Rollover Risk and Market Freezes[1]

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VIRAL V. ACHARYA, DOUGLAS GALE, and TANJU YORULMAZER

ABSTRACT

The debt capacity of an asset is the maximum amount that can be borrowed using the asset as collateral. We model a sudden collapse in the debt capacity of good collateral. We assume short term debt that must be frequently rolled over, a small transaction cost of selling collateral in the event of default, and a small probability of meeting a buy-to-hold investor. We then show that a small change in the asset's fundamental value can be associated with a catastrophic drop in the debt capacity, the kind of market freeze observed during the crisis of 2007-08.

[1] Acharya is at NYU-Stern, CEPR, ECGI, and NBER; Gale is at New York University; and Yorulmazer is at the Federal Reserve Bank of New York. The views expressed here are those of the authors and do not necessarily represent the views of the Federal Reserve Bank of New York or the Federal Reserve System. We are grateful to Dave Backus, Sudipto Bhattacharya, Alberto Bisin, Patrick Bolton, Markus Brunnermeier, John Geanakoplos, Ivalina Kalcheva, Todd Keister, Michael Manove, Martin Oehmke, Onur Ozgur, Matt Pritsker, Hyun Song Shin, S. "Vish" Viswanathan, Andy Winton, and to the participants of the Asset PricingWorkshop at Gerzensee 2009, Bundesbank-CEPR-CFS Conference, the Central Bank Liquidity Tools Conference at the New York Fed, the Bank of Korea International Conference 2009, the FIRS 2009 Conference in Prague, the WFA 2009 Meetings and seminar participants at Carnegie Mellon University, New York Fed, New York University, MIT, the EUI and the Ca' Foscari University of Venice for useful suggestions. Julia Dennett provided excellent research assistance. All errors remain our own.

One of the many striking features of the crisis of 2007-09 has been the sudden freeze in the market for the rollover of short-term debt. From an institutional perspective, the inability to borrow overnight against high-quality but long-term assets was a market failure that effectively led to the demise of a substantial part of investment banking in the United States. More broadly, it led to the collapse, in the United States, the United Kingdom, and other countries, of banks and other financial institutions that had relied on significant maturity mismatch between assets and liabilities, and, in particular, on the rollover of short-term wholesale debt in the asset-backed commercial paper (ABCP) and overnight sale and repurchase (repo) markets.

In this paper, we are interested in developing a model of a sudden collapse in the ability to borrow short-term against long-lived assets in the absence of obvious problems of asymmetric information or fears about the value of collateral. We refer to this phenomenon as a "market freeze." More precisely, a market freeze occurs when the *debt capacity*, the maximum amount of collateralized borrowing that can be supported by an asset, is a small fraction of the *fundamental value*, the economic value measured by the NPV of the stream of returns. An extreme form of a market freeze occurs when the fundamental value is close to the maximum possible value of the asset and the debt capacity is close to the minimum possible value of the asset. We develop a model of debt capacity and provide sufficient conditions for the occurrence of this extreme form of market freeze.

Three assumptions are crucial for our results:

(i) the debt has a much shorter tenor than the assets and needs to be rolled over frequently;

(ii) in the event of default by the borrower, the collateral is sold by the creditors and there is a (small) liquidation cost;

(iii) the probability of meeting an unconstrained buyer for the collateral, that is, a buyer who does not rely on short-term debt finance, is sufficiently low.

We take these features as given, without attempting to rationalize them as the result of equilibrium behavior. For example, we take the (short) tenor of the debt as exogenous. There is ample empirical evidence that financial institutions relied heavily on short-term finance prior to the crisis, but we do not attempt to explain why this was so.[2] We also take

[2] Using data on outstanding repurchase agreements of the US primary dealers (source: Federal Reserve Bank of New York), Morris and Shin (2009) document that during 2003–2007, term repo remained steady around \$1.5 trillion, whereas overnight repo contracts doubled from \$1.5 trillion to \$3 trillion; both shrank by over a trillion dollars by 2009. Acharya, Schnabl and Suarez (2009) show that outstanding ABCP typically had a maturity of less than one week and rose from \$650 billion to over \$1.2 trillion between 2004 and 2Q 2007, only to revert back to its 2004 level by 1Q 2009.

as given the liquidation costs incurred by the sellers of the assets. More precisely, we assume the costs are either fixed or proportional to debt capacity, but in either case the costs are exogenous. Finally, it is important to note that the tenor of the debt and the liquidation costs are assumed to be the same for all market participants. In particular, in our benchmark model, as the tenor of the debt becomes shorter for the owner of the asset, it also becomes shorter for the potential buyers. In an extension, we allow for buy-to-hold investors; our results continue to hold as long as the likelihood of finding such buyers is sufficiently small.

In efficient markets, the debt capacity of an asset is equal to its NPV or "fundamental" value. If there are liquidation costs and a positive probability of default, the debt capacity would naturally be expected to be somewhat less than the fundamental value. In our model we can derive the much stronger result that the gap between the fundamental value and the debt capacity can be significantly large even if the liquidation cost incurred in the event of default is tiny. More precisely, *when the tenor of the debt is sufficiently short*, other things being equal, the debt capacity can equal the *minimum possible future value of the asset*.

The intuition for the result is as follows. When the tenor of the debt is short, the probability of receiving good news about the asset before the next roll-over date is very small. Then it is very likely that the next refinancing will be undertaken with the same information as in the current period. The maximum amount that can be borrowed without a substantial risk of default is equal to the debt capacity at the next rollover date, assuming good news does not arrive in the interim. The borrower will choose to avoid a substantial risk of default because he wants to avoid the liquidation costs. This means that today's debt capacity is less than or equal to the debt capacity at the next rollover date. Applying this argument repeatedly shows that today's debt capacity must be less than or equal to the debt capacity at the maturity of the assets, assuming no arrival of good news.

We have described the market freeze as resulting from the lack of arrival of good news about the fundamental value of the debt when the tenor of the debt is very short and constant. We can also interpret the market freeze as resulting from a sudden *shortening* in the tenor of the debt. If the arrival of bad news, that perhaps signals a small change in the fundamental value of the assets, also causes lenders to restrict the tenor of the debt they are willing to hold, the fall in the debt capacity will be substantial as we have characterized. Thus, it is not necessary to assume that banks choose short-term finance from the outset. The freeze may result from lenders suddenly shortening the tenor of the paper they are willing to hold.[3]

Our model captures some of the elements of the collapse in short-term asset-backed financing witnessed during the crisis of 2007-09. The first such collapse occurred in the summer of 2007. While many special purpose vehicles financed by ABCP had purchased liquidity guarantees from third parties, the providers of these guarantees were themselves

[3] This interpretation was suggested to us by Arvind Krishnamurthy.

feared to be under-capitalized. The money market funds that provided ABCP thus faced the risk of liquidating assets, many of which were asset-backed securities that had little trading liquidity. Acharya, Schnabl and Suarez (2009) document a similar phenomenon in the case of the case of bank-sponsored conduits.[4] Goldsmith-Pinkham and Yorulmazer (2010) analyze a similar episode in which financing of long-term mortgages with short-term wholesale debt led to the near failure of Northern Rock in the United Kingdom in September 2007.[5] The failure of Bear Stearns due to sudden fall in its ability to roll over overnight repo financing in mid-March 2008 is another example of a market freeze. In his analysis of the failure of Bear Stearns, the Federal Reserve Chairman Ben Bernanke observed that "repo markets could be severely disrupted when investors believe they might need to sell the underlying collateral in illiquid markets" (Remarks to the Risk Transfer Mechanisms and Financial Stability Workshop at the Bank for International Settlements, May 29, 2008).[6]

In addition to helping us understand this recent past, our model may suggest ways to increase the stability of the financial system. Understanding the causes of market freezes is a necessary step toward creating a more stable and efficient financial system for the future. Following the crisis, the parallel ("shadow") banking system, consisting of special purpose vehicles such as SIVs and conduits, securities lending, repo financing etc., has shrunk significantly and reduced the financial system's lending capacity by several trillion dollars. While some of this collapse was driven by concerns regarding the quality of the assets, liquidity issues relating to the heavy reliance of a large part of the financial sector on short-term rollover debt also played an important role. Restoring the parallel banking system is seen by many as an important step in the reconstruction of the financial system to provide credit.[7] Our paper highlights the need to address the problem of rollover risk in short-term financing of long-term assets in order to avoid the instability of the past.

The rest of the paper is organized as follows. Section 1 provides an introduction to the

[4] There was a reduction in the rollover of ABCP, the cost of rolling over rose from 10 basis points relative to the Federal Funds rate prior to August 7, 2007, to over 150 bps, and many conduits had to be taken back by banks onto their balance-sheets.

[5] Northern Rock had a balance-sheet featuring significant maturity mismatch. Soon after Northern Rock's woes, other UK banks such as HBOS, Alliance and Leicester, and Bradford and Bingley, that had relied primarily on short-term wholesale debt, suffered too.

[6] Bear Stearns relied day-to-day on its ability to obtain short-term finance through repo borrowing. At this time, Bear was reported to be financing $85 billion of a pool of assets, mostly mortgage- and asset-backed securities, on the overnight market (Cohan, 2009). Beginning late Monday, March 10, even though Bear Stearns continued to have high quality collateral, counterparties became unwilling to lend on customary terms, likely fearing the cost of liquidating the collateral in an illiquid market. At the end of the week, the Federal Reserve stepped in and helped arrange a takeover bid by J.P. Morgan Chase. (Securities and Exchange Commission, 2008).

[7] In addition, many of these assets are now held directly by central banks or by commercial and investment banks relying on lending facilities provided by the central banks. Some day these holdings will have to find another home and the most likely place would be a revitalized and more stable parallel banking system.

model and results in terms of a simple numerical example. Section 2 derives the main result for the special case of the model with two states. It also illustrates, in terms of a numerical example, that market freezes can occur even if the debt maturity is not as "short" as our main result requires. Section 3 provides a complete characterization of the debt capacity for the general model and extends the limit result to an arbitrary number of states. The proof of the limit result is relegated to Appendix A. Section 4 discusses the related literature. Section 5 concludes.

1 Model and results

In this section, we introduce the essential ideas in terms of a numerical example. For concreteness, consider the case of a bank that wishes to repo an asset. The question we ask is: What is the maximum amount of money that the bank can borrow using the asset as collateral? There are two ways to interpret this exercise. We can imagine that a value maximizing bank is trying to maximize its return on equity by minimizing the amount of capital needed to finance the assets it owns. In this case, every bank that purchases the asset is assumed to have the same motive for maximizing leverage.[8] Alternatively, we can simply see our exercise as establishing a bound on the amount that can be borrowed, assuming that other buyers in the market are limited by a similar bound.

Time is represented by the unit interval $[0, 1]$. The asset is purchased at the initial date $t = 0$. The asset has a finite life (e.g., mortgages) which we normalize to one unit. To keep the analysis simple, we assume that the asset has a terminal value at $t = 1$, but generates no income at the intermediate dates $0 \leq t < 1$. We also assume that the risk-free interest rate is 0 and that all market participants are risk neutral.

The arrival of information is modeled as a continuous-time stochastic process. For simplicity, let us assume that there are two states, a low state L and a high state H. At any point in time, the state is publicly observed. Transitions between states are governed by a stationary Markov chain. Transition probabilities depend on the period of time during which the transition occurs, but not the dates. If we are considering a transition during the period $[0, t]$, the transition probability matrix is denoted by

$$\mathbf{P}(t) = \begin{bmatrix} p_{LL}(t) & p_{LH}(t) \\ p_{HL}(t) & p_{HH}(t) \end{bmatrix},$$

[8] This seems consistent with the evidence of Adrian and Shin (2010) that asset growth (shrinkage) of broker-dealers is coincident with equivalent growth (shrinkage) in their leverage, especially so for repo financing. Acharya, Schnabl and Suarez (2009) also describe how conduits had little equity of their own and were largely financed with extremely short-term ABCP. They also explain why the conduit activity is consistent with minimizing capital of sponsor banks.

where $p_{LH}(t)$ is the probability of a transition between the low state at time 0 and the high state at time t and $p_{HL}(t)$ is the probability of the transition from the high state at time 0 and the low state at time t. We assume that the transition matrix takes the form

$$\mathbf{P}(t) = e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!},$$

where the matrix $\mathbf{A}$ is the generator. Since the transition probabilities in any state must sum to 1 the rows of $\mathbf{A}$ must sum to 0. The crucial feature of the transition matrix is that the probability of a change of state converges to 0 as $t \to 0$. That is, as $t \to 0$, $\mathbf{P}(t) \to \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

The terminal value of the asset depends on the state of the economy at the terminal date $t = 1$. The terminal value of the asset is v^H in the high state and v^L in the low state, where $v^H > v^L > 0$.

We assume that the asset will be financed by debt that has to be rolled over repeatedly. The debt is assumed to have a fixed maturity, denoted by $0 < \tau < 1$, so that the debt must be rolled over N times, where

$$\tau = \frac{1}{N+1}.$$

The unit interval is divided into intervals of length τ by a series of dates denoted by t_n and defined by

$$t_n = n\tau, \ \ n = 0, 1, ..., N+1,$$

where t_0 is the date the asset is purchased, t_n is the date of the n-th rollover (for $n = 1, ..., N$), and t_{N+1} is the final date at which the asset matures and the terminal value is realized. This time-line is illustrated in Figure 1.

— Figure 1 here —

If the bank is forced to default, the lenders will seize and liquidate the collateral. In this event, we assume that the lenders incur a small liquidation cost, so that the net amount recovered is a fraction $\lambda \in [0, 1]$ of the sale price. This assumption has several components. In the first place, it implies that the seized collateral is liquidated, i.e., sold to another buyer. Secondly, the new buyers are also finance constrained so that the sale price is equal to the maximum amount of finance that can be raised using the asset as collateral. Thirdly, the process of seizing and disposing of collateral is not costless. For concreteness, we can think of the liquidation cost $1 - \lambda$ as a transaction cost (legal costs, commissions, fees, time delay, etc.), although other interpretations are possible (see Pedersen, 2009, for a discussion of variety of transactions costs and illiquidity in markets). Note that similar results could be obtained with a fixed liquidation cost (see the online appendix).

As an example, suppose that a bank has borrowed 90 and, when it comes time to refinance, finds that it can only raise 87 using the assets as collateral. The lender, say a Money Market Fund, cannot hold the collateral and is forced to dispose of it. A finance constrained buyer can borrow 87 using the assets as collateral, so this is the maximum that it can pay for the assets. However, the amount received by the lender will be a smaller amount, say, 86, because transaction costs have to be subtracted from the sale price.

It is crucial for our argument that the recovery rate λ is applied to the sale price rather than to the fundamental value of the assets. If the buyer of the assets were a wealthy investor who could *buy and hold* the assets until maturity, the fundamental value would be the relevant benchmark. The investor might well be willing to pay some fraction of the fundamental value, although he would presumably try to get the assets for less, recognizing the lender's eagerness to dispose of the collateral. What we are assuming here, by contrast, is that the buyer of the assets is another financial institution that must also issue short-term debt in order to finance the purchase. (We discuss an extension to allow for the presence of buy-to-hold investors in Section 1.2. Also see the online appendix). Hence, the buyer is constrained by the same forces that determined the debt capacity in the first place. Note that the buyer's subjective valuation of the assets might be much greater than the debt capacity, but the finance constraint prevents him from offering to pay his full value.

1.1 A numerical example

To illustrate the method of calculating debt capacity in the presence of rollover risk, we use the following parameter values: the recovery rate is $\lambda = 0.90$, the tenor of the repo is $\tau = 0.01$, the values of the asset are $v^H = 100$ and $v^L = 50$ in the high and low states, respectively, and the generator is

$$\mathbf{A} = \begin{bmatrix} -8.0 & 8.0 \\ 0.1 & -0.1 \end{bmatrix}. \tag{1}$$

To illustrate intuitively how the generator $\mathbf{A}$ might arise, consider an alternative formulation in which information events occur at discrete intervals. The arrival of an information event is governed by a Poisson process with parameter $\alpha > 0$. That is, the probability that an information event occurs in a short time interval $[t, t + \Delta t]$ is $\alpha \Delta t$. When an information event occurs, the state of the economy changes randomly according to a fixed probability transition matrix

$$\mathbf{P} = \begin{bmatrix} p_{LL} & p_{LH} \\ p_{HL} & p_{HH} \end{bmatrix},$$

where p_{LH} (p_{HL}) represents the probability of switching from state L (H) to state H (L) at an information event, and p_{LL} (p_{HH}) represents the probability of staying in state L (H). Note that the occurrence of an information event is itself random, so the number of information events in a given time interval is random and affects the probability of observing a transition from one state to another.

When the Poisson parameter is $\alpha = 10$ and the probability transition matrix is given as

$$\mathbf{P} = \begin{bmatrix} 0.20 & 0.80 \\ 0.01 & 0.99 \end{bmatrix},$$

we obtain the same transition probability matrix for any time interval with length $\tau \in (0,1]$ as in the stationary Markov chain formulation with the generator matrix $\mathbf{A}$ given in (1).[9]

The transition probability matrix for an interval of unit length can be calculated to be

$$\mathbf{P}(1) = \begin{bmatrix} 0.01265 & 0.98735 \\ 0.01234 & 0.98766 \end{bmatrix}. \tag{2}$$

At time 1, the fundamental values are 100 in state H and 50 in state L by assumption. So the fundamental values at time 0 can be calculated by using the terminal values and the transition probabilities in the matrix $\mathbf{P}(1)$. The fundamental value in state H at time 0 is

$$V_0^H = 0.98766 \times 100 + 0.01234 \times 50 = 99.383$$

since, starting in state H at time 0, there is a probability 0.98766 of being in state H and a probability 0.01234 of being in state L at time 1. Similarly, the fundamental value in state L at time 0 is

$$V_0^L = 0.98735 \times 100 + 0.01265 \times 50 = 99.367.$$

Note that the fundamental values are nearly identical. In spite of this, we shall find that the debt capacity of the asset, defined to be the maximum amount that can be borrowed using the asset as collateral, can be very different in the two states.

Whereas the fundamental value only depends on the state, debt capacity is determined by equilibrium in the repo market and has to be calculated for every one of the dates, $t_0, ..., t_{99}$, at which repo contracts mature. To do this, we first have to calculate the transition probabilities over an interval of length $\tau = 0.01$, that is, the length of the period between rollover dates. We find that

$$\mathbf{P}(0.01) = \begin{bmatrix} 0.92315 & 0.07685 \\ 0.00096 & 0.99904 \end{bmatrix}. \tag{3}$$

[9] In particular, the probability transition matrix for a time interval of length τ is given as $\mathbf{P}(\tau) = \sum_{k=0}^{\infty} \left\{ \left(\frac{e^{-\alpha\tau}(\alpha\tau)^k}{k!} \right) \begin{bmatrix} p_{LL} & p_{LH} \\ p_{HL} & p_{HH} \end{bmatrix}^k \right\}$. For the numerical example, we approximate this as $\mathbf{P}(\tau) = \sum_{k=0}^{200} \left\{ \left(\frac{e^{-10\tau}(10\tau)^k}{k!} \right) \begin{bmatrix} 0.20 & 0.80 \\ 0.01 & 0.99 \end{bmatrix}^k \right\}$.

Notice that the initial state has a much larger impact on the transition probabilities in $\mathbf{P}(0.01)$ than it does in $\mathbf{P}(1)$. For example, the probability of ending up in state H after an interval 0.01 has passed is almost 1 if you start in state H but is close to 0.077 if you start in state L. This is because the interval is so short that the state is unlikely to change before the next rollover date.

Consider now the debt capacities at the last rollover date $t_{99} = 0.99$. In what follows, we let D denote the face value of the debt issued and denote the value of D that maximizes the market value of debt at date t_n in state s by D_n^s. It is never desirable to choose $D > 100$ because this leads to default in both states, with associated liquidation costs, but without any increase in the payoff. For values of D between 50 and 100 or less than 50, the expected value of the debt is increasing in D holding constant the probability of default. Then it is clear that the relevant face values of debt (D) to consider are 50 and 100. For any other face value we could increase D without changing the probability of default.

If we set $D = 50$, the debt can be paid off at date 1 in both states and the expected value of the payoff is 50. So the market value of the debt with face value 50 is exactly 50.

Now suppose we set $D = 100$. There will be default in state L, but not in state H, at time $t = 1$. The payoff in state H will be 100 but the payoff in state L will be $(0.9)\,50 = 45.0$, because the recovery rate after default is 0.90. The market value of the debt at time t_{99} will depend on the state at time t_{99}, because the transition probabilities depend on the state. We can easily calculate the expected payoffs in each state:

$$\begin{aligned} \text{state H} &: \quad 0.99904 \times 100 + 0.00096 \times 0.9 \times 50 = 99.947; \\ \text{state L} &: \quad 0.07685 \times 100 + 0.92315 \times 0.9 \times 50 = 49.226. \end{aligned}$$

For example, if the state is H at date t_{99}, then with probability 0.99904 the state is H at date 1 and the debt pays off 100 and with probability 0.00096 the state is L at date 1, the asset must be liquidated and the creditors only realize 45.

Comparing the market values of the debt with the two different face values, we can see that the face value that maximizes the market value of debt will depend on the state. In state H, the expected value of the debt when $D = 100$ is $99.947 > 50$, so that $D_{99}^H = 100$. In state L, on the other hand, the expected value of the debt with face value $D = 100$ is only $49.226 < 50$, so the face value that achieves the debt capacity is $D_{99}^L = 50$. Thus, if we use the notation B_n^s to denote the debt capacity in state s at date t_n, we have shown that $B_{99}^H = 99.947$ and $B_{99}^L = 50$.

Next, consider the debt capacities at date $t_{98} = 0.98$. Now, the relevant face values to consider are 50 and 99.9470 (since these are the maximum amounts that can be repaid in each state at date t_{99} without defaulting and incurring the associated liquidation costs).

If $D = 50$, the expected payoff is 50 too, since the debt capacity at date t_{99} is greater than or equal to 50 in both states and, hence, the debt can always be rolled over. In contrast, if $D = 99.947$, the debt cannot be rolled over in state L at date t_{99} and the liquidation cost is incurred. Thus, the expected value of the debt depends on the state at date t_{98}:

$$\begin{aligned} \text{state H} &: \quad 0.99904 \times 99.9470 + 0.00096 \times 0.9 \times 50 = 99.894, \\ \text{state L} &: \quad 0.07685 \times 99.9470 + 0.92315 \times 0.9 \times 50 = 49.222. \end{aligned}$$

Comparing the expected value corresponding to different face values of the debt, we see that the face value that achieves the debt capacity is $D_{98}^H = 99.947$ in state H and $D_{98}^L = 50$ in state L, so that the debt capacities are $B_{98}^H = 99.894$ and $B_{98}^L = 50$. In fact, we did not really need to do the calculation again to realize that $B_{98}^L = 50$. The only change from the calculation we did at t_{99} is that the payoff in state H has gone down, so the expected payoff from setting $D = 99.947$ must have gone down too and, *a fortiori*, the face value that maximizes the market value of debt must be 50.

It is clear that we can repeat this argument indefinitely in state L. At each date t_n, the debt capacity in the high state is lower than it was at t_{N+1} and the debt capacity in the low state is the same as it was at t_{N+1}. These facts tells us that if the face value that achieves the debt capacity at t_{n+1} is $D_{n+1}^L = 50$, then *a fortiori* it will be $D_n^L = 50$ at date t_n. Thus, the debt capacity is equal to 50 at each date t_n, including the first date $t_0 = 0$.

What is the debt capacity in state H at t_0? The probability of staying in the high state from date 0 to date 1 is $(0.99904)^{100} = 0.90842$ and the probability of hitting the low state at some point is $1 - 0.90842 = 0.09158$ so the debt capacity at time 0 is

$$B_0^H = 0.90842 \times 100 + 0.09158 \times 0.9 \times 50 = 94.9603.$$

So the fall in debt capacity occasioned by a switch from the high to the low state at time 0 is $94.963 - 50 = 44.963$ compared to a change in the fundamental value of $99.383 - 99.367 = 0.016$. This fall is illustrated sharply in Figure 2, which shows that, while fundamental values in states H and L will diverge sharply at maturity, they are essentially the same at date 0. Nevertheless, debt capacity in state L is simply the terminal value in state L. Thus, a switch to state L from state H produces a sudden drop in debt capacity of the asset.

— Figure 2 here —

1.2 Discussion

The intuition for the market freeze result can be explained in terms of the tradeoff between the costs of default and the face value of the debt. Suppose we are in the low information state at date t_n. If the period length τ is sufficiently short, it is very likely that the information state at the next rollover date t_{n+1} will be the low state. Choosing a face value of the debt greater than B_{n+1}^L, the debt capacity in the same state at date t_{n+1}, will increase the payoff to the creditors if the state switches to H at the next date, but it will also lead to default if the state remains L. Since there is a liquidation (transaction) cost, issuing debt with face value greater than the debt capacity is always unattractive if the probability of switching to state H is sufficiently small. Then, the best the borrower can do is to issue debt with a face value equal to the debt capacity assuming the state remains L. But this implies that the debt capacity in the low state is v^L at every date. In other words, no matter how high the fundamental value is in state L, the borrower is forced to act as if the asset is only worth v^L in order to avoid default.

In the remainder of this section, we consider the role of the different assumptions of the model in driving the limit result on market freezes.

Credit risk If $v^H = v^L$, the terminal value of the asset is equal to the fundamental with probability one, so we can set the face value of the debt equal to $v^H = v^L$ without any risk of default. In this case, the debt capacity must be equal to the fundamental value regardless of any other assumptions. So one necessary assumption is the existence of credit risk, that is, a positive probability that the terminal value of the asset will be less than the initial fundamental value. However, this credit risk can be arbitrarily small, as we illustrated in the numerical example where, at time 0, the probability that the asset's terminal value is 50 is less than 0.01. We could obtain the same results for even smaller values of credit risk at the cost of increasing the number of rollovers.

Liquidation cost We need a liquidation cost in order to have a market freeze. If the recovery ratio is $\lambda = 1$, then regardless of the credit risk, the debt capacity will equal the fundamental value. To see this, simply put the face value of the debt equal to 100 at each date. The market value of the debt will equal the fundamental value of the asset, which must equal the debt capacity. So a necessary condition of the market freeze is $\lambda < 1$. The liquidation cost does not need to be large, however. In the numerical example, the loss ratio was 0.1 and it could be made even smaller with an appropriate reduction in the maturity of the debt.

Debt finance Among the key assumptions of our model, we take as given that asset purchases are entirely debt-financed, not just for the initial owner of the assets but for all potential buyers. In particular, this assumption rules out the presence of any long-term or buy-to-hold investors. However, this assumption can be relaxed.

Suppose that, when assets are being liquidated, the buyer found by the liquidating creditors is, with probability $1 - \beta$, short-term debt financed and, with probability β, he is financed by long-term debt or equity. We can think of the buyer with long-term finance as a buy-to-hold investor, such as Warren Buffett, who is willing and able to pay a fraction, possibly 100%, of the fundamental value. With this modification, we show in the online appendix that a market freeze occurs under the usual assumptions if the probability β is not too large. Intuitively, if liquidating creditors are certain to find a buyer who can pay the fundamental value of the asset, then our backward induction mechanism fails and there can be no market freeze. However, if such buyers are scarce, because the extent of free long-term capital in the financial sector is limited, then most buyers are also short-term debt financed and our mechanism is back at work. We show in the online appendix that our main result on the sharp drop in the debt capacity of the asset and the market freeze can easily be obtained for a reasonable set of parameter values such as the probability of meeting a buy-to-hold investor β being less than 10%.

Short-term debt As a practical matter, many financial firms are indeed funded with short-term rollover debt. There exist agency-based explanations in the literature (for example, Flannery, 1986, Diamond, 1989, 1991, 2004, Calomiris and Kahn, 1991, and Diamond and Rajan, 2001a, 2001b) for the existence of short-term debt as optimal financing in such settings. In contrast to this literature, Brunnermeier and Oehmke (2009) consider a model where a financial institution is raising debt from multiple creditors and argue that there may be excessive short-term debt in equilibrium as short-term debt issuance dilutes long-term debt values and creates among various creditors a "maturity rat race." Other reasons for the use of short-term debt are the attraction of betting on interest rates if bankers have short-term horizons and choose to shift risk (see, for example, Allen and Gale, 2000, and Acharya, Cooley, Richardson and Walter, 2009).

Rollover frequency We have highlighted the role of rollover risk and indeed our main result requires that the rate of refinancing be sufficiently high in order to obtain a market freeze. Figure 3 illustrates the role of rollover frequency on debt capacity in state L by varying the number of rollovers as $N = 10, 50$ and 100. Debt capacity with just 10 rollovers is over 90, but falls rapidly to just above 60 with 50 rollovers, and 100 rollovers are sufficient to obtain the limiting result that debt capacity is the terminal value of 50 in state L.

— Figure 3 here —

Even if the period length is longer than our result requires, so that the borrower sets the face value greater than the debt capacity (in the same state at the next rollover date), it is still possible that a market freeze occurs, as we show with a numerical example in Section 2.

Information structure The crucial property of the information structure is that $\mathbf{P}(\tau) \to \mathbf{I}$ as $\tau \to 0$, that is, the probability of a change in state in any rollover period gets smaller as the period length gets smaller. Since the number of rollovers N determines the period length τ, in fact, $\tau = \frac{1}{N+1}$, as the number of rollovers increases, τ gets smaller and information arrives slowly relative to rollovers.

Note that we do not make any special assumptions about the generator $\mathbf{A}$. In particular, we can impose a substantial amount of symmetry if desired. For example, the information state can be a symmetric random walk with reflecting barriers. The only essential property is that the probability of a change in states converges to zero as the period length converges to zero.

2 Debt capacity with two states

In this section we provide a proof for the market freeze result when there are two states. We make the same assumptions as for the numerical example but the parameters are otherwise arbitrary. For the time being, we treat the tenor of the commercial paper τ and the number of rollovers N as fixed. Later, we will be interested to see what happens when the tenor τ becomes very small and the number of rollovers N becomes correspondingly large.

There are two states, a "low" state L and a "high" state H. Transitions occur between the rollover dates t_n and are governed by a stationary transition probability matrix

$$\mathbf{P}(\tau) = \begin{bmatrix} p_{LL}(\tau) & p_{LH}(\tau) \\ p_{HL}(\tau) & p_{HH}(\tau) \end{bmatrix},$$

where $p_{HL}(\tau)$ $(p_{LH}(\tau))$ is the probability of a transition from state H (L) at time t_n to state L (H) at time t_{n+1}. The one requirement we impose on these probabilities is that the shorter the period length τ, the more likely it is that there is no change in states before the next rollover date:

$$\lim_{\tau \to 0} p_{HL}(\tau) = \lim_{\tau \to 0} p_{LH}(\tau) = 0.$$

The terminal value of the asset is v^H if the terminal state is H and v^L if the terminal state is L, where $0 < v^L < v^H$.

In the numerical example, we saw that the borrower chooses a low face value of the debt in the low state and a high face value of the debt in the high state. Here we will provide necessary and sufficient conditions under which choosing high and low face values in the high and low states, respectively, will achieve the debt capacity in those states. We begin by considering the low state.

The low state Suppose that the economy is in the low state at date t_N, which is the last of the rollover dates. Let D be the face value of the debt issued by the bank. If $D > v^H$, the bank will default in both states at date t_{N+1} and the creditors will receive λv^H in the high state and λv^L in the low state.[10] Clearly, the market value of the debt at date t_N would be greater if the face value were $D = v^H$, so the borrower will never choose $D > v^H$. Now suppose that the bank issues debt with face value D, where $v^L < D < v^H$. This will lead to default in the low state at date t_{N+1} and the creditors will receive D in the high state and λv^L in the low state. Clearly, this is dominated by choosing a higher value of D. Thus, either $D = v^H$ or $D \leq v^L$. An exactly similar argument shows that the borrower will never choose $D < v^L$, so we are left with only two possibilities, either $D = v^H$ or $D = v^L$. In the first case, the market value of the debt is $p_{LL}(\tau)\lambda v^L + p_{LH}(\tau) v^H$ and in the second case it is v^L. A necessary and sufficient condition for D_N^L to equal v^L is

$$p_{LL}(\tau)\lambda v^L + p_{LH}(\tau) v^H \leq v^L. \tag{4}$$

This condition will clearly be satisfied for all $\tau > 0$ sufficiently small, but for the time being we will simply assume that (4) is satisfied.

Now suppose that (4) is satisfied and that $B_{n'+1}^L = v^L$ for $n' = n, ..., N$. Consider what happens in the low state at date t_n. By the usual argument, the only candidates for the face value that maximizes the market value of debt are $D = v^L$ and $D = B_{n+1}^H$. If the face value is $D = v^L$, the creditors will receive v^L in both states at date t_{n+1} and the market value of the debt at date t_n will be v^L. On the other hand, if the face value of the debt is $D = B_{n+1}^H$, the creditors receive B_{n+1}^H in the high state and λv^L in the low state, so the market value of the debt at date t_n is

$$p_{LL}(\tau)\lambda v^L + p_{LH}(\tau) B_{n+1}^H \leq p_{LL}(\tau)\lambda v^L + p_{LH}(\tau) v^H,$$

since $B_{n+1}^H \leq v^H$. But (4) implies that $p_{LL}(\tau)\lambda v^L + p_{LH}(\tau) v^H \leq v^L$, so the debt capacity is $B_n^L = v^L$. In fact, this induction argument shows that the debt capacity is $B_n^L = v^L$ for all $n = 1, ..., N$.

The high state Now consider the high state. Again, our two candidates for the face value of the debt at each date t_n are B_{n+1}^H and v_L. Let us assume that at each date t_n the face value of the debt is set equal to the future debt capacity B_{n+1}^H, that is, we begin at date t_N by setting $D_N^H = v^H$ and $B_N^H = p_{HH}(\tau) v^H + p_{HL}(\tau)\lambda v^L$ and then recursively define $D_n^H = B_{n+1}^H$ and

$$B_n^H = p_{HH}(\tau) B_{n+1}^H + p_{HL}(\tau)\lambda v^L,$$

for $n = 1, ..., N-1$. It can easily be shown by backward induction that $B_n^H \leq B_{n+1}^H$ for any n, so in order to show that this strategy will be chosen, it is necessary and sufficient to show

[10] To simplify the argument, we are assuming that there is a liquidation cost at date t_{N+1}, too. None of the results depend on this.

that $B_0^H \geq v^L$. By repeated substitution we can show that

$$\begin{aligned}
B_0^H &= p_{HH}(\tau) B_1^H + p_{HL}(\tau) \lambda v^L \\
&= p_{HH}(\tau) \left\{ p_{HH}(\tau) B_2^H + (1 - p_{HH}(\tau)) \lambda v^L \right\} + (1 - p_{HH}(\tau)) \lambda v^L \\
&= (p_{HH}(\tau))^2 \left(B_2^H - \lambda v^L \right) + \lambda v^L \\
&\dots \\
&= (p_{HH}(\tau))^N \left(v^H - \lambda v^L \right) + \lambda v^L.
\end{aligned}$$

Then the face value that achieves the debt capacity is $D_n^H = B_{n+1}^H$ for all n if and only if

$$(p_{HH}(\tau))^N \left(v^H - \lambda v^L \right) + \lambda v^L \geq v^L$$

or

$$v^H - \lambda v^L \geq \frac{(1-\lambda) v^L}{(p_{HH}(\tau))^N}. \tag{5}$$

We have thus proved the following proposition.

Proposition 1 *Define* $\left\{ \left(B_n^H, D_n^H, B_n^L, D_n^L \right) \right\}_{n=0}^N$ *by setting*

$$D_n^H = B_{n+1}^H, \tag{6}$$

$$B_n^H = p_{HH}(\tau) B_{n+1}^H + p(\tau) \lambda v^L, \tag{7}$$

and

$$D_n^L = B_n^L = v^L, \tag{8}$$

for $n = 1, ..., N$. *The values defined by (6-8) constitute a solution to the problem of achieving debt capacity if and only if (4) and (5) are satisfied.*

The qualitative properties of the debt capacities characterized in Proposition 1 are the same as in the numerical example in Section 1.1. In the low state, the debt capacity B_n^L is constant and equal to the lowest possible terminal value, v^L. The fundamental value of the asset in the low state V_n^L is greater than the debt capacity at every date t_n except at the terminal date, when they are both equal to v^L. In the high state, the debt capacity B_n^H is always less than the fundamental value V_n^H, except at the terminal date when both are equal to v^H. We call this behavior of the debt capacity a "market freeze" since a switch in the information state from high state to the low state can produce a sudden, sharp drop in debt capacity that is much larger than the drop in the fundamental value associated with the same switch.

2.1 Satisfying the conditions for a market freeze

In the two-state model, there are two necessary and sufficient conditions for the existence of an equilibrium in which there is a market freeze. The first condition ensures that the debt capacity in the high state in the first period is achieved by setting the face value of the debt equal to the next period's debt capacity in the same state:

$$v^H - \lambda v^L \geq \frac{(1-\lambda)\, v^L}{\left(p_{HH}(\tau)\right)^N}, \tag{9}$$

where $\left(p_{HH}(\tau)\right)^N$ is the probability of remaining in the high state for N periods of length τ. If a_{HL} denotes the arrival rate of a switch from the high to the low state, then the properties of a Poisson process imply that $\left(p_{HH}(\tau)\right)^N \geq e^{-a_{HL}}$. A sufficient condition for (9), therefore, is

$$v^H - \lambda v^L \geq (1-\lambda)\, v^L e^{a_{HL}},$$

which can be rewritten as

$$\frac{v^H - v^L}{v^L} \geq (1-\lambda)\left(e^{a_{HL}} - 1\right). \tag{10}$$

Each of the terms in this condition has an intuitive interpretation: the term on the left is the (proportional) *upside* in the low state; the term $1-\lambda$ is the (proportional) *liquidation cost*; and a_{HL} is the *arrival rate* of a switch from the high state to the low state. The cost of setting the face value high is the expected liquidation cost (the right hand side) and the benefit is the upside that is captured if a switch does not occur (the left hand side). The maximum value of a_{HL} consistent with (10), is shown as a function of the liquidation cost in the figure below for various values of the upside.

– Figure 4a–

For the range of values of interest for the liquidation cost, condition (10) will be satisfied as long as a_{HL} is less than 1.79. This is equivalent to saying that the probability of a switch to the low state during the life of the asset is less than or equal to 0.83, a bound that seems pretty loose.

It is clear that condition (10) will be satisfied, for given values of the upside and the liquidation cost, if the arrival rate a_{HL} is chosen sufficiently small. And since we want the high state to be a situation in which investors are optimistic and the probability of anything going wrong is very small, it is natural to think of a_{HL} as a small number. Obviously, if the liquidation cost is also small and/or if the upside is large, it is even easier to satisfy condition (10). Thus, we can simply assume that a_{HL} is sufficiently small to satisfy (10).

We also note that the parameter a_{HL} does not enter into any of the other conditions that we will be considering.

The second of the conditions referred to above is the inequality

$$p_{LL}(\tau)\,\lambda v^{L}+p_{LH}(\tau)\,v^{H}\leq v^{L} \tag{11}$$

The expression on the left hand side of condition (11) is the expected value of the debt in the low state in the last period if the face value is set equal to v^H and the inequality tells us that setting the face value equal to v^L gives a higher market value than setting the face value equal to v^H. To understand the constraint this condition puts on the parameters, it is helpful to rewrite (11) as follows:

$$\frac{p_{LH}(\tau)}{1-p_{LH}(\tau)}\frac{v^{H}-v^{L}}{v_{L}}\leq 1-\lambda. \tag{12}$$

The term $p_{LH}(\tau)\,/\,(1-p_{LH}(\tau))$ is the *odds ratio* of switching to the high state to remaining in the low state. Condition (12) requires that the odds ratio times the upside be less than or equal to the liquidation cost.

Comparing (10) and (12), we can see that an increase in the upside will make it easier to satisfy (10) but harder to satisfy (12). Conversely, an increase in the liquidation cost will make it easier to satisfy (12) but harder to satisfy (10). So there is a tension between the two conditions; however, as noted above, (10) can be satisfied for any given values of the upside and the liquidation cost if the flow probability a_{HL} is sufficiently small, as we assume. So in what follows, we focus on the parameter values that satisfy (12).

The analysis of this case is more difficult because, in order to have a meaningful market freeze, we want the drop in the debt capacity, caused by a switch from the high to the low state at date 0, to be much greater than the change in the fundamental value. In other words, we want the fundamental values in the high and low states to be close together and close to v^H. Since we want the probability of remaining in the high state to be high and since condition (10) is easily satisfied in any case, we can simplify the problem by assuming that the high state is an absorbing state, that is, $a_{HL}=0$. This assumption makes it harder to satisfy (12) in two ways. First, it makes it more attractive to set the face value of the debt high in the low state. Second, it makes it harder to satisfy the constraint on the fundamental value.

When the high state is absorbing, the fundamental value in the low state beginning at date 0 is

$$\left(1-e^{-a_{LH}}\right)v^{H}+e^{-a_{LH}}v^{L}=v^{L}+\left(1-e^{-a_{LH}}\right)\left(v^{H}-v^{L}\right),$$

where a_{LH} is the flow probability of a switch from the low to the high state and, hence, $e^{-a_{LH}}$ is the probability of remaining in the low state forever (i.e., from date 0 until date 1). Then $1-e^{-a_{LH}}$ is a good measure of the fundamental in the low state at date 0, because the fundamental will be close to v^L if $1 - e^{-a_{LH}} \approx 0$ and the fundamental will be close to v^H if $1 - e^{-a_{LH}} \approx 1$. For the purposes of this exercise, we will assume that the gap between the low-state fundamental and the high-state fundamental should be at most 10% of $v^H - v^L$, so that $e^{-a_{LH}} \leq 0.1$ or

$$a_{LH} \geq -\ln(0.1) = 2.3026. \tag{13}$$

So our task now is to characterize the parameters that satisfy (12) and (13).

When τ is small, $p_{LH}(\tau)$ is approximately equal to a_{LH}/N. Substituting this expression into the inequality (12), we obtain

$$\frac{a_{LH}}{N - a_{LH}} \frac{v^H - v^L}{v^L} \leq 1 - \lambda,$$

and substituting the smallest value of a_{LH} that satisfies (13) into this inequality we get

$$\frac{2.3026}{N - 2.3026} \frac{v^H - v^L}{v^L} \leq 1 - \lambda. \tag{14}$$

So now we have a condition in terms of N, the number of rollovers, the upside and the liquidation cost, that is,

$$N^* = 2.3026 \left(1 + \frac{v^H - v^L}{v^L} \frac{1}{1 - \lambda}\right),$$

where N^* is the smallest number of rollovers that satisfies the inequality (14). The figure below shows the value of N^* as a function of $1 - \lambda$ for values of the upside equal to 0.5, 1.0 and 2.0.

– Figure 4b–

The characterization of the parameter values that are consistent with a market freeze is complicated for several reasons, including the number of parameters and the number of conditions that must be satisfied, and the condition that the fundamental values in the high and the low states be close and sufficiently high to make the drop in debt capacity large relative to the change in the fundamental when the state changes from high to low. Nonetheless, the preceding analysis suggests that the critical trade off is between the number of rollover dates N (equivalently, the length of the rollover period τ) and the liquidation cost $1 - \lambda$. If we fix the upside and the distance between the high and low fundamental as a proportion of the upside, we are left with the relation illustrated in the figure above that shows the minimum number of rollovers needed for any particular value of the liquidation cost. To give a sense of how reasonable the required parameter values are, we present the following concrete examples, where the upside is 1.0 and the gap between the low-state fundamental and the high-state fundamental is 10% of $v^H - v^L$.

Example 2 *The asset has a maturity of six months and is funded by overnight repos. So the debt must be rolled over approximately* 162 *times. In order for the market to freeze in the low state (debt capacity equal to* v^L*), the value of the liquidation cost must be at least* $1 - \lambda = 0.0144$. *or around* 1.5% *of the upside.*

Example 3 *The asset has a maturity of two years and is funded by short term loans that are rolled over weekly. In total the debt must be rolled over* 104 *times. In order for the market to freeze in the low state, the value of the liquidation cost must be at least* $1 - \lambda = 0.02264$ *or around* 2.25% *of the upside.*

Example 4 *The asset has a maturity of ten years and is funded by one month loans, so the debt must be rolled over* 120 *times. In order for the market to freeze in the low state, the value of the liquidation cost must be at least* $1 - \lambda = 0.01956$ *or around* 2.0% *of the upside.*

In a similar way, we can use the formula above to explore the relationship between the number of rollovers N and the *size* of the market freeze, defined as $1 - e^{-a_{LH}}$, for a given value of the liquidation cost. For example, suppose that the upside is 1.0 and the liquidation cost is 1% of the upside. Then the minimum number of rollovers needed for a market freeze is

$$N^* = a_{LH}\left(1 + 1.0 \times \frac{1}{0.01}\right) = a_{LH} \times 101,$$

and the size of the market freeze is $1 - e^{-a_{LH}} = 1 - e^{-\frac{N^*}{101}}$. We plot this relationship in the figure below, which shows that the size $1 - e^{-a_{LH}}$ increases towards 1 as N^* increases.

– Figure 4c–

2.2 Debt capacity with intermediate rollover risk

We can get similar results even if the period length is not short enough to generate the result stated in Proposition 1. A simple adaptation of the numerical example will illustrate a scenario in which a high face value of debt is chosen in the low state, with the result that the bank faces a positive probability of default if the economy remains in the low state. Suppose that the value of the asset in the low state is $v^L = 40$. All the other parameters remain the same. Now the loss from default in the low state is less than the gain from a high face value in the high state, so the face value of the debt that is equal to next period's debt capacity in the high state maximizes the market value of debt this period.

As before, we calculate the debt capacity, beginning with the last rollover date. The last rollover date is t_{99}. The transition probabilities are given by equation (3) as before. If the face value of the debt is set equal to $v_H = 100$ in the low state, the market value of the debt issued will be $0.07685 \times 100 + 0.92315 \times 0.90 \times 40 = 40.918$, which is higher than the face value obtained by setting the face value equal to 40. Thus, the face value that maximizes the market value of debt implies default if the economy remains in the low state. It is still the case that $D^H_{99} = 100$ in the high state, and the debt capacity is now $B^H_{99} = 99.939$.

As long as the face value of the debt is set equal to B^H_{n+1} in both states, the debt capacity satisfies

$$\begin{bmatrix} B^H_n \\ B^L_n \end{bmatrix} = \begin{bmatrix} 0.99904 & 0.90 \times 0.00096 \\ 0.07685 & 0.90 \times 0.92315 \end{bmatrix} \begin{bmatrix} B^H_{n+1} \\ B^L_{n+1} \end{bmatrix}. \tag{15}$$

However, this assumes that the borrower chooses to default in the low state at every rollover date, which is not necessarily true. Starting at the last rollover date, it can be shown that the debt capacity in state L rises as we go back in time, reaches a maximum at t_{80}, and then declines as we move to earlier and earlier dates (see Figure 5). The problem is that as the debt capacity rises, the liquidation costs (which are proportional to the debt capacity) also rise and eventually outweigh the upside potential of a switch to the high state.[11] At the point where the maximum is reached, the borrower changes the face value of the debt from B^H_{n+1} to B^L_{n+1} and avoid default in the low state. Then the debt capacity is given by the formula above for $n = 80, ..., 99$ and is given by $B^L_n = B^L_{80}$ for $n = 0, ..., 80$. We can use the formula in equation (15) to show that $B^L_{80} = 44.918$ and $B^H_{80} = 98.847$. The gap between the debt capacities in the two states is $94.469 - 44.918 = 49.551$, compared to the negligible difference in the fundamental values 99.2596 and 99.241 in the high and the low states, respectively. Thus, even the borrower wants to capture the upside potential of a switch to the high state, the debt capacity in the low state does not rise much above the minimum value of the asset, i.e., it is 44.918 rather than 40.

— Figure 5 here —

In the rest of the paper, we explore the determinants of debt capacity in a richer model with many states and a broad range of parameters.

3 Debt capacity in the general case

We allow for a finite number of information states or signals, denoted by $\mathcal{S} = \{s_1, ..., s_I\}$. The current information state is public information. Transitions among the states are governed by a stationary Markov transition probability $\mathbf{P}(\tau)$ given as

$$\mathbf{P}(\tau) = \begin{bmatrix} p_{11}(\tau) & \cdots & p_{1I}(\tau) \\ \vdots & \ddots & \vdots \\ p_{I1}(\tau) & \cdots & p_{II}(\tau) \end{bmatrix},$$

[11] It is also possible to extend this example to the case with fixed costs of liquidation. Details are available from authors upon request.

where τ is the interval over which the transitions take place. We assume that the transition matrix takes the form

$$\mathbf{P}(\tau) = e^{\mathbf{A}\tau} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}\tau)^k}{k!},$$

where the matrix $\mathbf{A}$ is the generator. The crucial feature of the transition matrix is that the probability of a change of state converges to 0 as $\tau \to 0$. That is, $\mathbf{P}(\tau) \to \mathbf{I}$ as $\tau \to 0$.

The information state is a stochastic process $\{S(t)\}$ but for our purposes all that matters is the value of this process at the rollover dates. We let S_n denote the value of the information state $S(t_n)$ at the rollover date t_n.

The terminal value of the assets is a function of the information state at date $t = 1$. We denote by v_i the value of the assets if the terminal state is $S_{N+1} = s_i$ and assume that the values $\{v_1, ..., v_I\}$ satisfy

$$0 < v_1 < \ldots < v_I.$$

Let V_n^i denote the fundamental value of the asset at date t_n in state i. Then clearly the values $\{V_n^i\}$ are defined by putting $V_{N+1}^i = v_i$, for $i = 1, ..., I$, and

$$V_n^i = \sum_{j=1}^{I} p_{ij}(1 - t_n) v_j, \text{ for } n = 0, ..., N \text{ and } i = 1, ..., I,$$

where $p_{ij}(1 - t_n)$ is, of course, the (i, j) entry of $\mathbf{P}(1 - t_n)$ denoting the probability of a transition from state i at date t_n to state j at date $t_{N+1} = 1$.

Figure 6 illustrates the fundamental values in a setup with $I = 6$ states where terminal values are $v_i = 40 + i10$, for $i = 1, ..., 6$. The transition matrix $\mathbf{P}$ is described in Appendix B. As in our two-state example, the fundamental values in different states are virtually identical at date 0 though they diverge in steps of 10 at maturity.

— Figure 6 here —

Let B_n^i denote the equilibrium debt capacity of the assets in state s_i at date t_n. By convention, we set $B_{N+1}^i = v_i$ for all i.

Proposition 5 *The equilibrium values of* $\{B_n^i\}$ *must satisfy*

$$B_n^i = \max_{k=1,...,I} \left\{ \sum_{\{j : B_{n+1}^k > B_{n+1}^j\}} p_{ij}(\tau) \lambda B_{n+1}^j + \sum_{\{j : B_{n+1}^k \leq B_{n+1}^j\}} p_{ij}(\tau) B_{n+1}^k \right\}$$

for $i = 1, ..., I$ *and* $n = 0, ..., N$.

The result is immediate once we apply the now familiar backward induction argument to show that the borrower sets D_n^i equal to B_{n+1}^j for some j. Although the result amounts to little more than the definition of debt capacity, it is very useful because it allows us to calculate the debt capacities by backward induction.

The main result on the downward bias of debt capacities is contained in the following proposition.

Proposition 6 *There exists $\tau^* > 0$ such that for all $0 < \tau < \tau^*$, for any $n = 0, ..., N$ and any $i = 1, ..., I$, the borrower chooses $D_n^i \leq B_{n+1}^i$. Thus,*

$$B_n^i = \sum_{\left\{j:B_{n+1}^k > B_{n+1}^j\right\}} p_{ij}(\tau) \lambda B_{n+1}^j + \sum_{\left\{j:B_{n+1}^k \leq B_{n+1}^j\right\}} p_{ij}(\tau) B_{n+1}^k.$$

for some k such that $B_{n+1}^k \leq B_{n+1}^j$.

Proof. See Appendix A. ■

Several properties follow immediately from Proposition 6 whenever $0 < \tau < \tau^*$. We provide these results in the form of three corollaries. First, in the lowest state, s_1, the debt capacity is constant and equal to v_1, the lowest possible terminal value.

Corollary 7 *$B_n^1 = v_1$ for all n.*

Proof. From the formula in Proposition 6, for some k,

$$\begin{aligned} B_n^1 &= \sum_{\left\{j:B_{n+1}^k > B_{n+1}^j\right\}} p_{ij}(\tau) \lambda B_{n+1}^j + \sum_{\left\{j:B_{n+1}^k \leq B_{n+1}^j\right\}} p_{ij}(\tau) B_{n+1}^k \\ &\leq \sum_{j=1}^{I} p_{ij}(\tau) B_{n+1}^1 = B_{n+1}^1, \end{aligned}$$

since $B_{n+1}^k \leq B_{n+1}^1$. Since this inequality holds for $n = 0, ..., N$ and, by convention, $B_{N+1}^1 = v_1$, it follows that $B_n^1 \leq v_1$, for any n.

We can also show that $B_n^1 \geq v_1$. To see this, note that $B_{N+1}^i = v_1$ for all i. Moreover, if the same condition holds for $n+1$, it must be true that $B_n^i \geq v_1$, because we can always choose $D_n^i = v_1$.

Thus, we have shown that $B_n^1 = v_1$ for all n. ■

Second, the debt capacity B_n^i is monotonically non-decreasing in n, that is, debt capacity increases as the asset matures, holding the state constant. This follows directly from the fact that, if the face value of the debt equals B_{n+1}^i, the debt capacity B_n^i cannot be greater than B_{n+1}^i.

Corollary 8 $B^i_n \leq B^i_{n+1}$*, for any* $i = 1, ..., I$ *and* $n = 0, ..., N$*.*

Proof. The inequality follows directly from the formula in Proposition 6:

$$\begin{aligned} B^i_n &= \sum_{\left\{j:B^k_{n+1}>B^j_{n+1}\right\}} p_{ij}(\tau)\,\lambda B^j_{n+1} + \sum_{\left\{j:B^k_{n+1}\leq B^j_{n+1}\right\}} p_{ij}(\tau)\,B^k_{n+1} \\ &\leq \sum_{\left\{j:B^k_{n+1}>B^j_{n+1}\right\}} p_{ij}(\tau)\,B^i_{n+1} + \sum_{\left\{j:B^k_{n+1}\leq B^j_{n+1}\right\}} p_{ij}(\tau)\,B^i_{n+1} \\ &= \sum_{j=1}^{I} p_{ij}(\tau)\,B^i_{n+1} = B^i_{n+1}, \end{aligned}$$

since $B^j_{n+1} < B^k_{n+1} \leq B^i_{n+1}$ implies that $\lambda B^j_{n+1} < B^i_{n+1}$. ■

Third, since $B^i_{N+1} = v_i$ by convention, the preceding result immediately implies that the debt capacity B^i_n is less than or equal to v_i.

Corollary 9 $B^i_n \leq v_i$ *for all* $i = 1, ..., I$ *and* $n = 0, ..., N$*.*

Finally, we can confirm that the debt capacity in state s_i at any date t_n is less than the fundamental value V^i_n. This follows directly from the formula in Proposition 6 for $n = N+1$ and any i, so suppose that it holds for $n+1, ..., N$ and any $i = 1, ..., I$. Then the formula in Proposition 6 implies that, if $D^i_n = B^k_{n+1}$, say,

$$\begin{aligned} B^i_n &= \sum_{\left\{j:B^k_{n+1}>B^j_{n+1}\right\}} p_{ij}(\tau)\,\lambda B^j_{n+1} + \sum_{\left\{j:B^k_{n+1}\leq B^j_{n+1}\right\}} p_{ij}(\tau)\,B^k_{n+1} \\ &\leq \sum_{\left\{j:B^k_{n+1}>B^j_{n+1}\right\}} p_{ij}(\tau)\,B^j_{n+1} + \sum_{\left\{j:B^k_{n+1}\leq B^j_{n+1}\right\}} p_{ij}(\tau)\,B^j_{n+1} \\ &\leq \sum_{j=1}^{I} p_{ij}(\tau)\,V^j_{n+1} = V^i_n, \end{aligned}$$

for any $i = 1, ..., I$, so by induction the claim holds for any $n = 0, ..., N$ and any $i = 1, ..., I$.

Some of these properties are illustrated in Figures 7a and 7b which show the debt capacities in the six states of our numerical example for $N = 10$ and $N = 100$ rollovers, respectively. For 10 rollovers, τ is not sufficiently small to obtain our limit result even in the worst state and debt capacity in each state is in fact higher than the terminal value in that state. Nevertheless, it is still the case that there is a drop in debt capacity of between 5 and 10 as the state changes to the next worse one, without much change in the fundamental value. By contrast, with 100 rollovers, the limit result is obtained and in fact debt capacity in the two worst states (states 1 and 2) is (essentially) the minimum possible value of the asset which is 50. Furthermore, as we go from the best state (state 6) to the second-best state (state 5), debt capacity falls roughly by a magnitude of 25 even though the fundamental value (Figure 6) has hardly changed. Thus, the market freeze is substantially worse with 100 rollovers compared to 10.

— Figure 7a and 7b here —

Proposition 6 shows that, when the period length is sufficiently short (the rollover rate is sufficiently high), there is a downward bias in the debt capacity, because the face value of the debt is bounded above by the future debt capacity in the same state. This is an important step toward proving the existence of a market freeze but two further requirements are needed. First, we need to show that the fundamental values are uniformly high and that the debt capacities are high in some states and low in others.

Consider the debt capacities first. The proposition shows that $B_n^1 = v_1$ for all $n = 0, ..., N$ so it is enough to show that the debt capacity is high in some states. The following proposition does just that.

Proposition 10 *The initial debt capacity in the highest state, I, satisfies the inequality*

$$B_0^I \geq e^{-\alpha} v_I + \left(1 - e^{-\alpha}\right) \lambda v_1,$$

where $\alpha = \sum_{j \neq I} a_{Ij}$.

So, as long as α is sufficiently small, the debt capacity in the high state will be close to v_I. We know that the fundamental value V_0^I lies between B_0^I and v_I, so the fundamental value will also be close to v_I. Then in order to show that a market freeze is possible it is only necessary to ensure that the fundamental value for the lowest state, V_0^1, is also close to v_I. This will be true as long as the probability of a transition from the lowest to the highest state is high enough. In general, this probability will depend on the entire matrix $\mathbf{A}$ so it is hardly worth trying to write down a sufficient condition in terms of the individual parameters, but the numerical examples have illustrated that this is clearly possible.

4 Related research

At a general level, our result on market freezes can be considered a generalization of the Shleifer and Vishny (1992) and Allen and Gale (1994) results that when potential buyers of assets of a defaulted firm are themselves financially constrained, there is a reduction in the ex-ante debt capacity of the industry as a whole. We expand on their insight by considering short-term debt financing of long-term assets with rollovers to be met mostly by new short-term financing or liquidations to other buyers also financed mostly through short-term debt. Our "market freeze" result can be considered as a particularly perverse dynamic arising through the Shleifer and Vishny (1992) and the Allen and Gale (1994) channel at *each* rollover date, that through backward induction, can in the worst case drive short-term debt capacity of an asset to its minimum possible cash flow.

More specifically, our paper is related to the literature on freezes and runs in financial markets. Rosenthal and Wang (1993) use a model where owners occasionally need to sell their assets for exogenous liquidity reasons through auctions with private information. Because of the informational rents earned by the privately informed bidders, sellers may not be able to extract the full value of the asset and this liquidation cost gets built into the market price of the asset, making the market price systematically lower than the fundamental value. In our model, the reason for the debt capacity being lower than the fundamental value is not the private information of potential buyers, rather it is the rollover risk and the liquidation cost associated with defaults.

He and Xiong (2009) consider a model of dynamic debt runs in which creditors have supplied debt maturing at differing maturities and each creditor faces the risk, at the time of rolling over the debt, that fundamentals may deteriorate before the remaining debt matures, causing a fire sale of assets. In their model, the volatility of fundamentals plays a key role in driving the runs, even when the average value of fundamentals has not been affected. Our model is complementary to theirs and somewhat different in the sense that *both* average value and uncertainty about fundamentals are held constant in our model. It is the rate at which information arrives relative to the rollovers that determines whether there is rollover risk in short-term debt.

Huang and Ratnovski (2008) model the behavior of short-term wholesale financiers who prefer to rely on noisy public signals such as market prices and credit ratings, rather than producing costly information about the institutions they lend to. Hence, wholesale financiers run on other institutions based on imprecise public signals, triggering potentially inefficient runs. While their model is about runs in the wholesale market, as is ours, their main focus is to challenge the peer-monitoring role of wholesale financiers, whereas our main focus is the role of rollover and liquidation risk in generating such runs.

An alternative modelling device to generate market freezes is to employ the notion of Knightian uncertainty (see Knight, 1921) and agents' overcautious behavior towards such uncertainty. Gilboa and Schmeidler (1989) build a model where agents become extremely cautious and consider the worst-case among the possible outcomes, that is, agents are uncertainty averse and use maxmin strategies when faced with Knightian uncertainty. Dow and Werlang (1992) apply the framework of Gilboa and Schmeidler (1989) to the optimal portfolio choice problem and show that there is an interval of prices within which uncertainty-averse agents neither buy nor sell the asset. Routledge and Zin (2004) and Easley and O'Hara

(2009, 2010) use Knightian uncertainty and agents that use maxmin strategies to generate widening bid-ask spreads and freeze in financial markets. Caballero and Krishnamurthy (2008) also use the framework of Gilboa and Schmeidler (1989) to develop a model of financial crises: During periods of increased Knightian uncertainty, agents refrain from making risky investments and hoard liquidity, leading to flight to quality and freezes in markets for risky assets. While ambiguity aversion leads to a market freeze in these models, in our model agents maximize expected utility and the main source of the market freeze is rollover and liquidation risk.

We regard our approach as complementary to Knightian uncertainty. Knightian uncertainty is appropriate when investors have very limited information about the nature of the risks they face. We are interested, by contrast, in explaining the drying up of liquidity in the absence of obvious problems of asymmetric information or fears about the value of collateral. For this purpose, it would seem to be an advantage to appeal to standard assumptions about preferences and beliefs.

5 Conclusion

In this paper, we have attempted to provide a simple information-theoretic model for freezes in the market for short-term financing of finitely lived assets. The key ingredients of our model were rollover risk, liquidation risk, rapid rate of refinancing relative to the arrival of news, and similarity of financial institutions in their degree of maturity mismatch. In particular, our model could be interpreted as a micro-foundation for the funding risk arising in capital structures of financial institutions or special purpose vehicles that have extreme maturity mismatch between assets and liabilities.

In future work, it would be interesting to embed an agency-theoretic role for short-term debt, which we assumed as given, and see how the desirability of such rollover finance is affected when information problems can lead to complete freeze in its availability. While we took the release of information about the underlying asset as ordained by nature, it seems worthwhile to reflect on its deeper foundations, and thereby assess whether a strategic disclosure of information by agents in charge of the asset can alleviate (or aggravate) the problem of freezes.

Appendix A: Proofs

We can solve for the equilibrium debt capacities in the model of Section 3 by backward induction. Let D denote the face value of the debt issued in state s_i at date t_n. This debt will pay off D in state s_j at date t_{n+1} if $D \leq B^j_{n+1}$ and λB^j_{n+1} otherwise. In other words, the market value of the debt is given by the formula

$$\sum_{B^j_{n+1}<D} p_{ij}(\tau)\,\lambda B^j_{n+1} + \sum_{B^j_{n+1}\geq D} p_{ij}(\tau)\,D$$

and the debt capacity is given by

$$B^i_n = \max_D \left\{ \sum_{B^j_{n+1}<D} p_{ij}(\tau)\,\lambda B^j_{n+1} + \sum_{B^j_{n+1}\geq D} p_{ij}(\tau)\,D \right\}.$$

Let D^i_n denote the face value of the debt that maximizes the market value of debt in state i at date t_n. It is clear that the market value of the debt is maximized by setting the face value $D = B^j_{n+1}$, for some value of $j = 1, ..., I$. Thus, we can write the equilibrium condition as

$$B^i_n = \max_{k=1,...,I} \left\{ \sum_{\left\{j:B^k_{n+1}>B^j_{n+1}\right\}} p_{ij}(\tau)\,\lambda B^j_{n+1} + \sum_{\left\{j:B^k_{n+1}\leqslant B^j_{n+1}\right\}} p_{ij}(\tau)\,B^k_{n+1} \right\},$$

for $i = 1, ..., I$ and $n = 0, ..., N$.

Proof of Proposition 6: For a fixed but arbitrary date t_n and state s_i, we compare the strategy of setting $D = B^i_{n+1}$ with the strategy of setting $D = B^k_{n+1}$, where $B^k_{n+1} > B^j_{n+1}$. Consider the difference in the expected values of the debt:

$$\begin{aligned}
& \sum_{\left\{j:B^i_{n+1}>B^j_{n+1}\right\}} p_{ij}(\tau)\,\lambda B^j_{n+1} + \sum_{\left\{j:B^i_{n+1}\leqslant B^j_{n+1}\right\}} p_{ij}(\tau)\,B^i_{n+1} \\
& - \sum_{\left\{j:B^k_{n+1}>B^j_{n+1}\right\}} p_{ij}(\tau)\,\lambda B^j_{n+1} - \sum^{I}_{\left\{j:B^k_{n+1}\leqslant B^j_{n+1}\right\}} p_{ij}(\tau)\,B^k_{n+1} \\
= & \sum_{\left\{j:B^i_{n+1}\leqslant B^j_{n+1}<B^k_{n+1}\right\}} p_{ij}(\tau)\left(B^i_{n+1} - \lambda B^j_{n+1}\right) + \sum_{\left\{j:B^k_{n+1}\leqslant B^j_{n+1}\right\}} p_{ij}(\tau)\left(B^i_{n+1} - B^k_{n+1}\right) \\
= & \; p_{ii}(\tau)\left(B^i_{n+1} - \lambda B^i_{n+1}\right) + \sum_{\left\{j:B^i_{n+1}<B^j_{n+1}<B^k_{n+1}\right\}} p_{ij}(\tau)\left(B^i_{n+1} - \lambda B^j_{n+1}\right) \\
& + \sum_{\left\{j:B^k_{n+1}\leqslant B^j_{n+1}\right\}} p_{ij}(\tau)\left(B^i_{n+1} - B^k_{n+1}\right) \\
\geq & \; p_{ii}(\tau)(1-\lambda)\,v_1 + \sum_{\left\{j:B^i_{n+1}<B^j_{n+1}<B^k_{n+1}\right\}} p_{ij}(\tau)(v_1 - v_I) + \sum_{\left\{j:B^k_{n+1}\leqslant B^j_{n+1}\right\}} p_{ij}(\tau)(v_1 - v_I) \\
= & \; p_{ii}(\tau)(1-\lambda)\,v_1 + \sum_{\left\{j:B^i_{n+1}<B^j_{n+1}\right\}} p_{ij}(\tau)(v_1 - v_I).
\end{aligned}$$

since $B^i_{n+1} \geq v_1$, $B^i_{n+1} - \lambda B^j_{n+1} \geq B^i_{n+1} - B^j_{n+1} \geq (v_1 - v_I)$ for $j = i+1, ..., I$ and $B^i_{n+1} - B^k_{n+1} \geq v_1 - v_I$. Then it is clear that, for τ sufficiently small (i.e., $p_{ii}(\tau)$ sufficiently close to 1), the last expression above is positive. Since the last expression is independent of n, the bound is uniform, i.e., there exists a constant $\tau^* > 0$ such that, for $\tau < \tau^*$, $D^i_n = B^i_{n+1}$, for all i and n.

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Appendix B: Numerical parameters for the example with I = 6 states

The terminal values for the 6 states are chosen as $v_i = 10i$, for $i \in \{5,...,10\}$. The generator matrix A and the unconditional transition matrices $\mathbf{P}(N = 10)$ and $\mathbf{P}(N = 100)$ that is, the transition matrices with 10 and 100 rollovers, respectively, are given below.

$$A = \begin{bmatrix} -8 & 8 & 0 & 0 & 0 & 0 \\ 1 & -10 & 9 & 0 & 0 & 0 \\ 0 & 1 & -10 & 9 & 0 & 0 \\ 0 & 0 & 1 & -10 & 9 & 0 \\ 0 & 0 & 0 & 1 & -10 & 9 \\ 0 & 0 & 0 & 0 & 0.1 & -0.1 \end{bmatrix}$$

$$\mathbf{P}(N = 10) = \begin{bmatrix} 0.498446 & 0.328747 & 0.129744 & 0.034727 & 0.007006 & 0.00133 \\ 0.041093 & 0.432477 & 0.341745 & 0.138156 & 0.037333 & 0.009196 \\ 0.001802 & 0.037972 & 0.433411 & 0.342035 & 0.13748 & 0.0473 \\ 5.36\text{x}10^{-5} & 0.001706 & 0.038004 & 0.433336 & 0.338412 & 0.188488 \\ 1.20\text{x}10^{-6} & 5.12\text{x}10^{-5} & 0.001697 & 0.037601 & 0.420155 & 0.540494 \\ 2.53\text{x}10^{-9} & 1.40\text{x}10^{-7} & 6.49\text{x}10^{-6} & 0.000233 & 0.006005 & 0.993755 \end{bmatrix}$$

$$\mathbf{P}(N = 100) = \begin{bmatrix} 0.923483 & 0.073136 & 0.00328 & 9.82\text{x}10^{-5} & 2.21\text{x}10^{-6} & 4.04\text{x}10^{-8} \\ 0.009142 & 0.905609 & 0.081471 & 0.003666 & 0.00011 & 2.52\text{x}10^{-6} \\ 4.56\text{x}10^{-5} & 0.009052 & 0.905652 & 0.081472 & 0.003665 & 0.000113 \\ 1.52\text{x}10^{-7} & 4.53\text{x}10^{-5} & 0.009052 & 0.905652 & 0.081461 & 0.003789 \\ 3.79\text{x}10^{-10} & 1.51\text{x}10^{-7} & 4.53\text{x}10^{-5} & 0.009051 & 0.905287 & 0.085617 \\ 7.69\text{x}10^{-14} & 3.85\text{x}10^{-11} & 1.55\text{x}10^{-8} & 4.68\text{x}10^{-6} & 0.000951 & 0.999044 \end{bmatrix}$$

Figures

Figure 1: Timeline (illustrating *N*+1 state transitions and *N* rollovers).

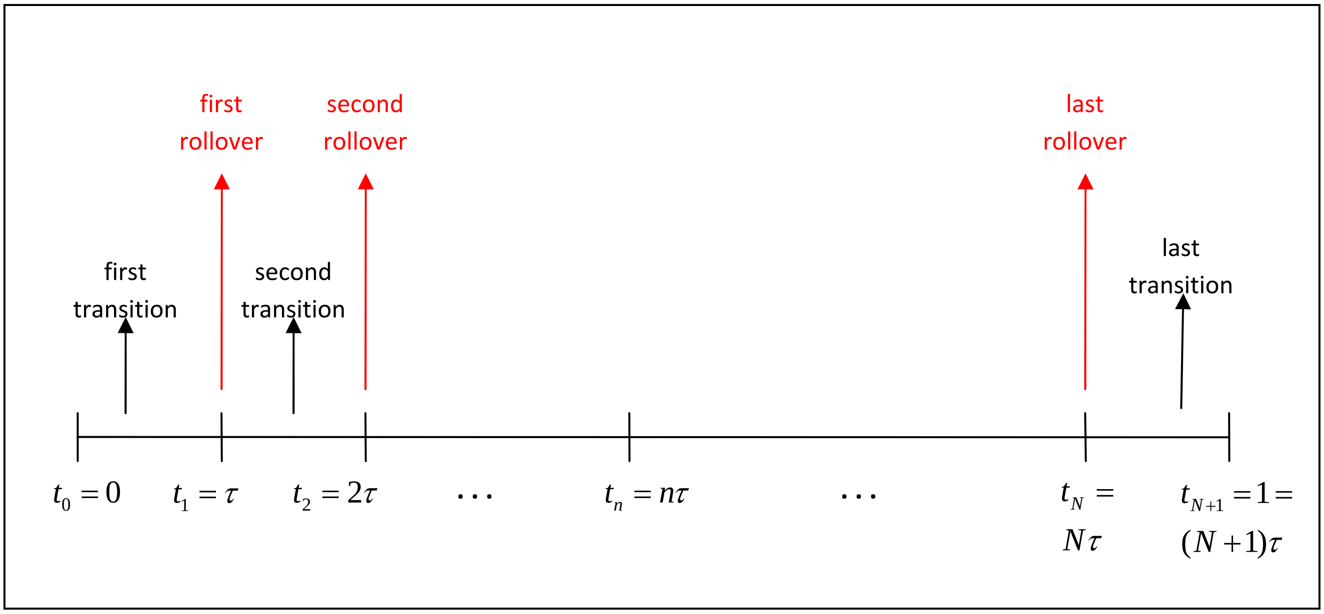


Figure 2: Fundamental value (V) and debt capacity (B) in high (v^H=100) and low (v^L=50) states as a function of time

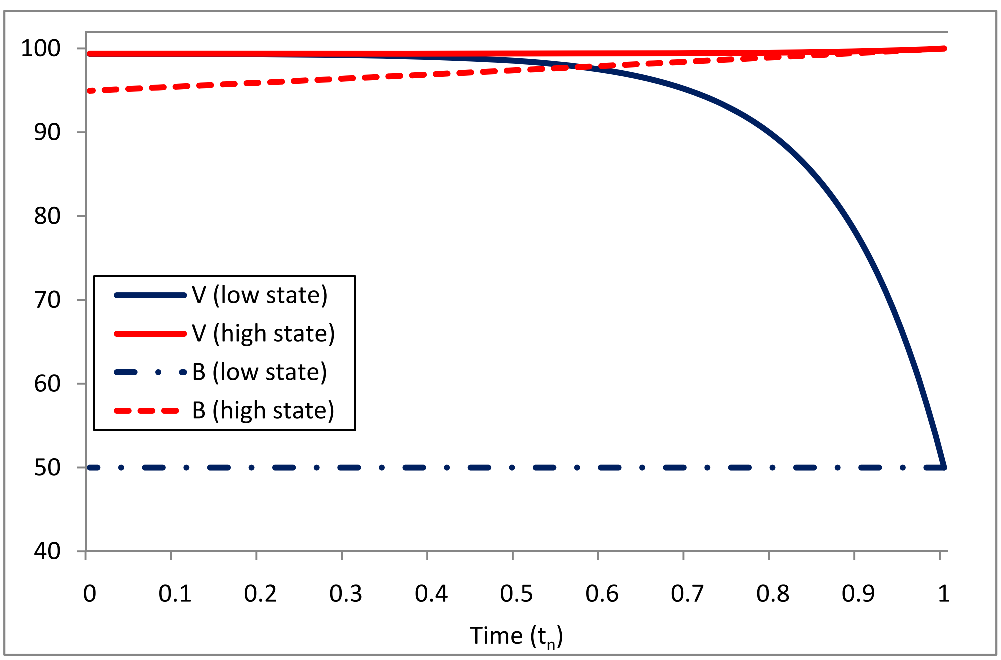


Figure 3: Fundamental value (V) and debt capacity (B) in low (v^L=50) state for different number of rollovers (N)

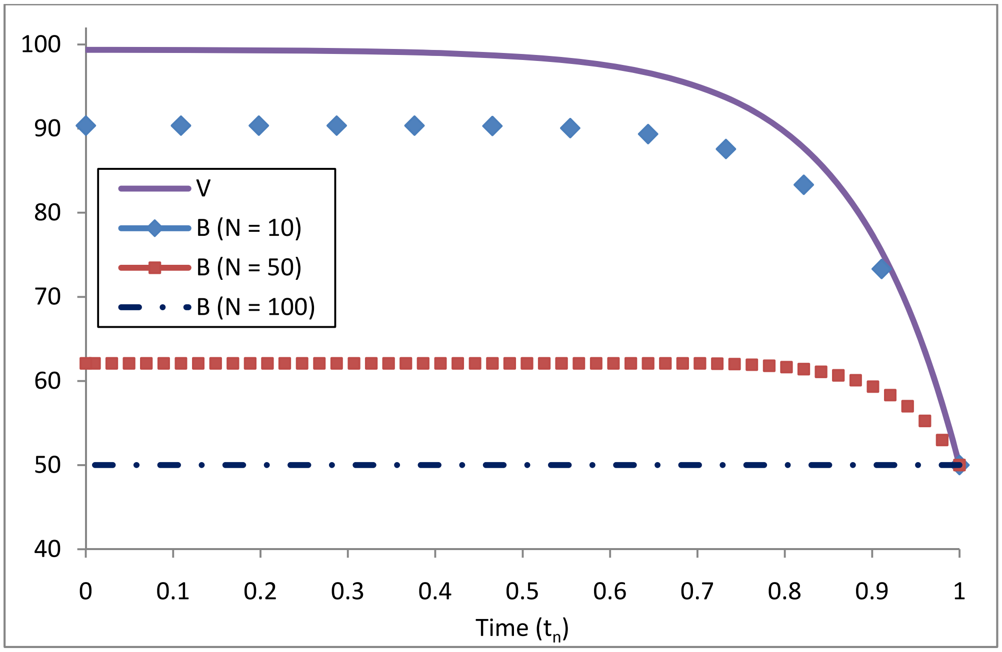


Figure 4a: a_{HL} as a function of liquidation cost

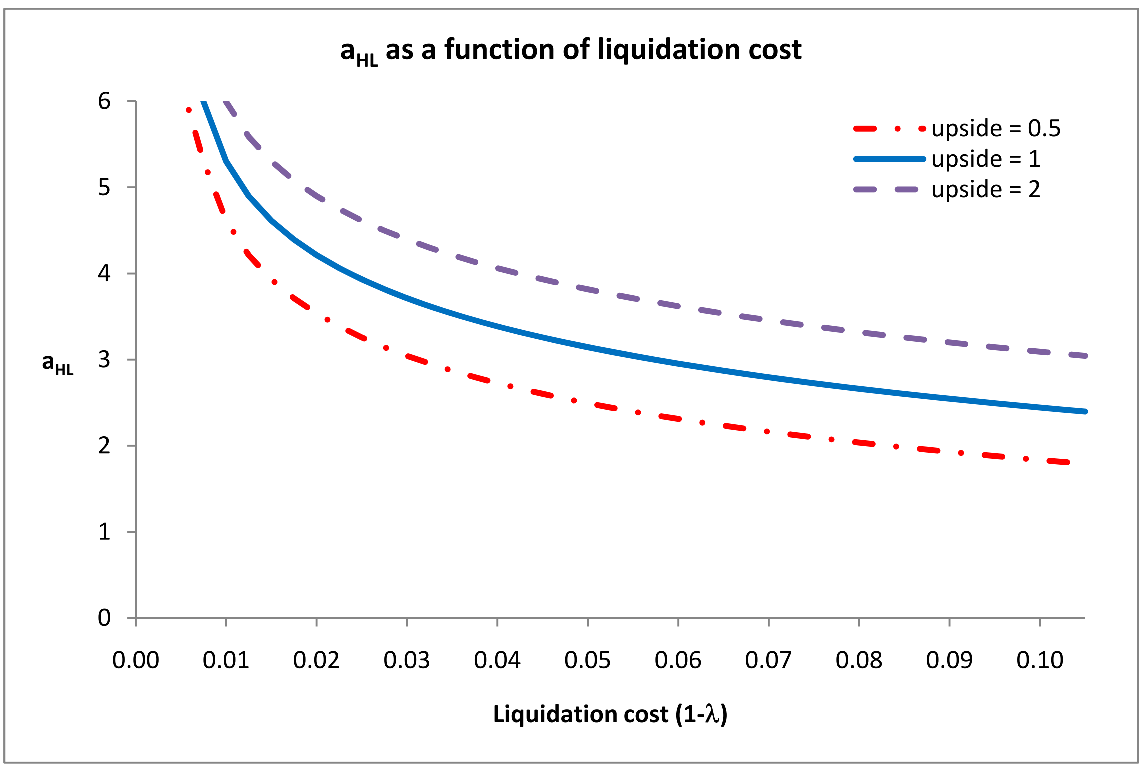


Figure 4b: N^* as a function of liquidation cost

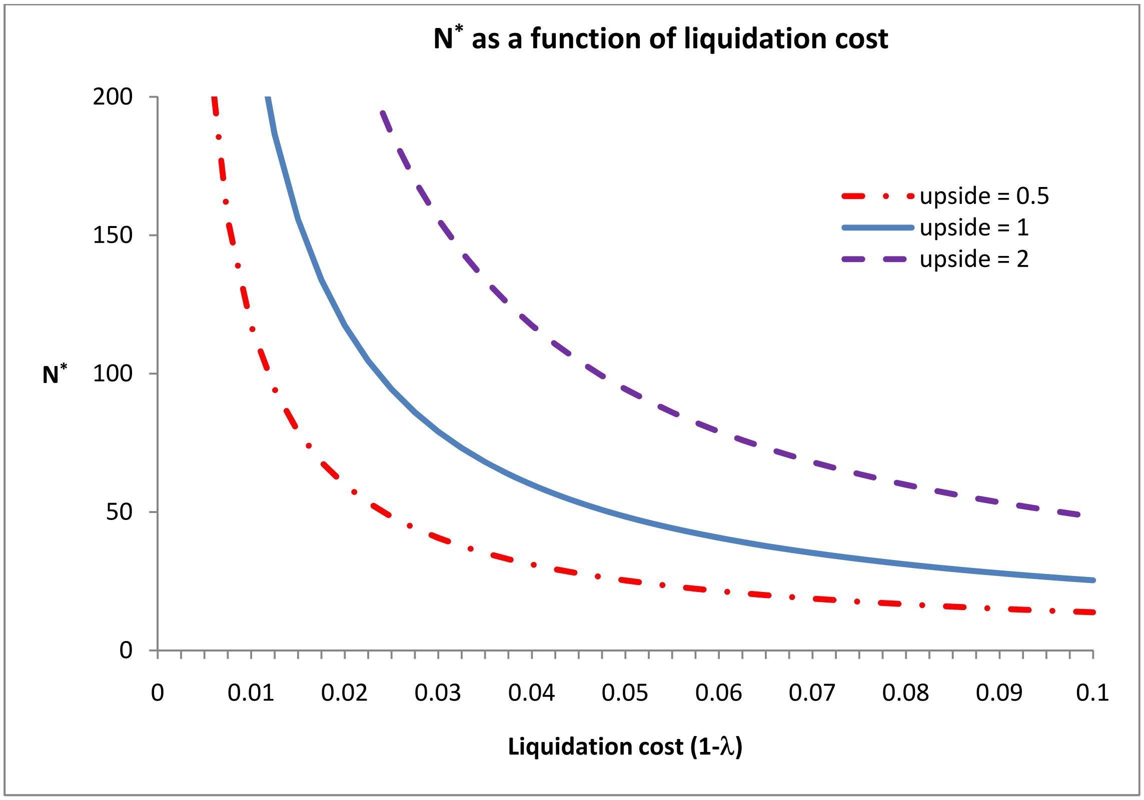


Figure 4c: Size of market freeze as a function of N^* for different liquidation cost

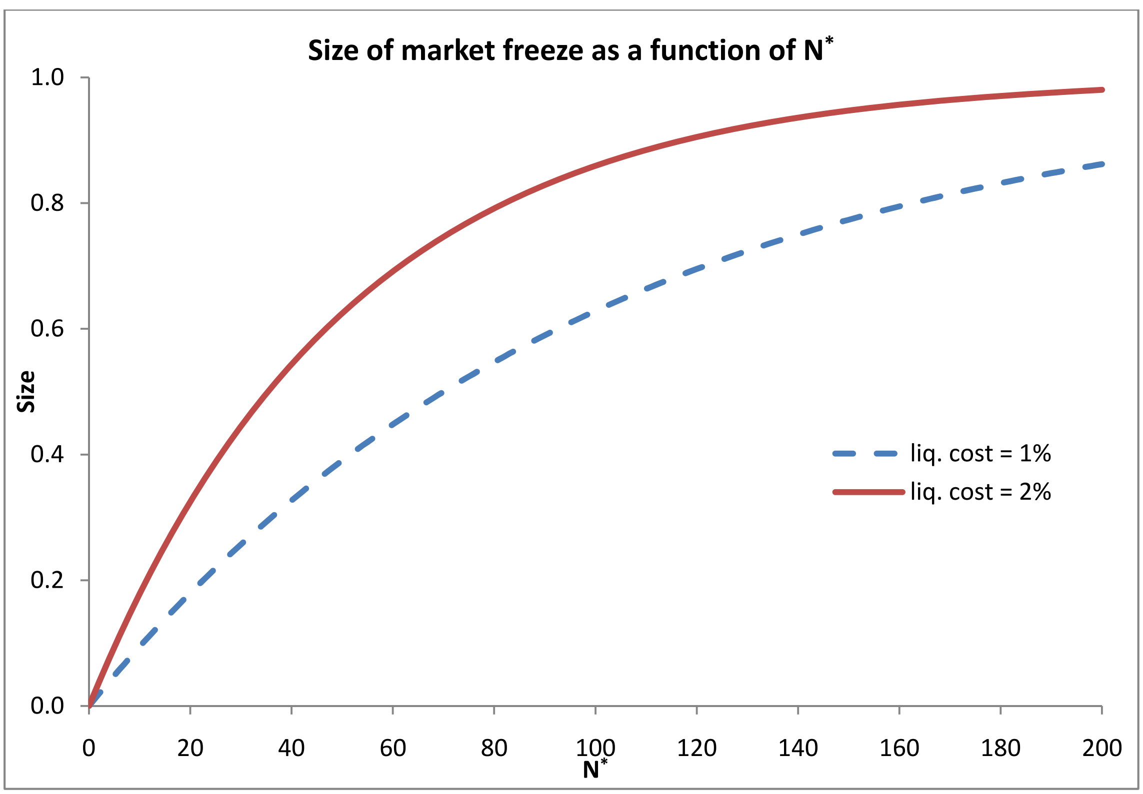


Figure 5: Debt capacity (B) in high (v^H=100) and low (v^L=40) states as a function of time

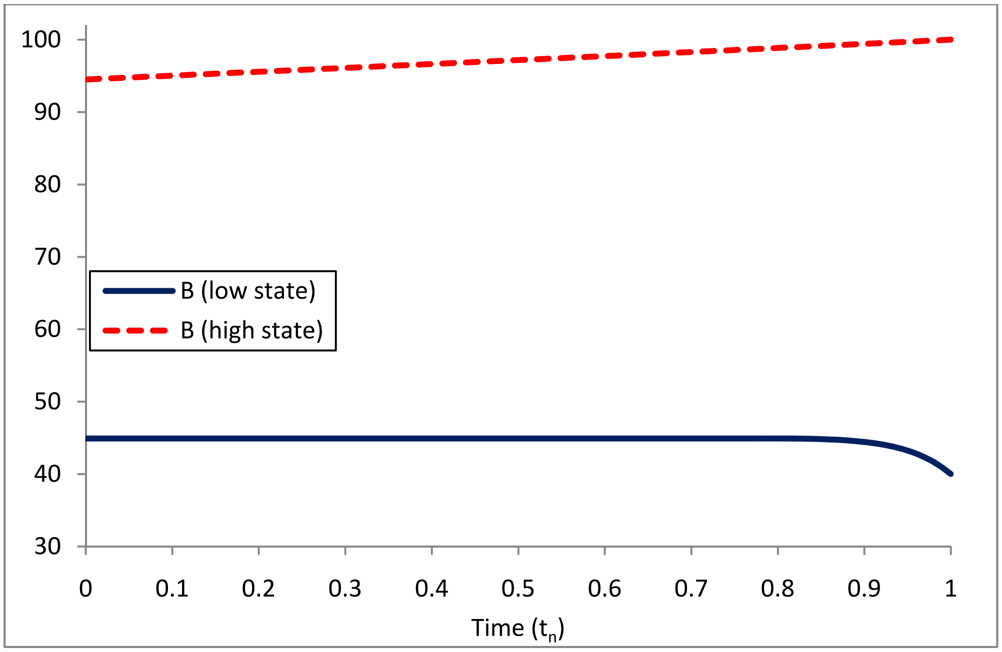


Figure 6: Fundamental values (V) as a function of time

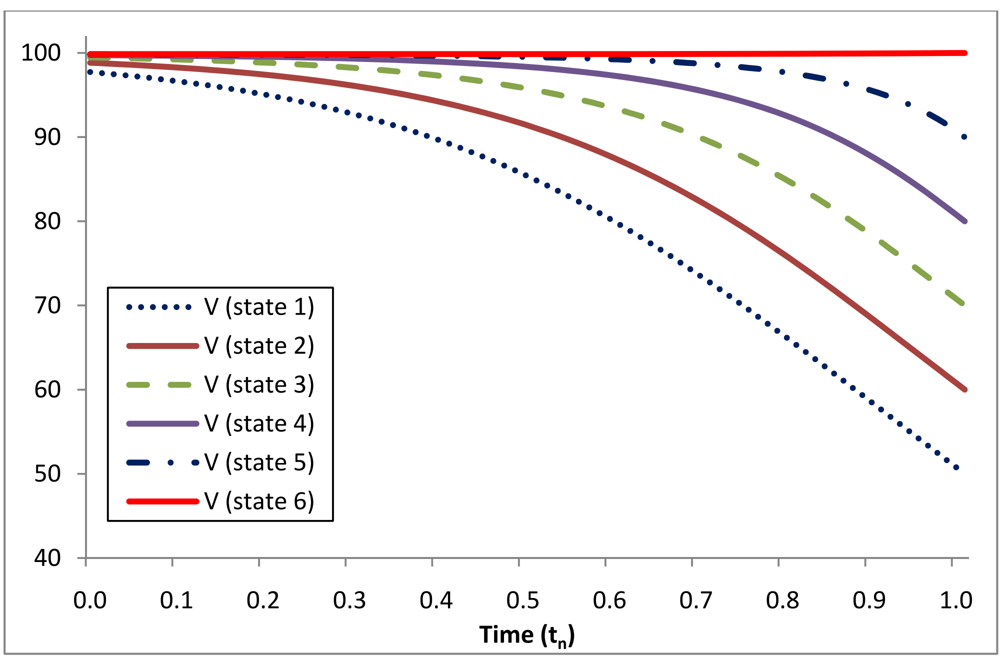


Figure 7a: Debt capacity (B) as a function of time for rollover frequency N=10

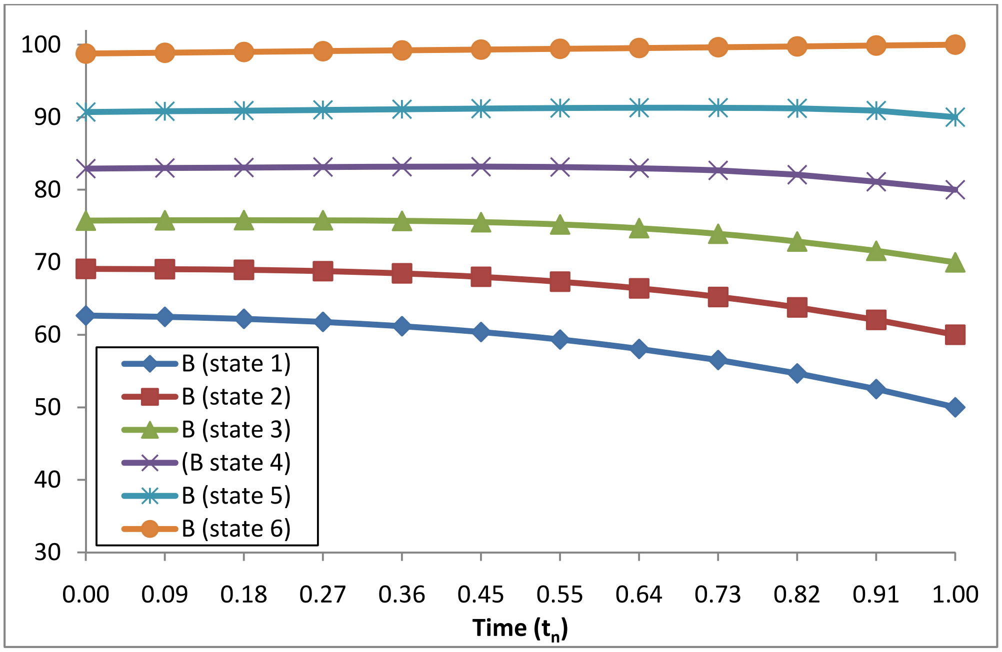
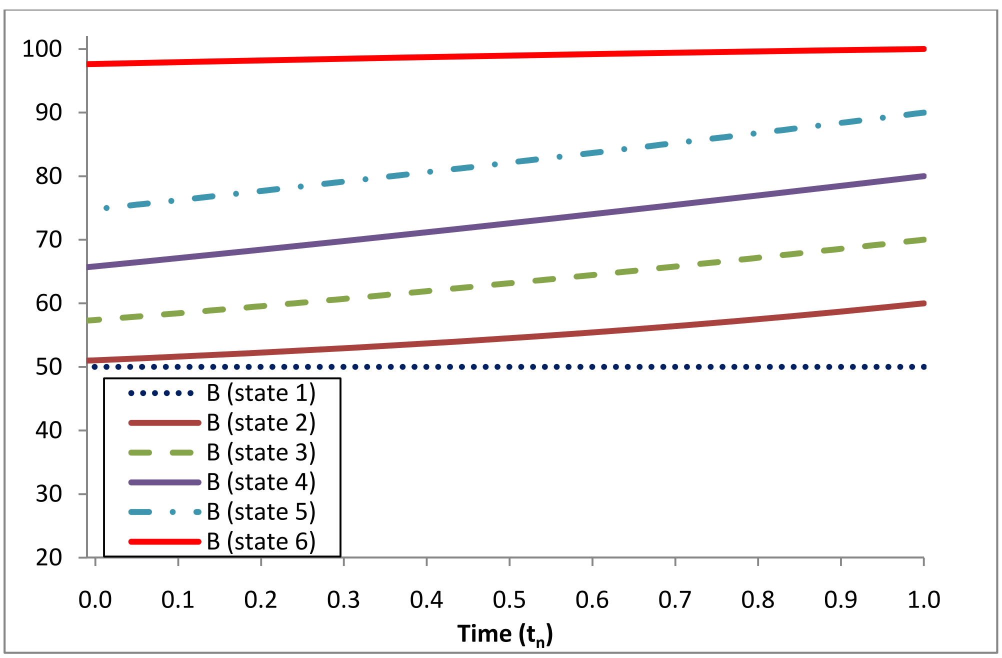


Figure 7b: Debt capacity (B) as a function of time for rollover frequency N=100



Internet Appendix[1]

Rollover Risk and Market Freezes

VIRAL V. ACHARYA, DOUGLAS GALE, and TANJU YORULMAZER

The appendix provides two extensions of the benchmark model: (i) extension with buy-to-hold investors; and (ii) extension with a fixed cost of liquidation.

[1] Viral Acharya, Douglas Gale, and Tanju Yorulmazer, 20XX, Internet Appendix to "Rollover Risk and Market Freezes," Journal of Finance [Vol. #], [pages], http://www.afajof.org/IA/[year].asp.

1 Extension with buy-to-hold investors

In this extension, we analyze the case where, in the event of default, with probability β the asset can be sold to a buy-to-hold investor who holds the assets until maturity, whereas with probability $(1-\beta)$ the asset must be sold to an investor financed by short-term debt, as in the benchmark case. We show that for reasonable parameter values our results in the benchmark case hold. While the fundamental values in the high and the low states are high and very close to each other, we observe a significant difference in the debt capacities in the high and the low states, and the debt capacity in the low state can be as low as the minimum value of the assets.

We assume that the buy-to-hold investors arrive in the market randomly according to a Poisson process with parameter $b > 0$. Hence, the probability β of one or more buy-to-hold investor(s) arriving in a time-interval $[t, t+\tau]$ is $\left(1 - e^{-\tau b}\right)$. Note that the probability of a buy-to-hold investor arriving is increasing in the tenor τ.

We assume that when the borrower meets a buy-to-hold investor, she can sell the assets to the buy-to-hold investor without any transaction cost.[2] Furthermore, we assume that the seller can extract the fundamental value of the asset from the buy-to-hold investor, as long as there is at least one buy-to-hold investor in the market.[3] Note that these two assumptions increase the debt capacity and make it harder to get a market freeze.

As in the benchmark case, we can calculate the debt capacity starting from the last rollover date going backwards. To simplify the notation in the following analysis, we use $p(\tau)$ instead of $p_{HL}(\tau)$, and $q(\tau)$ instead of $p_{LH}(\tau)$. We make the reasonable assumption (A1) that $q(\tau) < 1 - p(\tau)$ so that the state is more likely to be high at date t_{n+1} when it is high at date t_n as opposed to it is being low at date t_n. Formally, $\{p(\tau), 1-p(\tau)\}$ strictly dominates $\{1-q(\tau), q(\tau)\}$ in the sense of first order stochastic dominance.

In addition, we make the following two assumptions:

A2) $p < p^* = \frac{v^H - v^L}{v^H - \lambda v^L}$.

A3) $q < q^* = \frac{V_n^L - B_{n+1}^L}{\max\left\{V_{n+1}^L, B_{n+1}^H\right\} - B_{n+1}^L}$.[4]

Before we derive the expressions for the debt capacities in the high and the low states, we show that the following four inequalities hold for all $n = 1, ..., N$:

1) $V_n^H > V_n^L$ so that the fundamental value is higher in state H than in state L.

2) $V_n^H > B_n^H$ so that the fundamental value is higher than the debt capacity in state H.

[2] Alternatively, one can assume that there is a transaction cost associated with sales to buy-to-hold investors. This would make our results only stronger.

[3] When there is only one (or few) buy-to-hold investor(s), this may create some market power for the buy-to-hold investors, where they can acquire the asset at a price lower than the fundamental value. Hence, this assumption biases the results against a market freeze.

[4] Note that for sufficiently short period length τ both assumptions are easily satisfied.

3) $V_n^L > B_n^L$ so that the fundamental value is higher than the debt capacity in state L.

4) $B_n^H > B_n^L$ so that the debt capacity is higher in state H than in state L.

First, we show that $V_n^H > V_n^L$ for all $n = 0, ...N$.[5] We prove the result by induction. At date t_N, we have

$$V_N^H = pv^L + (1-p)v^H > (1-q)v^L + qv^H = V_N^L,$$

since $q < 1 - p$. Now, suppose $V_k^H > V_k^L$ for all $k = n+1, ...N$. Then we obtain

$$V_n^H = pV_{n+1}^L + (1-p)V_{n+1}^H > (1-q)V_{n+1}^L + qV_{n+1}^H = V_n^L.$$

Next, we show that the fundamental value will be higher than the debt capacity in each state and the debt capacity is higher in the high state than in the low state, that is, $V_n^H > B_n^H$, $V_n^L > B_n^L$ and $B_n^H > B_n^L$ for all $n = 0, ...N$. We prove the result by induction.

At date t_N, we have

$$\begin{aligned} V_N^H &= pv^L + (1-p)v^H > \max\left\{v^L, \left(p\lambda v^L + (1-p)v^H\right)\right\} = B_N^H, \text{ and} \\ V_N^L &= (1-q)v^L + qv^H > \max\left\{v^L, \left((1-q)\lambda v^L + qv^H\right)\right\} = B_N^L, \end{aligned}$$

for $\lambda \in [0,1)$, and by assumptions A1 and A2,

$$B_N^H = p\lambda v^L + (1-p)v^H > \max\left\{v^L, \left((1-q)\lambda v^L + qv^H\right)\right\} = B_N^L.$$

Hence, the inequalities hold for the last rollover date t_N. Now, suppose that the inequalities hold for all the dates t_{n+1} through t_N. Next, we will show that the inequalities hold for date t_n.

Using the same argument as in the benchmark model, we can eliminate a wide range of values for the face value of debt at date t_n, which leaves us with four candidates: V_{n+1}^H, V_{n+1}^L, B_{n+1}^H, B_{n+1}^L. We analyze each case below:

(i) $D = V_{n+1}^H$: Note that V_{n+1}^H is the highest of the four candidates. Hence, the borrower will be able to pay the face value at date t_{n+1} when it is the high state and she meets a buy-to-hold investor. In all other states, the borrower will default. However, by assumption, if the state is low and the borrower meets a buy-to-hold investor, there is no transaction cost. Hence, we obtain:

$$\begin{aligned} B_n^L &= \beta\left[qV_{n+1}^H + (1-q)V_{n+1}^L\right] + (1-\beta)\left[q\lambda B_{n+1}^H + (1-q)\lambda B_{n+1}^L\right] \\ B_n^H &= \beta\left[(1-p)V_{n+1}^H + pV_{n+1}^L\right] + (1-\beta)\left[(1-p)\lambda B_{n+1}^H + p\lambda B_{n+1}^L\right]. \end{aligned}$$

[5] Note that at date t_{N+1}, the result is obvious.

(ii) $D = V_{n+1}^L$: We have $V_{n+1}^H > V_{n+1}^L > B_{n+1}^L$. However, we do not know the relation between V_{n+1}^L and B_{n+1}^H.

For $V_{n+1}^L \geqslant B_{n+1}^H$, we obtain:

$$\begin{aligned} B_n^L &= \beta\left[qV_{n+1}^L + (1-q)V_{n+1}^L\right] + (1-\beta)\left[q\lambda B_{n+1}^H + (1-q)\lambda B_{n+1}^L\right] \\ &= \beta V_{n+1}^L + (1-\beta)\left[q\lambda B_{n+1}^H + (1-q)\lambda B_{n+1}^L\right] \\ B_n^H &= \beta\left[(1-p)V_{n+1}^L + pV_{n+1}^L\right] + (1-\beta)\left[(1-p)\lambda B_{n+1}^H + p\lambda B_{n+1}^L\right] \\ &= \beta V_{n+1}^L + (1-\beta)\left[(1-p)\lambda B_{n+1}^H + p\lambda B_{n+1}^L\right]. \end{aligned}$$

For $V_{n+1}^L < B_{n+1}^H$, we obtain:

$$\begin{aligned} B_n^L &= \beta\left[qV_{n+1}^L + (1-q)V_{n+1}^L\right] + (1-\beta)\left[qV_{n+1}^L + (1-q)\lambda B_{n+1}^L\right] \\ &= \left[\beta + (1-\beta)q\right]V_{n+1}^L + (1-\beta)(1-q)\lambda B_{n+1}^L \\ B_n^H &= \beta\left[(1-p)V_{n+1}^L + pV_{n+1}^L\right] + (1-\beta)\left[(1-p)V_{n+1}^L + p\lambda B_{n+1}^L\right] \\ &= \left[\beta + (1-\beta)(1-p)\right]V_{n+1}^L + (1-\beta)p\lambda B_{n+1}^L. \end{aligned}$$

(iii) $D = B_{n+1}^H$: We have $V_{n+1}^H > B_{n+1}^H > B_{n+1}^L$. However, we do not know the relation between V_{n+1}^L and B_{n+1}^H.

For $B_{n+1}^H > V_{n+1}^L$, we obtain:

$$\begin{aligned} B_n^L &= \beta\left[qB_{n+1}^H + (1-q)V_{n+1}^L\right] + (1-\beta)\left[qB_{n+1}^H + (1-q)\lambda B_{n+1}^L\right] \\ &= qB_{n+1}^H + (1-q)\left[\beta V_{n+1}^L + (1-\beta)\lambda B_{n+1}^L\right] \\ B_n^H &= \beta\left[(1-p)B_{n+1}^H + pV_{n+1}^L\right] + (1-\beta)\left[(1-p)B_{n+1}^H + p\lambda B_{n+1}^L\right] \\ &= (1-p)B_{n+1}^H + p\left[\beta V_{n+1}^L + (1-\beta)\lambda B_{n+1}^L\right]. \end{aligned}$$

For $B_{n+1}^H \leqslant V_{n+1}^L$, we obtain:

$$\begin{aligned} B_n^L &= \beta\left[qB_{n+1}^H + (1-q)B_{n+1}^H\right] + (1-\beta)\left[qB_{n+1}^H + (1-q)\lambda B_{n+1}^L\right] \\ &= \left[\beta + (1-\beta)q\right]B_{n+1}^H + (1-\beta)(1-q)\lambda B_{n+1}^L \\ B_n^H &= \beta\left[(1-p)B_{n+1}^H + pB_{n+1}^H\right] + (1-\beta)\left[(1-p)B_{n+1}^H + p\lambda B_{n+1}^L\right] \\ &= \left[\beta + (1-\beta)(1-p)\right]B_{n+1}^H + (1-\beta)p\lambda B_{n+1}^L. \end{aligned}$$

(iv) $D = B_{n+1}^L$: We obtain:

$$\begin{aligned} B_n^L &= B_{n+1}^L \\ B_n^H &= B_{n+1}^L. \end{aligned}$$

At each date t_n one of the four candidates for the face value of debt is chosen to achieve the debt capacities B_n^L and B_n^H.

First, we show that $V_n^H > B_n^H$. From the expressions in (i)-(iv), we have

$$B_n^H \leqslant \beta \underbrace{\left[(1-p)V_{n+1}^H + pV_{n+1}^L\right]}_{=V_n^H} + (1-\beta)\underbrace{\left[(1-p)\max\left\{V_{n+1}^L, B_{n+1}^H\right\} + pB_{n+1}^L\right]}_{<V_n^H} < V_n^H.$$

Second, we show that $V_n^L > B_n^L$. From the expressions in (i)-(iv), we have

$$B_n^L < \beta \underbrace{\left[qV_{n+1}^H + (1-q)V_{n+1}^L\right]}_{=V_n^L} + (1-\beta)\left[q\max\left\{V_{n+1}^L, B_{n+1}^H\right\} + (1-q)B_{n+1}^L\right].$$

Using assumption A3, we obtain

$$B_n^L < \beta V_n^L + (1-\beta)\left[q\max\left\{V_{n+1}^L, B_{n+1}^H\right\} + (1-q)B_{n+1}^L\right] < V_n^L.$$

Finally, from the expressions in (i)-(iv), using assumption A1, we can easily see that $B_n^H > B_n^L$. This completes our proof by induction so that the debt capacities in the high and the low states (B_n^H and B_n^L) are given as in the expressions (i)-(iv) for all $n = 1, ...N$.

We can show that the debt capacity B_n^L in cases (i), (ii) and (iii) satisfies:

$$B_n^L \leqslant \beta\left[qV_{n+1}^H + (1-q)V_{n+1}^L\right] + (1-\beta)\left[qB_{n+1}^H + (1-q)\lambda B_{n+1}^L\right].$$

We observe a market freeze in state L at date t_n when

$$\beta\left[qV_{n+1}^H + (1-q)V_{n+1}^L\right] + (1-\beta)\left[qB_{n+1}^H + (1-q)\lambda B_{n+1}^L\right] \leqslant v^L.$$

Starting from the last rollover date t_N, going backwards, we can find the threshold probability of meeting a buy-to-hold investor β_n^* such that for

$$\beta \leqslant \beta_n^* = \frac{v^L - qB_{n+1}^H - (1-q)\lambda B_{n+1}^L}{q\left[V_{n+1}^H - B_{n+1}^H\right] + (1-q)\left[V_{n+1}^L - \lambda B_{n+1}^L\right]},$$

we observe a market freeze at date t_n. Furthermore, for $\beta < \beta^* = \min_{n=0,\ldots,N}\{\beta_n^*\}$, we observe a market freeze in state L for all dates and the face value of debt is set equal to v^L, the lowest possible value of the asset.[6]

Note that the debt capacities B_n^L and B_n^H are increasing in the probability β of meeting a buy-to-hold investor. Hence, a *sufficient* condition for setting the face value of debt greater than v^L in the high state at date t_n is $v^H - v^L > \frac{(1-\lambda)v^L}{(1-p(\tau))^N}$ (the same condition as in Proposition 1, where $\beta = 0$).

[6]From the benchmark model ($\beta = 0$), we know that for sufficiently short period lenght τ, we obtain a market freeze where the debt capacity is equal to v^L in state L. Note that the debt capacity is continous and (weakly) increasing in β so that for sufficiently short period lenght τ, we obtain $\beta^* > 0$.

Proposition 1 *Let $\beta \leqslant \beta^*$ and $v^H - v^L > \frac{(1-\lambda)v^L}{(1-p(\tau))^N}$. For $n = 0, ..., N$, the debt capacities in the low and the high states satisfy:*

$$\begin{aligned} B_n^L &= v^L, \\ B_n^H &\geqslant \beta\left[(1-p)B_{n+1}^H + p\min\left\{B_{n+1}^H, V_{n+1}^L\right\}\right] + (1-\beta)\left[(1-p)B_{n+1}^H + p\lambda B_{n+1}^L\right]. \end{aligned}$$

Next, we provide numerical analysis and figures to show that for reasonable parameter values our results hold. Unless we state otherwise, in the following analysis we use the following values for the parameters of the model: $v^H = 100$, $v^L = 50$, $\lambda = 0.9$ and $\tau = 0.01$. The fundamental values in states H and L at $t = 0$ are $V_0^H = 99.3829$ and $V_0^L = 99.3677$, respectively. Note that only changes in the parameters v^H and v^L and the transition probabilities have an effect on the fundamental value. Figures A1-A5 provide debt capacities in the high and the low states at $t = 0$ for various parameter values.

Figure A1 uses the benchmark values and shows that $\beta^* = 0.0192$ (that is $b^* = 1.936$ for the Poisson process that governs the arrival of buy-to-hold investors). For $\beta = 0.019$ the debt capacities in the high and the low states at $t = 0$ are $B_0^H = 94.9946$ and $B_0^L = 50$, respectively. Furthermore, for $\beta = 0.05$ and 0.10, even though the debt capacity in the low state is higher than 50, we still observe large differences between the debt capacities in the low and the high states ($B_0^L = 57.1207$ (65.7564) and $B_0^H = 95.5946$ (96.3507) for $\beta = 0.05$ (0.10)).

–Figure A1 –

In the following figures, we change various parameters of the model and show that our market freeze result holds. Figure A2 provides the debt capacities for $\lambda = 0.8$. We can see that for $\beta = 0.10$, the debt capacities in the high and the low states at $t = 0$ are $B_0^H = 94.9284$ and $B_0^L = 50$, respectively.

–Figure A2 –

Figure A3 shows the case with $v^L = 60$. In this case, the fundamental value in the high and the low states are at $t = 0$ are $V_0^H = 99.5063$ and $V_0^L = 99.4942$, respectively. We calculate the debt capacities in the low and the high states to be $B_0^L = 60$ (65.8774) and $B_0^H = 95.9271$ (96.4541) for $\beta = 0.05$ (0.10).

–Figure A3 –

Figure A4 illustrates the debt capacities for $\tau = 0.005$. We calculate the debt capacities in the low and the high states to be $B_0^L = 50$ (59.7402) and $B_0^H = 95.0599$ (95.8739) for $\beta = 0.05$ (0.10).

–Figure A4 –

In Figure A5 we alter the transition probabilities in the low state where the generator matrix is

$$\mathbf{A} = \begin{bmatrix} 5 & 5 \\ 0.1 & -0.1 \end{bmatrix}.$$

The debt capacities in the low and the high states are $B_0^L = 50$ (60.8361) and $B_0^H = 95.0453$ (95.8385) for $\beta = 0.05$ (0.10), respectively.

–Figure A5 –

The numerical analysis and the figures show that our theoretical result on market freeze holds for reasonable parameter values even in the presence of buy-to-hold investors who are willing to pay their fundamental value for the assets. In particular, while the fundamental values in the high and the low states are high, with only a miniscule difference between the two, the difference between debt capacities in states H and L is large. We also show that for reasonable parameter values the debt capacity in the low state is equal to the lowest possible terminal value of the asset.

Figure A1: Debt capacities for benchmark values

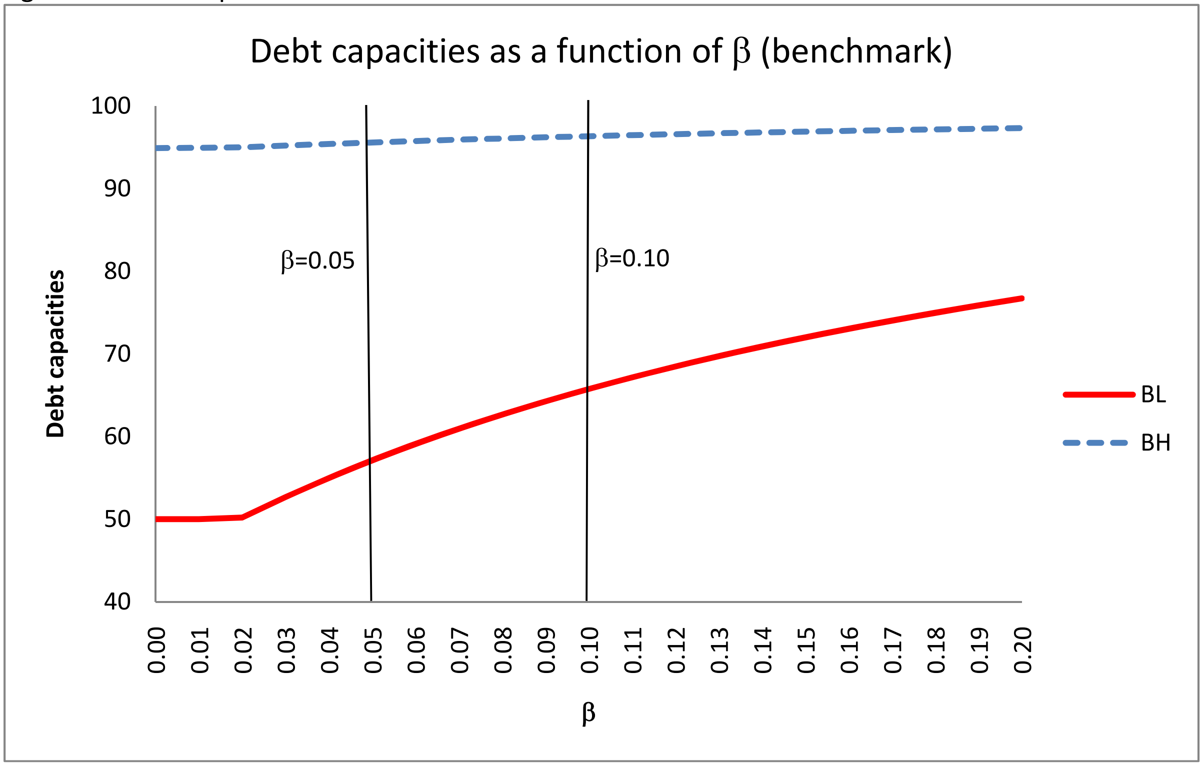


Figure A2: Debt capacities (λ=0.8)

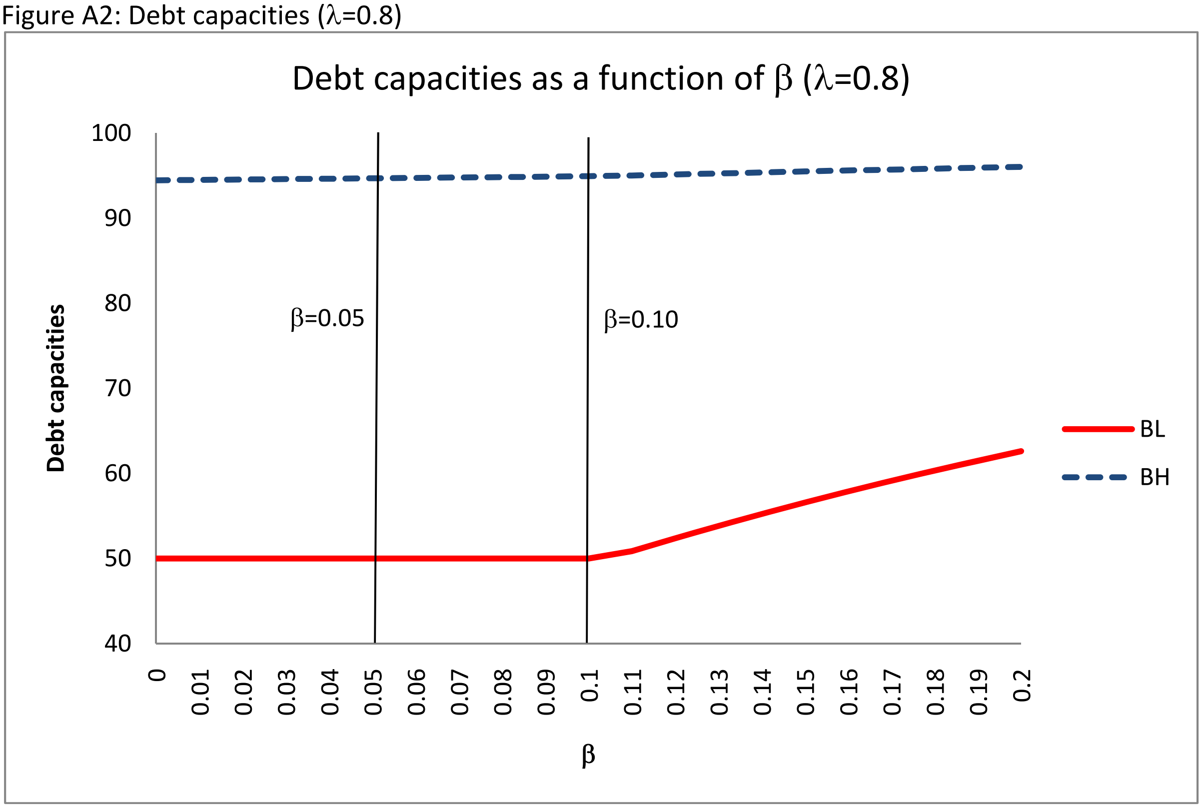


Figure A3: Debt capacities (v^L=60)

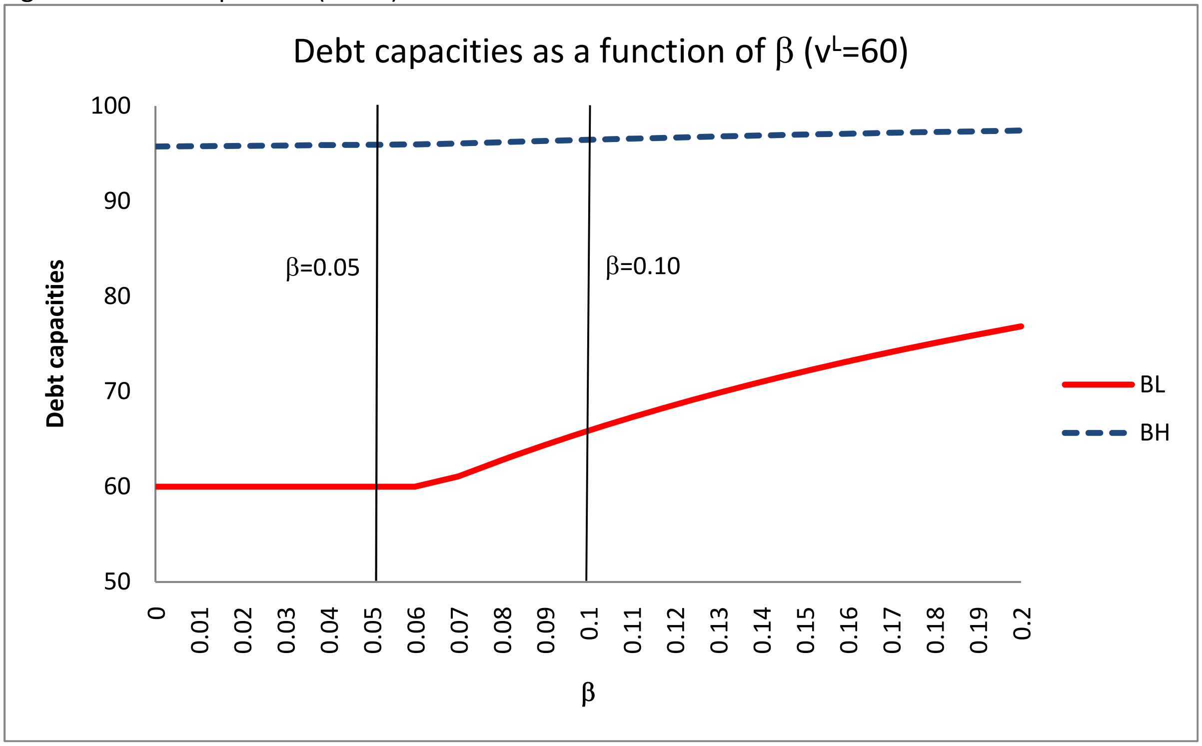


Figure A4: Debt capacities (τ=0.005)

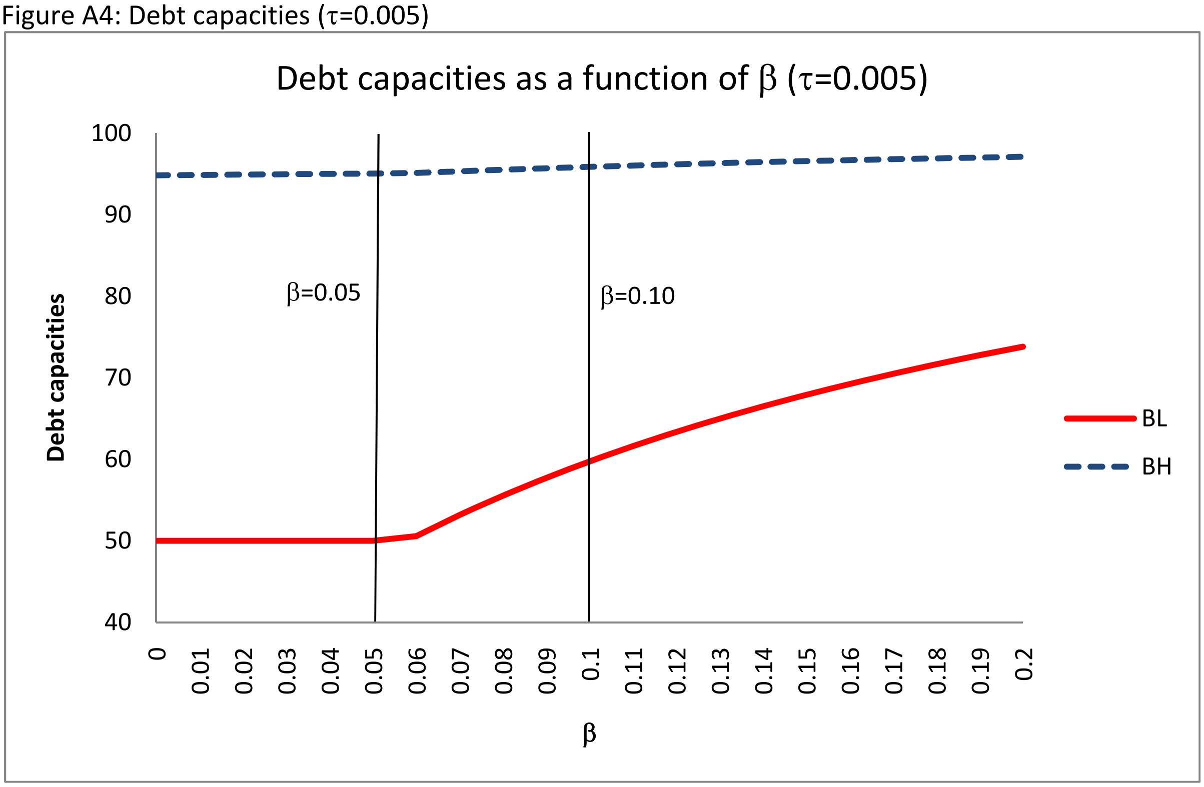
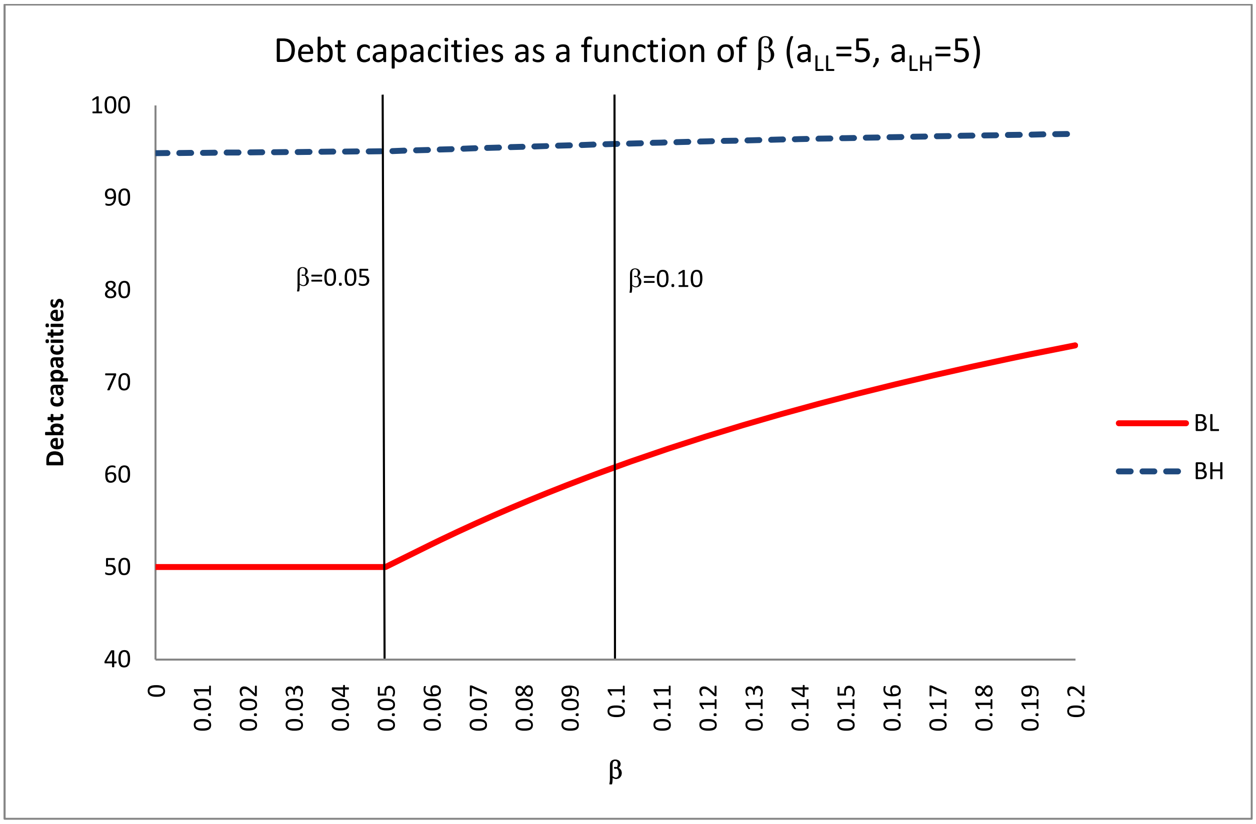


Figure A5: Debt capacities for different transition probabilities



2 Extension with Fixed Cost

2.1 Numerical example with fixed cost

Here, we provide a numerical example with constant liquidation cost, where $c = 4.25$ so that each time there is a default and the assets are liquidated, there is a fixed liquidation cost of 4.25. As in the example in the text with proportional cost of liquidation, we use the following parameter values: the tenor of the repo is $\tau = 0.01$, the values of the asset are $v^H = 100$ and $v^L = 50$ in the high and low states, respectively,[7] and the generator is

$$\mathbf{A} = \begin{bmatrix} -8.0 & 8.0 \\ 0.1 & -0.1 \end{bmatrix}.$$

The transition probability matrix for an interval of unit length can be calculated to be

$$\mathbf{P}(1) = \begin{bmatrix} 0.01265 & 0.98735 \\ 0.01234 & 0.98766 \end{bmatrix}. \tag{1}$$

Note that, we can calculate the fundamental values as in the numerical example with proportional liquidation cost so that we obtain $V_0^H = 99.383$ and $V_0^L = 99.367$.

As in the numerical example in the text, we obtain

$$\mathbf{P}(0.01) = \begin{bmatrix} 0.92315 & 0.07685 \\ 0.00096 & 0.99904 \end{bmatrix}. \tag{2}$$

Consider now the debt capacities at the last rollover date $t_{99} = 0.99$. If we set $D = 50$, the debt can be paid off at date 1 in both states and the expected value of the payoff is 50. So the market value of the debt with face value 50 is exactly 50.

Now suppose we set $D = 100$, there will be default in state L but not in state H at time 1. The payoff in state H will be 100 but the payoff in state L will be $50 - 4.25 = 45.75$. Then the market value of the debt at time t_{99} will depend on the state at time t_{99}, because the transition probabilities depend on the state. We can easily calculate the expected payoffs in each state:

$$\begin{aligned} \text{state H} &: \; 0.99904 \times 100 + 0.00096 \times (50 - 4.25) = 99.948; \\ \text{state L} &: \; 0.07685 \times 100 + 0.92315 \times (50 - 4.25) = 49.919. \end{aligned}$$

For example, if the state is H at date t_{99}, then with probability 0.99904 the state is H at date 1 and the debt pays off 100 and with probability 0.00096 the state is L at date 1, the asset must be liquidated and the creditors only realize 45.25.

Comparing the market values of the debt with the two different face values, we can see that the face value that maximizes the market value of debt will depend on the state. In

[7] The numerical example with the constant liquidation for $v^L = 40$ is omitted. We obtain qualitatively similar results to the results with proportional liquidation cost as in the paper. However, we provide Figure A7 to illustrate the debt capacities for the case with constant liquidation and $v^L = 40$.

state H, the expected value of the debt when $D = 100$ is $99.948 > 50$, so that $D_{99}^H = 100$. In state L, on the other hand, the expected value of the debt with face value $D = 100$ is only $49.919 < 50$, so that $D_{99}^L = 50$. Thus, if we use the notation B_n^s to denote the debt capacity in state s at date t_n, we have shown that $B_{99}^H = 99.948$ and $B_{99}^L = 50$.

Next, consider the debt capacities at date $t_{98} = 0.98$. Now, the relevant face values to consider are 50 and 99.948.

If $D = 50$, the expected payoff is 50 too, since the debt capacity at date t_{99} is greater than or equal to 50 in both states and, hence, the debt can always be rolled over. In contrast, if $D = 99.948$, the debt cannot be rolled over in state L at date t_{99} and the liquidation cost is incurred. Thus, the expected value of the debt depends on the state at date t_{98}:

$$\begin{aligned} \text{state H} &: \quad 0.99904 \times 99.948 + 0.00096 \times (50 - 4.25) = 99.896, \\ \text{state L} &: \quad 0.07685 \times 99.948 + 0.92315 \times (50 - 4.25) = 49.915. \end{aligned}$$

Comparing the expected value corresponding to different face values of the debt, we see that the face value that achieves the debt capacity is $D_{98}^H = 99.948$ in state H and $D_{98}^L = 50$ in state L, so that the debt capacities are $B_{98}^H = 99.896$ and $B_{98}^L = 50$.

At each date t_n, the debt capacity in the high state is lower than it was at t_{N+1} and the debt capacity in the low state is the same as it was at t_{N+1}. These facts tells us that if $D_{n+1}^L = 50$ at t_{n+1}, then *a fortiori* $D_n^L = 50$ at date t_n. Thus, the debt capacity is equal to 50 at each date t_n, including the first date $t_0 = 0$.

What is the debt capacity in state H at t_0? The probability of staying in the high state from date 0 to date 1 is $(0.99904)^{100} = 0.90842$ and the probability of hitting the low state at some point is $1 - 0.90842 = 0.09158$ so the debt capacity at time 0 is

$$B_0^H = 0.90842 \times 100 + 0.09158 \times (50 - 4.25) = 95.032.$$

So the fall in debt capacity occasioned by a switch from the high to the low state at time 0 is $95.032 - 50 = 45.032$ compared to a change in the fundamental value of $99.383 - 99.367 = 0.016$. This fall is illustrated sharply in Figure A6 which shows that while fundamental values in states H and L will diverge sharply at maturity, they are essentially the same at date 0. Nevertheless, debt capacity in state L is simply the terminal value in state L. Thus, a switch to state L from state H produces a sudden drop in debt capacity of the asset.

— Figure A6 here —

2.2 Debt capacity with two states

In this section we provide a proof for the market freeze result when there are two states. We make the same assumptions as for the numerical example but the parameters are otherwise arbitrary. For the time being, we treat the tenor of the commercial paper τ and the number of rollovers N as fixed. Later, we will be interested to see what happens when the tenor τ becomes very small and the number of rollovers N becomes correspondingly large.

There are two states, a "low" state L and a "high" state H. Transitions between states occur at the dates t_n and are governed by a stationary transition probability matrix

$$\mathbf{P}(\tau) = \left[\begin{array}{cc} p_{LL}(\tau) & p_{LH}(\tau) \\ p_{HL}(\tau) & p_{HH}(\tau) \end{array} \right],$$

where $p_{HL}(\tau)$ $(p_{LH}(\tau))$ is the probability of a transition from state H (L) at time t_n to state L (H) at time t_{n+1}. The one requirement we impose on these probabilities is that the shorter the period length, the more likely it is that there is no change in the information state before the next rollover date:

$$\lim_{\tau \to 0} p_{HL}(\tau) = \lim_{\tau \to 0} p_{LH}(\tau) = 0.$$

The terminal value of the asset is v^H if the terminal state is H and v^L if the terminal state is L, where $0 < v^L < v^H$.

In the numerical example, we saw that the borrower chose a low face value of the debt in the low state and a high face value of the debt in the high state. Here we will provide necessary and sufficient conditions under which choosing high and low face values in the high and the low states, respectively, will achieve the debt capacity in those states. We begin by considering the low state.

The low state Suppose that the economy is in the low state at date t_N, which is the last of the rollover dates. Let D be the face value of the debt issued by the bank. If $D > v^H$, the bank will default in both states at date t_{N+1} and the creditors will receive $\left(v^H - c\right)$ in the high state and $\left(v^L - c\right)$ in the low state.[8] Clearly, the market value of the debt at date t_N would be greater if the face value were $D = v^H$, so the borrower will never choose $D > v^H$. Now suppose that the bank issues debt with face value D, where $v^L < D < v^H$. This will lead to default in the low state at date t_{N+1} and the creditors will receive D in the high state and λv^L in the low state. Clearly, this is dominated by choosing a higher value of D. Thus, either $D = v^H$ or $D \leq v^L$. An exactly similar argument shows that a face value $D < v^L$ will not be chosen, so we are left with only two possibilities, either $D = v^H$ or $D = v^L$. In the first case, the market value of the debt is $p_{LL}(\tau)\left(v^L - c\right) + p_{LH}(\tau)\, v^H$ and in the second case it is v^L. A necessary and sufficient condition for D_N^L to equal v^L is

$$p_{LL}(\tau)\left(v^L - c\right) + p_{LH}(\tau)\, v^H \leq v^L. \tag{3}$$

This condition will clearly be satisfied for all $\tau > 0$ sufficiently small, but for the time being we will simply assume that (3) is satisfied.

Now suppose that (3) is satisfied and that $B^L_{n'+1} = v^L$ for $n' = n, ..., N$. Consider what happens in the low state at date t_n. By the usual argument, the only candidates for the face value that achieves the debt capacity are $D = v^L$ and $D = B^H_{n+1}$. If the face value is

[8] To simplify the argument, we are assuming that there is a liquidation cost at date t_{N+1}, too. None of the results depend on this.

$D = v^L$, the creditors will receive v^L in both states at date t_{n+1} and the market value of the debt at date t_n will be v^L. On the other hand, if the face value of the debt is $D = B^H_{n+1}$, the creditors receive B^H_{n+1} in the high state and $\left(v^L - c\right)$ in the low state, so the market value of the debt at date t_n is

$$p_{LL}(\tau)\left(v^L - c\right) + p_{LH}(\tau) B^H_{n+1} \leq p_{LL}(\tau)\left(v^L - c\right) + p_{LH}(\tau) v^H,$$

since $B^H_{n+1} \leq v^H$. But (3) implies that $p_{LL}(\tau)\left(v^L - c\right) + p_{LH}(\tau) v^H \leq v^L$, so the debt capacity is $B^L_n = v^L$ for all $n = 1, ..., N$.

The high state Now consider the high state. Again, our two candidates for the face value of the debt at each date t_n are B^H_{n+1} and v_L. Let us assume that at each date t_n the face value of the debt is set equal to the future debt capacity B^H_{n+1}, that is, we begin at date t_N by setting $D^H_N = v^H$ and $B^H_N = p_{HH}(\tau) v^H + p_{HL}(\tau)\left(v^L - c\right)$ and then recursively define $D^H_n = B^H_{n+1}$ and

$$B^H_n = p_{HH}(\tau) B^H_{n+1} + p_{HL}(\tau)\left(v^L - c\right),$$

for $n = 1, ..., N-1$. It can easily be shown by backward induction that $B^H_n \leq B^H_{n+1}$ for any n, so in order to show that this strategy will be chosen, it is necessary and sufficient to show that $B^H_0 \geq v^L$. By repeated substitution we can show that

$$\begin{aligned} B^H_0 &= p_{HH}(\tau) B^H_1 + p_{HL}(\tau)\left(v^L - c\right) \\ &= p_{HH}(\tau)\left\{p_{HH}(\tau) B^H_2 + \left[1 - p_{HH}(\tau)\right]\left(v^L - c\right)\right\} + \left[1 - p_{HH}(\tau)\right]\left(v^L - c\right) \\ &= p_{HH}(\tau)^2\left(B^H_2 - \left(v^L - c\right)\right) + \left(v^L - c\right) \\ &\cdots \\ &= \left(p_{HH}(\tau)\right)^N\left(v^H - \left(v^L - c\right)\right) + \left(v^L - c\right). \end{aligned}$$

Then $D^H_n = B^H_{n+1}$ for all n if and only if

$$\left(p_{HH}(\tau)\right)^N\left(v^H - \left(v^L - c\right)\right) + \left(v^L - c\right) \geq v^L$$

or

$$v^H - v^L + c \geq \frac{c}{\left(p_{HH}(\tau)\right)^N}. \tag{4}$$

We have proved the following proposition.

Proposition 2 *Define* $\left\{\left(B^H_n, D^H_n, B^L_n, D^L_n\right)\right\}_{n=0}^N$ *by setting*

$$D^H_n = B^H_{n+1}, \tag{5}$$

$$B^H_n = p_{HH}(\tau) B^H_{n+1} + p_{HL}(\tau)\left(v^L - c\right), \tag{6}$$

and

$$D^L_n = B^L_n = v^L, \tag{7}$$

for $n = 1, ..., N$. *The values defined by (5-7) constitute a solution to the problem of achieving the debt capacity if and only if (3) and (4) are satisfied.*

The qualitative properties of the debt capacities characterized in Proposition 2 are the same as in the numerical example in Section 2.1. In the low state, the debt capacity B_n^L is constant and equal to the lowest possible terminal value, v^L. The fundamental value of the asset in the low state V_n^L is greater than the debt capacity at every date t_n except at the terminal date, when they are both equal to v^L. In the high state, the debt capacity B_n^H is always less than the fundamental value V_n^H, except at the terminal date when both are equal to v^H. We call this behavior of the debt capacity a "market freeze" since a switch in the information state from high state to the low state can produce a sudden, sharp drop in debt capacity that is much larger than the drop in fundamental value associated with the switch.

2.3 Debt capacity in the general case

We allow for a finite number of information states or signals, denoted by $\mathcal{S} = \{s_1, ..., s_I\}$. The current information state is public information. Transitions among the states are governed by a stationary Markov transition probability $\mathbf{P}(\tau)$ given as

$$\mathbf{P}(\tau) = \begin{bmatrix} p_{11}(\tau) & \cdots & p_{1I}(\tau) \\ \vdots & \ddots & \vdots \\ p_{I1}(\tau) & \cdots & p_{II}(\tau) \end{bmatrix},$$

where τ is the interval over which the transitions take place. We assume that the transition matrix takes the form

$$\mathbf{P}(\tau) = e^{\mathbf{A}\tau} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}\tau)^k}{k!},$$

where the matrix $\mathbf{A}$ is the generator. The crucial feature of the transition matrix is that the probability of a change of state converges to 0 as $\tau \to 0$. That is, $\mathbf{P}(\tau) \to \mathbf{I}$ as $\tau \to 0$.

The information state is a stochastic process $\{S(t)\}$ but for our purposes all that matters is the value of this process at the rollover dates. We let S_n denote the value of the information state $S(t_n)$ at the rollover date t_n.

The terminal value of the assets is a function of the information state at date $t = 1$. We denote by v_i the value of the assets if the terminal state is $S_{N+1} = s_i$ and assume that the values $\{v_1, ..., v_I\}$ satisfy

$$0 < v_1 < \ldots < v_I.$$

Let V_n^i denote the fundamental value of the asset at date t_n in state i. Then clearly the values $\{V_n^i\}$ are defined by putting $V_{N+1}^i = v_i$, for $i = 1, ..., I$, and

$$V_n^i = \sum_{j=1}^{I} p_{ij}(1 - t_n) v_j, \text{ for } n = 0, ..., N \text{ and } i = 1, ..., I,$$

where $p_{ij}(1-t_n)$ is, of course, the (i,j) entry of $\mathbf{P}(1-t_n)$ denoting the probability of a transition from state i at date t_n to state j at date $t_{N+1}=1$.

Let B_n^i denote the equilibrium debt capacity of the assets in state s_i at date t_n. By convention, we set $B_{N+1}^i = v_i$ for all i.

Proposition 3 *The equilibrium values of* $\{B_n^i\}$ *must satisfy*

$$B_n^i = \max_{k=1,\ldots,I}\left\{\sum_{\left\{j:B_{n+1}^k>B_{n+1}^j\right\}} p_{ij}(\tau)\left(B_{n+1}^j-c\right)+\sum_{\left\{j:B_{n+1}^k\le B_{n+1}^j\right\}} p_{ij}(\tau)\,B_{n+1}^k\right\}$$

for $i=1,\ldots,I$ *and* $n=0,\ldots,N$.

The result is immediate once we apply the now familiar backward induction argument to show that the borrower sets D_n^i equal to B_{n+1}^j for some j. Although the result amounts to little more than the definition of debt capacity, it is very useful because it allows us to calculate the debt capacities by backward induction.

The main result on the downward bias of debt capacities is contained in the following proposition.

Proposition 4 *There exists* $\tau^*>0$ *such that for all* $0<\tau<\tau^*$, *for any* $n=0,\ldots,N$ *and any* $i=1,\ldots,I$, *the borrower chooses the face value* $D_n^i\le B_{n+1}^i$. *Thus,*

$$B_n^i = \sum_{\left\{j:B_{n+1}^k>B_{n+1}^j\right\}} p_{ij}(\tau)\left(B_{n+1}^j-c\right)+\sum_{\left\{j:B_{n+1}^k\le B_{n+1}^j\right\}} p_{ij}(\tau)\,B_{n+1}^k.$$

for some k *such that* $B_{n+1}^k\le B_{n+1}^i$.

Proof. See Appendix A. ■

Several properties follow immediately from Proposition 4 whenever $0<\tau<\tau^*$. We provide these results in the form of three corollaries. First, in the lowest state, s_1, the debt capacity is constant and equal to v_1, the lowest possible terminal value.

Corollary 5 $B_n^1=v_1$ *for all* n.

Proof. From the formula in Proposition 4, for some k,

$$\begin{aligned} B_n^1 &= \sum_{\left\{j:B_{n+1}^k>B_{n+1}^j\right\}} p_{ij}(\tau)\left(B_{n+1}^j-c\right)+\sum_{\left\{j:B_{n+1}^k\le B_{n+1}^j\right\}} p_{ij}(\tau)\,B_{n+1}^k \\ &\le \sum_{j=1}^{I} p_{ij}(\tau)\,B_{n+1}^1 = B_{n+1}^1, \end{aligned}$$

since $B_{n+1}^k \leq B_{n+1}^1$. Since this inequality holds for $n = 0, ..., N$ and, by convention, $B_{N+1}^1 = v_1$, it follows that $B_n^1 \leq v_1$, for any n.

We can also show that $B_n^1 \geq v_1$. To see this, note that $B_{N+1}^i = v_1$ for all i. Moreover, if the same condition holds for $n+1$, it must be true that $B_n^i \geq v_1$, because we can always choose $D_n^i = v_1$.

Thus, we have shown that $B_n^1 = v_1$ for all n. ∎

Second, the debt capacity B_n^i is monotonically non-decreasing in n, that is, debt capacity increases as the asset matures, holding the state constant. This follows directly from the fact that, if the face value of the debt equals B_{n+1}^i, the debt capacity B_n^i cannot be greater than B_{n+1}^i.

Corollary 6 *$B_n^i \leq B_{n+1}^i$, for any $i = 1, ..., I$ and $n = 0, ..., N$.*

Proof. The inequality follows directly from the formula in Proposition 4:

$$\begin{aligned}
B_n^i &= \sum_{\left\{j: B_{n+1}^k > B_{n+1}^j\right\}} p_{ij}(\tau)\left(B_{n+1}^j - c\right) + \sum_{\left\{j: B_{n+1}^k \leq B_{n+1}^j\right\}} p_{ij}(\tau) B_{n+1}^k \\
&\leq \sum_{\left\{j: B_{n+1}^k > B_{n+1}^j\right\}} p_{ij}(\tau) B_{n+1}^i + \sum_{\left\{j: B_{n+1}^k \leq B_{n+1}^j\right\}} p_{ij}(\tau) B_{n+1}^i \\
&= \sum_{j=1}^{I} p_{ij}(\tau) B_{n+1}^i = B_{n+1}^i,
\end{aligned}$$

since $B_{n+1}^j < B_{n+1}^k \leqslant B_{n+1}^i$ implies that $B_{n+1}^j - c < B_{n+1}^i$. ∎

Third, since $B_{N+1}^i = v_i$ by convention, the preceding result immediately implies that the debt capacity B_n^i is less than or equal to v_i.

Corollary 7 *$B_n^i \leq v_i$ for all $i = 1, ..., I$ and $n = 0, ..., N$.*

Finally, we can confirm that the debt capacity in state s_i at any date t_n is less than the fundamental value V_n^i. This follows directly from the formula in Proposition 4 for $n = N+1$ and any i, so suppose that it holds for $n+1, ..., N$ and any $i = 1, ..., I$. Then the formula in Proposition 4 implies that, if $D_n^i = B_{n+1}^k$, say,

$$\begin{aligned}
B_n^i &= \sum_{\left\{j: B_{n+1}^k > B_{n+1}^j\right\}} p_{ij}(\tau)\left(B_{n+1}^j - c\right) + \sum_{\left\{j: B_{n+1}^k \leq B_{n+1}^j\right\}} p_{ij}(\tau) B_{n+1}^k \\
&\leq \sum_{\left\{j: B_{n+1}^k > B_{n+1}^j\right\}} p_{ij}(\tau) B_{n+1}^j + \sum_{\left\{j: B_{n+1}^k \leq B_{n+1}^j\right\}} p_{ij}(\tau) B_{n+1}^j \\
&\leq \sum_{j=1}^{I} p_{ij}(\tau) V_{n+1}^j = V_n^i,
\end{aligned}$$

for any $i = 1, ..., I$, so by induction the claim holds for any $n = 0, ..., N$ and any $i = 1, ..., I$.

Proposition 4 shows that, when the period length is sufficiently short (the rollover rate is sufficiently high), there is a downward bias in the debt capacity, because the face value of the debt is bounded above by the future debt capacity in the same state. This is an important step toward proving the existence of a market freeze but two further requirements are needed. First, we need to show that the fundamental values are uniformly high and that the debt capacities are high in some states and low in others.

Consider the debt capacities first. The proposition shows that $B_n^1 = v_1$ for all $n = 0, ..., N$ so it is enough to show that the debt capacity is high in some states. The following proposition does just that.

Proposition 8 *The initial debt capacity in the highest state, I, satisfies the inequality*

$$B_0^I \geq e^{-\alpha} v_I + \left(1 - e^{-\alpha}\right)(v_1 - c),$$

where $\alpha = \sum_{j \neq I} a_{Ij}$.

So, as long as α is sufficiently small, the debt capacity in the high state will be close to v_I. We know that the fundamental value V_0^I lies between B_0^I and v_I, so the fundamental value will also be close to v_I. Then in order to show that a market freeze is possible it is only necessary to ensure that the fundamental value for the lowest state, V_0^1, is also close to v_I. This will be true as long as the probability of a transition from the lowest to the highest state is high enough. In general, this probability will depend on the entire matrix $\mathbf{A}$ so it is hardly worth trying to write down a sufficient condition in terms of the individual parameters. However, note that the numerical examples in the main text for the case with the proportional liquidation cost have illustrated that this is clearly possible.

2.4 Proofs

We can solve for the equilibrium debt capacities in the model of Section 2.3 by backward induction. Let D denote the face value of the debt issued in state s_i at date t_n. This debt will pay off D in state s_j at date t_{n+1} if $D \leq B_{n+1}^j$ and λB_{n+1}^j otherwise. In other words, the market value of the debt is given by the formula

$$\sum_{B_{n+1}^j < D} p_{ij}(\tau)\left(B_{n+1}^j - c\right) + \sum_{B_{n+1}^j \geq D} p_{ij}(\tau) D$$

and the debt capacity is given by

$$B_n^i = \max_D \left\{ \sum_{B_{n+1}^j < D} p_{ij}(\tau)\left(B_{n+1}^j - c\right) + \sum_{B_{n+1}^j \geq D} p_{ij}(\tau) D \right\}.$$

Let D_n^i denote the face value that maximizes the market value of debt in state i at date t_n. It is clear that the market value of the debt is maximized by setting the face value $D = B_{n+1}^j$, for some value of $j = 1, ..., I$. Thus, we can write the equilibrium condition as

$$B_n^i = \max_{k=1,...,I} \left\{ \sum_{\left\{j:B_{n+1}^k > B_{n+1}^j\right\}} p_{ij}(\tau)\left(B_{n+1}^j - c\right) + \sum_{\left\{j:B_{n+1}^k \leqslant B_{n+1}^j\right\}} p_{ij}(\tau) B_{n+1}^k \right\},$$

for $i = 1, ..., I$ and $n = 0, ..., N$.

Proposition 9 *For some $\tau^* > 0$ and all $\tau < \tau^*$, $D_n^i \leq B_{n+1}^i$ for all $i = 1, ..., I$ and $n = 0, ..., N$.*

Proof. For a fixed but arbitrary date t_n and state s_i, we compare the strategy of setting $D = B_{n+1}^i$ with the strategy of setting $D = B_{n+1}^k$, where $B_{n+1}^k > B_{n+1}^j$. Consider the difference in the expected values of the debt:

$$\begin{aligned}
& \sum_{\left\{j:B_{n+1}^i > B_{n+1}^j\right\}} p_{ij}(\tau)\left(B_{n+1}^j - c\right) + \sum_{\left\{j:B_{n+1}^i \leqslant B_{n+1}^j\right\}} p_{ij}(\tau) B_{n+1}^i \\
& - \sum_{\left\{j:B_{n+1}^k > B_{n+1}^j\right\}} p_{ij}(\tau)\left(B_{n+1}^j - c\right) - \sum_{\left\{j:B_{n+1}^k \leqslant B_{n+1}^j\right\}}^{I} p_{ij}(\tau) B_{n+1}^k \\
= & \sum_{\left\{j:B_{n+1}^i \leqslant B_{n+1}^j < B_{n+1}^k\right\}} p_{ij}(\tau)\left(B_{n+1}^i - \left(B_{n+1}^j - c\right)\right) + \sum_{\left\{j:B_{n+1}^k \leqslant B_{n+1}^j\right\}} p_{ij}(\tau)\left(B_{n+1}^i - B_{n+1}^k\right) \\
= & \; p_{ii}(\tau)\left(B_{n+1}^i - \left(B_{n+1}^j - c\right)\right) + \sum_{\left\{j:B_{n+1}^i < B_{n+1}^j < B_{n+1}^k\right\}} p_{ij}(\tau)\left(B_{n+1}^i - \left(B_{n+1}^j - c\right)\right) \\
& + \sum_{\left\{j:B_{n+1}^k \leqslant B_{n+1}^j\right\}} p_{ij}(\tau)\left(B_{n+1}^i - B_{n+1}^k\right) \\
\geq & \; p_{ii}(\tau) c + \sum_{\left\{j:B_{n+1}^i < B_{n+1}^j < B_{n+1}^k\right\}} p_{ij}(\tau)\left(v_1 - v_I\right) + \sum_{\left\{j:B_{n+1}^k \leqslant B_{n+1}^j\right\}} p_{ij}(\tau)\left(v_1 - v_I\right) \\
= & \; p_{ii}(\tau) c + \sum_{\left\{j:B_{n+1}^i < B_{n+1}^j\right\}} p_{ij}(\tau)\left(v_1 - v_I\right).
\end{aligned}$$

since $B_{n+1}^i \geq v_1$, $B_{n+1}^i - \left(B_{n+1}^j - c\right) \geq B_{n+1}^i - B_{n+1}^j \geq (v_1 - v_I)$ for $j = i+1, ..., I$ and $B_{n+1}^i - B_{n+1}^k \geq v_1 - v_I$. Then it is clear that, for τ sufficiently small (i.e., $p_{ii}(\tau)$ sufficiently close to 1), the last expression above is positive. Since the last expression is independent of n, the bound is uniform, i.e., there exists a constant $\tau^* > 0$ such that, for $\tau < \tau^*$, $D_n^i = B_{n+1}^i$ for all i and n. ■

Figure A6: Fundamental value (V) and debt capacity (B) in high (v^H=100) and low (v^L=50) states as a function of time

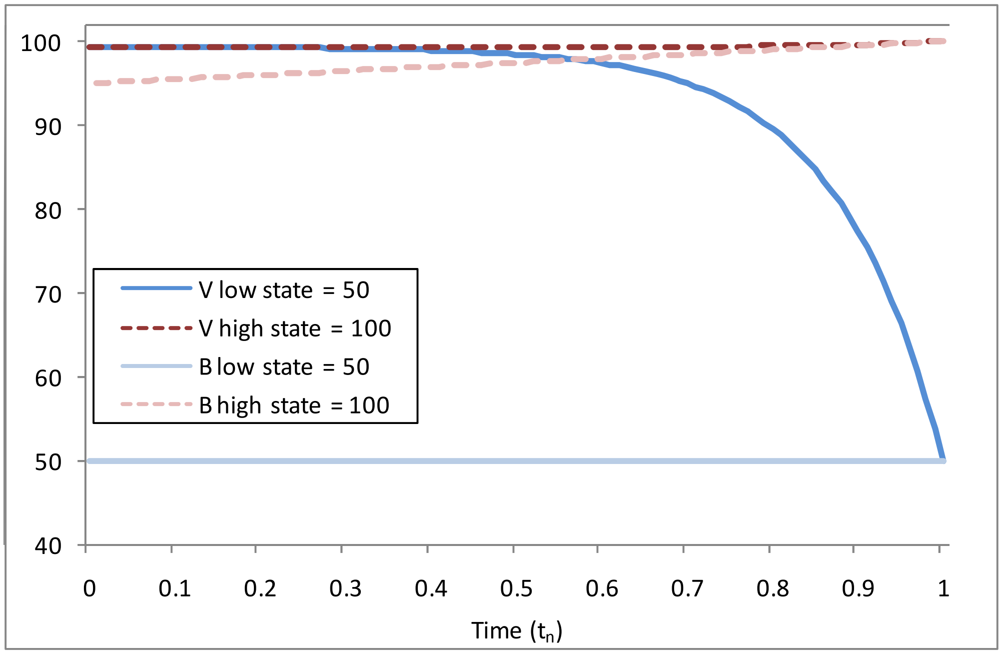


Figure A7: Debt capacity (B) in high (v^H=100) and low (v^L=40) states as a function of time