Fire sales forensics: measuring endogenous risk

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Abstract

We propose a tractable framework for quantifying the impact of fire sales on the volatility and correlations of asset returns in a multi-asset setting. Our results enable to quantify the impact of fire sales on the covariance structure of asset returns and provide a quantitative explanation for spikes in volatility and correlations observed during liquidation of large portfolios. These results allow to test for the presence of fire sales during a given period of time and to estimate the impact and magnitude of fire sales from observation of market prices: we give conditions for the identifiability of model parameters from time series of asset prices, propose an estimator for the magnitude of fire sales in each asset class and study the consistency and large sample properties of the estimator. We illustrate our estimation methodology with two empirical examples: the hedge fund losses of August 2007 and the Great Deleveraging following the default of Lehman Brothers in Fall 2008.

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1 Introduction

Fire sales or, more generally, the sudden deleveraging of large financial portfolios, have been recognized as a destabilizing factor in recent (and not-so-recent) financial crises, contributing to unexpected spikes in volatility and correlations of asset returns and resulting in spirals of losses for investors (Carlson, 2006; Brunnermeier, 2008; Khandani and Lo, 2011). In particular, unexpected increases in correlations across asset classes have frequently occurred during market downturns (Cont and Wagalath, 2012; Bailey et al., 2012), leading to a loss of diversification benefits for investors, precisely when such benefits were desirable.

For instance, during the first week of August 2007, when a large fund manager deleveraged his/her positions in long-short market neutral equity strategies, other long-short market neutral equity funds experienced huge losses, while in the meantime, index funds were left unaffected (Khandani and Lo, 2011). On a larger scale, the Great Deleveraging of financial institutions' portfolios subsequent to the default of Lehman Brothers in fall 2008 led to an unprecedented peak in correlations across asset returns (Fratzscher, 2011).

The importance of fire sales as a factor of market instability is recognized in the economic literature. Shleifer and Vishny (1992, 2011) characterize an asset fire sale by a financial institution as a forced sale in which potential high valuation buyers are affected by the same shocks as the financial institution, resulting in a sale of the asset at a discounted price to non specialist buyers. They underline the fact that in the presence of fire sales, losses by financial institutions with overlapping holdings become self-reinforcing, leading to downward spirals for asset prices and, ultimately, to systemic risk. Pedersen (2009) describes qualitatively the effects of investors running for the exit and the spirals of losses and spillover effects they generate. Shin (2010) and Ozdenoren and Yuan (2008) propose equilibrium models which takes into account the supply and demand generated by investors reacting to a price move and show how feedback effects contribute to the amplification of volatility and market instability. Boyer et al. (2006) emphasize the role of institutional investors in price-mediated contagion, suggesting that crisis spread through the asset holdings of international investors rather than through changes in fundamentals. Brunnermeier (2008) describes the channel through which losses in mortgage backed securities during the recent financial crisis led to huge losses in equity markets, although those two assets classes had been historically uncorrelated.

The empirical link between fire sales and increase in correlation across asset returns has been documented in several recent studies. Coval and Stafford (2007) give empirical evidence for fire sales by open-end mutual funds by studying the transactions caused by capital flows. They show that funds in distress experience outflows of capital by investors which result in fire sales in existing positions, creating a price pressure in the securities held in common by distressed funds. Jotikasthira et al. (2011) lead an empirical investigation on the effects of fund flows from developed countries to emerging markets. They show that such investment flows generate forced trading by fund managers, affecting asset prices and correlations between emerging markets and creating a new channel through which shocks are transmitted from developed markets to emerging markets. Anton and Polk (2008) find empirically that common active mutual fund ownership predicts cross-sectional variation in return realized covariance.

However, although the empirical examples cited above are related to liquidation of large *portfolios*, most theoretical studies focus for simplicity on fire sales in a single asset market and thus are not able to investigate the effect of fire sales on asset return correlations and the resulting limits to diversification alluded to above.

Kyle and Xiong (2001) propose an equilibrium model, which takes into account the supply and demand of three categories of traders: noise traders, long-term investors and convergence traders, in a market with two risky assets and find that convergence traders, who are assumed to trade using a logarithmic utility function, can react to a price shock in one asset by deleveraging their positions in both markets, leading to contagion effects. Greenwood and Thesmar (2011) propose a simple framework for modeling price dynamics which takes into account the ownership structure of financial assets, considered as given exogenously. Cont and Wagalath (2012) model the systematic supply and demand generated by investors exiting a large distressed fund and quantify its impact on asset returns.

We propose here a tractable framework for modeling and estimating the impact of fire sales in multiple funds on the volatility and correlations of asset returns in a multi-asset setting. We explore the mathematical properties of the model in the continuous-time limit and derive analytical results relating the realized covariance of asset returns to the parameters describing the volume of fire sales. In particular, we show that, starting from homoscedastic inputs, feedback effects from fire sales naturally generate heteroscedasticity in the covariance structure of asset returns, thus providing an economic interpretation for various multivariate models of heteroscedasticity in the recent literature (Engle, 2002; Da Fonseca et al., 2008; Gouriéroux et al., 2009; Stelzer, 2010). Our results allow for a structural explanation for the variability observed in measures of cross sectional dependence in asset returns (Bailey et al., 2012), by linking such increases in cross-sectional correlation to the deleveraging of large portfolios.

The analytically tractable nature of these results allows to explore in detail the problem of *estimating* these parameters from empirical observations of price series; we explore the corresponding identification problem and propose a method for estimating the magnitude of distressed selling in each asset class, and study the consistency and large sample properties of the proposed estimator. These results provide a quantitative framework for the 'forensics analysis' of the impact of fire sales and distressed selling, which we illustrate with two empirical examples: the August 2007 hedge fund losses and the Great Deleveraging of bank portfolios following the default of Lehman Brothers in September 2008.

Our framework links large shifts in the realized covariance structure of asset returns with the liquidation of large portfolios, in a framework versatile enough to be amenable to empirical data. This provides a toolbox for risk managers and regulators in view of investigating unusual market events and their impact on the risk of portfolios in a systematic way, moving a step in the direction proposed by Fielding et al. (2011), who underlined the importance of systematically investigating all 'systemic risk' events in financial markets, as done by the National Transportation Safety Board for major civil transportation accidents.

Outline This paper is organized as follows. Section 2 presents a simple framework for modeling the impact of fire sales in various funds on asset returns. Section 3 resolves the question of the identification and estimation of the model parameters, characterizing the fire sales. Section 4 displays the results of our estimation procedure on liquidations occurring after the collapse of Lehman Brothers while Section 5 is focused on the study of the positions liquidated during the first week of August 2007.

2 Fire sales and endogenous risk

2.1 Impact of fire sales on price dynamics: a multiperiod model

Consider a financial market where n assets/financial strategies are traded at discrete dates $t_k = \frac{k}{N}$, multiples of a time step $\frac{1}{N}$ (taken to be a trading day in the empirical examples: $N = 250$). The value of asset/financial strategy i at date t_k is denoted S_k^i.

We consider J institutional investors trading in these assets: fund j initially holds α_i^j units of asset i. The value of this (benchmark) portfolio at date t_k is denoted

$$V_k^j = \sum_{i=1}^{n} \alpha_i^j S_k^i \tag{1}$$

The impact of (exogenous) economic factors ('fundamentals') on prices is modeled through an IID sequence $(\xi_k)_{k\geq 1}$ of $\mathbb{R}^n$-valued centered random variables such that, in the absence of fire sales, the return of asset i during period $[t_k, t_{k+1}]$ is given by

$$\exp\left(\frac{1}{N}\left(m_i - \frac{\Sigma_{i,i}}{2}\right) + \sqrt{\frac{1}{N}}\xi_{k+1}^i\right) - 1$$

Here m_i represents the expected return of asset i in the absence of fire sales and the 'fundamental' covariance matrix Σ, defined by

$$\Sigma_{i,j} = \mathrm{cov}(\xi_k^i, \xi_k^j)$$

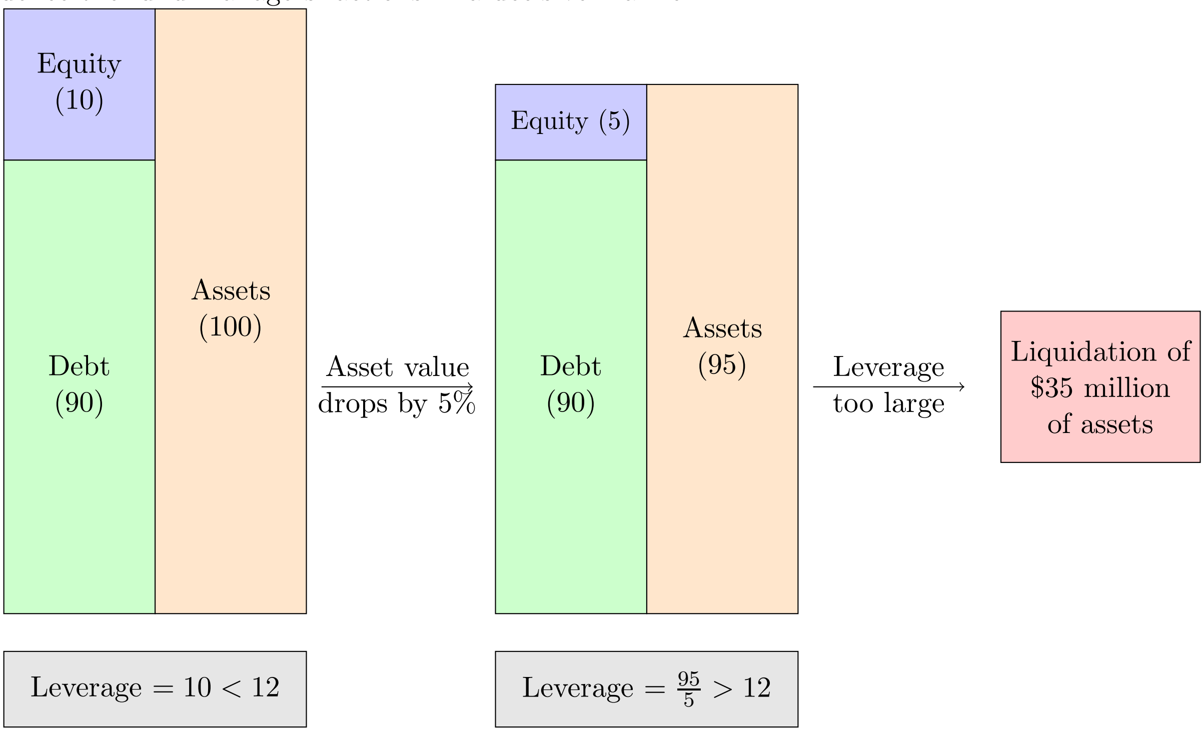
represents the covariance structure of returns in the absence of large systematic trades by institutional investors.

Typically, over short time horizons of a few days, institutional investors do not alter their portfolio allocations. However, the occurrence of large losses typically leads the fund to sell off part of its assets (Coval and Stafford, 2007; Jotikasthira et al., 2011; Shleifer and Vishny, 2011). Such *distressed selling* may be triggered endogenously by

- capital requirements set by regulators or target leverage ratios set by fund managers, which lead financial institutions to deleverage their portfolios when faced

with trading losses (Danielsson et al., 2004; Greenwood and Thesmar, 2011). Consider the simple example of a fund whose maximal leverage ratio is 12. Initially this fund possesses \$10 million of equity and borrows \$90 million to build a portfolio of assets worth \$100 million. The initial leverage of this fund is hence equal to $\frac{Assets}{Assets-Debt} = \frac{100}{100-90} = 10 < 12$.
A decline of d (expressed in percent) in the value of the assets held by the fund modifies the fund's leverage to a value of $\frac{100\times(1-d)}{100\times(1-d)-90}$. As a consequence, a decline in asset value of more than 1.8% leads to a spike in the fund's leverage ratio above the maximum leverage ratio of 12. In order to maintain such maximum leverage ratio, the fund can either raise equity (which can be costly, especially at a time when its portfolio value is decreasing) or, most likely, engage in fire sales. The diagram below illustrates such endogenous mechanism for distressed selling when asset value drops by 5%, leading to liquidation of \$35 million of assets. On the contrary, as long as the drop in asset value is lower than 1.8%, the leverage of the fund remains below 12 and there is no distressed selling.

Note that this mechanism is asymmetric with respect to losses/gains: large losses trigger fire sales, but large gains do not necessarily result in massive buying. Once the capital requirement constraints or leverage constraints are not binding, they may cease to influence the fund managers' actions in a decisive manner.



Fire sales may be also due to:

- investors redeeming (or expanding) their positions depending on the performance of the funds, causing inflows and outflows of capital. This mechanism is described by

Coval and Stafford (2007), who show empirically that funds in distress experience outflows of capital by investors and explain that, as the ability of borrowing is reduced for distressed funds and regulation and self-imposed constraints prevent them from short-selling other securities, such outflows of capital result in fire sales in existing positions.

- rule based strategies –such as portfolio insurance– which result in selling when a fund underperforms (Gennotte and Leland, 1990),
- sale of assets held as collateral by creditors of distressed funds (Shleifer and Vishny, 2011).

The impact of fire sales may also be exacerbated by short-selling and predatory trading: Brunnermeier and Pedersen (2005) show that, in the presence of fire sales in a distressed fund, the mean-variance optimal strategy for other investors is to short-sell the assets held by the distressed fund and buy them back after the period of distress. A common feature of these mechanisms is that they react to a (negative) change in fund value.

Here we do not attempt to model each of these mechanisms in detail but focus instead on their aggregate effect. This aggregate effect may be modeled in a parsimonious manner by introducing a *deleveraging schedule*, represented by a function f_j which measures the systematic supply/demand generated by the fund j as a function of the fund's return: when, due to market shocks, the value of the portfolio j moves over $[t_k, t_{k+1}]$ from V_k^j to

$$\sum_{l=1}^{n} \alpha_l^j S_k^l \exp\left(\frac{1}{N}(m_l - \frac{\Sigma_{l,l}}{2}) + \sqrt{\frac{1}{N}}\xi_{k+1}^l\right)$$

a portion

$$f_j\left(\frac{V_k^j}{V_0^j}\right) - f_j\left(\frac{1}{V_0^j}\sum_{l=1}^{n} \alpha_l^j S_k^l \exp\left(\frac{1}{N}(m_l - \frac{\Sigma_{l,l}}{2}) + \sqrt{\frac{1}{N}}\xi_{k+1}^l\right)\right) \tag{2}$$

of fund j is liquidated between t_k and t_{k+1}, proportionally in each asset detained by the fund.

As shown in the previous example and by Jotikasthira et al. (2011), negative returns for a fund lead to outflows of capital from this fund: this implies that f_j is an increasing function. Fire sales occur when a fund underperforms significantly and its value goes below a threshold and it ends when the fund is entirely liquidated: as a consequence, we choose f_j to be constant for small and large values of its argument (i.e. constant outside an interval $[\beta_j^{liq}, \beta_j]$) with $\beta_j < 1$. Furthermore, we choose f_j to be concave, capturing the fact that fire sales accelerate as the fund exhibits larger losses. Figure 1 displays an example of such a deleveraging schedule f_j. As long as fund j's value remains above $\beta_j V_0^j$, the portion liquidated, given in (2), is equal to zero, as f_j is constant on $[\beta_j, +\infty[$: there are no fire sales. A drop in fund value below that threshold generates fire sales of a portion of fund j, described in (2).

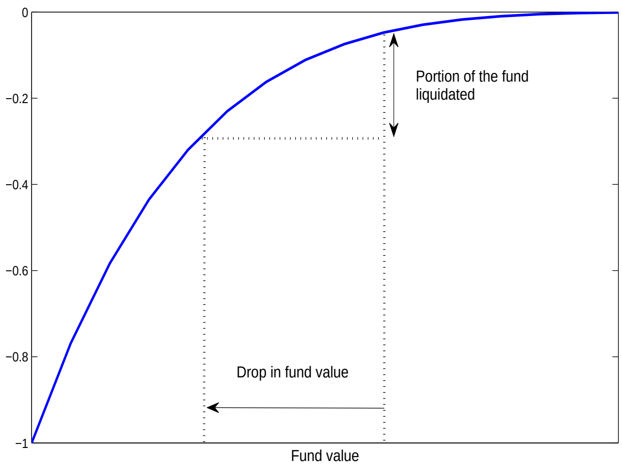


Figure 1: Example of a deleveraging schedule f_j

When the trades are sizable with respect to the average trading volume, the supply/demand generated by this deleveraging strategy impacts asset prices. We introduce, for each asset i, a *price impact* function $\phi_i(.)$ which captures this effect: the impact of buying v shares (where $v < 0$ represents a sale) on the return of asset i is $\phi_i(v)$. We assume that $\phi_i : \mathbb{R} \mapsto \mathbb{R}$ is increasing and $\phi_i(0) = 0$.

The impact of fire sales on the return of asset i is then equal to

$$\phi_i \left[\sum_{j=1}^{J} \alpha_i^j \left(f_j\left(\frac{1}{V_0^j} \sum_{l=1}^{n} \alpha_l^j S_k^l \exp\left(\frac{1}{N}(m_l - \frac{\Sigma_{l,l}}{2}) + \sqrt{\frac{1}{N}} \xi_{k+1}^l\right)\right) - f_j\left(\frac{V_k^j}{V_0^j}\right) \right) \right]$$

The price dynamics can be summed up as follows:

$$\boxed{S_k^i} \xrightarrow[\text{factors } (\xi_{k+1})]{\text{exogenous}} \boxed{S_k^i \exp\left(\frac{1}{N}(m_i - \frac{\Sigma_{i,i}}{2}) + \sqrt{\frac{1}{N}} \xi_{k+1}^i\right)} \xrightarrow{\text{fire sales}} \boxed{S_{k+1}^i}$$

$$S_{k+1}^i = S_k^i \exp\left(\frac{1}{N} \left(m_i - \frac{\Sigma_{i,i}}{2} \right) + \sqrt{\frac{1}{N}} \xi_{k+1}^i \right) \times$$

$$\left(1 + \phi_i \left[\sum_{j=1}^{J} \alpha_i^j \left(f_j\left(\frac{1}{V_0^j} \sum_{l=1}^{n} \alpha_l^j S_k^l \exp\left(\frac{1}{N}(m_l - \frac{\Sigma_{l,l}}{2}) + \sqrt{\frac{1}{N}} \xi_{k+1}^l\right)\right) - f_j\left(\frac{V_k^j}{V_0^j}\right) \right) \right] \right) \quad (3)$$

where V_k^j is the benchmark portfolio value of fund j at date t_k, defined in (1).

At each period, the return of asset i can be decomposed into a fundamental component, which is independant from the past, and an endogenous component due to the impact of fire sales. Note that when there are no fire sales, this endogenous term is equal to zero and the return of asset i is equal to its fundamental return.

Assumption 2.1 $S_0 \in \left(\mathbb{R}_+^*\right)^n$ *and* $\min_{1\leq i\leq n} \phi_i \left(-2 \sum_{j=1}^{n} |\alpha_i^j| \times \|f_j\|_\infty \right) > -1.$

Proposition 2.2 *Under Assumption 2.1,* (1)–(3) *define a price dynamics S which is a discrete-time Markov process in* $\left(\mathbb{R}_+^*\right)^n$.

Proof Equations (1) and (3) show that S_{k+1} depends only on its value at t_k and on ξ_{k+1}, which is independent of events previous to t_k. The price vector S is thus a discrete-time Markov process. Furthermore, when $\min_{1\leq i\leq n} \phi_i \left(-2 \sum_{j=1}^{n} |\alpha_i^j| \times \|f_j\|_\infty \right) > -1$, the endogenous price impact due to fire sales, is strictly larger than -1, which ensures that the Markov process stays in $\left(\mathbb{R}_+^*\right)^n$.

This multiperiod model exhibits interesting properties: in particular, as shown in (Cont and Wagalath, 2012), the presence of distressed selling induces an endogenous, heteroscedastic component in the covariance structure of returns, which leads to path-dependent realized correlations, even in the absence of any heteroscedasticity in the fundamentals.

Figure 2 shows an example of such endogenous correlations: we simulated 10^6 price trajectories of this multiperiod model with the parameters used in (Cont and Wagalath, 2012, Section 3) and for each trajectory, we computed the realized correlation between all pairs of assets. We find that even in the case where the exogenous shocks driving the asset values are independent (i.e. the 'fundamental' covariance matrix Σ is diagonal), the presence of distressed selling leads to significant realized correlations, thereby increasing the volatility experienced by investors holding the fund during episodes of fire sales. This phenomenon may substantially decrease the benefits of diversification.

Our goal is to explore such effects systematically and propose a method for estimating their impact on price dynamics.

2.2 Continuous-time limit

The multiperiod model described above is rather cumbersome to study directly; in the sequel we focus on its continuous-time limit, which is analytically tractable and more easily related to commonly used diffusion models for price dynamics. This will allow us to compute realized covariances between asset returns in the presence of feedback effects from distressed selling.

For two n-dimensional vectors x and y, we denote $x.y = \sum_{1\leq i\leq n} x_i y_i$ the scalar product between vectors x and y. For $M \in \mathcal{M}_n(\mathbb{R})$, M^t is the transpose of matrix M. $\mathcal{S}_n(\mathbb{R})$ (resp. $\mathcal{S}_n^+(\mathbb{R})$) denotes the set of real-valued symmetric matrices (resp. real-valued symmetric positive semi-definite matrices). For a sequence $X^{(N)}$ of random variables indexed by integers N, we denote the fact that $X^{(N)}$ converges in law (resp. in probability) to X

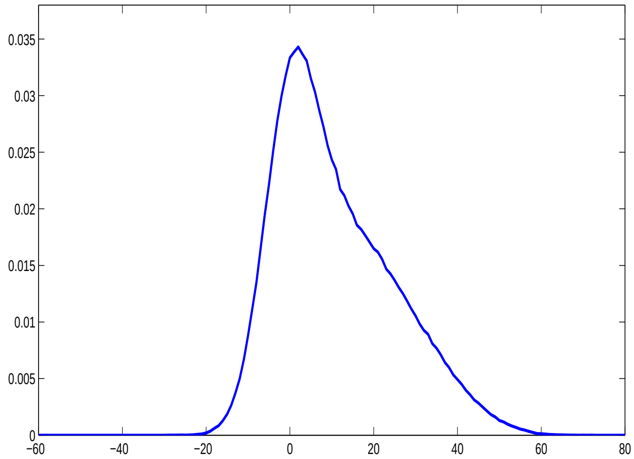


Figure 2: Distribution of realized correlation between two securities in the presence of distressed selling (case of zero fundamental correlation)

when N goes to infinity by $X^{(N)} \underset{N\to\infty}{\Rightarrow} X$ (resp. $X^{(N)} \underset{N\to\infty}{\overset{\mathbb{P}}{\to}} X$). For $(a,b) \in \mathbb{R}^2$, we denote $a \wedge b = \min(a,b)$.

In order to study the continuous-time limit of the multiperiod model described in the previous section, we make the following assumption.

Assumption 2.3 *For $i = 1..n,\ j = 1..J$,*

$$\phi_i \in \mathcal{C}^3(\mathbb{R})\ , \quad f_j \in \mathcal{C}_0^3(\mathbb{R}) \ \ \textit{and} \ \ \alpha_i^j \geq 0$$

$$\exists \eta > 0, \mathbb{E}(\|\exp(\eta\xi)\|) < \infty \ \ \textit{and} \ \ \mathbb{E}(\|\xi\|^{\eta+4}) < \infty$$

where $\mathcal{C}_0^p(\mathbb{R})$ denotes the set of real-valued, p-times continuously differentiable maps whose first derivative has compact support.

Note that if $f_j \in \mathcal{C}_0^p(\mathbb{R})$, all its derivatives of order $1 \leq l \leq p$ have compact support. In particular f_j is constant for large values and very small values of its argument. This assumption has a natural interpretation in our context: fire sales occur when funds underperform, i.e. when the value of the fund relative to a benchmark falls below a threshold, and cease when the fund defaults, i.e. when the value of the fund relative to the benchmark decreases below a default threshold.

Theorem 2.4 *Under Assumptions 2.1 and 2.3, the process $(S_{\lfloor Nt \rfloor})_{t\geq 0}$ converges weakly on the Skorokhod space $D([0,\infty[,\mathbb{R}^n)$, as $N \to \infty$, to a diffusion process $(P_t)_{t\geq 0}$ solution of the stochastic differential equation*

$$\frac{dP_t^i}{P_t^i} = \mu_i(P_t)dt + (\sigma(P_t)dW_t)_i \qquad 1 \leq i \leq n \tag{4}$$

where μ (resp., σ) is a $\mathbb{R}^n$-valued (resp. matrix-valued) mapping defined by

$$\sigma_{i,k}(P_t) = A_{i,k} + \phi_i'(0) \sum_{j=1}^{J} \alpha_i^j f_j'(\frac{V_t^j}{V_0^j}) \frac{\left(A\pi_t^j\right)_k}{V_0^j} \tag{5}$$

$$\mu_i(P_t) = m_i + \frac{\phi_i'(0)}{2} \sum_{j=1}^{J} \frac{\alpha_i^j}{(V_0^j)^2} f_j''(\frac{V_t^j}{V_0^j}) \pi_t^j.\Sigma\pi_t^j \tag{6}$$

$$+ \sum_{j=1}^{J} \phi_i'(0) \frac{\alpha_i^j}{V_0^j} f_j'(\frac{V_t^j}{V_0^j}) \left(\pi_t^j.\overline{m} + (\Sigma\pi_t^j)_i\right) + \frac{\phi_i''(0)}{2} \sum_{j,r=1}^{J} \frac{\alpha_i^j \alpha_i^r}{V_0^j V_0^r} f_j'(\frac{V_t^j}{V_0^j}) f_r'(\frac{V_t^r}{V_0^r}) \pi_t^j.\Sigma\pi_t^r$$

Here W_t is an n-dimensional Brownian motion, $\pi_t^j = \begin{pmatrix} \alpha_1^j P_t^1 \\ \vdots \\ \alpha_n^j P_t^n \end{pmatrix}$ is the (dollar) allocation of fund j, $V_t^j = \sum_{k=1}^{n} \alpha_k^j P_t^k$ is the value of fund j, $\overline{m}_i = m_i - \frac{\Sigma_{i,i}}{2}$ and A is a square-root of the fundamental covariance matrix: $AA^t = \Sigma$.

The proof of this Theorem is given in Appendix 6.1.

Remark 2.5 *The limit price process that we exhibit in Theorem 2.4 depends on the price impact functions only through their first and second derivatives in 0, $\phi_i'(0)$ and $\phi_i''(0)$. In particular, the expression of σ in (5) shows that realized volatilities and realized correlations of asset returns depend only on the slope $\phi_i'(0)$ of the price impact function. As a consequence, under our assumptions, a linear price impact function would lead to the same realized covariance structure for asset returns in the continuous-time limit.*

In the remainder of this paper, which is dedicated to the study of the impact of fire sales on the covariance structure of asset returns, we hence use the assumption of linear price impact: $D_i = \frac{1}{\phi_i'(0)}$ then corresponds to the market depth for asset i and is interpreted as the number of shares an investor has to buy in order to increase the price of asset i by 1%.

Corollary 2.6 (Case of linear price impact) *When $\phi_i(x) = \frac{x}{D_i}$, the drift and volatility of the stochastic differential equation (4) verified by the continuous-time price process are:*

$$\sigma_{i,k}(P_t) = A_{i,k} + \frac{1}{D_i} \sum_{1 \le j \le J} \alpha_i^j f_j'(\frac{V_t^j}{V_0^j}) \frac{\left(A\pi_t^j\right)_k}{V_0^j} \tag{7}$$

$$\mu_i(P_t) = m_i + \frac{1}{D_i} \sum_{j=1}^{J} \left(\frac{\alpha_i^j}{2(V_0^j)^2} f_j''(\frac{V_t^j}{V_0^j}) \pi_t^j.\Sigma\pi_t^j + \frac{\alpha_i^j}{V_0^j} f_j'(\frac{V_t^j}{V_0^j}) \left(\pi_t^j.\overline{m} + (\Sigma\pi_t^j)_i\right) \right) \tag{8}$$

where W_t, π_t^j, V_t^j, $\overline{m}$ and A are defined in Theorem 2.4.

When market depths are infinite, the price dynamics follows a multivariate exponential Brownian motion. In the presence of fire sales by distressed sellers, the fundamental dynamics of the assets is modified.

2.3 Realized covariance in the presence of fire sales

The *realized covariance* (Andersen et al., 2003; Barndorff-Nielsen and Shephard, 2004) between dates t_1 and t_2 computed on a time grid with step $\frac{1}{N}$ is defined as

$$\widehat{C}^{(N)}_{[t_1,t_2]} = \frac{1}{t_2 - t_1}([X,X]^{(N)}_{t_2} - [X,X]^{(N)}_{t_1}) \tag{9}$$

where X is the log price process defined by $X^i_t = \ln P^i_t$ and $[X,X]^{(N)}_t = \left([X^i,X^k]^{(N)}_t\right)_{1\leq i,k\leq n}$ with

$$[X^i,X^k]^{(N)}_t = \sum_{1\leq l\leq \lfloor tN \rfloor} (X^i_{l/N} - X^i_{(l-1)/N})(X^k_{l/N} - X^k_{(l-1)/N}) \tag{10}$$

As N goes to infinity, the process $\left([X,X]^{(N)}_t\right)_{t\geq 0}$ converges in probability on the Skorokhod space $D([0,\infty[,\mathbb{R}^n)$ to an increasing, $\mathcal{S}^+_n(\mathbb{R})$-valued process $([X,X]_t)_{t\geq 0}$, the quadratic covariation of X (Jacod and Protter, 2012, Theorem 3.3.1). We define the $\mathcal{S}^+_n(\mathbb{R})$-valued process $c = (c_t)_{t\geq 0}$, which corresponds intuitively to the 'instantaneous covariance' of returns, as the derivative of the quadratic covariation process. The realized covariance matrix of returns between t_1 and t_2 is denoted $C_{[t_1,t_2]}$.

$$[X,X]_t = \int_0^t c_s ds \qquad C_{[t_1,t_2]} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} c_t \,\mathrm{d}t \tag{11}$$

Theorem 2.6 allows to compute the realized covariance matrix for the n assets.

Proposition 2.7 *The instantaneous covariance matrix of returns, c_t, defined in* (11), *is given by:*

$$c_t = \Sigma + \sum_{j=1}^{J}\left[\frac{1}{V^j_0} f_j'(\frac{V^j_t}{V^j_0})\left(\Lambda_j(\pi^j_t)^t\Sigma + \Sigma\pi^j_t\Lambda^t_j\right)\right] + \sum_{j,k=1}^{J}\frac{\pi^j_t.\Sigma\pi^k_t}{V^j_0 V^k_0} f_j'(\frac{V^j_t}{V^j_0}) f_k'(\frac{V^k_t}{V^k_0})\Lambda_j\Lambda^t_k$$

where $\pi^j_t = \begin{pmatrix} \alpha^j_1 P^1_t \\ \vdots \\ \alpha^j_n P^n_t \end{pmatrix}$ *denotes the (dollar) holdings of fund j and* $\Lambda_j = \begin{pmatrix} \frac{\alpha^j_1}{D_1} \\ \vdots \\ \frac{\alpha^j_n}{D_n} \end{pmatrix}$ *represents the positions of fund j in each market as a fraction of the respective market depth.*

Fire sales impact realized covariances between assets. In the presence of fire sales, realized covariance is the sum of the fundamental covariance matrix Σ and an excess realized covariance which is liquidity-dependent and path-dependent. The magnitude of this endogenous impact is measured by the vectors Λ_j, which represent the positions of each fund as a fraction of asset market depths. The volume generated by fire sales in fund j on each asset i is equal to $\alpha_i^j \times f_j'$ and its impact on the return of asset i is equal to $\frac{\alpha_i^j}{D_i} \times f_j'$. This impact can be significant even if the asset is very liquid, when the positions liquidated are large enough compared to the asset's market depth. Thus, even starting with homoscedastic inputs, fire sales naturally lead to endogenous patterns of heteroscedasticity in the covariance structure of asset returns –in particular spikes or plateaux of high correlation during liquidation periods– similar to those observed in empirical data.

More precisely, we observe that the excess realized covariance terms due to fire sales contain a term of order one in $\|\Lambda\|$ plus higher order terms:

$$c_t = \Sigma + \sum_{j=1}^{J} \left[\frac{1}{V_0^j} f_j' \left(\frac{V_t^j}{V_0^j} \right) \left(\Lambda_j (\pi_t^j)^t \Sigma + \Sigma \pi_t^j \Lambda_j^t \right) \right] + O(\|\Lambda\|^2) \tag{12}$$

where

$$\Lambda = (\Lambda_1, ..., \Lambda_J) \in \mathcal{M}_{n \times J}(\mathbb{R}) \tag{13}$$

where Λ_j is defined in Proposition 2.7 and $\frac{O(\|\Lambda\|^2)}{\|\Lambda\|^2}$ is bounded as $\|\Lambda\| \to 0$. This result is due to the fact that under Assumption 2.3, the second order terms $\frac{\pi_t^j . \Sigma \pi_t^k}{V_0^j V_0^k} f_j'(\frac{V_t^j}{V_0^j}) f_k'(\frac{V_t^k}{V_0^k})$ in the expression of c_t in Proposition 2.7 are bounded because for all $1 \leq j \leq n$, f_j' has a compact support.

In addition, if we denote γ_j the average rate of liquidation (for example $\gamma_j = \frac{f_j(\beta_j) - f_j(\beta_j^{liq})}{\beta_j - \beta_j^{liq}}$), we can approximate the terms of order one in $\|\Lambda\|$ in (12) as follows:

$$\sum_{j=1}^{J} \left[\frac{1}{V_0^j} f_j' \left(\frac{V_t^j}{V_0^j} \right) \left(\Lambda_j (\pi_t^j)^t \Sigma + \Sigma \pi_t^j \Lambda_j^t \right) \right] = \sum_{j=1}^{J} \left[\frac{\gamma_j}{V_0^j} \left(\Lambda_j (\pi_t^j)^t \Sigma + \Sigma \pi_t^j \Lambda_j^t \right) \right] + O(\|f''\|)$$

where $\|f''\| = \sum_{j=1}^{J} \|f_j''\|_\infty$.

As a consequence, Proposition 2.7 may be interpreted as follows: if there are no fire sales between 0 and T, the realized covariance of returns between 0 and T is given by

$$C_{[0,T]} = \frac{1}{T} \int_0^T c_t \, \mathrm{d}t = \Sigma$$

while the realized covariance between T and $T + \tau_{liq}$ (where liquidations could have occurred) contains an endogenous component, whose leading terms will be

$$C_{[T, T+\tau_{liq}]} = \frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} c_t \, \mathrm{d}t = \Sigma + L M_0 \Pi \Sigma + \Sigma \Pi M_0 L + O(\|\Lambda\|^2, \|f''\|) \tag{14}$$

where the remainder is composed of higher order corrections in $\|\Lambda\|^2$ and $\|f''\|$, and

$$M_0 = \sum_{j=1}^{J} \frac{\gamma_j}{V_0^j} \times \alpha^j (\alpha^j)^t \tag{15}$$

where $\alpha^j = \begin{pmatrix} \alpha_1^j \\ \vdots \\ \alpha_n^j \end{pmatrix}$ is the vector of positions of fund j and L and Π are diagonal matrices with i-th diagonal term equal respectively to $\frac{1}{D_i}$ and $\frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} P_t^i \, \mathrm{d}t$. In practice, as shown by simulation studies in (Cont and Wagalath, 2012), this first order approximation is precise enough and we will focus on this approximation in the numerical examples.

In the absence of distressed selling between 0 and T, the realized covariances between asset returns during this period are equal to their fundamental value. Between T and $T + \tau_{liq}$, fire sales can affect the realized covariance between asset returns. The excess realized covariance is characterized by a matrix M_0, defined in (15), which reflects the magnitude of the fire sales. Note that we do not assume that all the funds are liquidating between T and $T + \tau_{liq}$. A fund j which is not subject to fire sales during this period of time has a rate of liquidation γ_j equal to zero.

In (15), $\alpha^j(\alpha^j)^t$ is a $n \times n$ symmetric matrix representing an orthogonal projection on fund j's positions and hence M_0 is a sum of projectors. The symmetric matrix M_0 captures the direction and intensity of liquidations in the J funds.

2.4 Spillover effects: price-mediated contagion

Consider now the situation where a reference fund with positions $(\alpha_1, ..., \alpha_n)$ is subject to distressed selling. As argued above, this leads to endogenous volatility and correlations in asset prices, which then modifies the volatility experienced by any other fund holding the same assets.

Proposition 2.7 allows to compute the magnitude of this *volatility spillover* effect (Cont and Wagalath, 2012). The following result shows that the realized variance of a (small) fund with positions $(\mu_t^i, i = 1..n)$ is the sum of the realized variance in the absence of distressed selling and an endogenous term which represents the impact of fire sales in the reference fund.

Corollary 2.8 (Spillover effects) *In the presence of fire sales in a reference fund with positions $(\alpha_1, ..., \alpha_n)$, the realized variance for a small fund with positions $(\mu_t^i)_{1 \le i \le n}$ between t_1 and t_2 is equal to $\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \gamma_s \, \mathrm{d}s$ where*

$$\gamma_s M_s^2 = \pi_s^\mu . \Sigma \pi_s^\mu + \frac{2f'(\frac{V_s}{V_0})}{V_0} (\pi_s^\mu . \Sigma \pi_s^\alpha)(\Lambda . \pi_s^\mu) + \frac{f'(\frac{V_s}{V_0})^2}{V_0^2} (\pi_s^\alpha . \Sigma \pi_s^\alpha)(\Lambda . \pi_s^\mu)^2 \tag{16}$$

where $\pi_s^\alpha = \begin{pmatrix} \alpha_1 P_s^1 \\ \vdots \\ \alpha_n P_s^n \end{pmatrix}$ *and* $\pi_s^\mu = \begin{pmatrix} \mu_t^1 P_s^1 \\ \vdots \\ \mu_t^n P_s^n \end{pmatrix}$ *denote the (dollar) holdings of the reference fund and the small fund respectively,* $M_s = \sum_{i=1}^{n} \mu_s^i P_s^i$ *is the small fund's value, and* $\Lambda = (\frac{\alpha_1}{D_1}, ..., \frac{\alpha_n}{D_n})^t$ *represents the positions of the reference fund in each market as a fraction of the respective market depth.*

The second and third term in (16), which represent the price-mediated contagion of endogenous risk from the distressed fund to other funds holding the same assets, are maximal for funds whose positions are colinear to those of the distressed fund. On the other hand, these endogenous terms are zero if the two portfolios verify an 'orthogonality condition':

$$\Lambda.\pi_t^\mu = \sum_{i=1}^{n} \frac{\alpha_i}{D_i} \mu_t^i P_t^i = 0, \tag{17}$$

in which case the fund with positions μ_t is not affected by the fire sales of assets by the distressed fund.

3 Identification and estimation

Theorem 2.4 describes the convergence of the multiperiod model to its diffusion limit under the assumption that the funds liquidate long positions. However, the continuous-time model given in Theorem 2.4 makes sense in a more general setting where we relax the constraint on the sign of α_i^j i.e. when long-short portfolios are liquidated: in this case, the coefficients of the stochastic differential equation are still locally Lipschitz, so by (Ikeda and Watanabe, 1981, Theorem 3.1, Ch.4) the equation still has a unique strong solution on some interval $[0, \tau[$, where τ is a stopping time (possibly infinite).

In the sequel, we consider the continuous-time model given in Theorem 2.4 in this more general setting which allows for the liquidation of long-short portfolios. Note that the expressions for covariances and spillover effects are not modified.

3.1 Inverse problem and identifiability

Equation (14) describes the leading term in the impact of fire sales on the realized covariance matrix of returns. Conversely, given that realized covariances can be estimated from observation of prices series, one can use this relation to recover information about the volume of liquidation during a fire sales episode.

We now consider the inverse problem of explaining 'abnormal' patterns in realized covariance and volatility in the presence of fire sales and estimating the parameters of the liquidated portfolio from observations of prices. Mathematically, this boils down to answering the following question: for a given time period $[T, T+\tau_{liq}]$ where liquidations could have occurred, is it possible, given Σ, $C_{[T,T+\tau_{liq}]}$, L and Π, to find M such that

$$C_{[T,T+\tau_{liq}]} = \Sigma + LM\Pi\Sigma + \Sigma\Pi ML \tag{18}$$

The following proposition gives conditions under which this inverse problem is well-posed i.e. the parameter M is identifiable:

Proposition 3.1 (Identifiability) *Let L and Π be diagonal matrices with*

$$L_{ii} = \frac{1}{D_i} \qquad \Pi_{ii} = \frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} P_t^i \,\mathrm{d}t$$

If $\Pi\Sigma L^{-1}$ is diagonalizable and there exists an invertible matrix Ω and ϕ_1,...,ϕ_n such that

$$\Omega^{-1}\Pi\Sigma L^{-1}\Omega = \begin{pmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_n \end{pmatrix}$$

and for all $1 \le p, q \le n$

$$\phi_p + \phi_q \neq 0$$

then there exists a unique symmetric $n \times n$ matrix M verifying (18) which is given by

$$M = \Phi(\Sigma, C_{[T,T+\tau_{liq}]}) \tag{19}$$

where $\Phi(\Sigma, C)$ is a $n \times n$ matrix defined by

$$\left[\Omega^t \Phi(\Sigma, C)\Omega\right]_{p,q} = \frac{1}{\phi_p + \phi_q} \times \left[\Omega^t L^{-1}(C - \Sigma)L^{-1}\Omega\right]_{p,q} \tag{20}$$

In this case, the unique solution M of (18) *verifies*

$$M = M_0 + O(\|\Lambda\|^2, \|f''\|) \tag{21}$$

where M_0 is defined in (15)*.*

The proof of this proposition is given in Appendix 6.2. Thanks to (21), we deduce the following corollary:

Corollary 3.2 *The knowledge of M allows to estimate, up to an error term of order one in $\|\Lambda\|$ and zero in $\|f''\|$, the volume of fire sales in asset class i between T and $T + \tau_{liq}$:*

$$\sum_{j=1}^{J} \frac{\alpha_i^j P_T^i}{V_T^j} \times \gamma_j \times \left(\frac{V_T^j - V_{T+\tau_{liq}}^j}{V_0^j}\right) \times V_T^j$$
$$= (0, ..., 0, P_T^i, 0, ..., 0)M(P_T - P_{T+\tau_{liq}}) + O(\|\Lambda\|^2, \|f''\|)$$

Note that the knowledge of M does not allow in general to reconstitute the detail of fire sales in each fund. Indeed, the decomposition of M given in (15) is not always unique. Nevertheless, when different funds engage in similar patterns of fire sales, the common component of these patterns may be recovered from the principal eigenvector of M. In the empirical examples, we find that M has one large eigenvalue, meaning that liquidations were concentrated in one direction.

3.2 Consistency and large sample properties

In the remainder of the paper, we make the following assumption, which guarantees that the identification problem is well-posed in the sense of Proposition 3.1:

Assumption 3.3 $\Pi\Sigma L^{-1}$ *is diagonalisable with distinct eigenvalues* $\phi_1, ..., \phi_n$ *such that for all* $1 \leq p, q \leq n$*:*

$$\phi_p + \phi_q \neq 0$$

As a consequence, (19) (20) (21) hold. We require that the eigenvalues of $\Pi\Sigma L^{-1}$ are distinct so that the set of matrices Σ verifying Assumption 3.3 is an open subset of $\mathcal{S}_n(\mathbb{R})$ which allows for the study of the differentiability of Φ defined in (20).

Proposition 3.1 states that if we know $L = diag(\frac{1}{D_i})$, $\Pi = diag(\frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} P_t^i \, \mathrm{d}t)$, the fundamental covariance matrix, Σ, and the realized covariance matrix between T and $T + \tau_{liq}$, $C_{[T,T+\tau_{liq}]}$, we can reconstitute M and hence the aggregate characteristics of the liquidation between T and $T + \tau_{liq}$, according to Corollary 3.2.

The market depth parameters (L) may be estimated using intraday data, following the methods outlined in Obizhaeva (2011); Cont et al. (2010). This is further discussed in Section 4. Π may be computed from time series of prices.

Σ and $C_{[T,T+\tau_{liq}]}$ are estimated using the realized covariance matrices computed on a time-grid with step $\frac{1}{N}$, defined in (9). In order to estimate Σ, we have to identify a period of time with no fire sales. Denote

$$\tau = \inf \{t \geq 0 \mid \exists\, 1 \leq j \leq J, V_t^j < \beta_j V_0^j\} \wedge T. \tag{22}$$

τ is the first time, prior to T, when fire sales occur. In our model, fire sales begin when the value of a fund j drops below a certain threshold $\beta_j V_0^j$, with $\beta_j < 1$. Given Corollary 2.6, asset prices and hence fund values are continuous, which implies that τ is a stopping time, bounded by T. Furthermore, as $\beta_j < 1$ for all $1 \leq j \leq J$, τ is strictly positive almost surely: $\mathbb{P}(\tau = 0) = 0$. As a consequence, we estimate the fundamental covariance matrix Σ using the sample realized covariance matrix on $[0, \tau]$, denoted $\widehat{\Sigma}^{(N)}$. In addition, a natural estimator for $C_{[T,T+\tau_{liq}]}$ is the sample realized covariance matrix between T and $T + \tau_{liq}$, denoted $\widehat{C}^{\;(N)}$. By (Jacod and Protter, 2012, Theorem 3.3.1), we find that the estimators of Σ and $C_{[T,T+\tau_{liq}]}$ are consistent:

$$\widehat{\Sigma}^{(N)} = \frac{1}{\tau}[X, X]_\tau^{(N)} \underset{N\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} \Sigma \tag{23}$$

$$\widehat{C}^{(N)} = \frac{1}{\tau_{liq}} \left([X, X]_{T+\tau_{liq}}^{(N)} - [X, X]_T^{(N)} \right) \underset{N\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} C_{[T,T+\tau_{liq}]} \tag{24}$$

where the process $[X, X]^{(N)}$ is defined in (10) and τ is defined in (22). We can hence define an estimator $\widehat{M}^{(N)}$ of M by:

$$\widehat{M}^{(N)} = \Phi(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)}) \tag{25}$$

where Φ is defined in (20).

Proposition 3.4 (Consistency) *$\widehat{M}^{(N)}$ defined in* (25) *is a consistent estimator of M:*

$$\widehat{M}^{(N)} = \Phi(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)}) \underset{N\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} M.$$

The proof of this proposition is given in Appendix 6.2. Proposition 3.4 shows that $\widehat{M}^{(N)}$ defined in 25 is a consistent estimator of M, which contains the information on liquidations between T and $T + \tau_{liq}$. The following proposition gives us the rate of this estimator $\widehat{M}^{(N)}$ and its asymptotic distribution.

Proposition 3.5 (Asymptotic distribution of estimator)

$$\sqrt{N}\left(\widehat{M}^{(N)} - M\right) \underset{N\to\infty}{\Rightarrow} \nabla\Phi\left(\Sigma, C_{[T,T+\tau_{liq}]}\right) \cdot \begin{pmatrix} \frac{1}{\tau}\overline{Z}_\tau \\ \frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) \end{pmatrix} \tag{26}$$

where τ is defined in (22)*, $\nabla\Phi$ is the gradient of Φ, defined in* (20)*, and*

$$\overline{Z}_t^{ij} = \frac{1}{\sqrt{2}} \sum_{1\le k,l\le n} \int_0^t \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl}\right) \mathrm{d}\tilde{W}_s^{kl} \tag{27}$$

where $\tilde{W}$ is a n^2-dimensional Brownian motion independent from W and $\tilde{V}$ is a $\mathcal{M}_{n^2\times n^2}(\mathbb{R})$-valued process verifying

$$(\tilde{V}_t\tilde{V}_t^t)^{ij,kl} = [\sigma\sigma^t(P_t)]_{i,k}[\sigma\sigma^t(P_t)]_{j,l} \tag{28}$$

where σ is defined in (7)*.*

The proof of this proposition is given in Appendix 6.3. The Brownian motion $\tilde{W}$ describes the estimation errors in (25): the fact that it is asymptotically independent from the randomness W driving the path of the price process allows to compute the asymptotic distribution of the estimator, conditioned on a given price path and derive confidence intervals, as explained below.

3.3 Testing for the presence of fire sales

Proposition 3.5 allows to test whether $M \neq 0$ i.e. if significant fire sales occurred between T and $T + \tau_{liq}$. Consider the null hypothesis

$$M = 0 \qquad (H_0)$$

Under hypothesis (H_0), there are no fire sales between T and $T + \tau_{liq}$. The central limit theorem given in Proposition 3.5 can be simplified as follows:

Proposition 3.6 *Under the null hypothesis (H_0), the estimator $\widehat{M}^{(N)}$ verifies the following central limit theorem:*

$$\sqrt{N}\widehat{M}^{(N)} \underset{N\to\infty}{\Rightarrow} \Phi\left(\Sigma, \Sigma + \frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) - \frac{1}{\tau}\overline{Z}_\tau\right)$$

where $\overline{Z}$ *is a* n^2*-dimensional Brownian motion with covariance*

$$cov(\overline{Z}^{i,j}, \overline{Z}^{k,l}) = \Sigma_{i,k}\Sigma_{j,l} + \Sigma_{i,l}\Sigma_{j,k}$$

and Φ *and* τ *are defined in* (20) *and* (22) *respectively.*

The proof of this proposition is given in Appendix 6.4. τ is given in (22) and can be simulated thanks to Corollary 2.6. This result allows to test whether the variability in the realized covariance of asset returns during $[T, T + \tau_{liq}]$ may be explained by the superposition of homoscedastic fundamental covariance structure and feedback effects from fire sales. To do this, we estimate the matrix M and test the nullity of the liquidation volumes derived in Corollary (3.2). In practice, it may be possible, for economic reasons, to identify a period $[0, T]$ with no fire sales and hence test the presence of fire sales during $[T, T + \tau_{liq}]$.

Corollary 3.7 *Under the null hypothesis* (H_0) *and if there are no fire sales between* 0 *and* T*,*

$$\sqrt{N}\left(P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}})\right) \underset{N\to\infty}{\Rightarrow} \mathcal{N}\left(0, \left(\frac{1}{T} + \frac{1}{\tau_{liq}}\right) \sum_{i,j,k,l=1}^{n} m_{ij} m_{kl} \left(\Sigma_{ik}\Sigma_{jl} + \Sigma_{jk}\Sigma_{il}\right)\right)$$

with $m_{ij} = \displaystyle\sum_{p,q=1}^{n} \frac{[\Omega^{-1}P_T]_p[\Omega^{-1}(P_T - P_{T+\tau_{liq}})]_q}{\phi_p + \phi_q}\Omega_{ip}\Omega_{jq}D_iD_j$ *where* Ω *and* $(\phi_i)_{1\le i\le n}$ *are defined in Proposition 3.1,* P_t *is the vector of prices at date* t *and* $(D_i)_{1\le i\le n}$ *are the asset market depths.*

The proof of this corollary is given in Appendix 6.4. Corollary 3.7 gives the asymptotic law of $\left(P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}})\right)$, the estimated volume of liquidations, under the null hypothesis (H_0) and if there are no fire sales during $[0, T]$. We can then define a level l such that

$$\mathbb{P}\left(\left|P_T^t \widehat{M}^{(\ N)}(P_T - P_{T+\tau_{liq}})\right| > l\right) \le 1 - p_l$$

where p_l is typically equal to 95% or 99%. If we find that $\left|P_T^t \widehat{M}^{(\ N)}(P_T - P_{T+\tau_{liq}})\right| > l$ and *if we know that there were no fire sales during* $[0, T]$, then the null hypothesis of no fire sales between T and $T + \tau_{liq}$ may be rejected at confidence level p_l.

3.4 Numerical experiments

To assess the accuracy of these estimators in samples of realistic size, we first apply this test to a simulated discrete-time market. We consider the case of one fund investing in $n = 20$ assets, with fundamental volatility 30% and zero fundamental correlation. Furthermore, we assume that all assets have the same market depth D and that the fund is initially equally weighted across these assets: $\frac{\alpha_i P_0^i}{V_0} = \frac{1}{n}$. The size of the fund can be captured by the vector Λ, defined in Proposition 2.7, which represents the size of the fund's position in each asset as a fraction of the asset's market depth. In our simulations, we choose this ratio equal to 20%.

We examine the results of our estimation method in the two following cases:

- the fund is not subject to distressed selling
- the fund is subject to distressed selling: when the fund value drops below $\beta_0 = 95\%$ of its initial value, the manager deleverages the fund portfolio.

Figure 3 displays a trajectory for the fund's value, where the fund was subject to distressed selling between T=116 days and $T + \tau_{liq} = 127$ days.

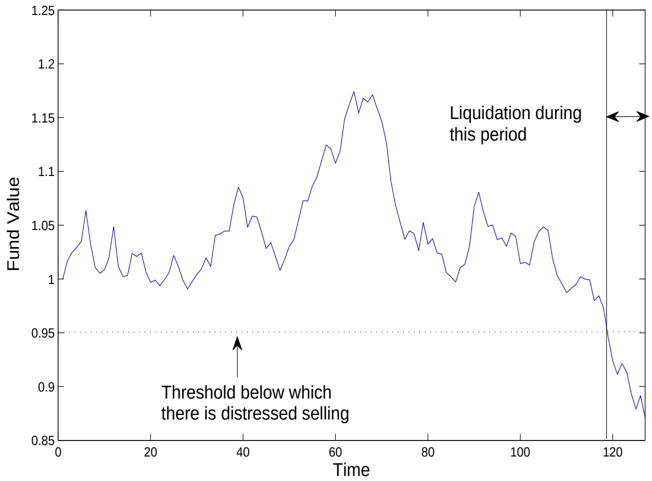


Figure 3: Fund value

We consider a market where trading is possible every day ($\frac{1}{N} = \frac{1}{250}$). We calculate $\widehat{\Sigma}^{(N)}$ and $\widehat{C}^{(N)}$ and we apply our estimation procedure and calculate in each case (no liquidation and liquidation cases) an estimate for the volume of liquidations. Using 3.7, we can determine, at confidence level 95%, for example, whether there has been a liquidation or not.

Under the assumption (H_0) that $M = 0$ and using Lemma 3.7 we find that

$$\mathbb{P}\left(\left|P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}})\right| > 3.2 \times 10^3\right) \leq 5\%$$

We find that

- when there are no fire sales, $P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}}) = 203 < 3.2 \times 10^3$ and we cannot reject assumption (H_0)
- when fire sales occur, $P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}}) = 7 \times 10^3 > 3.2 \times 10^3$ and we reject (H_0) at a 95% confidence level.

Let us now focus on the results of our estimation procedure in the case where there were liquidations and check whether it allows for a proper reconstitution of the liquidated portfolio. We find that the estimates for the proportions liquidated $\frac{\alpha_i P_0^i}{V_0}$ are all positive and ranging from 2% to 10%, around the true value which is $\frac{1}{20} = 5\%$.

4 The Great Deleveraging of Fall 2008

Lehman Brothers was the fourth largest investment bank in the USA. During the year 2008, it experienced severe losses, caused mainly by the subprime mortgage crisis, and on September, 15^{th}, 2008, it filed for chapter 11 bankruptcy protection, citing bank debt of \$613 billion, \$155 billion in bond debt, and assets worth \$639 billion, becoming the largest bankruptcy filing in the US history.

The failure of Lehman Brothers generated liquidations and deleveraging in all asset classes all over the world. The collapse of this huge institution was such a shock to financial markets - major equity indices all lost around 10% on that day - that it triggered stop loss and deleveraging strategies among a remarkable number of financial institutions worldwide. Risk measures of portfolios, for example the value at risk, increased sharply, obliging financial institutions to hold more cash, which they got by deleveraging their portfolios, rather than by issuing debt which would have been very costly at such distressed times.

This massive deleveraging has been documented in several empirical studies. Fratzscher (2011) studies the effect of key events, such as the collapse of Lehman Brothers, on capital flows. He uses a dataset on portfolio capital flows and performance at the fund level, from EPFR, and containing daily, weekly and monthly flows for more than 16000 equity funds and 8000 bond funds, domiciled in 50 countries. He aggregates the net capital flows (ie net of valuation changes) for each country and finds that they are negative for all the countries of the study. This means that fund managers of such funds deleveraged their positions after the collapse of Lehman Brothers, sometimes in dramatic proportions: in some cases, the ouflows can represent up to 30% of the assets under management by the funds.

Our method allows to estimate the net effect of liquidations during this period. We report below the result of the estimation method described in Section 3 SPDRs and components of the Eurostoxx 50 index. Figure 4 shows that the increase of average correlation in these two equity baskets lasted for around three months after September, 15^{th}, 2008. As a consequence, we examine liquidations that occurred between September, 15^{th}, 2008 and December, 31^{st}, 2008.

We calculate the realized covariance matrices respectively between 02/01/2008 and $T = 09/15/2008$ and between $T = 09/15/2008$ and $T + \tau_{liq} = 12/31/2008$ and apply the estimation procedure described in Section 3. We use a linear price impact model Obizhaeva (2011); Cont et al. (2010). To calibrate the market depth parameters D_i, we follow the approach proposed in Obizhaeva (2011): denoting by σ_i the average daily volatility of asset i and ADV_i the average daily trading volume, it was shown in Obizhaeva (2011) for a large panel of US stocks that the ratio $\frac{1}{D}\frac{ADV}{\sigma_r}$ does not vary

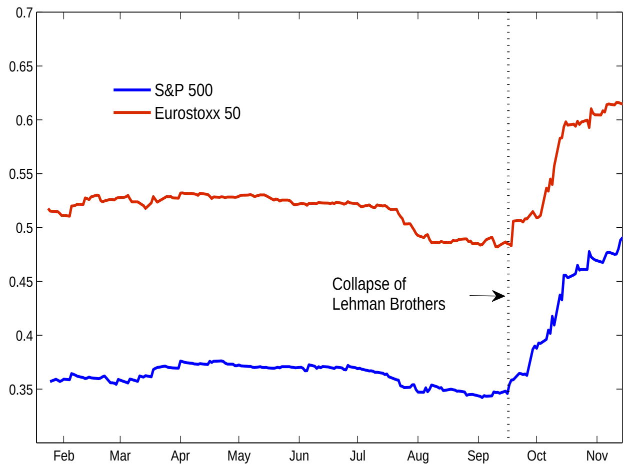


Figure 4: One-year EWMA estimator of average pairwise correlations of daily returns in S&P500 and EuroStoxx 50 index

significantly from one asset to another and

$$\frac{1}{D}\frac{ADV}{\sigma_r} \approx 0.33. \tag{29}$$

Obizhaeva (2011) also argues empirical evidence that the difference in price impact of buy-originated trades and sell-originated trades is not statistically significant. We use average daily volumes and average daily volatility to estimate the market depth of each asset, using (29). Alternatively one could use intraday data, following the methodology proposed in Cont et al. (2010).

4.1 Sector ETFs

We first study fire sales among sector SPDRs, which are sector sub indices of the S&P 500. There exist nine sector SPDRs: Financials (XLF), Consumer Discretionary (XLY), Consumer Staples (XLP), Energy (XLE), Health Care (XLV), Industrials (XLI), Materials (XLB), Technology (XLK) and Utilities (XLU) and our goal is to determine how economic actors investing in those SPDRs liquidated their portfolios following the collapse of Lehman Brothers.

In order to compute our estimation procedure, we need to know the market depth of each SPDR, which we can estimate as described in the previous section. Market depths are given in Table 1. We find that financials have the highest market depth and that other SPDRs have similar market depths.

We can then apply the estimation method described in Section 3 and find the magnitude of fire sales in each SPDR between September, 15^{th}, 2008 and December, 31^{st}, 2008.

Sector SPDR	Estimated Market Depth $\times 10^8$ shares
Financials	34.8
Consumer Discretionary	4.4
Consumer Staples	6.2
Energy	8.8
Health Care	6.4
Industrials	8.1
Materials	7.0
Technology	7.9
Utilities	7.1

Table 1: Estimated market depth for SPDRs.

Our method yields an estimate of 86 billion dollars for fire sales afffecting SPDRs between September, 15^{th}, 2008 and December, 31^{st}, 2008. Using Corollary 3.7, we can reject the hypothesis of no liquidation at a 95% confidence level for this period. The liquidation volume that we find is equivalent to a daily liquidation volume of 1.2 billion dollars per day. In comparison, the average volume on SPDRs before Lehman Brother's collapse was 5.1 billion dollars per day. This shows how massive the liquidations were after this market shock.

Corollary 3.2 allows us to determine the aggregate composition of liquidations between September 15^{th} 2008 and December, 31^{st}, 2008. The daily liquidated volumes and the proportions of each SPDR are given in Table 2. This shows that the aggregate portfolio liquidated after Lehman Brother's collapse was a long portfolio. This is consistent with the observation that many financial institutions liquidated equity holdings in order to meet capital requirements during this period, due to the increase of the risk associated with Lehman Brother's collapse. The highest volume of liquidations are associated with financial stocks, followed by the energy sector. Those two sectors represent 60% of the liquidations and more that 50 billion dollars liquidated before December, 31^{st}, 2008.

As discussed in Section 3.1, the principal eigenvector of M reflects the common patterns of fire sales. Table 3 gives the proportions of fire sales associated to the principal eigenvector of M. We see that this portfolio is essentially made of financials, which have a weight of 78%. The large weight of XLF, the financial sector index, may be explained in terms of the loss of investor confidence in banks in the aftermath of the Lehman's collapse.

4.2 Eurostoxx 50

We now conduct our analysis on stocks belonging to the Eurostoxx 50 in order to determine the average composition of portfolios diversified among the components of the Eurostoxx 50 and that were liquidated after Lehman Brother's filing for bankruptcy.

Sector SPDR	Daily amount liquidated $\times 10^6$\$	Weight
Financials	320	28%
Consumer Discretionary	55	5%
Consumer Staples	38	3.5%
Energy	300	26%
Health Care	63	5.5%
Industrials	90	8%
Materials	110	9.5%
Technology	65	5.5%
Utilities	100	9%

Table 2: Daily volume and proportions of fire sales for SPDR between September 15^{th}, 2008 and Dec 31,2008.

Sector SPDR	Weight
Financials	78%
Consumer Discretionary	0%
Consumer Staples	2.5%
Energy	4%
Health Care	0%
Industrials	0%
Materials	2.5%
Technology	10%
Utilities	3%

Table 3: Proportions of fire sales between September 15^{th}, 2008 and December, 31^{st}, 2008 associated to the principal eigenvector of M

The Eurostoxx 50 is an equity index regrouping the 50 largest capitalizations of the Euro zone. It is the most actively traded index in Europe and is used as a benchmark to measure the financial health of the euro zone.

We use the same methodology as in the previous section (choice of dates, estimation of Σ and market depths). Note that we restricted our study to 45 stocks of the index, for which we had clean data. The 5 stocks left correspond to the lowest capitalizations among the index components, with very low liquidity.

We find that 350 billion euros were liquidated on stocks belonging to the Eurostoxx 50 between September, 15^{th}, 2008 and December, 31^{st}, 2008. Our statistical test described in Corollary 3.7 allows us to reject the null hypothesis of no liquidation at a 99% confidence level. Our estimate for the liquidated volume is equivalent to a daily liquidation of 5 billion euros, which is equal to one third of the average daily volume of the index components before September, 15^{th}, 2008.

Figure 5, where each bar represents the weight of a stock in the aggregate liquidated portfolio, shows that most of the liquidations following Lehman Brother's collapse involved liquidation of long positions in stocks.

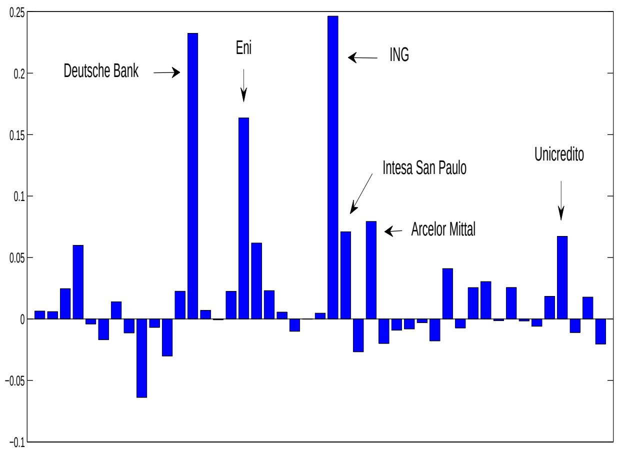


Figure 5: Fire sales in Eurostoxx 50 stocks in Fall 2008: each bar represents the weight of one stock in the aggregate liquidated portfolio

Figure 5 shows that fire sales are more intense for some stocks than others. Table 4 gives the detail of those stocks. As suggested by the previous section, we see that the fire sales in the Eurostoxx 50 index were concentrated in the financial and energy sectors. ING and Deutsche Bank account for almost half of the volume liquidated on the whole index.

Stock	Amount liquidated $\times 10^6$ €	Weight
ING	1100	25%
Deutsche Bank	1000	23%
Eni	750	16%
Arcelor Mittal	350	8%
Intesa San Paolo	320	7%
Unicredito	300	6.5%

Table 4: Most liquidated stocks in the Eurostoxx 50 during the three months following September, 15^{th}, 2008

5 The hedge fund losses of August 2007

From August 6^{th} to August 9^{th} 2007, long-short market-neutral equity funds experienced large losses: many funds lost around 10% per day and experienced a rebound of around 15% on August 10^{th}, 2007. During this week, as documented by Khandani and Lo (2011), market-neutral equity funds whose returns previously had a low historical volatility exhibited negative returns exceeding 20 standard deviations, while no major move was observed in equity market indices.

Khandani and Lo (2011) suggested that this event was due to a large market-neutral fund deleveraging its positions. They simulate a contrarian long-short equity market neutral strategy implemented on all stocks in the CRSP Database and were able to reconstitute qualitatively the empirically observed profile of returns of quantitative hedge funds : low volatility before August 6^{th}, huge losses during three days and a rebound on August 10^{th}. We reconstituted empirically the returns for Khandani and Lo's equity market neutral strategy on the S&P500 for the first three quarters of 2007. Figure 6 shows that this strategy underperforms significantly during the second week of August 2007, while no major move occurred in the S&P 500. Such empirical results tend to confirm the hypothesis of the unwind of a large portfolio, which generated through price impact large losses across similar portfolios, as predicted by our model.

Using historical data on returns of 487 stocks from the S&P500 index, we have reconstituted the composition of the fund that deleveraged its positions during the second week of August 2007 using the estimation procedure described in Section 3 for the periods $[0, T] = [08/03/2006, 08/03/2007]$ and $[T, T + \tau_{liq}] = [08/06/2007, 08/09/2007]$.

Figure 7 displays the composition of the aggregate portfolio liquidated on the S&P500 during this period and found by our estimation method. The first and striking difference with the case of the deleveraging after Lehman Brother's collapse is that, during this quant event, the liquidated portfolio was a long-short portfolio. We clearly see in Figure 7 that for some stocks the liquidated position is significantly negative, meaning that a short position is being exited. More precisely, 250 stocks have positive weights in the liquidated portfolio, whereas 237 have negative weights. Furthermore, we find that

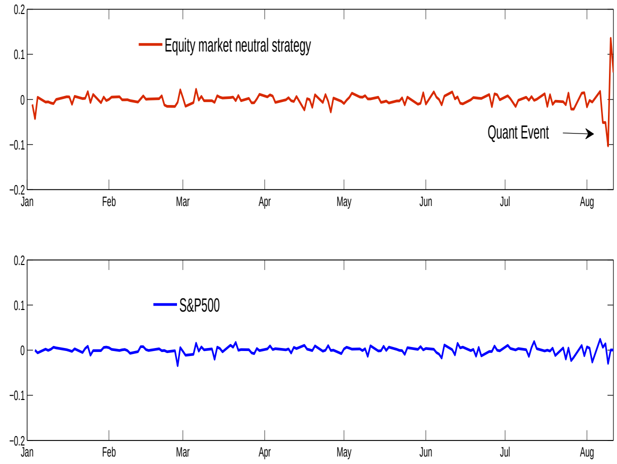


Figure 6: Returns of an market-neutral equity portfolio in 2007, compared with S&P500 returns.

the liquidated portfolio was highly leveraged: for each dollar of capital, 15 dollars are invested in long positions and 14 dollars are invested in short positions.

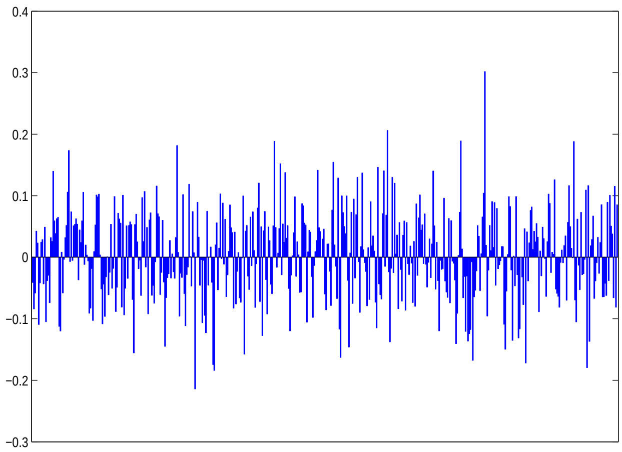


Figure 7: Equity positions liquidated during the 2^{nd} week of August 2007.

Importantly, the estimated portfolio is market-neutral in the sense of Equation (16):

using the notations of Section 2.4 we find

$$\frac{\hat{\Lambda}.\pi_t^{\hat{\mu}}}{\|\hat{\Lambda}\|\|\pi_t^{\hat{\mu}}\|} = \frac{\displaystyle\sum_{i=1}^{n} \frac{\alpha_i}{D_i}\mu_t^i P_t^i}{\|\hat{\Lambda}\|\|\pi_t^{\hat{\mu}}\|} = 0.0958$$

which corresponds to an angle of 0.47π between the vectors $\hat{\Lambda}$ and $\pi_t^{\hat{\mu}}$, i.e. very close to orthogonality. This provides a quantitative explanation for the fact that, although massive liquidations occurred in the equity markets, index funds were not affected by this event. Note that, unlike other explanations proposed at the time, this explanation does not involve any assumption of liquidity drying up during the period of hedge fund turbulence.

6 Appendices

6.1 Proof of Theorem 2.4

We work under Assumptions 2.1 and 2.3. We denote $Z_{k+1} = \frac{1}{N}\overline{m} + \sqrt{\frac{1}{N}}\xi_{k+1} \in \mathbb{R}^n$ where $\overline{m_i} = m_i - \frac{\Sigma_{i,i}}{2}$. We can write the price dynamics (3) as follows:

$$S_{k+1}^i = S_k^i \exp\left(Z_{k+1}^i\right)\left[1 + \phi_i\left(\sum_{j=1}^{J}\alpha_i^j\left(f_j(\sum_{l=1}^{n}\frac{\alpha_l^j S_k^l}{V_0^j}\exp\left(Z_{k+1}^l\right)) - f_j(\sum_{l=1}^{n}\frac{\alpha_l^j S_k^l}{V_0^j})\right)\right)\right]$$

As a consequence, we have $S_{k+1} = \theta(S_k, Z_{k+1})$ where $\theta : \left(\mathbb{R}_+^*\right)^n \times \mathbb{R}^n \mapsto \left(\mathbb{R}_+^*\right)^n$ is $\mathcal{C}^3(\mathbb{R})$ as f_j and ϕ_i are $\mathcal{C}^3(\mathbb{R})$ for all $1 \leq j \leq J$ and $1 \leq i \leq n$.

Define now a (resp., b) a $\mathcal{M}_n(\mathbb{R})$-valued (resp. $\mathbb{R}^n$-valued) mapping such that

$$a_{i,j}(S) = \sum_{l=1}^{n}\frac{\partial\theta_i}{\partial z_l}(S,0) \times A_{l,j} \tag{30}$$

$$b_i(S) = \sum_{j=1}^{n}\frac{\partial\theta_i}{\partial z_j}(S,0)\overline{m_j} + \frac{1}{2}\sum_{j,l=1}^{n}\frac{\partial^2\theta_i}{\partial z_j \partial z_l}(S,0)\Sigma_{j,l} \tag{31}$$

In order to show Theorem 2.4, we first show the following lemma:

Lemma 6.1 *Under Assumptions 2.1 and 2.3, for all $\epsilon > 0$ and $r > 0$:*

$$\lim_{N\to\infty}\sup_{\|S\|\leq r} N \times \mathbb{P}\left(\|S_{k+1} - S_k\| \geq \epsilon | S_k = S\right) = 0 \tag{32}$$

$$\lim_{N\to\infty}\sup_{\|S\|\leq r} \|N \times \mathbb{E}\left(S_{k+1} - S_k | S_k = S\right) - b(S)\| = 0 \tag{33}$$

$$\lim_{N\to\infty}\sup_{\|S\|\leq r} \left\|N \times \mathbb{E}\left((S_{k+1} - S_k)(S_{k+1} - S_k)^t | S_k = S\right) - aa^t(S)\right\| = 0 \tag{34}$$

where a and b are defined respectively in (30) *and* (31)*.*

Proof Fix $\epsilon > 0$ and $r > 0$. As θ is $\mathcal{C}^1$, for $\|S\| \leq r$, there exists $C > 0$ such that for all $Z \in \mathbb{R}^n$

$$\|\theta(S, Z) - \theta(S, 0)\| \leq C\|Z\|$$

As $S_{k+1} = \theta(S_k, Z_{k+1})$ and $S_k = \theta(S_k, 0)$, we find that:

$$\mathbb{P}\left(\|S_{k+1} - S_k\| \geq \epsilon | S_k = S, \|S\| \leq r\right) \leq \mathbb{P}\left(C\|Z_{k+1}\| \geq \epsilon\right)$$

$$\leq \mathbb{P}\left(C\|\frac{\overline{m}}{N} + \sqrt{\frac{1}{N}}\xi_{k+1}\| \geq \epsilon\right) \leq \mathbb{P}\left(\|\xi_{k+1}\| \geq \frac{\epsilon - \frac{\|\overline{m}\|C}{N}}{C\sqrt{\frac{1}{N}}}\right)$$

$$\leq \mathbb{E}\left[\left(\|\xi_{k+1}\| \frac{C\sqrt{\frac{1}{N}}}{\epsilon - \frac{\|\overline{m}\|C(r)}{N}}\right)^{2+\eta}\right] \leq \frac{1}{N^{1+\frac{\eta}{2}}}\mathbb{E}\left[(\|\xi_{k+1}\|)^{2+\eta}\right] \times \left(\frac{C}{\epsilon - \frac{\|\overline{m}\|C(r)}{N}}\right)^{2+\eta}$$

which implies (32).

As θ is $\mathcal{C}^2$, we can write the Taylor expansion of θ_i in 0, for $1 \leq i \leq n$:

$$\theta_i(S, Z) - \theta_i(S, 0) = \frac{\partial \theta_i}{\partial z}(S, 0)Z + \frac{1}{2}Z.\frac{\partial^2 \theta_i}{\partial z \partial z'}(S, 0)Z + Z.R_i(S, Z)Z$$

where R_i converges uniformly to 0 when Z goes to 0, when $\|Z\| \leq \epsilon$ and $\|S\| \leq r$. We have:

$$\mathbb{E}\left(\frac{\partial \theta_i}{\partial z}(S, 0)Z_{k+1}\right) = \frac{1}{N}\sum_{j=1}^{n} \frac{\partial \theta_i}{\partial z_j}(S, 0)\overline{m}_j$$

and

$$\mathbb{E}\left(Z_{k+1}.\frac{\partial^2 \theta_i}{\partial z \partial z'}(S, 0)Z_{k+1}\right) = \frac{1}{N}\sum_{j,l=1}^{n} \frac{\partial^2 \theta_i}{\partial z_l z_j}\Sigma_{j,l} + o(\frac{1}{N})$$

Recalling that $S^i_{k+1} - S^i_k = \theta(S_k, Z_{k+1}) - \theta(S_k, 0)$, we find that:

$$\lim_{N\to\infty} \sup_{\|S\| \leq r} \|N\mathbb{E}\left[(S_{k+1} - S_k)|S_k = S, \|Z_{k+1}\| \leq \epsilon\right] - b(S)\| = 0 \tag{35}$$

We remark that:

$$\left\|N\mathbb{E}\left((S_{k+1} - S_k)\mathbf{1}_{\|Z_{k+1}\| \leq \epsilon}|S_k = S\right) - b(S)\right\|$$

$$\leq \|(N\mathbb{E}\left((S_{k+1} - S_k)|S_k = S, \|Z_{k+1}\| \leq \epsilon\right) - b(S))\| \,\mathbb{P}(\|Z_{k+1}\| \leq \epsilon) + \|b(S)\|\mathbb{P}(\|Z_{k+1}\| \geq \epsilon)$$

As we saw that $\mathbb{P}(\|Z_{k+1}\| \geq \epsilon) \leq \frac{1}{N^{1+\frac{\eta}{2}}}\mathbb{E}\left[(\|\xi_{k+1}\|)^{2+\eta}\right] \times \left(\frac{1}{\epsilon - \frac{\|\overline{m}\|}{N}}\right)^{2+\eta}$ and given (35) and the fact that b is continuous, we find that:

$$\lim_{N\to\infty} \sup_{\|S\| \leq r} \left\|N\mathbb{E}\left((S_{k+1} - S_k)\mathbf{1}_{\|Z_{k+1}\| \leq \epsilon}|S_k = S\right) - b(S)\right\| = 0 \tag{36}$$

Similarly, we show that

$$\lim_{N\to\infty}\sup_{\|S\|\le r}\left\|N\mathbb{E}\left((S_{k+1}-S_k)(S_{k+1}-S_k)^t\mathbf{1}_{\|Z_{k+1}\|\le\epsilon}|S_k=S\right)-aa^t(S)\right\|=0 \tag{37}$$

Given (3), we have the following inequality:

$$S^i_{k+1}\le S^i_k\exp\left(\frac{\overline{m}_i}{N}+\sqrt{\frac{1}{N}}\xi^i_{k+1}\right)\left(1+\phi_i\left(2\sum_{j=1}^J\frac{\alpha^j_i}{D_i}\|f_j\|_\infty\right)\right)$$

which implies that, conditional on $S_k=S$ and for $p>0$ such that $p\sqrt{\frac{1}{N}}<\eta$, $S_{k+1}\in L^p$. Using this result for $p=2$, we find that for $\sqrt{\frac{1}{N}}<\frac{\eta}{2}$, $S_{k+1}\in L^2$ and we can use Cauchy Schwarz inequality:

$$\begin{aligned}&\left|\mathbb{E}\left((S^i_{k+1}-S^i_k)\mathbf{1}_{\|Z_{k+1}\|\ge\epsilon}|S_k=S\right)\right|\\
\le&\sqrt{\mathbb{E}\left((S^i_{k+1}-S^i_k)^2|S_k=S\right)\mathbb{P}\left(\|Z_{k+1}\|\ge\epsilon\right)}\\
\le&\frac{1}{N^{1+\frac{\eta}{4}}}\sqrt{\mathbb{E}\left((S^i_{k+1}-S^i_k)^2|S_k=S\right)}\sqrt{\mathbb{E}\left(\frac{\|\xi_{k+1}\|}{\epsilon-\frac{\|\overline{m}\|}{N}}\right)^{4+\eta}}\end{aligned}$$

As $\mathbb{E}(\|\xi_{k+1}\|^{4+\eta})<\infty$, $S_{k+1}\in L^2$ and S_{k+1} stays L^2 bounded conditional on $S_k=S$ and $\|S\|\le r$. As a consequence, we obtain:

$$\lim_{N\to\infty}\sup_{\|S\|\le r}\left\|N\mathbb{E}\left((S_{k+1}-S_k)\mathbf{1}_{\|Z_{k+1}\|\ge\epsilon}|S_k=S\right)\right\|=0 \tag{38}$$

Using the same property with p=4, we show that

$$\lim_{N\to\infty}\sup_{\|S\|\le r}\left\|N\mathbb{E}\left((S_{k+1}-S_k)(S_{k+1}-S_k)^t\mathbf{1}_{\|Z_{k+1}\|\ge\epsilon}|S_k=S\right)\right\|=0 \tag{39}$$

(36) and (38) give (33). Similarly, (37) and (39) give (34).

The following lemma gives the expressions of a and b by direct computation of (30) – (31).

Lemma 6.2 (30) *and* (31) *respectively can be written as*

$$a_{i,k}(S)=S^i\left[A_{i,k}+\phi'_i(0)\sum_{j=1}^J\frac{\alpha^j_i}{V^j_0}f'\left(\frac{V_j(S)}{V^j_0}\right)(A^t\pi_j(S))_k\right] \tag{40}$$

$$b_i(S)=S^im_i+S^i\frac{\phi''_i(0)}{2}\sum_{j=1}^J\frac{\alpha^j_i}{(V^j_0)^2}f''_j\left(\frac{V_j(S)}{V^j_0}\right)\pi_j(S).\Sigma\pi_j(S) \tag{41}$$

$$+S^i\phi_i'(0)\sum_{j=1}^{J}\frac{\alpha_i^j}{V_0^j}f_j'\left(\frac{V_j(S)}{V_0^j}\right)(\pi_j(S).\overline{m}+(\Sigma\pi_j(S))_i)$$

$$+S^i\frac{\phi_i''(0)}{2}\sum_{j,r=1}^{J}\frac{\alpha_i^j\alpha_i^r}{V_0^jV_0^r}f_j'\left(\frac{V_j(S)}{V_0^j}\right)f_r'\left(\frac{V_r(S)}{V_0^r}\right)\pi_j(S).\Sigma\pi_r(S)$$

where $\pi_j(S)=\begin{pmatrix}\alpha_1^jS^1\\ \vdots\\ \alpha_n^jS^n\end{pmatrix}$ *and* $V_j(S)=\sum_{l=1}^{n}\alpha_l^jS^l$.

Because f_j is $\mathcal{C}^3$ for $1\le j\le J$, a and b are $\mathcal{C}^2$ and $\mathcal{C}^1$ respectively. Furthermore, because f_j', and hence f_j'' and $f_j^{(3)}$, have a compact support, there exists $R>0$ such that, for all $1\le j\le J$, when $\|S\|\ge R$, $f_j'\left(\frac{V_j(S)}{V_0^j}\right)=f_j''\left(\frac{V_j(S)}{V_0^j}\right)=f_j^{(3)}\left(\frac{V_j(S)}{V_0^j}\right)=0$. As a consequence, there exists $K>0$ such that for all $S\in\left(\mathbb{R}_+^*\right)^n$:

$$\|a(S)\|+\|b(S)\|\le K\|S\| \tag{42}$$

Furthermore, as the first derivatives of a and b are bounded, a and b are Lipschitz.

Define the differential operator $G: C_0^\infty\left(\mathbb{R}_+^*\right)^n\mapsto C_0^1\left(\mathbb{R}_+^*\right)^n$ by

$$Gh(x)=\frac{1}{2}\sum_{1\le i,j\le n}(aa^t)_{i,j}(x)\partial_i\partial_jh+\sum_{1\le i\le n}b_i(x)\partial_ih$$

As a and b verify (42), one can apply (Ethier and Kurtz, 1986, Theorem 2.6, Ch.8) to conclude that the martingale problem associated to (G,δ_{S_0}) is well-posed. In fact, as a and b are Lipschitz, the solution of this martingale problem is given by the unique strong solution of the stochastic differential equation:

$$dP_t=b(P_t)dt+a(P_t)dW_t\qquad with\quad P_0=S_0.$$

As we have shown Lemma 6.1, by (Ethier and Kurtz, 1986, Theorem 4.2, Ch.7), when $N\to\infty$, $\left(S_{\lfloor Nt\rfloor}\right)_{t\ge0}$ converges in distribution to the solution of the martingale problem associated to (G,δ_{S_0}), which concludes the proof of Theorem 2.4.

6.2 Proofs of Propositions 3.1 and 3.4

Let us invert (18) under the assumptions of Proposition 3.1. Denote

$$\Omega^{(i)}=\begin{pmatrix}\Omega_{1,i}\\ \vdots\\ \Omega_{n,i}\end{pmatrix}$$

the i-th column of the matrix Ω. By definition, we know that $\Pi\Sigma L^{-1}\Omega^{(p)}=\phi_p\Omega^{(p)}$ which is equivalent to $(\Omega^{(p)})^tL^{-1}\Sigma\Pi=\phi_p(\Omega^{(p)})^t$. As (18) is equivalent to $M\Pi\Sigma L^{-1}+$

$L^{-1}\Sigma\Pi M = L^{-1}(C_{[T,T+\tau_{liq}]} - \Sigma)L^{-1}$ and multiplying this equality on the left by $(\Omega^{(p)})^t$ and on the right by $\Omega^{(q)}$, we find that

$$(\phi_p + \phi_q)[\Omega^t M \Omega]_{p,q} = [\Omega^t L^{-1}(C_{[T,T+\tau_{liq}]} - \Sigma)L^{-1}\Omega]_{p,q}$$

which gives the matrix $\Omega^t M \Omega$ as a function of Σ and $C_{[T,T+\tau_{liq}]}$. As Ω is invertible, this characterizes the matrix M, as a function, denoted Φ of Σ and $C_{[T,T+\tau_{liq}]}$, proving (19) and (20).

Furthermore, notice that $M_0 = \Phi\left(\Sigma, C_{[T,T+\tau_{liq}]} + O(\|\Lambda\|^2, \|f''\|)\right)$. Given the expression for Φ in (20), (21) follows directly. This concludes the proof of Proposition 3.1.

Lemma 6.3 *The mapping Φ defined in* (20) *is $\mathcal{C}^\infty$ in a neighborhood of* (Σ, C).

Proof The following map

$$F : \mathcal{S}_n^3(\mathbb{R}) \mapsto \mathcal{S}_{\mathrm{n}}(\mathbb{R}), (\mathrm{S}, \mathrm{C}, \mathrm{N}) \mapsto \mathrm{LN\Pi S} + \mathrm{S\Pi NL} + \mathrm{S} - \mathrm{C} \tag{43}$$

is infinitely differentiable, its gradient with respect to N given by

$$\frac{\partial F}{\partial N}(S, C, N).H_3 = LH_3\Pi S + S\Pi H_3 L.$$

As Σ verifies Assumption 3.3, we showed that $\frac{\partial F}{\partial N}(\Sigma, C, N)$ is invertible for all C. As $\Phi(\Sigma, C)$ is defined as the only matrix verifying $F(\Sigma, C, \Phi(\Sigma, C)) = 0$, the implicit function theorem states that Φ is C^∞ in a neighborhood of (Σ, C).

As convergence in probability implies that a subsequence converges almost surely, we assume from now on that the estimators defined in (23) and (24) converge almost surely. As a consequence, for N large enough, $\widehat{\Sigma}^{(N)}$ also verifies Assumption 3.3. This is possible because the set of matrices Σ verifying this assumption is an open set and $\widehat{\Sigma}^{(N)}$ converges almost surely to Σ when N goes to infinity. We can hence define $\widehat{M}^{(N)}$ as in (25).

Lemma 6.3 implies in particular that Φ is continuous and hence that $\Phi(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)})$ converges almost surely, and hence in probability, to $\Phi(\Sigma, C_{[T,T+\tau_{liq}]})$. As a consequence, $\Phi(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)})$ is a consistent estimator of $\Phi(\Sigma, C_{[T,T+\tau_{liq}]})$, meaning that $\widehat{M}^{(N)}$ is a consistent estimator of M. This shows Proposition 3.4.

6.3 Proof of Proposition 3.5

Using Theorem 2.6 and Ito's formula, we deduce that the log price $X_t^i = \ln(P_t^i)$ verifies the following stochastic differential equation:

$$dX_t^i = \left(\mu_i(e^{X_t}) - \frac{1}{2}(\sigma(e^{X_t})\sigma(e^{X_t})^t)_{i,i}\right)dt + \left(\sigma(e^{X_t})dW_t\right)_i$$

where σ, μ and W are defined in Theorem 2.6 and e^{X_t} is a n-dimensional column vector with i-th term equal to $\exp X_t^i$. As a consequence, X is an Ito process which verifies, for $t \geq 0$,

$$\int_0^t \left(\sum_{1 \leq i \leq n} \left(\mu_i(P_t) - \frac{1}{2}(\sigma(P_t)\sigma_t(P_t)^t)_{i,i} \right)^2 + \|\sigma\sigma^t(P_t)\|^2 \right) \mathrm{d}s < \infty$$

We are thus in the setting of (Jacod and Protter, 2012, Theorem 5.4.2, Ch.5) which describes the asymptotic distribution of the quadratic covariation of an Ito process with well-behaved coefficients. We need to extend $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ to a larger probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$. There exists $\tilde{W}$ a n^2-dimensional Brownian motion, defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ and independent from W, such that

$$\sqrt{N}\left([X,X]^{(N)} - [X,X]\right) \underset{N\to\infty}{\overset{s.l.}{\longrightarrow}} \overline{Z}$$

where the $n \times n$ dimensional process $\overline{Z}$ is defined in (27) and s.l. means stable convergence in law (see (Jacod and Protter, 2012, Section 2.2.1)). The auxiliary Brownian motion $\tilde{W}$ represents the estimation error. Furthermore, (Jacod and Protter, 2012, Equation 2.2.5) shows that

$$\left(\sqrt{N}\left([X,X]^{(N)} - [X,X]\right), \tau\right) \underset{N\to\infty}{\Rightarrow} (\overline{Z}, \tau)$$

This implies that the estimators $(\widehat{\Sigma}^{(\ N)}, \widehat{C}^{(\ N)})$ defined in (23) and (24) verify the following central limit theorem:

$$\sqrt{N}\left[\begin{pmatrix} \widehat{\Sigma}^{(\ N)} \\ \widehat{C}^{(\ N)} \end{pmatrix} - \begin{pmatrix} \Sigma \\ C_{[T,T+\tau_{liq}]} \end{pmatrix}\right] \underset{N\to\infty}{\Rightarrow} \begin{pmatrix} \frac{1}{\tau}\overline{Z}_\tau \\ \frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) \end{pmatrix} \tag{44}$$

Since $\Phi \in \mathcal{C}^1$, one can then apply the 'delta method' to $(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)})$ to derive the result in Proposition 3.5.

6.4 Proof of Proposition 3.6 and Corollary 3.7

Under the null hypothesis (H_0), $\frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} c_t \,\mathrm{d}t = \Sigma$ and hence

$$\Phi\left(\Sigma, \frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} c_t \,\mathrm{d}t\right) = \Phi(\Sigma, \Sigma) = 0$$

Let us calculate now the first derivative of Φ on (Σ, Σ). Recall that $\Phi(\Sigma, C)$ is defined as the only element of $\mathcal{S}_n(\mathbb{R})$ such that $F(\Sigma, C, \Phi(\Sigma, C)) = 0$, where F is defined in (43). F is affine in each component and as a consequence is $\mathcal{C}^\infty$ and we can define its derivatives on (S, C, N), $\frac{\partial F}{\partial S}(S, C, N)$, $\frac{\partial F}{\partial C}(S, C, N)$ and $\frac{\partial F}{\partial N}(S, C, N)$ which are linear mappings from $\mathcal{S}_n(\mathbb{R})$ to $\mathcal{S}_n(\mathbb{R})$ defined by:

$$\frac{\partial F}{\partial S}(S, C, N).H_1 = LN\Pi H_1 + H_1 \Pi N L + H_1$$

$$\frac{\partial F}{\partial C}(S,C,N).H_2 = -H_2$$

$$\frac{\partial F}{\partial N}(S,C,N).H_3 = LH_3\Pi S + S\Pi H_3 L$$

As a consequence, we have

$$\nabla F(S,C,N).(H_1,H_2,H_3) = LN\Pi H_1 + H_1\Pi NL + H_1 - H_2 + LH_3\Pi S + S\Pi H_3 L$$

In the proof of Lemma 6.3, we showed that $\frac{\partial F}{\partial N}(\Sigma, C, N)$ is invertible. As a consequence we can apply the implicit function theorem in order to compute the gradient of Φ. As $F(\Sigma, C, \Phi(\Sigma, C)) = 0$ and $\Phi(\Sigma, \Sigma) = 0$, we find that $\frac{\partial F}{\partial S}(\Sigma, \Sigma, 0).H_1 = H_1$, $\frac{\partial F}{\partial C}(\Sigma, \Sigma, 0).H_2 = -H_2$ and $\frac{\partial F}{\partial N}(\Sigma, \Sigma, 0).H_3 = LH_3\Pi\Sigma + \Sigma\Pi H_3 L$ and hence the derivative of Φ on (Σ, Σ) is given by:

$$\nabla\Phi(\Sigma,\Sigma).(H_1,H_2) = \left(\frac{\partial F}{\partial N}(\Sigma,\Sigma,0)\right)^{-1}(H_2 - H_1)$$

which is equivalent to

$$\nabla\Phi(\Sigma,\Sigma).(H_1,H_2) = \Phi(\Sigma, \Sigma + H_2 - H_1)$$

Using Proposition 3.5, we find that

$$\sqrt{N}\widehat{M}^{(\ N)} \overset{\mathcal{L}}{\Rightarrow} \Phi\left(\Sigma, \Sigma + \frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) - \frac{1}{\tau}\overline{Z}_\tau\right)$$

which concludes the proof of Proposition 3.6.

If there are no fire sales between 0 and T, then $\tau = T$ almost surely. In addition, under (H_0), we have $\sigma\sigma^t = \Sigma$ and the expression for the process $\tilde{V}_t$ defined in (28) is simplified as

$$(\tilde{V}_t\tilde{V}_t^t)^{ij,kl} = \Sigma_{i,k}\Sigma_{j,l} \tag{45}$$

which implies that the process $\overline{Z}$ defined in (27) is a Brownian motion.

Furthermore, given Proposition 3.6, under (H_0), $\sqrt{N}\left(P_T^t\widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}})\right)$ converges in law when N goes to infinity to the random variable

$$P_T^t\Phi\left(\Sigma, \Sigma + \frac{1}{\tau_{liq}}(\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) - \frac{1}{T}\overline{Z}_T\right)(P_T - P_{T+\tau_{liq}})$$

Given the expression for Φ given in (20), we find the expression for:

$$P_T^t\Phi(\Sigma, C)(P_T - P_{T+\tau_{liq}})$$

$$= \sum_{1\le p,q\le n}(\Omega^{-1}P_T)_p\frac{\left[\Omega^t L^{-1}(C-\Sigma)L^{-1}\Omega\right]_{p,q}}{\phi_p + \phi_q}(\Omega^{-1}(P_T - P_{T+\tau_{liq}}))_q$$

Given the fact that $L^{-1} = diag(D_i)$, we have $(\Omega^t L^{-1})_{p,i} = \Omega_{i,p} D_i$ and $(L^{-1}\Omega)_{j,q} = \Omega_{j,q} D_j$. As a consequence, denoting

$$m_{i,j} = \sum_{1 \le p,q \le n} \frac{[\Omega^{-1} P_T]_p [\Omega^{-1}(P_T - P_{T+\tau_{liq}})]_q}{\phi_p + \phi_q} \Omega_{ip} \Omega_{jq} D_i D_j$$

we can write $P_T^t \Phi(\Sigma, C)(P_T - P_{T+\tau_{liq}})$ as $\sum_{1 \le i,j \le n} m_{ij}(C_{i,j} - \Sigma_{i,j})$. Hence the limit of $\sqrt{N}\left(P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}})\right)$ is equal to

$$\sum_{1 \le i,j \le n} m_{ij} \left(\frac{1}{\tau_{liq}} (\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) - \frac{1}{T} \overline{Z}_T \right)_{i,j}$$

Under the assumptions of Corollary 3.7, $\overline{Z}$ is a Brownian motion on $[0, T + \tau_{liq}]$ (see (45)), so the limit process is a mean-zero Gaussian process. To compute its variance, we first compute the variance of $\sum_{1 \le i,j \le n} m_{ij} \overline{Z}_t^{i,j}$ which, given the expression of $\overline{Z}$ in (27), can be written as

$$\sum_{1 \le k,l \le n} \int_0^t \frac{1}{\sqrt{2}} \sum_{1 \le i,j \le n} m_{i,j} \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl} \right) \mathrm{d}\tilde{W}_s^{kl}.$$

Using the Ito isometry formula, its variance is thus equal to

$$\begin{aligned}
&\sum_{1 \le k,l \le n} \int_0^t \left(\sum_{1 \le i,j \le n} \frac{1}{\sqrt{2}} m_{i,j} \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl} \right) \right)^2 \mathrm{d}s \\
&= \frac{t}{2} \sum_{1 \le k,l \le n} \left(\sum_{1 \le i,j,p,q \le n} m_{i,j} m_{p,q} \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl} \right) \left(\tilde{V}_s^{pq,kl} + \tilde{V}_s^{qp,kl} \right) \right) \\
&= \frac{t}{2} \sum_{1 \le i,j,p,q \le n} m_{i,j} m_{p,q} \left(\sum_{1 \le k,l \le n} \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl} \right) \left(\tilde{V}_s^{pq,kl} + \tilde{V}_s^{qp,kl} \right) \right) \\
&= t \sum_{1 \le i,j,p,q \le n} m_{i,j} m_{p,q} \left(\Sigma_{i,p} \Sigma_{j,q} + \Sigma_{i,q} \Sigma_{j,p} \right)
\end{aligned}$$

using the fact that $\sum_{1 \le k,l \le n} \tilde{V}_s^{ij,kl} \tilde{V}_s^{pq,kl} = \Sigma_{i,p} \Sigma_{j,q}$ as $\tilde{V}$ verifies (45). Given the fact that $\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T$ and $\overline{Z}_T$ are independent, we find that the variance of the limit $\sum_{1 \le i,j \le n} m_{ij} \left(\frac{1}{\tau_{liq}} (\overline{Z}_{T+\tau_{liq}} - \overline{Z}_T) - \frac{1}{T} \overline{Z}_T \right)_{i,j}$ is equal to

$$\left(\frac{1}{T} + \frac{1}{\tau_{liq}} \right) \sum_{1 \le i,j,k,l \le n} m_{ij} m_{kl} \left(\Sigma_{ik} \Sigma_{jl} + \Sigma_{jk} \Sigma_{il} \right)$$

which concludes the proof of Corollary 3.7.

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