this paper. If $m_1 = \bar{m}$, then add the assumption $B'(m_1) = V'(m_1)$. Shorten the notation $\phi(m_1)$ as ϕ_1 and $b^*(m_1)$ as b_1^*. As a preliminary step, we prove $V'(\phi_1) > V'(m_1)$ so that V must be strictly concave in some sections of $[\phi_1, m_1]$. By the construction of m_1, $V'(m_1)$ satisfies (3.10) and $V'(\phi_1)$ satisfies (3.9). Subtracting the two relations yields:

$$V'(\phi_1) - V'(m_1) = \frac{1 - \beta}{\beta\left[1 - \beta\left(1 - b_1^*\right)\right]} u_1(x_1^*, b_1^*) > 0.$$

Next, we prove that $b^*(.)$ is strictly increasing at m_1. Let m_2 be sufficiently close to m_1 so that $\phi(m_2) > 0$ and $b^*(m_2) > 0$ (which is feasible because $\phi(m)$ and $b(m)$ are continuous functions). Shorten the notation $(x^*(m_i), b^*(m_i), \phi(m_i))$ to (x_i^*, b_i^*, ϕ_i), where $x_i^* = m_i - \phi_i$ and $i = 1, 2$. Since the proofs of strict monotonicity of $b^*(m_1)$ at m_1 are similar in the cases $m_2 > m_1$ and $m_2 < m_1$, let us consider only the case $m_2 > m_1$. By Lemma 3.3, $x_2^* \geq x_1^*$, $b_2^* \geq b_1^*$ and $\phi_2 \geq \phi_1$. We prove that $b_2^* > b_1^*$. Because $b_i^* > 0$, the first-order condition for b, (3.8), yields:

$$u(x_i^*, b_i^*) + \beta[V(\phi_i) - V(m_i)] + b_i^* u_2(x_i^*, b_i^*) = 0.$$

Subtracting these conditions for $i = 1, 2$, and re-organizing, we get:

$$
\begin{array}{rl}
& u(x_1^*, b_1^*) - u(x_1^*, b_2^*) - b_2^* u_2(x_2^*, b_2^*) + b_1^* u_2(x_1^*, b_1^*) \\
= & \{u(x_2^*, b_2^*) + \beta[V(m_2 - x_2^*) - V(m_2)] - u(x_1^*, b_2^*) - \beta[V(m_2 - x_1^*) - V(m_2)]\} \\
& +\beta \{V(m_1) - V(m_1 - x_1^*) + V(m_2 - x_1^*) - V(m_2)\}.
\end{array}
$$

The term in the first pair of braces on the right-hand side is equal to $[R(x_2^*, b_2^*, m_2) - R(x_1^*, b_2^*, m_2)]$, where R is defined by (3.13). This term is non-negative because x_2^* maximizes $R(x, b_2^*, m_2)$ and x_1^* is a feasible choice of x in this maximization problem (as $x_1^* \leq x_2^*$). The term in the second pair of braces is strictly positive because V is strictly concave in some sections of $[\phi_1, m_1] \subset [\phi_1, m_2]$, as proven earlier. Thus, the left-hand side of the above equation must be strictly positive. Noting that $u_2(x_2^*, b_2^*) \geq u_2(x_1^*, b_2^*)$ (because $u_{12} > 0$ and $x_2^* \geq x_1^*$), we get:

$$u(x_1^*, b_1^*) - u(x_1^*, b_2^*) - b_2^* u_2(x_2^*, b_2^*) + b_1^* u_2(x_1^*, b_1^*) > 0.$$

The left-hand side of this inequality is a strictly increasing function of b_2^*, and it is equal to 0 when $b_2^* = b_1^*$. Thus, the inequality implies $b_2^* > b_1^*$.

Let us complete the proof of part (4). Since (3.10) and (3.9) hold at $m = m_1$, we can combine the two relations to obtain:

$$V'(\phi_1) = V'(m_1) \left[\frac{1-\beta}{\beta b^*(m_1)} + 1 \right].$$

Because $b^*(m_1)$ is strictly increasing at m_1 and V is concave, the right-hand side above is strictly decreasing in m_1. Thus, the above equation requires V to be strictly concave at $\phi(m_1)$ and ϕ to be strictly increasing at m_1. **QED**

D. Proof of Theorem 3.5

We prove each part of Theorem 3.5 in turn.

Part (i). Let us express the functional equation (2.1) in a worker's problem as $W(m) = T_W V(m)$ for $m \in [0, \bar{m}]$, and express the functional equation (2.2) in a buyer's problem as $B(m) = T_B V(m)$ for $m \in [0, \bar{m}]$. Substituting these expressions into (2.3) to obtain $\tilde{V}$, we can rewrite (2.4) as $V(m) = TV(m)$, where

$$TV(m) \equiv \max_{(z_1, z_2, \pi_1, \pi_2)} [\pi_1 \max\{T_W V(z_1), T_B V(z_1)\} + \pi_2 \max\{T_W V(z_2), T_B V(z_2)\}] \tag{D.1}$$

$$
\begin{array}{ll}
\text{s.t.} & \pi_1 z_1 + \pi_2 z_2 = m, \quad \pi_1 + \pi_2 = 1, \quad z_2 \geq z_1, \\
& \pi_j \in [0, 1] \text{ and } z_j \geq 0 \text{ for } j = 1, 2.
\end{array}
$$

Lemma 3.1 proves that T_W maps $\mathcal{V}[0, \bar{m}]$ into $\mathcal{V}[0, \bar{m}]$; i.e., T_W maps the set of continuous, increasing and concave functions on $[0, \bar{m}]$ into itself. Theorem 3.2 proves that T_B maps $\mathcal{V}[0, \bar{m}]$

into $\mathcal{C}[0,\bar{m}]$ (but not necessarily into $\mathcal{V}[0,\bar{m}]$). Thus, the objective function in (D.1) is a continuous function of z_1 and z_2. Also, the objective function is increasing in π_1 and π_2, and the feasibility set in the above problem is increasing in m. These features of the maximization problem above imply that T maps $\mathcal{V}[0,\bar{m}]$ into $\mathcal{C}[0,\bar{m}]$. Moreover, since the function $\max\{T_W V(z), T_B V(z)\}$ is continuous in z on a closed interval $[0,\bar{m}] \ni z$, the lottery in (2.4) makes $TV(m)$ a concave function (see Appendix F in Menzio and Shi, 2010, for a proof). Thus, T is a self-map on $\mathcal{V}[0,\bar{m}]$.

It is evident from (2.1) and (2.2) that T_W and T_B are monotone mappings, and so T is a monotone mapping. It is also easy to verify that T_W and T_B feature discounting with a factor $\beta \in (0,1)$, and so does T. Hence, T satisfies Blackwell's sufficient conditions for a monotone contraction mapping. Moreover, because $\mathcal{V}[0,\bar{m}]$ is a closed subset of the complete metric space $\mathcal{C}[0,\bar{m}]$, T has a unique fixed point $V \in \mathcal{V}[0,\bar{m}]$ (see Stokey et al., 1989).

Part (ii). For a worker with any balance m, the choice of working zero hours yields the value $\beta V(m)$. Because this choice is always feasible, $W(m) \geq \beta V(m)$ for all m. For a buyer who holds $m \leq k$, the value is $B(m) = \beta V(m) \leq W(m)$. It is clear that $V(0) = \tilde{V}(0) = W(0)$. Also, $V(0) \geq 0$, because an individual without money can always choose not to trade. To prove $V(0) > 0$, suppose $V(0) = 0$, to the contrary. In this case, $0 = V(0) \geq W(0) \geq \beta V(0) = 0$, and so $W(0) = V(0) = 0$. Using the definition of $W(0)$, we have $\beta V(\hat{m}) - h(\hat{m}) = 0$. Since this equation has a unique solution and since $\hat{m} = 0$ satisfies the equation, then $\hat{m} = 0$. Recall that $\hat{m} = \ell^*(0)$ is the optimal labor supply of an individual without money and that the policy function $\ell^*(m)$ is decreasing in m. Thus, $\hat{m} = 0$ implies that $\ell^*(m) = 0$ for all $m \geq 0$. In this case, no individual will work for money, and so a monetary equilibrium does not exist. Therefore, for a monetary equilibrium to exist, it must be the case that $V(m) \geq W(m) \geq W(0) = V(0) > 0$ for all m.

We now prove that $V(m) > W(m)$ for all $m > 0$. For all $m > 0$ such that the constraint $y^* \geq m$ is binding for a worker, (3.2) yields $W(m) = \beta V(m) < V(m)$. Now consider $m > 0$ such that the constraint $y^* \geq m$ is not binding for a worker. Contrary to the result in this part, suppose $V(\tilde{m}) = W(\tilde{m})$ for some $\tilde{m} > 0$ such that $y^*(\tilde{m}) > \tilde{m}$. Since $\beta V'(y^*(\tilde{m})) = W'(\tilde{m})$ by (3.1) in Lemma 3.1, then $V'(y^*(\tilde{m})) > 0$, and concavity of V implies $V'(\tilde{m}^-) > 0$. In this case,

$$V'\left(\tilde{m}^-\right) \leq W'(\tilde{m}) = \beta V'(y^*(\tilde{m})) \leq \beta V'(\tilde{m}^-) < V'(\tilde{m}^-).$$

The first inequality follows from the hypothesis $V(\tilde{m}) = W(\tilde{m})$ and the fact $V(m) \geq W(m)$ for all $m < \tilde{m}$. The equality is from (3.1). The second inequality follows from concavity of V, and the last inequality from $V'(\tilde{m}^-) > 0$. Since the above result is a contradiction, we conclude that $V(m) > W(m)$ for all $m > 0$.

Part (iii). We prove first that there is some $m' \in (0,\infty)$ such that $B(m') > W(m')$. Suppose, to the contrary, that $B(m) \leq W(m)$ for all $m \in (0, \infty)$. Then, $\tilde{V}(m) = W(m)$ for all m. Since $W(m)$ is concave (see Lemma 3.1), $\tilde{V}(.)$ is concave in this case, and so $V(m) = \tilde{V}(m) = W(m)$ for all m. In this case, (3.2) yields

$$V(m) = \max_{y \geq m} \left[\beta V(y) - h(y - m)\right], \quad \text{all } m > 0.$$

If $y^*(m) = m$, the above equation yields $V(m) = 0$, which contradicts part (ii) above. If $y^*(m) > m$, (3.1) in Lemma 3.1 implies that W is differentiable at m, with $W'(m) = \beta V'(y^*(m)) > 0$. Since $W(m) = V(m)$ for all $m > 0$ in this case, $V'(m) = W'(m) = \beta V'(y^*(m)) \leq \beta V'(m)$. This implies $V'(m) = 0 = V'(y^*(m))$, which contradicts $V'(y^*(m)) > 0$.

Next, we prove that there exists $m_0 \in (k, \bar{m}]$ with $V(m_0) = B(m_0)$ such that an individual with $m < m_0$ will play the lottery with the prize m_0. For an individual with a balance $m \in (0, k)$, the lottery with $z_1 = 0$ and $z_2 = m'$ yields a value higher than $\tilde{V}(m)$, where m' is described above. Thus, these individuals will participate in lotteries. However, m' may not necessarily be the optimal prize of the lottery for these individuals. The optimal prize is m_0, defined by (3.16). Clearly, $m_0 > k > 0$, $V(m_0) = \tilde{V}(m_0) = B(m_0)$, and $V(m) \geq \tilde{V}(m)$ for all $m \in [0, m_0]$.

Now we prove that $b^*(m_0) > 0$ and $\phi(m_0) = 0$. Suppose $b^*(m_0) = 0$ to the contrary, and so $B(m_0) = \beta V(m_0)$. Since $V(m_0) = B(m_0)$, as shown above, then $V(m_0) = 0$, which contradicts the above result in part (ii) that $V(m) > 0$ for all $m \geq 0$. Thus, it must be true that $b^*(m_0) > 0$. Since $V(m_0) = B(m_0)$, (C.5) holds for $m = m_0$ which, together with $b^*(m_0) > 0$, implies $V'(m_0^-) < u_1(x^*(m_0), b^*(m_0))/\beta$. Since $V(m)$ is linear for $m \in [0, m_0]$, then $V'(\phi(m_0)) = V'(m_0^-) < u_1(x^*(m_0), b^*(m_0))/\beta$. If $\phi(m_0) > 0$, then (3.9) holds for $m = m_0$, which yields the contradiction that $V'(\phi(m_0)) = u_1(x^*(m_0), b^*(m_0))/\beta$. Thus, it must be true that $\phi(m_0) = 0$.

Finally, since $V(m_0) = B(m_0)$ and $b^*(m_0) > 0$, m_0 satisfies the hypotheses in part (iv) of Theorem 3.2 if $m_0 < \bar{m}$. Thus, if $m_0 < \bar{m}$, then (3.10) holds for $m = m_0$, which implies $V'(m_0) = B'(m_0) > 0$.

Part (iv). We first prove that $V'(m)$ exists for all $m \in [0, \bar{m})$ and $B'(m)$ exists for all $m \in [k, \bar{m})$ such that $b^*(m) > 0$. If $V'(m)$ exists for all $m \in [0, \bar{m})$, then part (iii) of Theorem 3.2 implies that $B'(m)$ exists for all $m \in [k, \bar{m})$ such that $b^*(m) > 0$. To prove that $V'(m)$ exists for all $m \in [0, \bar{m})$, note that the lottery with the prize m_0 implies that $V'(m)$ exists for all $m \in [0, m_0)$. If $m_0 = \bar{m}$, then $V'(m)$ exists for all $m \in [0, \bar{m})$. If $m_0 < \bar{m}$, then $V'(m_0)$ also exists, as shown in part (iii) above. What remains to be proven is that $V'(m)$ exists for all $m \in (m_0, \bar{m})$. Suppose to the contrary that $V'(\tilde{m})$ does not exist for some $\tilde{m} \in (m_0, \bar{m})$. In this case, $V'(\tilde{m}^-) > V'(\tilde{m}^+)$, and so V is strictly concave at $\tilde{m}$. Because $V(m) > W(m)$ for all $m > 0$, as proven in part (ii) above, we must have $V(\tilde{m}) = B(\tilde{m})$. Also, $b^*(\tilde{m}) \geq b^*(m_0) > 0$. Thus, the hypotheses in part

(iv) of Theorem 3.2 are true for $m = \tilde{m}$, and so $V'(\tilde{m})$ exists. This contradicts the supposition that $V'(\tilde{m})$ does not exist.

Next, we prove that $V'(m) > 0$ for all $m \in [0, \bar{m})$. For all $m \in [0, m_0)$, $V(m)$ is linear and $V'(m) = V'(m_0^-) > 0$. If $m_0 = \bar{m}$, then $V'(m) > 0$ for all $m \in [0, \bar{m})$. If $m_0 < \bar{m}$, then $V'(m_0) = B'(m_0) > 0$, as proven in part (iii) above. We need to prove $V'(m) > 0$ for all $m \in [m_0, \bar{m})$. Consider any $m > m_0$. Since $b^*(m_0) > 0$ and $b^*(m)$ is an increasing function (see part (i) of Theorem 3.2), then $b^*(m) > 0$, which further implies that $B(m)$ is strictly increasing (see part (iii) of Theorem 3.2). Because $\tilde{V}(m) = B(m)$ for all $m \geq m_0$, then $\tilde{V}(m)$ is strictly increasing over such m. Recall that $V(m)$ is constructed with lotteries over $\tilde{V}(m)$. If $V(m_1) = V(m_2)$ for some $m_2 > m_1 > m_0$, contrary to the claimed result, then $V(m)$ must be constant for all $m \in [m_1, m_2]$. Extend this interval to $[m_1', m_2']$, with $m_1' \leq m_1$ and $m_2' \geq m_2$, so that $V(m_1') = \tilde{V}(m_1')$ and $V(m_2') = \tilde{V}(m_2')$. Then, $\tilde{V}(m_2') = V(m_2) = V(m_1) = \tilde{V}(m_1')$, which contradicts strict monotonicity of $\tilde{V}$.

Part (v). For each exogenous upper bound on individuals' real balances, the policy and value functions are characterized by the results in section 3 up to part (iv) of the current theorem. Now we vary the upper bound, possibly to ∞, and prove that individuals' real balances in the equilibrium are indeed bounded above by a finite $\bar{m}$ that satisfies the current part of the theorem. Note first that the balance obtained by a worker is $\hat{m} = \ell^*(0) \leq 1$, which is clearly bounded above. If $B(\hat{m}) = V(\hat{m})$ and $B'(\hat{m}) = V'(\hat{m})$, then $z_2^*(\hat{m}) = \hat{z}_2 = \hat{m}$, in which case we can set $\bar{m} = \hat{m}$ as the upper bound to satisfy the properties stated in the current part of the theorem. In the remainder of this proof, suppose $B(\hat{m}) < V(\hat{m})$, and so a lottery is played at $\hat{m}$. Set the upper bound $\bar{m}$ in the analysis up to part (iv) of the theorem to an arbitrary finite number $\bar{m} > \hat{m}$. Given this arbitrary bound $\bar{m}$, it may or may not be true that $z_2^*(\hat{m}) < \bar{m}$. By varying the arbitrary bound, we can re-define $\bar{m}$ as the least upper bound above which $z_2^*(\hat{m}) < \bar{m}$. If this least upper bound is finite, then it satisfies the properties stated in the current part of the theorem. If this least upper bound is infinite, then $z_2^*(\hat{m}) = \bar{m}$ for all $\bar{m} > \hat{m}$, in which case the lottery at $\hat{m}$ is not well-defined for endogenously determined $\bar{m}$. It suffices to show that this unbounded case does not arise in the equilibrium. The unbounded case occurs only if there exists a finite $m_1 > \hat{m}$ such that the following two conditions are satisfied:

(A) $B(m)$ is strictly increasing and (weakly) convex for all $m \geq m_1$;

(B) for every $m_2 \geq m_1$, there exists $z_1 < \hat{m}$ such that for all $m < m_2$, $B(m)$ lies below or on the extension of the line connecting $B(z_1)$ and $B(m_2)$.

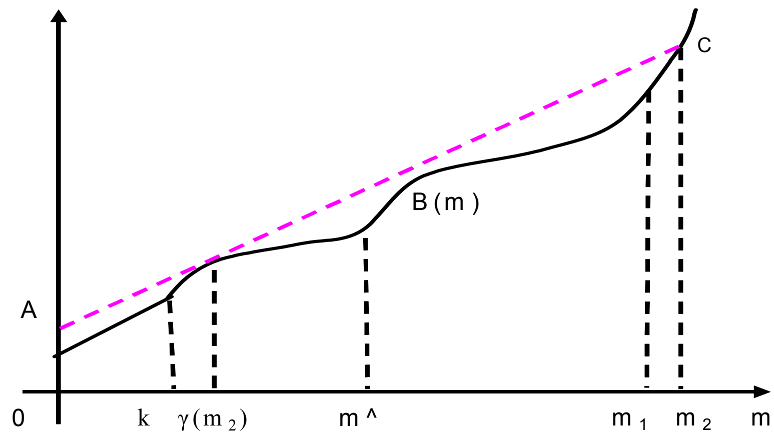

Figure 3.1

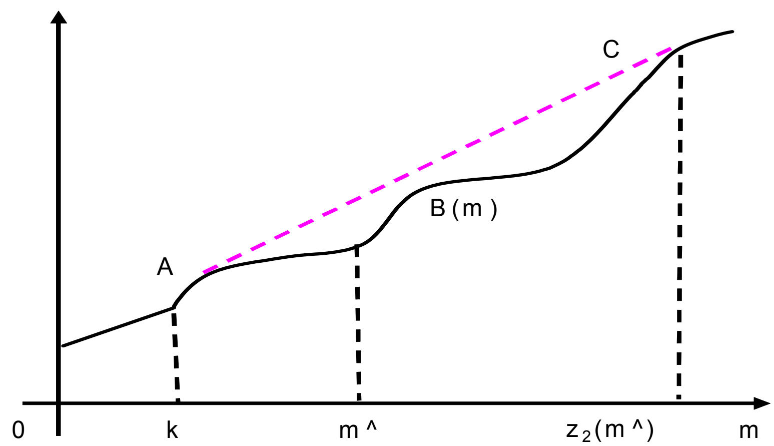

Figure 3.2

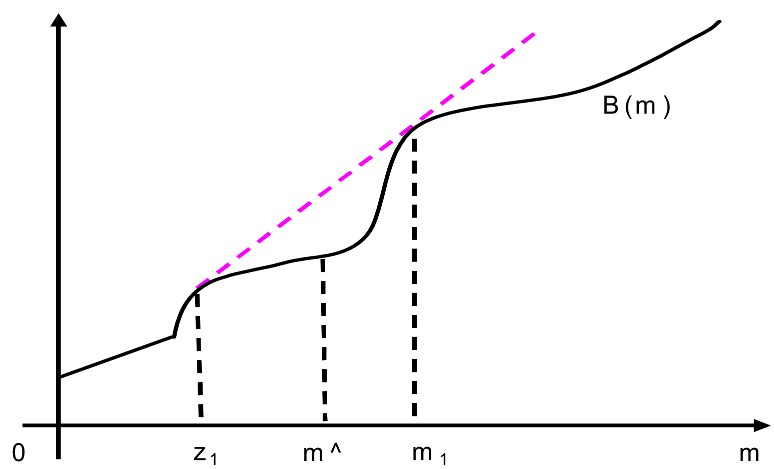

Figure 3.3

Figure 3.1 depicts this unbounded case. If (A) is violated, as depicted in Figure 3.2, then there must exist a finite number $m_1 > \hat{m}$ such that $B(m)$ is concave for all $m \geq m_1$. In this case, the high prize of the lottery at $\hat{m}$ is $z_2^*(\hat{m}) < \infty$, and so we can set $\bar{m} = z_2^*(\hat{m})$ as the upper bound stated in the current part of Theorem 3.5. If (B) is violated, as depicted in Figure 3.3, then there must exist a finite $m_1 > \hat{m}$ and an associated $z_1 < \hat{m}$ such that the low prize of the lottery at $\hat{m}$ is z_1 and the high prize is m_1 and that, for all $m > m_1$, the function $B(m)$ lies below or on the extension of the line connecting $B(z_1)$ and $B(m_1)$. In this case, $\bar{m} = m_1$ satisfies the properties in this part of Theorem 3.5. Note that in the case depicted in Figure 3.3, $B(m)$ can still be strictly increasing and convex for sufficiently large m, but such a section of B is irrelevant in the equilibrium because an individual's balance never goes above m_1. Also note that the requirement $z_1 < \hat{m}$ in (B) is important, since the case depicted in Figure 3.3 would not violate (B) if this requirement were not imposed.

Suppose, to the contrary, that there exists a finite $m_1 > \hat{m}$ such that (A) and (B) above are satisfied, as depicted in Figure 3.1. We prove that this leads to the contradiction that $B(m)$ is uniformly bounded. Consider any arbitrary $m_2 \geq m_1$. When individuals' real balances are exogenously capped by m_2, the lottery at $\hat{m}$ is well-defined, with m_2 as the high prize, and all characterizations of the policy and value functions that we have obtained so far (including parts (i)-(iv) of the current Theorem 3.5) remain valid with $\bar{m} = m_2$. However, since $B'(m_2) > V'(m_2)$ in this case, we have $B'(\bar{m}) > V'(\bar{m})$. Denote the low prize of the lottery at $\hat{m}$ as $z_1^*(\hat{m}) = \gamma(m_2)$ so as to emphasize its dependence on the exogenous upper bound m_2. Without loss of generality, assume that $\hat{m} \geq m_0$, i.e., $B(\gamma(m_2)) = V(\gamma(m_2))$. (If $B(\gamma(m_2)) < V(\gamma(m_2))$, then $\gamma(m_2) = 0$, in which case the proof is still valid after replacing $V(\gamma(m_2))$ below with $V(0)$.) Denote

$$\alpha(m_2) = \frac{B(m_2) - V(\gamma(m_2))}{m_2 - \gamma(m_2)},$$

$$\hat{V}(m, m_2) = B(m_2) - \alpha(m_2)(m_2 - m), \quad m \in [0, m_2].$$

Here, $\alpha(m_2)$ is the slope of the line connecting $B(\gamma(m_2))$ and $B(m_2)$, and $\hat{V}$ is the extension of this line to the domain $[0, m_2]$ (the dashed line from point A to point C in Figure 3.1).

We prove that $\alpha(m_2)$ is increasing for all $m_2 \geq m_1$. Take any arbitrary $m' > m_2 \geq m_1$. By definition, $\gamma(m') < \hat{m} \leq m_2$. So, $B(\gamma(m'))$ lies below or on the line $\hat{V}(m, m_2)$; i.e., $\hat{V}(\gamma(m'), m_2) \geq B(\gamma(m')) = V(\gamma(m'))$. Also, since $B(m)$ is increasing and convex for all $m \geq m_1$, and $m' > m_2$, then $B(m') \geq \hat{V}(m', m_2)$. Using these two results, we have:

$$\alpha(m') \geq \frac{\hat{V}(m', m_2) - V(\gamma(m'))}{m' - \gamma(m')} \geq \frac{\hat{V}(m', m_2) - \hat{V}(\gamma(m'), m_2)}{m' - \gamma(m')} = \alpha(m_2),$$

where the last equality comes from substituting the expression for $\hat{V}$. Thus, $\alpha(.)$ is increasing.

Now we derive a uniform upper bound on $B(m_2)$ for all $m_2 \geq m_1$. It is clear that $V(m) \leq \hat{V}(m, m_2)$ for $m \in [0, m_2]$, with equality if $m \in [\gamma(m_2), m_2]$. We have:

$$\begin{aligned}
B(m_2) &= \max_{b\in[0,1],x\in[0,m_2]} \{b\left[u(x,b) + \beta V(m_2 - x)\right] + (1-b)\beta V(m_2)\} \\
&\leq \max_{b\in[0,1],x\in[0,m_2]} \left\{b\left[u(x,b) + \beta \hat{V}(m_2 - x, m_2)\right] + (1-b)\beta \hat{V}(m_2, m_2)\right\} \\
&= \beta B(m_2) + \max_{b\in[0,1],x\in[0,m_2]} b\left[u(x,b) - \beta\alpha(m_2)x\right]. \\
&\leq \beta B(m_2) + \max_{b\in[0,1],q\geq 0} b\left\{U(q) - \beta\alpha(m_2)\left[q + \tfrac{k}{\mu(b)}\right]\right\}.
\end{aligned}$$

The first inequality follows from $V(m) \leq \hat{V}(m, m_2)$ for all $m \in [0, m_2]$. The ensuing equality comes from the linearity of $\hat{V}$ and $\hat{V}(m_2, m_2) = B(m_2)$. The last inequality comes from the fact that if we ignore the constraint $x \leq m_2$ in the maximization problem, the resulted maximum cannot be smaller. Here, we have substituted the relationship $x = q + \frac{k}{\mu(b)}$ and used (q, b) as the choices. Similarly, because $b \leq 1$ and $\mu(b) \leq 1$, the resulted maximum cannot be smaller if we set $b = 1$ and $\mu(b) = 1$. Thus,

$$B(m_2) \leq \frac{D(\alpha(m_2))}{1-\beta} \quad \text{where } D(\alpha(m_2)) \equiv \max_{q\geq 0}\left[U(q) - \beta\alpha(m_2)(q+k)\right]. \tag{D.2}$$

The notation $D(\alpha(m_2))$ emphasizes the fact that D depends on m_2 only through $\alpha(m_2)$. Because $U'(q)$ is strictly decreasing and $U'(\infty) = 0$, the solution for q to the maximization problem in (D.2) is unique, strictly positive and finite for all $\alpha < \infty$. So, $D(\alpha) < \infty$ for all $\alpha < \infty$. Applying the envelope condition, we have $D'(\alpha) < 0$. Because $\alpha(.)$ is increasing, as shown above, then $\alpha(m_2) \geq \alpha(m_1) > 0$ and $D(\alpha(m_2)) \leq D(\alpha(m_1)) < \infty$. Therefore, $B(m_2) \leq D(\alpha(m_1))/(1-\beta) < \infty$ for all $m_2 \geq m_1$. This result contradicts the supposition that $B(m)$ is strictly increasing and convex for all $m \geq m_1$. **QED**

E. Proof of Lemma 4.1

Part (i) of the lemma is implied by part (iii) of Theorem 3.5, with $m = \hat{m}$. Part (ii) of the lemma is obvious if $\hat{m} < m_0$ and, if $\hat{m} \geq m_0$, it is implied by part (iii) of the lemma. In particular, since part (iii) implies that $B(\phi^i(\hat{z}_j)) = V(\phi^j(\hat{z}_j))$, $B'(\phi^i(\hat{z}_j)) = V'(\phi^i(\hat{z}_j))$ and V is strictly concave at $\phi^i(\hat{z}_j)$ for all i in the set $\{0, 1, 2, ..., \hat{n}_j - 1\}$, then $\phi^i(\hat{z}_j) \geq m_0$ and a buyer with the balance $\phi^i(\hat{z}_j)$ has no need for a lottery for any i in the aforementioned set. Thus, the only lottery possibly played in the steady state is the one at $\hat{m}$.

We use induction to prove parts (a) and (b) of part (iii) of the lemma. Assume $\hat{m} \geq m_0$, as is required in part (iii), and take $\hat{z}_j$ as either prize of the lottery at $\hat{m}$. Start with $i = 1$. Because $\hat{m} \geq m_0$, then $\hat{z}_j \geq m_0$, and so $b^*(\hat{z}_j) \geq b^*(m_0) > 0$, where the strict inequality comes from part (iii) of Theorem 3.5. Thus, part (a) holds true for $i = 1$. Moreover, by the construction of the lottery at $\hat{m}$, $B(\hat{z}_j) = V(\hat{z}_j)$ and $B'(\hat{z}_1) = V'(\hat{z}_1)$. Also, since $\hat{z}_2 = \bar{m}$, part (v) of Theorem 3.5 implies $B'(\hat{z}_2) = V'(\hat{z}_2)$. Thus, $m = \hat{z}_j$ satisfies the hypotheses in part (v) of Theorem 3.2 which implies that, if $\phi(\hat{z}_j) > 0$, then V is strictly concave at $\phi(\hat{z}_j)$. Strict concavity of V at $\phi(\hat{z}_j)$ implies $V(\phi(\hat{z}_j)) = B(\phi(\hat{z}_j))$: if $B(\phi(\hat{z}_j)) < V(\phi(\hat{z}_j))$, V around $\phi(\hat{z}_j)$ would be a linear segment generated by the lottery in (2.4), which would contradict strict concavity of V at $\phi(\hat{z}_j)$. Thus, $m = \phi(\hat{z}_j)$ satisfies the hypotheses in part (iv) of Theorem 3.2 which implies $V'(\phi(\hat{z}_j)) = B'(\phi(\hat{z}_j))$. Moreover, strict concavity of V at $\phi(\hat{z}_j)$ implies that $\phi(\hat{z}_j) \geq m_0$, because V is linear for all $m < m_0$. Thus, parts (b) holds true for $i = 1$ if $\phi(\hat{z}_j) > 0$. If $\phi(\hat{z}_j) = 0$, on the other hand, part (b) is vacuous.

Suppose that parts (a) and (b) hold for an arbitrary $i \in \{1, 2, ..., \hat{n}_j - 1\}$, we prove that they hold for $i + 1$ and, by induction, they hold for all $i \in \{1, 2, ..., \hat{n}_j - 1\}$. Because $\phi^i(\hat{z}_j) \geq m_0$ by the supposition, $b^*(\phi^i(\hat{z}_j)) \geq b^*(m_0) > 0$, and so part (a) holds for $i + 1$. If $i = \hat{n}_j - 1$, then part (b) is vacuous for $i + 1$. If $i < \hat{n}_j - 1$, then $\phi^{i+1}(\hat{z}_j) > 0$. Since $V(\phi^i(\hat{z}_j)) = B(\phi^i(\hat{z}_j))$ and V is strictly concave at $\phi^i(\hat{z}_j)$, by the supposition, then $m = \phi^i(\hat{z}_j)$ satisfies the hypotheses in part (v) of Theorem 3.2 which implies that V is strictly concave at $\phi^{i+1}(\hat{z}_j)$. Strict concavity of V at $\phi^{i+1}(\hat{z}_j)$ implies $V(\phi^{i+1}(\hat{z}_j)) = B(\phi^{i+1}(\hat{z}_j))$ and $\phi^{i+1}(\hat{z}_j) \geq m_0$. Thus, $m = \phi^{i+1}(\hat{z}_j)$ satisfies the hypotheses in part (iv) of Theorem 3.2 which implies $V(\phi^{i+1}(\hat{z}_j)) = B(\phi^{i+1}(\hat{z}_j))$. Hence, part (b) holds for $i + 1$.

If $i = \hat{n}_j$, part (a) follows from the same proof as above, and part (b) is vacuous.

Finally, suppose $\phi^{\hat{n}_j}(\hat{z}_j) > 0$, contrary to part (c). Because part (b) holds for $i = \hat{n}_j - 1$, then $m = \phi^{\hat{n}_j - 1}(\hat{z}_j)$ satisfies all the hypotheses in part (v) of Theorem 3.2 which implies that V is strictly concave at $\phi^{\hat{n}_j}(\hat{z}_j)$. A contradiction. **QED**

F. Proof of Theorem 4.2

The text preceding the theorem has established that a unique monetary steady state exists, the steady state is block recursive, and the frequency function g is independent of ω. These results imply that money is neutral in the steady state. Turn to the result that from either $\hat{z}_j$ ($j = 1, 2$), the frequency function, $g(\phi^i(\hat{z}_j))$, is decreasing in $\phi^i(\hat{z}_j)$. To prove this result, note that $\phi^i(\hat{z}_j) = \phi^{i-1}(\hat{z}_j) - x^*(\phi^{i-1}(\hat{z}_j)) < \phi^{i-1}(\hat{z}_j)$ for all $1 \leq i \leq \hat{n}_j$ and $j = 1, 2$. By part (iii) of Theorem 3.5, $b^*(m_0) > 0$. For each $j \in \{1, 2\}$, Lemma 4.1 implies that $\phi^i(\hat{z}_j) \geq m_0$ and $b^*(\phi^i(\hat{z}_j)) > 0$ for all $1 \leq i \leq \hat{n}_j - 1$. Thus, for all $1 \leq i \leq \hat{n}_j - 1$, $\phi^i(\hat{z}_j)$ satisfies part (v) of Theorem 3.2, which implies that $b^*(.)$ is strictly increasing at $\phi^i(\hat{z}_j)$ for each i and j. With this feature, (4.4) implies that $g(\phi^i(\hat{z}_j)) > g(\phi^{i-1}(\hat{z}_j))$ for all $i = 1, 2, ..., \hat{n}_j - 1$ and $j = 1, 2$.

Next, we prove that there exists $\beta_0 > 0$ such that if $\beta \leq \beta_0$, then $\hat{m} < m_0$ and $\phi(\hat{z}_2) = 0$. Let us shorten the notation $b^*(m_0)$ to b_0 and $q^*(m_0)$ to q_0. Define $\bar{q}(\beta)$ and $\underline{q}$ by

$$\frac{U(\underline{q})}{U'(\underline{q})} - \underline{q} = k, \quad U'(\bar{q}(\beta)) = \frac{1}{\beta} h'(\bar{q}(\beta)) + k). \tag{F.1}$$

We then define β_0 as

$$\beta_0 = \max_{\beta \in [0,1]} \left\{ \beta : \bar{q}(\beta) \leq \underline{q} \right\}. \tag{F.2}$$

For any $\beta \in (0, 1]$, the assumptions on U and h imply that $\bar{q}(\beta)$ and $\underline{q}$ are well defined. In particular, the assumptions on U imply that $[\frac{U(q)}{U'(q)} - q]$ is a strictly increasing function of q whose value at $q = 0$ is 0. Moreover, $\bar{q}(\beta)$ and $\underline{q}$ have the following features:

(a) $\bar{q}'(\beta) > 0$ and $\lim_{\beta \to 0} \bar{q}(\beta) = 0 < \underline{q}$: These follow from the assumptions on U and h.

(b) $\underline{q} < q_0$: To verify this feature, note that $V'(0) = V'(m_0)$. Since $V'(m_0)$ satisfies part (v) of Theorem 3.2 with $m = m_0$, we have:

$$V'(0) = \frac{b_0 U'(q_0)}{1 - \beta + \beta b_0}, \tag{F.3}$$

where we have substituted $u_1(m_0, b_0) = U'(q_0)$. Also, the lottery in (3.16) implies $V(m) = V(0) + mV'(0)$ for all $m \in [0, m_0]$. Substituting $V(m_0)$ from this result and $V'(0)$ from (F.3) into the Bellman equation for $B(m_0)$ $(= V(m_0))$, we obtain:

$$b_0 \left[U(q_0) - m_0 U'(q_0) \right] = (1 - \beta) V(0). \tag{F.4}$$

Here, we have substituted $u_1(m_0, b_0) = U'(q_0)$ and $u(m_0, b_0) = U(q_0)$. Because $V(0) > 0$ by part (ii) of Theorem 3.5 and $b_0 > 0$, (F.4) implies $U(q_0) > m_0 U'(q_0)$. Because $m_0 > q_0 + k$ (as $\mu(b_0) < 1$), this result further implies $\frac{U(q_0)}{U'(q_0)} - q_0 > k = \frac{U(\underline{q})}{U'(\underline{q})} - \underline{q}$, which is equivalent to $\underline{q} < q_0$.

(c) $\bar{q}(\beta) > q^*(\hat{m})$ for all $\beta \in (0, 1]$: By the definition of $\bar{q}(\beta)$ in (F.1), $\bar{q}(\beta) > q^*(\hat{m})$ if and only if $U'(q^*(\hat{m})) > \frac{1}{\beta} h'(q^*(\hat{m})) + k)$. The latter relation is verified as follows:

$$U'(q^*(\hat{m})) \geq U'(q^*(\hat{z}_2)) > V'(\hat{z}_2) = V'(\hat{m}) = \frac{1}{\beta} h'(\hat{m}) > \frac{1}{\beta} h'(q^*(\hat{m})) + k).$$

The first inequality comes from the fact $q^*(\hat{m}) \leq q^*(\hat{z}_2)$. To obtain the second inequality, we apply (3.10) for $m = \hat{z}_2$, which yields $V'(\hat{z}_2) = \frac{b^*(\hat{z}_2) U'(q^*(\hat{z}_2))}{1 - \beta + \beta b^*(\hat{z}_2)}$. The first equality above comes from the fact that V is linear between $\hat{m}$ and $\hat{z}_2$, and the second equality above from the definition of $\hat{m}$. The last inequality comes from $\hat{m} = q^*(\hat{m}) + \frac{k}{\mu(b^*(\hat{m}))}$ and $\mu(b^*(\hat{m})) < 1$.

Feature (a) implies that the set $\{\beta \in [0, 1] : \bar{q}(\beta) \leq \underline{q}\}$ is non-empty and that $\beta_0 > 0$ is well-defined. Moreover, $\bar{q}(\beta) \leq \underline{q}$ for all $\beta \leq \beta_0$. Using features (b) and (c), we conclude that if

$\beta \le \beta_0$, then $q^*(\hat{m}) < \bar{q}(\beta) \le \underline{q} < q_0$. Recall that $q^*(m)$ is an increasing function. Thus, if $\beta \le \beta_0$ then $\hat{m} < m_0$, in which case $\phi(\hat{z}_2) = \phi(m_0) = 0$.

As a preliminary step toward finding a condition for $\phi(\hat{z}_2) > 0$, we consider the limit $\beta \to 1$ and characterize the optimal choices in more detail. This exercise is guided by the above result that $\phi(\hat{z}_2) = 0$ if β is small. Note that although $\lim_{\beta \to 1} V(m) = \infty$, the limit of $(1-\beta)V(m)$ is strictly positive and finite for all $m \in [0, \infty)$. Also, the limit of $[V(m) - V(0)]$ is finite for all $m < \infty$. We characterize in detail the optimal choices of a buyer with the balance m_0 in the limit $\beta \to 1$. First, taking the limit $\beta \to 1$ on (F.3) and (F.4) yields:

$$V'(0) = U'(q_0), \tag{F.5}$$

$$b_0 \left[U(q_0) - m_0 U'(q_0) \right] = \lim_{\beta \to 1} \left[(1-\beta) V(0) \right]. \tag{F.6}$$

Second, since $u_2 = u_1 k \mu' / \mu^2$, the first-order condition of b_0 (see (3.8)) yields:

$$\frac{U(q_0)}{U'(q_0)} - m_0 + \frac{k\mu'(b_0) b_0}{[\mu(b_0)]^2} = 0, \tag{F.7}$$

where we have used (F.5) for $V'(0)$. Substituting $b_0 = \mu^{-1}(\frac{k}{m_0 - q_0})$ into (F.7), we can prove that $q_0 = q^*(m_0)$ is a strictly increasing function of m_0.

We are now ready to prove that $\phi(\hat{z}_2) > 0$ in the limit $\beta \to 1$ if and only if $m_0 < \hat{z}_2$. The "only if" part of this statement is apparent, because $m_0 \ge \hat{z}_2$ implies $\phi(\hat{z}_2) = \phi(m_0) = 0$. To prove the "if" part of the statement, we verify that $\phi(\hat{z}_2) = 0$ implies $m_0 \ge \hat{z}_2$ in the limit $\beta \to 1$. Suppose $\phi(\hat{z}_2) = 0$. Using part (ii) of Theorem 3.2, we deduce that $\beta V'(0) \le U'(q^*(\hat{z}_2))$. Taking the limit $\beta \to 1$ and using (F.5), we write this condition as $q_0 \ge q^*(\hat{z}_2)$. Because $q^*(m)$ is strictly increasing at $m = m_0$, then $m_0 \ge \hat{z}_2$.

The above procedure leads to the conclusion that when β is sufficiently close to one, $\phi(\hat{z}_2) > 0$ if and only if $m_0 < \hat{z}_2$. To characterize the condition $m_0 < \hat{z}_2$ explicitly, we suppose that the opposite, $m_0 \ge \hat{z}_2$, is true. After solving q_0 from (F.8) as $q_0(\hat{m})$ and b_0 from (F.9) as $b_0(\hat{m})$, we will solve the number $\hat{m}$ from (F.10) as m_c. Because the supposition $m_0 \ge \hat{z}_2$ implies $\hat{m} \le m_0$, the supposition leads to a contradiction if $\hat{m} = m_c$ satisfies $\hat{m} > m_0$, i.e., if (4.6) holds. Therefore, if (4.6) holds, then $m_0 < \hat{z}_2$ and $\phi(\hat{z}_2) > 0$ for β sufficiently close to one.

To carry out the procedure described above, we suppose $m_0 \ge \hat{z}_2$ and consider the limit $\beta \to 1$. Since $m_0 \ge \hat{m}$ in this case, the lottery for low balances implies $V'(0) = V'(\hat{m})$. Because the definition of $\hat{m}$ in the limit $\beta \to 1$ implies $V'(\hat{m}) = h'(\hat{m})$, then $V'(0) = h'(\hat{m})$. Substituting this result into (F.5), we solve $q_0 = q_0(\hat{m})$ where

$$q_0(\hat{m}) \equiv U'^{-1}(h'(\hat{m})). \tag{F.8}$$

Substituting $m_0 = q_0 + \frac{k}{\mu(b_0)}$ and $q_0 = q_0(\hat{m})$ into (F.7) yields:

$$\frac{k}{\mu(b_0)} - \frac{k\mu'(b_0)b_0}{[\mu(b_0)]^2} = \frac{U(q_0(\hat{m}))}{h'(\hat{m})} - q_0(\hat{m}). \tag{F.9}$$

Since $\mu'(b) < 0$ and $1/\mu(b)$ is strictly convex in b, the left-hand side of (F.9) is strictly increasing in b_0. Thus, for any given $\hat{m}$, (F.9) solves for a unique b_0. Denote this solution as $b_0(\hat{m})$.

Moreover, since $\hat{m} \leq m_0$ under the supposition $m_0 \geq \hat{z}_2$, the lottery for low balances implies that $V(\hat{m})$ is linear in $\hat{m}$ and the slope of the line is $V'(\hat{m}) = h'(\hat{m})$ in the limit $\beta \to 1$. That is, $V(\hat{m}) - V(0) = \hat{m}h'(\hat{m})$. On the other hand, in the limit $\beta \to 1$, a worker's Bellman equation yields $V(\hat{m}) - V(0) = h(\hat{m}) + \lim_{\beta\to 1}[(1-\beta)V(0)]$. Thus, $\lim_{\beta\to 1}[(1-\beta)V(0)] = \hat{m}h'(\hat{m}) - h(\hat{m})$. Substituting this result and $b_0 = b_0(\hat{m})$, we rewrite (F.6) as

$$-\frac{k\mu'(b_0)(b_0)^2}{[\mu(b_0)]^2}\bigg|_{b_0=b_0(\hat{m})} = \hat{m} - \frac{h(\hat{m})}{h'(\hat{m})}. \tag{F.10}$$

The right-hand side of (F.10) is a strictly increasing function of $\hat{m}$. From (F.8) and (F.9), we can verify that $q_0'(\hat{m}) < 0$, $b_0'(\hat{m}) < 0$, $q_0(0) = \infty$, $b_0(0) = 1$, $q_0(\infty) = 0$ and $\mu(b_0(\infty)) > 0$. With these properties and the maintained assumptions on the function $\mu(b)$, we can verify that the left-hand side of (F.10) is a strictly decreasing function of $\hat{m}$ and that there is a unique solution to (F.10) for the number $\hat{m}$. This solution, denoted as m_c, is the one used in (4.6). **QED**

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