Estimation of non-Gaussian Affine Term Structure Models

Drew D. Creal Jing Cynthia Wu

University of Chicago, Booth School of Business

New York University, Stern School of Business

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Motivation

Affine term structure models are useful for

- understanding the joint dynamics of the yield curve;
- describing the discount factor for financial markets;
- bridging macroeconomics and finance together;
- informing monetary policy.

Why do we need to go beyond Gaussian models?

- zero lower bound
- time-varying volatility

Estimation of ATSMs

- **Kim and Orphanides (2005):**

  ... the likelihood function seems to have multiple inequivalent local maxima which have similar likelihood values but substantially different implications...

- **Duffee (2002):**

  The QML functions for these models have a large number of local maxima.

- **Ang and Piazzesi (2003):**

  ... difficulties associated with estimating a model with many factors using maximum likelihood when yields are highly persistent.

Literature

Earlier work on estimating non-Gaussian models

- Dai and Singleton (2000), Duffee (2002)
- Collin-Dufresne, Goldstein and Jones (2008/2009), Ait-Sahalia and Kimmel (2010)
- Le, Singleton, and Dai (2010)

Recent work on estimating Gaussian models

- Joslin, Singleton, and Zhu (2011)
- Christensen, Diebold, and Rudebusch (2011)
- Hamilton and Wu (2012)

Contribution

Propose a new estimation approach for non-Gaussian ATSMs.

- Reduce parameter space by concentrating out $\mathbb{P}$ parameters.
- Provide analytical gradient to improve numerical behavior.
- Works for ANY rotation/identification scheme.
- For Gaussian models, this approach generalizes Joslin, Singleton, and Zhu (2011), and Hamilton and Wu (2012).

Extensions:

- Imposing constraints on parameters is straightforward.
- Adding macroeconomic variables is simple.
- Accommodating more general dynamics, e.g. AR(p).

Outline

Outline

Dynamics of the state vector under $\mathbb{P}$

State vector $x_t = (g_t', h_t')'$ has dynamics:

$$
\begin{aligned}
g_{t+1} &= \mu_g + \Phi_g g_t + \Phi_{gh} h_t + \Sigma_{gh}\varepsilon_{h,t+1} + \varepsilon_{g,t+1}, \\
\Sigma_{g,t}\Sigma_{g,t}' &= \Sigma_g \Sigma_g' + \sum_{i=1}^{H} \Sigma_{g,i}\Sigma_{g,i}' h_{it}, \\
h_{i,t+1} &\sim \text{N.C.-Gamma}(\nu_i, \Phi_{h,i}' h_t, \sigma_{h,i}), \qquad i = 1, \dots, H.
\end{aligned}
$$

- $\varepsilon_{g,t+1} \sim \mathrm{N}\left(0, \Sigma_{g,t}\Sigma_{g,t}'\right)$
- g_t and h_t are $G \times 1$ and $H \times 1$ vectors.
- Σ_g and $\Sigma_{g,i}$ are lower triangular.
- h_{it} is a discrete-time Cox Ingersoll Ross (1985) process.

Autoregressive gamma process

The non-Gaussian state variables h_t follow

$$h_{i,t+1}|h_t \sim \text{N.C.-Gamma}(\nu_i, \Phi'_{h,i}h_t, \sigma_{h,i}), \qquad i = 1, \ldots, H.$$

Details

Conditional mean and variance are linear in h_t:

$$\begin{aligned} \mathbb{E}[h_{i,t+1}|h_t] &= \nu_i \sigma_{h,i} + \Phi'_{h,i}h_t \qquad i = 1, \ldots, H \\ \mathbb{V}[h_{i,t+1}|h_t] &= \nu_i \sigma^2_{h,i} + 2\sigma_{h,i}\Phi'_{h,i}h_t \end{aligned}$$

Autoregressive gamma process

Conditional mean and variance are linear in h_{t-1}

$$
\begin{aligned}
\mathbb{E}[h_{i,t+1}|h_t] &= \nu_i \sigma_{h,i} + \Phi'_{h,i} h_t \qquad\qquad i = 1, \ldots, H \\
\mathbb{V}[h_{i,t+1}|h_t] &= \nu_i \sigma_{h,i}^2 + 2\sigma_{h,i} \Phi'_{h,i} h_t
\end{aligned}
$$

Stacked together

$$
\begin{aligned}
\mathbb{E}[h_{t+1}|h_t] &= \mu_h + \Phi_h h_t \\
\mathbb{V}[h_{t+1}|h_t] &= \begin{pmatrix} \nu_1 \sigma_{h,1}^2 + 2\sigma_{h,i} \Phi'_{h,1} h_t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_H \sigma_{h,H}^2 + 2\sigma_{h,H} \Phi'_{h,H} h_t \end{pmatrix}
\end{aligned}
$$

Note: $\mu_{h,i} = \nu_i \sigma_{h,i}$

Autoregressive gamma process

AG(1) process has the same transition density as the CIR process

$$dh_t = \kappa(\theta - h_t)dt + \sigma\sqrt{h_t}dW_t$$

Details

with the following mapping:

$$\Phi_h = 1 - \kappa\tau \qquad \nu = \frac{2\kappa\theta}{\sigma^2} \qquad \sigma_h = \frac{\sigma^2\tau}{2}$$

- $\nu_i > 1$ is the Feller condition.
- It guarantees that $h_{it} > 0$.

Dynamics of the state vector under $\mathbb{Q}$

$$
\begin{aligned}
g_{t+1} &= \mu_g^{\mathbb{Q}} + \Phi_g^{\mathbb{Q}} g_t + \Phi_{gh}^{\mathbb{Q}} h_t + \Sigma_{gh}\varepsilon_{h,t+1}^{\mathbb{Q}} + \varepsilon_{g,t+1}^{\mathbb{Q}}, \\
\Sigma_{g,t}\Sigma_{g,t}' &= \Sigma_g\Sigma_g' + \sum_{i=1}^{H} \Sigma_{g,i}\Sigma_{g,i}' h_{it}, \\
h_{i,t+1} &\sim \text{N.C.-Gamma}(\nu_i^{\mathbb{Q}}, \Phi_{h,i}^{\prime,\mathbb{Q}} h_t, \sigma_{h,i}), \qquad i = 1, \ldots, H.
\end{aligned}
$$

- $\varepsilon_{g,t+1}^{\mathbb{Q}} \overset{\mathbb{Q}}{\sim} \mathrm{N}\left(0, \Sigma_{g,t}\Sigma_{g,t}'\right)$.
- $\Sigma_g, \Sigma_{gh}, \Sigma_{g,i}, \sigma_{h,i}$ are the same under $\mathbb{P}$ and $\mathbb{Q}$.
- Drift changes but variance does not (approximately).

Bond prices

Short rate is a linear function of the state vector

$$r_t = \delta_0 + \delta_1' x_t$$

Bond prices are exponentially affine

$$P_t^n \;=\; \mathbb{E}_{t-1}^{\mathbb{Q}}\left[\exp\left(-r_t\right) P_{t+1}^{n-1}\right] \;=\; \exp\left(\bar{a}_n + \bar{b}_{n,g}' g_t + \bar{b}_{n,h}' h_t\right)$$

Yields (log-prices) are linear

$$y_t^n \;=\; a_n + b_n' x_t \;=\; a_n + b_{n,g}' g_t + b_{n,h}' h_t$$

with $a_n = -\frac{1}{n}\bar{a}_n$, $b_{n,g} = -\frac{1}{n}\bar{b}_{n,g}$ and $b_{n,h} = -\frac{1}{n}\bar{b}_{n,h}$.

Factor loadings

The bond loadings can be expressed recursively as

$$
\begin{aligned}
\bar{a}_n &= -\delta_0 + \bar{a}_{n-1} - \sum_{i=1}^{H} \nu_i^{\mathbb{Q}} \log\left(1 - \sigma_{h,i}\bar{b}_{n-1,gh,i}\right) + \left(\mu_g^{\mathbb{Q}} - \Sigma_{gh}\mu_h^{\mathbb{Q}}\right)' \bar{b}_{n-1,g} \\
&\quad + \frac{1}{2}\bar{b}_{n-1,g}' \Sigma_g \Sigma_g' \bar{b}_{n-1,g} \\
\bar{b}_{n,g} &= \Phi_g^{\mathbb{Q}\prime} \bar{b}_{n-1,g} - \delta_{1,g} \\
\bar{b}_{n,h} &= \sum_{i=1}^{H} \frac{\bar{b}_{n-1,gh,i}}{1 - \sigma_{h,i}\bar{b}_{n-1,gh,i}} \Phi_{h,i}^{\mathbb{Q}} + \left(\Phi_{gh}^{\mathbb{Q}} - \Sigma_{gh}\Phi_h^{\mathbb{Q}}\right)' \bar{b}_{n-1,g} - \delta_{1,h} \\
&\quad + \frac{1}{2}\left(I_H \otimes \bar{b}_{n-1,g}'\right) \bar{\bar{\Sigma}} \left(\iota_H \otimes \bar{b}_{n-1,g}\right)
\end{aligned}
$$

where $\bar{b}_{n,gh} = \Sigma_{gh}' \bar{b}_{n,g} + \bar{b}_{n,h}$ and with the restriction that $\bar{b}_{n-1,gh,i} < \frac{1}{\sigma_{h,i}}$.

Initial conditions:

$$
\bar{a}_1 = -\delta_0, \quad \bar{b}_{1,g} = -\delta_{1,g}, \quad \bar{b}_{1,h} = -\delta_{1,h}
$$

Observed yields

The measurement equations are

$$\begin{aligned} Y_t^{(1)} &= A_1 + B_1 x_t \\ Y_t^{(2)} &= A_2 + B_2 x_t + \eta_t \qquad \eta_t \sim \mathrm{N}(0, \Omega) \end{aligned}$$

$A_1 = (a_{n_1}, \ldots, a_{n_{N_1}})'$, $A_2 = (a_{n_{N_1+1}}, \ldots, a_{n_{N_1+N_2}})'$, $B_1 = (b_{n_1}, \ldots, b_{n_{N_1}})'$, and $B_2 = (b_{n_{N_1+1}}, \ldots, b_{n_{N_1+N_2}})'$.

- $Y_t^{(1)}$ is a $N_1 \times 1$ vector of yields priced without error.
- $Y_t^{(2)}$ is a $N_2 \times 1$ vector of yields priced with error.
- $N_1 = G + H$.
- Factors are observable: $x_t = B_1^{-1}\left(Y_t^{(1)} - A_1\right)$.

Stochastic discount factor

The SDF is

$$m_{t+1} = \frac{\mathrm{N}\left(g_{t+1}|h_{t+1}, g_t, h_t; \theta^{\mathbb{Q}}\right) \mathrm{NCG}\left(h_{t+1}|h_t; \theta^{\mathbb{Q}}\right)}{\mathrm{N}\left(g_{t+1}|h_{t+1}, g_t, h_t; \theta^{\mathbb{P}}\right) \mathrm{NCG}\left(h_{t+1}|h_t; \theta^{\mathbb{P}}\right)}$$

- The ratio of normals: log-normal
- The ratio of non-central gammas: doubly non-central F

Market price of risk

$$\Lambda_t = \Sigma_{t-1}^{-1}\left(\mathbb{E}^{\mathbb{P}}[x_t|x_{t-1}] - \mathbb{E}^{\mathbb{Q}}[x_t|x_{t-1}]\right)$$

$$\mathbb{E}^{\mathbb{P}}[x_t|x_{t-1}] = \begin{pmatrix} \mu_h \\ \mu_g \end{pmatrix} + \begin{pmatrix} \Phi_h & 0 \\ \Phi_{gh} & \Phi_g \end{pmatrix}\begin{pmatrix} h_{t-1} \\ g_{t-1} \end{pmatrix}$$

$$\mathbb{E}^{\mathbb{Q}}[x_t|x_{t-1}] = \begin{pmatrix} \mu_h^{\mathbb{Q}} \\ \mu_g^{\mathbb{Q}} \end{pmatrix} + \begin{pmatrix} \Phi_h^{\mathbb{Q}} & 0 \\ \Phi_{gh}^{\mathbb{Q}} & \Phi_g^{\mathbb{Q}} \end{pmatrix}\begin{pmatrix} h_{t-1} \\ g_{t-1} \end{pmatrix}$$

$$\Sigma_{t-1} = \begin{pmatrix} I_H & 0 \\ \Sigma_{gh} & I_G \end{pmatrix}\begin{pmatrix} \sqrt{\nu_1\sigma_{h,1}^2 + 2\sigma_{h,1}\Phi_{h,1}' h_{t-1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Sigma_{g,t-1} \end{pmatrix}$$

Continuous-time limit: $\mathbb{V}[h_{it}|h_{t-1}] = \nu_i\sigma_{h,i}^2 + 2\sigma_{h,i}\Phi_{h,i}' h_{t-1} \approx 2\sigma_{h,i} h_{i,t-1}$

Reduced form: dynamics

$$
\begin{aligned}
Y_{t+1}^{(1)} &= \mu_1^* + \Phi_1^* Y_t^{(1)} \\
&+\Sigma^* \begin{pmatrix} \sqrt{\alpha_1^* + \gamma_1^* Y_t^{(1)}} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & \sqrt{\alpha_{N_1}^* + \gamma_{N_1}^* Y_t^{(1)}} & 0 \\ 0 & \cdots & 0 & \Sigma_{g,t}^* \end{pmatrix} \begin{pmatrix} \varepsilon_{h,t+1} \\ \varepsilon_{g,t+1} \end{pmatrix}
\end{aligned}
$$

where $\varepsilon_{h,t+1}$ is a standardized NCG r.v. and

$$
\mu_1^* = B_1 \begin{pmatrix} \mu_h \\ \mu_g \end{pmatrix} + (I_{N_1} - \Phi_1^*)\, A_1 \qquad \Phi_1^* = B_1 \begin{pmatrix} \Phi_h & 0 \\ \Phi_{gh} & \Phi_g \end{pmatrix} B_1^{-1}
$$

$$
\Sigma^* = B_1 \begin{pmatrix} I_H & 0 \\ \Sigma_{gh} & I_G \end{pmatrix} \qquad \begin{aligned} \alpha_i^* &= \nu_i \sigma_{h,i}^2 - 2\sigma_{h,i} \Phi_{h,i}' S_{h_i} A_1 \\ \Sigma_g^* &= \Sigma_g \end{aligned} \quad \begin{aligned} \gamma_i^* &= 2\sigma_{h,i} \Phi_{h,i}' S_{h_i} B_1^{-1} \\ \Sigma_{g,i}^* &= \Sigma_{g,i} \end{aligned}
$$

Reduced form: cross-section

$$Y_t^{(2)} = \mu_2^* + \Phi_2^* Y_t^{(1)} + \eta_t, \qquad \eta_t \sim \mathrm{N}\left(0, \Omega^*\right)$$

$$\mu_2^* = A_2 - \Phi_2^* A_1 \qquad \Phi_2^* = B_2 B_1^{-1} \quad \Omega^* = \Omega$$

- Estimation of the reduced form is not as straightforward as Gaussian models.
- QML of reduced form may be possible. Parameters can be concentrated out.
- Hamilton and Wu (2012 JoE)

Conditional moments of observed yields

$$
\begin{aligned}
\mathbb{E}\left[Y_{t+1}^{(1)} | Y_t^{(1)}\right] &= \mu_1^* + \Phi_1^* Y_t^{(1)} \\
\mathbb{V}\left[Y_{t+1}^{(1)} | Y_t^{(1)}\right] &= \Sigma^* \begin{pmatrix} \alpha_1^* + \gamma_1^* Y_t^{(1)} & \cdots & \cdots & 0 \\ \vdots & & \vdots & \vdots \\ \vdots & \cdots & \alpha_H^* + \gamma_H^* Y_t^{(1)} & 0 \\ 0 & \cdots & 0 & \Sigma_t^* \Sigma_t^{*\prime} \end{pmatrix} \Sigma^{*\prime}
\end{aligned}
$$

- Reduced form allows us to calculate moments of *yields*.
- Autocorrelation of $Y_{t+1}^{(1)} | Y_t^{(1)}$ is a VAR(1).
- The conditional variance is a function of a single lag of $Y_t^{(1)}$.
- In an ATSM, autocorrelations of volatility are restricted.

Outline

Model

Estimation

Identifying restrictions

Results

Conclusion

Likelihood function

$$
\begin{aligned}
p\left(Y^{(1)}_{1:T}, Y^{(2)}_{1:T}; \theta\right) &= \left|\det\left(J\right)\right|^{-(T-1)} \prod_{t=2}^{T} p\left(Y^{(2)}_t | x_t; \theta\right) \\
&\quad \prod_{t=2}^{T} p\left(g_t | g_{t-1}, h_t, h_{t-1}; \theta\right) \prod_{t=2}^{T} \prod_{i=1}^{H} p\left(h_{it} | h_{t-1}; \theta\right)
\end{aligned}
$$

▸ Details

- Divide θ into two sub-vectors: $\theta = (\theta_c, \theta_m)$.
- $\theta_c = (\mu_g, \Phi_g, \Phi_{gh}, \Omega)$.
- Given θ_m, we can calculate the bond loadings (A_1, B_1) and

$$
g_t = S_g B_1^{-1}\left(Y^{(1)}_t - A_1\right) \quad h_t = S_h B_1^{-1}\left(Y^{(1)}_t - A_1\right)
$$

- KEY POINT: Concentrate θ_c out of the log-likelihood using GLS.

Can we concentrate out more parameters?

- These are the only parameters that can be "exactly" concentrated out.
- *Approximately* concentrate out μ_h, Φ_h by GLS:

$$h_{t+1} = \mu_h + \Phi_h h_t + \varepsilon_{h,t+1}$$

- *Approximately* concentrate out Σ_{gh} by GLS:

$$g_{t+1} = \mu_g + \Phi_g g_t + \Phi_{gh} h_t + \Sigma_{gh}\varepsilon_{h,t+1} + \Sigma_{g,t}\varepsilon_{g,t+1}$$

- For Gaussian models, *approximately* concentrate out Σ_g using the residuals:

$$g_{t+1} = \mu_g + \Phi_g g_t + \Sigma_g \varepsilon_{g,t+1}$$

Discussion

- Need to invert B_1 to calculate the factors:

$$x_t \ = \ B_1^{-1} \left(Y_t^{(1)} - A_1 \right)$$

- θ_m must satisfy $h_{it} > 0$ for all i and t.
- Need to calculate the GLS/OLS estimator.
- The parameters θ_m must satisfy the restriction $\bar{b}_{n-1,gh,i} < \frac{1}{\sigma_{h,i}}$ in order for bond prices to exist.

Gradients

$$\frac{\partial \ell}{\partial \theta'} = \frac{\partial \ell_{\text{dynamics}}}{\partial \theta'} + \frac{\partial \ell}{\partial A'}\frac{\partial A}{\partial \theta'} + \frac{\partial \ell}{\partial \text{vec}\left(B'\right)'}\frac{\partial \text{vec}\left(B'\right)}{\partial \theta'}$$

- We calculate the gradient analytically.
- Gradients have a simple recursive structure.
- Greatly improves numerical stability.
- Speeds calculations.

Gradients: example # 1

$$\frac{\partial \ell}{\partial \text{vech}\left(\Sigma_g\right)'} = \frac{\partial \ell_{\text{dynamics}}}{\partial \text{vech}\left(\Sigma_g\right)'} + \frac{\partial \ell}{\partial A'} \frac{\partial A}{\partial \text{vech}\left(\Sigma_g\right)'}$$

- The parameter Σ_g enters the dynamics and the bond loadings.
- $\frac{\partial \text{vec}(B')}{\partial \text{vech}(\Sigma_g)'} = 0$.
- The derivative of A w.r.t. Σ_g can be computed recursively.
- Initial condition: $\bar{a}'_{1,\Sigma_g} = 0$.

$$\bar{a}'_{n,\Sigma_g} = \bar{a}'_{n-1,\Sigma_g} + \text{vec}\left(\bar{b}_{n-1,g}\bar{b}'_{n-1,g}\Sigma_g\right)' \mathcal{D}^L_G$$

Gradients: example # 2

$$\frac{\partial \ell}{\partial \mathrm{vec}\left(\Phi_g^{\mathbb{Q}}\right)'} = \frac{\partial \ell}{\partial A'} \frac{\partial A}{\partial \mathrm{vec}\left(\Phi_g^{\mathbb{Q}}\right)'} + \frac{\partial \ell}{\partial \mathrm{vec}\left(B'\right)'} \frac{\partial \mathrm{vec}\left(B'\right)}{\partial \mathrm{vec}\left(\Phi_g^{\mathbb{Q}}\right)'}$$

- The parameter $\Phi_g^{\mathbb{Q}}$ only enters the bond loadings.
- Initial conditions: $\bar{a}'_{1,\Phi_g^{\mathbb{Q}}} = 0 \quad \bar{b}'_{1,\Phi_g^{\mathbb{Q}}} = 0$.

$$\begin{aligned} \bar{a}'_{n,\Phi_g^{\mathbb{Q}}} &= \bar{a}'_{n-1,\Phi_g^{\mathbb{Q}}} + \bar{y}'_{n-1} \bar{b}_{n-1,gh,\Phi_g^{\mathbb{Q}}} + \left(\mu_g^{\mathbb{Q}} - \Sigma_{gh} \mu_h^{\mathbb{Q}}\right)' \bar{b}_{n-1,g,\Phi_g^{\mathbb{Q}}} \\ &\quad + \bar{b}'_{n-1,g} \Sigma_g \Sigma'_g \bar{b}_{n-1,g,\Phi_g^{\mathbb{Q}}} \end{aligned}$$

$$\bar{b}_{n,g,\Phi_g^{\mathbb{Q}}} = \Phi_g^{\mathbb{Q}\prime} \bar{b}_{n-1,g,\Phi_g^{\mathbb{Q}}} + \left(I_G \otimes \bar{b}'_{n-1,g}\right)$$

Extension # 1: macro variables

$$\begin{pmatrix} Y_t^{(1)} \\ Y_t^{(m)} \end{pmatrix} = \begin{pmatrix} A_1 \\ 0 \end{pmatrix} + \begin{pmatrix} B_{1,g} & B_{1,h} & B_{1,m} \\ 0 & 0 & I_M \end{pmatrix} \begin{pmatrix} g_t \\ h_t \\ Y_t^{(m)} \end{pmatrix},$$

$$\begin{pmatrix} g_{t+1} \\ Y_{t+1}^{(m)} \end{pmatrix} = \begin{pmatrix} \mu_g \\ \mu_m \end{pmatrix} + \begin{pmatrix} \Phi_g & \Phi_{gh} & \Phi_{gm} \\ \Phi_{mg} & \Phi_{mh} & \Phi_m \end{pmatrix} \begin{pmatrix} g_t \\ h_t \\ Y_t^{(m)} \end{pmatrix}$$

$$+ \begin{pmatrix} \Sigma_{gh} \\ \Sigma_{mh} \end{pmatrix} \varepsilon_{h,t+1} + \varepsilon_{t+1},$$

$$\varepsilon_{t+1} \sim \mathrm{N}\left(0, \Sigma_t \Sigma_t'\right), \qquad \Sigma_t \Sigma_t' = \Sigma\Sigma' + \sum_{i=1}^{H} \Sigma_i \Sigma_i' h_{it}$$

- Add $M \times 1$ vector of macro variables $Y_t^{(m)}$.
- $\theta_c = (\mu_g, \mu_m, \Phi_g, \Phi_m, \Phi_{mh}, \Phi_{gh}, \Phi_{gm}, \Phi_{mg}, \Omega)$

Extension # 2: estimation under constraints

- It is possible to add linear equality constraints between $\mathbb{P}$ and $\mathbb{Q}$ parameters.
- Concentrating parameters out is constrained least squares.
- Ridge regression of μ_g, Φ_g, Φ_{gh} is straightforward.

Outline

Rotation

$$
\begin{aligned}
r_t &= \delta_0 + \delta_1 x_t & (1) \\
x_t &= \mu + \Phi x_{t-1} + \Sigma \varepsilon_t & (2)
\end{aligned}
$$

Define

$$
\tilde{x}_t = D_0 + D_1 x_t \quad \Rightarrow \quad x_t = D_1^{-1}(\tilde{x}_t - D_0)
$$

Rewrite (1) and (2)

$$
\begin{aligned}
r_t &= \delta_0 + \delta_1 D_1^{-1}(\tilde{x}_t - D_0) \\
D_1^{-1}(\tilde{x}_t - D_0) &= \mu + \Phi D_1^{-1}(\tilde{x}_{t-1} - D_0) + \Sigma \varepsilon_t
\end{aligned}
$$

Equations (1) and (2) are observationally equivalent to:

$$
\begin{aligned}
r_t &= \tilde{\delta}_0 + \tilde{\delta}_1 \tilde{x}_t \\
\tilde{x}_t &= \tilde{\mu} + \tilde{\Phi} \tilde{x}_{t-1} + \tilde{\Sigma} \varepsilon_t
\end{aligned}
$$

Identifying restrictions

Gaussian part

- $\mu_g^{\mathbb{Q}} = 0$
- $\Phi_g^{\mathbb{Q}}$ in ordered Jordan form
- $\Phi_{gh}^{\mathbb{Q}} = 0$
- $\Sigma_g, \Sigma_{g,i}$ lower triangular
- $\delta_{1g} = \iota_G$ a column vector of ones

Non-Gaussian part

- $\delta_{1h,i} = \pm 1 \quad \forall \quad i = 1, \ldots, H$
- $\Phi_h^{\mathbb{Q}}$ in ordered Jordan form
- $h_t > 0 \; \forall \; t$

Local maxima

- $\delta_{1,h}$ is positive or negative?
- Σ_{gh} positive or negative?
- How are the elements across $\Phi_g^{\mathbb{Q}}$ and $\Phi_h^{\mathbb{Q}}$ ordered?
- For non-Gaussian models, you must intentionally find and compare local maxima.

Example 1: $G = 3, H = 0$

Gaussian model

- $\mu_g^{\mathbb{Q}} = 0$
- $\Phi_g^{\mathbb{Q}}$ is diagonal with diagonal elements in descending order
- $\Phi_g^{\mathbb{Q}}$ does/does not have repeated eigenvalues
- Σ_g lower triangular
- $\delta_{1,g} = \iota_G$ a column vector of ones

Example 2: $G = 2, H = 1$

Gaussian part

- $\mu_g^{\mathbb{Q}} = 0$
- $\Phi_g^{\mathbb{Q}}$ is diagonal with diagonal elements in descending order
- $\Phi_g^{\mathbb{Q}}$ does/does not have repeated eigenvalues
- $\Phi_{gh}^{\mathbb{Q}} = 0$
- Σ_g, Σ_{g1} lower triangular
- $\delta_{1g} = \iota_G$ a column vector of ones

Non-Gaussian part

- $\delta_{1h} = 1$
- $h_t > 0 \;\; \forall \;\; t$
- $\nu > 1, \nu^{\mathbb{Q}} > 1$
- $0 < \Phi_h < 1$
- $\bar{b}_{n-1,gh} < \frac{1}{\sigma_h}$

Number of parameters for 3 factor models

[22(10)] Gaussian

- [3]μ_g
- [9]Φ_g
- [0]$\mu_g^{\mathbb{Q}}$
- [3]$\Phi_g^{\mathbb{Q}}$
- [6]Σ_g
- [0]δ_1
- [1]δ_0

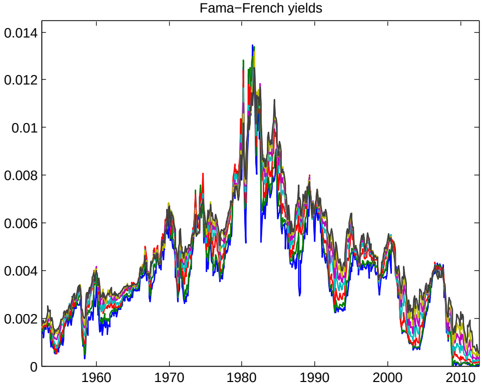
[24(16)]$\mathbb{A}_1(3)$

- [3]$\mu_g + \nu$
- [7]$\Phi_g + \Phi_{gh} + \Phi_h$: minus 2
- [1]$\mu_g^{\mathbb{Q}} + \nu^{\mathbb{Q}}$: plus 1
- [3]$\Phi_g^{\mathbb{Q}} + \Phi_{gh}^{\mathbb{Q}} + \Phi_h^{\mathbb{Q}}$
- [9]$\Sigma_g + \Sigma_{g1} + \Sigma_{gh} + \sigma_h$: plus 3
- [0]δ_1
- [1]δ_0

Comments: Gaussian model has more free parameters in the conditional mean, whereas the non-Gaussian model has more free parameters for the higher moments.

Outline

Data

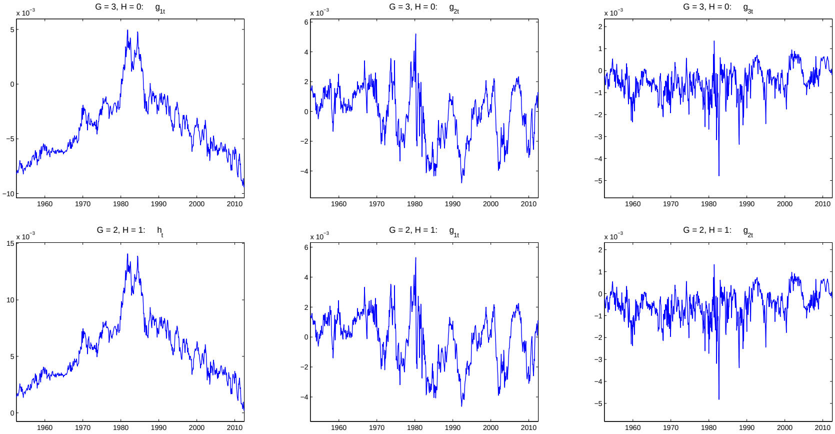


- Fama-Bliss zero-coupon bonds
- $6/1952$ to $6/2012$: $T = 721$ months
- Maturities: 1m, 3m, 1yr, 2yr, 3yr, 4yr, 5yr

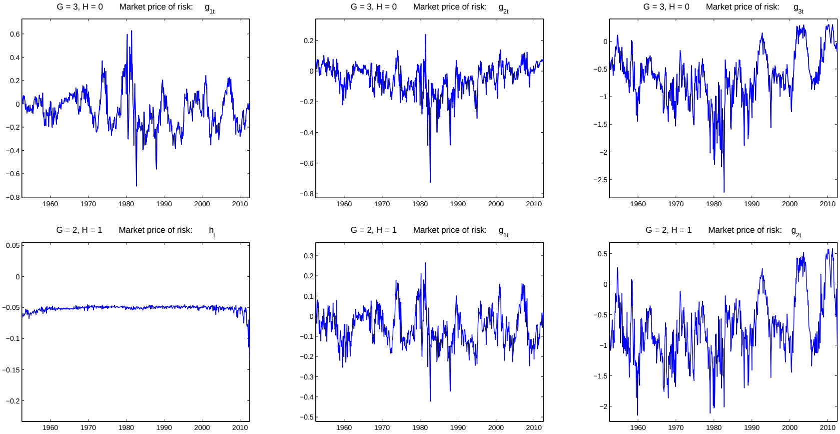
Estimates: 3 factor models

μ_g			μ_h	μ_g		ν
6.97e-05	-4.85e-05	-3.37e-04	2.97e-05	-1.32e-05	3.32e-05	1.934
(6.43e-05)	(4.65e-05)	(6.48e-05)	—	(1.102e-04)	(1.53e-05)	(0.124)
Φ_g			Φ_h			
1.007	0.048	0.067	0.994			
(0.011)	(0.016)	(0.040)	(0.004)			
			Φ_{gh}	Φ_g		
-0.012	0.938	0.0192	0.008	0.984	0.066	
(0.008)	(0.016)	(0.047)	(0.039)	(0.055)	(0.143)	
-0.037	-0.059	0.631	-0.041	-0.073	0.643	
(0.010)	(0.018)	(0.051)	(0.036)	(0.036)	(0.087)	
$\mu_g^{\mathbb{Q}}$			$\mu_h^{\mathbb{Q}}$	$\mu_g^{\mathbb{Q}}$		$\nu^{\mathbb{Q}}$
0	0	0	4.09e-05	0	0	2.637
—	—	—	—	—	—	(0.417)
$\Phi_g^{\mathbb{Q}}$			$\Phi_h^{\mathbb{Q}}$	$\Phi_g^{\mathbb{Q}}$		
0.995	0.954	0.530	0.996	0.951	0.536	
(0.0007)	(0.003)	(0.029)	(0.0009)	(0.003)	(0.033)	
Σ_g			σ_h			
3.99e-04	0	0	1.55e-05			
(2.52e-05)	—	—	(1.60e-06)			
			Σ_{gh}	Σ_g		
-3.09e-04	5.09E-04	0	-0.893	8.20e-09	0	
(3.83e-05)	(3.71E-05)	—	(0.104)	(5.58e-08)	—	
-4.50e-06	-2.52E-04	3.78E-04	0.054	-1.90e-09	1.08e-08	
(9.60e-06)	(2.69E-05)	(2.38E-05)	(0.1004)	(1.23e-07)	(7.33e-08)	
δ_0			δ_0	$\Sigma_{g,1}$		
0.0083			-0.0011	0.0063	0	
(0.0005)			(0.0004)	(0.0005)	—	
				-0.0035	0.0046	
				(0.0003)	(0.0003)	

Factors for 3 factor models



Risk pricing λ_t for 3 factor models



Decomposition

Risk neutral rate

$$\tilde{y}_t^n = \frac{1}{n}\left[r_t + \mathbb{E}\left(r_{t+1}\right) + \ldots + \mathbb{E}\left(r_{t+n-1}\right)\right]$$

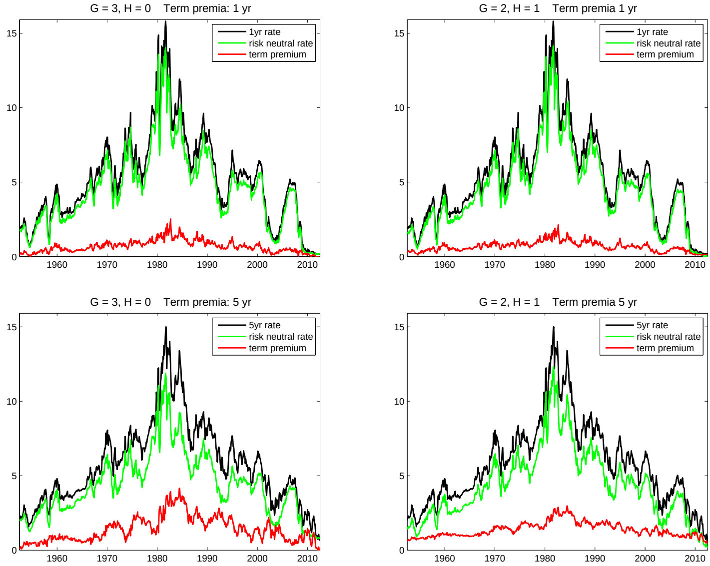
Model implied yield

$$y_t^n = a_n + b_n' x_t$$

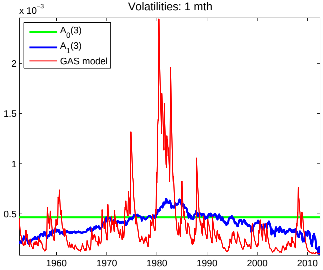
Term premium

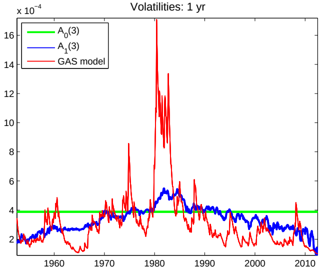
$$tp_t^n = y_t^n - \tilde{y}_t^n$$

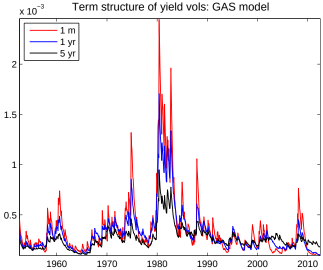
Term premiums: 1 and 5 yrs.

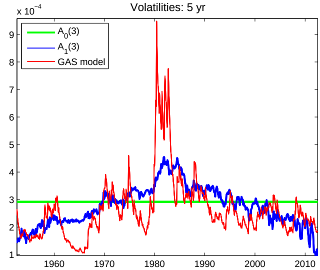


Volatility: Comparing with GAS (Creal, Koopman, Lucas, JAE 2012)









Estimates: 4 factor models

μ_g				μ_h	μ_g			ν		
3.89E-04	-9.73E-04	1.28E-03	-8.19E-04	5.64E-05	3.07E-04	-3.76E-05	-3.10E-04	2.6778		
(1.68E-04)	(2.20E-04)	(2.66E-04)	(3.64E-04)		(3.13E-04)	(3.39E-05)	(3.15E-04)	(2.2549)		
Φ_g				Φ_h						
1.034	0.087	-0.014	0.091	0.990						
(0.038)	(0.058)	(0.016)	(0.034)	(0.008)						
				Φ_{gh}	Φ_g					
-0.078	0.834	0.204	-0.079	0.004	0.867	1.170	0.091			
(0.085)	(0.111)	(0.120)	(0.083)	(0.006)	(0.041)	(1.210)	(0.224)			
0.0890	0.154	0.694	0.1926	0.002	0.015	0.816	0.0028			
(0.112)	(0.177)	(0.174)	(0.148)	(0.003)	(0.003)	(0.106)	(0.017)			
-0.085	-0.141	-0.040	0.546	-0.035	-0.009	-0.124	0.651			
(0.073)	(0.118)	(0.093)	(0.124)	(0.011)	(0.039)	(1.050)	(0.199)			
$\mu_g^{\mathbb{Q}}$				$\mu_h^{\mathbb{Q}}$	$\mu_g^{\mathbb{Q}}$			ν^{Q}		
0	0	0		2.71E-05	0	0	0	1.284		
								(0.338)		
$\Phi_g^{\mathbb{Q}}$				$\Phi_h^{\mathbb{Q}}$	$\Phi_g^{\mathbb{Q}}$					
0.992	0.960	0.876	0.696	0.995	0.912	–	0.702			
(0.003)	(0.013)	(0.033)	(0.043)	(0.002)	(0.015)	–	(0.054)			
Σ_g				σ_h						
6.94E-04	0	0	0	2.11E-05						
(2.29E-04)				(6.28E-06)						
				Σ_{gh}	Σ_g			$\Sigma_{g,1}$		
-1.47E-03	9.77E-04	0	0	0.925	8.37E-04	0	0	1.04E-02	0	0
(3.24E-04)	(4.71E-04)			(0.826)	(5.71E-04)			(3.42E-03)		
1.66E-03	-1.39E-03	8.74E-04	0	-0.217	-9.03E-05	6.85E-13	0	-6.75E-04	8.15E-04	0
(2.24E-04)	(4.43E-04)	(3.23E-04)		(0.057)	(5.25E-05)	(2.24E-05)		(2.97E-04)	(1.01E-04)	
-8.06E-04	5.65E-04	-7.18E-04	4.04E-04	-1.563	-7.96E-04	9.43E-11	4.04E-10	-9.01E-03	1.12E-03	4.68E-03
(3.47E-04)	(1.90E-04)	(3.53E-04)	(2.91E-05)	(0.723)	(5.51E-04)	(9.31E-05)	(3.77E-05)	(3.57E-03)	(7.67E-04)	(3.59E-04)

Conclusion

Methodology: Propose a new estimation approach for non-Gaussian ATSMs.

- Reduce parameter space
- Provide analytical gradient

Applications: Improve performance for any problem with two conditions

- can invert factors from observed yields
- (part of) the factor dynamics has analytical solutions

Caveat and Future directions

Caveat:

- Multiple local maxima due to the nature of the model
- Try to search in each region, and compare likelihood.

What do we use to replace $\mathbb{A}_0(3)$, $\mathbb{A}_1(3)$?

- $\mathbb{A}_1(4)$?, $\mathbb{A}_2(4)$? $\mathbb{A}_2(5)$?
- Unspanned volatility?

Extensions:

- Imposing constraints on parameters is straightforward.
- Adding macroeconomic variables is simple.
- Accommodating more general dynamics, e.g. AR(p).
- Correcting for bias (Bauer, Rudebusch and Wu (JBES 2012)) for the Gaussian dynamics is possible.

Appendix

Vector autoregressive gamma process

Autoregressive gamma process of Gouriéroux and Jasiak (2006)

$$
\begin{aligned}
h_{i,t+1} &\sim \text{Gamma}\left(\nu_i + z_{i,t+1}, \sigma_{h,i}\right) \qquad i = 1, \ldots, H \\
z_{i,t+1} &\sim \text{Poisson}\left(\frac{\Phi'_{h,i} h_t}{\sigma_{h,i}}\right)
\end{aligned}
$$

The transition density is

$$
\begin{aligned}
p(h_{t+1}|h_t;\theta) = & \prod_{i=1}^{H} \left(\frac{h_{i,t+1}}{\Phi_{h,i} h_t}\right)^{\frac{\nu_i - 1}{2}} \exp\left(-\frac{h_{i,t+1} + \Phi'_{h,i} h_t}{\sigma_{h,i}}\right) \\
& \left(\frac{1}{\sigma_{h,i}}\right) I_{\nu_i - 1}\left(2\sqrt{\frac{h_{i,t+1}\Phi'_{h,i} h_t}{\sigma^2_{h,i}}}\right)
\end{aligned}
$$

Back

Convergence to the continuous-time limit

Write the univariate AG(1) as a linear process

$$h_{t+1} \quad = \quad \nu\sigma_h + \Phi_h h_t + \sqrt{\nu\sigma_h^2 + 2\sigma_h \Phi_h h_t}\; \varepsilon_{h,t+1}$$

where ε_{ht} is a standardized NCG r.v. with $\mathbb{E}\left[\varepsilon_{ht}\right] = 0, \mathbb{V}\left[\varepsilon_{ht}\right] = 1$.

Let τ be an interval of time and define

$$\Phi_h \;=\; 1 - \kappa\tau \qquad \nu \;=\; \frac{2\kappa\theta}{\sigma^2} \qquad \sigma_h \;=\; \frac{\sigma^2\tau}{2}$$

The discrete-time process implies

$$\begin{aligned}
\mathbb{E}\left[h_{t+\tau}|h_t\right] &= \nu\sigma_h + \Phi_h h_t = \kappa\theta\tau + (1-\kappa\tau)h_t \\
\mathbb{V}\left[h_{t+\tau}|h_t\right] &= \nu\sigma_h^2 + 2\sigma_h\Phi_h h_t = \frac{\kappa\theta\sigma^2\tau^2}{2} + \sigma^2\tau(1-\kappa\tau)h_t
\end{aligned}$$

Back

Log-likelihood function

$$
\begin{aligned}
\ell_\theta \quad = \quad & \text{CONST} - (T-1)\log\left|\det(B_1)\right| - \frac{T-1}{2}\log|\Omega| - \frac{1}{2}\sum_{t=2}^{T}\text{tr}\left(\Omega^{-1}\eta_t\eta_t'\right) \\
& -\frac{1}{2}\sum_{t=2}^{T}\log\left|\Sigma_{g,t-1}\Sigma_{g,t-1}'\right| - \frac{1}{2}\sum_{t=2}^{T}\text{tr}\left(\left(\Sigma_{g,t-1}\Sigma_{g,t-1}'\right)^{-1}\varepsilon_{gt}\varepsilon_{gt}'\right) \\
& +\sum_{t=2}^{T}\sum_{i=1}^{H}\left[\frac{(\nu_i-1)}{2}\log(h_{it}) - \frac{h_{it}+\Phi_{h,i}'h_{t-1}}{\sigma_{h,i}} - \frac{(\nu_i-1)}{2}\log\left(\Phi_{h,i}'h_{t-1}\right)\right. \\
& \left. -\log(\sigma_{h,i}) + \log\left(I_{\nu_i-1}\left(2\sqrt{\frac{h_{it}\Phi_{h,i}'h_{t-1}}{\sigma_{h,i}^2}}\right)\right)\right]
\end{aligned}
$$

- $\varepsilon_{gt} = g_t - \mu_g - \Phi_g g_{t-1} - \Phi_{gh}h_{t-1} - \Sigma_{gh}\left[h_t - (\mu_h + \Phi_h h_{t-1})\right]$
- Likelihood is (conditionally) quadratic in $\mu_g, \Phi_g, \Phi_{gh}, \Omega$.

Back