

Fire Sales Forensics: Measuring Endogenous Risk

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Spike in correlations after Lehman Brothers collapse

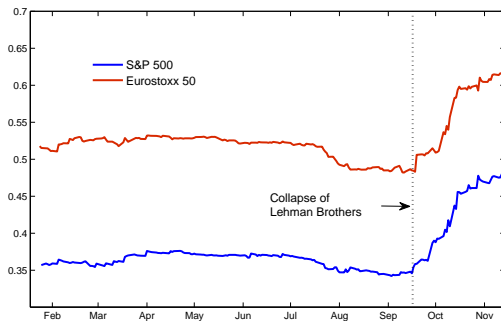


Figure : One-year average pairwise EWMA correlation in the S&P 500 (blue line) and in the Eurostoxx 50 (orange line) in 2008

Level of the S&P500 and volume for the S&P futures

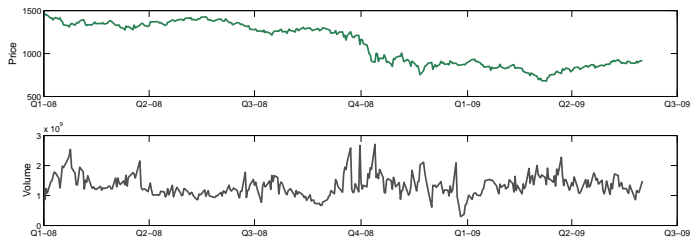


Figure : Level of the S&P500 and volume for the S&P futures

The economic origin of correlations in returns

Exogenous representations of correlation cannot explain spikes in realized correlations following the liquidation of a large fund.

→ two different origins for 'correlations' in returns:

- correlation in fundamentals
- correlation from trading: generated by systematic supply/demand from investors

Objectives:

- present a tractable framework for modeling endogenous correlation and its relation with 'systematic' trading and liquidity
- develop econometric tools which allow to investigate abnormal correlation regimes in terms of liquidation flows

Fire sales in distressed funds

Times of distress → systematic supply and demand generated by *fire sales*

Fire sales are "forced sales of assets in which high-valuation bidders are sidelined, typically due to debt overhang problems afflicting many specialist bidders simultaneously" (Shleifer & Vishny, 2001)

Fire sales can be triggered by:

- capital ratio constraints (Danielsson and Shin, 2003; Shin et al., 2004; Brunnermeier and Pedersen, 2004; Pedersen 2009; Shin 2010)
- investors redeeming their positions (Coval and Stafford, 2007; Anton and Polk, 2008; Jotikasthira et al., 2011)
- rule-based strategies (Platen and Schweizer, 1994, 1998)
- sales of assets held as collateral by creditors of distressed funds (Shleifer and Vishny 2011)

Common patterns during fire sales episodes

Common patterns observed during fire sales episodes:

- fire sales triggered when fund value decreases below a threshold
- more intense fire sales as fund value decreases
- fire sales do not take place through orderly liquidation and often impact market prices

Supply and demand from fire sales can be amplified by short-selling and predatory trading (Brunnermeier and Pedersen, 2005)

Framework

- discrete trading dates: $t_k = k\Delta t$
- n assets; vector of prices at t_k : $S_k = (S_k^1, \dots, S_k^n)$
- at each period, the value of the assets moves due to exogenous economic factors. In the absence of other effects, the return of asset i at period k would be:

$$\exp\left(\Delta t \left(m_i - \frac{\Sigma_{i,i}}{2}\right) + \sqrt{\Delta t} \xi_{S_{k+1}}^i\right) - 1$$

- $\xi_k = (\xi_k^1, \dots, \xi_k^n)_{k \geq 1}$ iid n -dimensional centered random variables, with covariance matrix Σ
- Σ and $m_i \rightarrow$ (fundamental) covariance structure of returns and expected return of asset i respectively, *in the absence of large systematic trades*

Modeling the supply and demand generated by fire sales

- J funds; fund j holds α_i^j units of asset i ; benchmark value of fund j at date t_k :

$$V_k^j = \sum_{i=1}^n \alpha_i^j S_k^i$$

- exogenous moves in asset prices \rightarrow moves in fund values \rightarrow possible fire sales
- fund $j \rightarrow$ 'deleveraging schedule' f_j ; supply by fund j on asset i at each period as

$$\alpha_i^j \left(f_j \left(\frac{V_k^j}{V_0^j} \right) - f_j \left(\frac{1}{V_0^j} \sum_{l=1}^n \alpha_l^j S_k^l \exp \left(\Delta t \left(m_l - \frac{\Sigma_{l,l}}{2} \right) + \sqrt{\Delta t} \xi_{k+1}^l \right) \right) \right)$$

We assume $f_j : \mathbb{R} \rightarrow \mathbb{R}$ to be:

- constant on $[\beta_0^j, +\infty[$ and increasing
- concave

Example of a deleveraging schedule

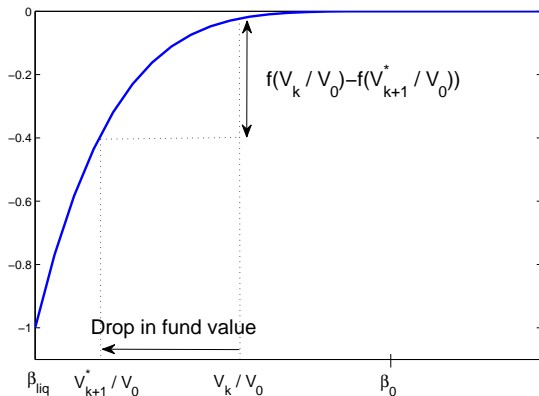


Figure : Example of a deleveraging schedule

Proportional liquidations

- in our study → hypothesis of proportional liquidations
- justified by empirical studies (Jotikasthira et al., 2011) when assets are of equivalent liquidity (typically equity indices and ETF)
- occurs when a fund has to be liquidated entirely
- allows to obtain empirical results
- hypothesis can be relaxed using the same framework and introducing a rate of liquidation for each asset (→ no empirical result possible without making an assumption on such asset liquidation rates)

Price impact of fire sales

- Fire sales impact prices in a non-random manner. Some empirical studies (Obizhaeva 2008; Cont Kukanov Stoikov 2010) provide evidence for the linearity of this price impact at daily and intraday frequencies.
- Impact on asset i 's return due to fire sales in fund j is equal to

$$\frac{\alpha_i^j}{D_i} \left(f_j \left(\frac{V_k^j}{V_0^j} \right) - f_j \left(\frac{1}{V_0^j} \sum_{l=1}^n \alpha_l^j S_k^l \exp \left(\Delta t \left(m_l - \frac{\Sigma_{l,l}}{2} \right) + \sqrt{\Delta t} \xi_{k+1}^l \right) \right) \right)$$

- D_i represents the *depth* of the market in asset i : a net demand of $\frac{D_i}{100}$ shares for security i moves i 's price by one percent
- Note that the case of a general (non linear) price impact function is studied in the paper

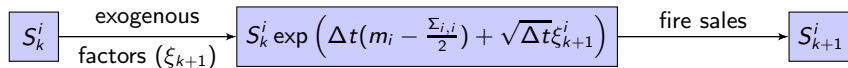
Price dynamics

Price dynamics summed up as follows:

$$S_{k+1}^i = S_k^i \exp \left(\Delta t \left(m_i - \frac{\Sigma_{i,i}}{2} \right) + \sqrt{\Delta t} \xi_{k+1}^i \right) \times$$

$$\left(1 - \frac{1}{D_i} \left[f_j \left(\frac{V_k^j}{V_0^j} \right) - \sum_{j=1}^J \alpha_i^j \left(f_j \left(\frac{1}{V_0^j} \sum_{l=1}^n \alpha_l^j S_k^l \exp \left(\Delta t \left(m_l - \frac{\Sigma_{l,l}}{2} \right) + \sqrt{\Delta t} \xi_{k+1}^l \right) \right) \right) \right] \right)$$

systematic impact due to fire sales



Assumption 1

For $i = 1..n$, $j = 1..J$,

$$f_j \in C_0^3(\mathbb{R}) \text{ and } \alpha_i^j \geq 0$$

$$\exists \eta > 0, \mathbb{E}(\|\exp(\eta\xi)\|) < \infty \text{ and } \mathbb{E}(\|\xi\|^{\eta+4}) < \infty$$

$$\min_{1 \leq i \leq n} \frac{1}{D_i} \left(2 \sum_{j=1}^n |\alpha_i^j| \times \|f_j\|_\infty \right) < 1$$

where $C_0^p(\mathbb{R})$ denotes the set of real-valued, p -times continuously differentiable maps whose first derivative has compact support.

Theorem 1

Under Assumption 1, $(S_{\lfloor Nt \rfloor})_{t \geq 0} \xrightarrow[N \rightarrow \infty]{} (P_t)_{t \geq 0}$, solution of the stochastic differential equation $\frac{dP_t^i}{P_t^i} = \mu_i(P_t)dt + (\sigma(P_t)dW_t)_i$ for $1 \leq i \leq n$, where μ (resp., σ) is a \mathbb{R}^n -valued (resp. matrix-valued) mapping defined by

$$\mu_i(P_t) = m_i + \frac{1}{D_i} \sum_{j=1}^J \left(\frac{\alpha_j^i}{2(V_0^j)^2} f_j'' \left(\frac{V_t^j}{V_0^j} \right) \pi_t^j \cdot \Sigma \pi_t^j + \frac{\alpha_j^i}{V_0^j} f_j' \left(\frac{V_t^j}{V_0^j} \right) \left(\pi_t^j \cdot \bar{m} + (\Sigma \pi_t^j)_i \right) \right)$$

$$\sigma_{i,k}(P_t) = A_{i,k} + \frac{1}{D_i} \sum_{j=1}^J \alpha_j^i f_j' \left(\frac{V_t^j}{V_0^j} \right) \frac{(A \pi_t^j)_k}{V_0^j}$$

- A : square root of the fundamental covariance matrix $\rightarrow AA^t = \Sigma$
- $\pi_t^j = (\alpha_1^j P_t^1, \dots, \alpha_n^j P_t^n)^t$ (dollar) allocation of fund j
- W n -dimensional Brownian motion
- $V_t^j = \sum_{k=1}^n \alpha_k^j P_t^k$ value of fund j at date t

Realized covariance

Proposition 2

The realized covariance matrix of returns between t_1 and t_2 is $C_{[t_1, t_2]} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} c_t dt$, where c_t , the instantaneous covariance matrix of returns, is given by

$$c_t = \Sigma + \underbrace{\sum_{1 \leq j \leq J} \frac{1}{V_0^j} f_j' \left(\frac{V_t^j}{V_0^j} \right) \left(\Lambda_j (\pi_t^j)^t \Sigma + \Sigma \pi_t^j \Lambda_j^t \right)}_{\text{order 1 in } \Lambda}$$

$$+ \underbrace{\sum_{1 \leq j, k \leq J} \frac{\pi_t^j \cdot \Sigma \pi_t^k}{V_0^j V_0^k} f_j' \left(\frac{V_t^j}{V_0^j} \right) f_k' \left(\frac{V_t^k}{V_0^k} \right) \Lambda_j \Lambda_k^t}_{\text{order 2 in } \Lambda}$$

where $\Lambda_j \in \mathbb{R}^n$ represents the positions of fund j in each market as a fraction of the respective market depth: $[\Lambda_j]_i = \frac{\alpha_i^j}{D_i}$.

Realized covariance

- realized covariance = fundamental covariance + *path-dependent and liquidity-dependent* excess covariance
- impact of fund liquidation *computable* under our model assumptions.
- even starting with homoscedastic inputs: feedback effects from fire sales → heteroscedasticity in the covariance structure of asset returns
- infinite market depths ($\forall i, D_i = \infty$) or no fire sales ($V_t^j \geq \beta_0^j V_0^j$) → realized covariance matrix = fundamental covariance matrix

Impact on fund variance

Proposition 3

In the presence of feedback effects from one large fund ($J=1$), the fund's realized variance between 0 and t is equal to $\frac{1}{t} \int_0^t \Gamma_s ds$ where Γ_s , the instantaneous variance of the fund, is given by:

$$\Gamma_s V_s^2 = \pi_s \cdot \Sigma \pi_s + \frac{2}{V_0} f' \left(\frac{V_s}{V_0} \right) (\pi_s \cdot \Sigma \pi_s) (\Lambda \cdot \pi_s) + \frac{1}{V_0^2} (f' \left(\frac{V_s}{V_0} \right))^2 (\pi_s \cdot \Sigma \pi_s) (\Lambda \cdot \pi_s)^2$$

with

- $\pi_t = (\alpha_1 P_t^1, \dots, \alpha_n P_t^n)^t$ (dollar) holdings of the fund
- $\Lambda = \left(\frac{\alpha_1}{D_1}, \dots, \frac{\alpha_n}{D_n} \right)^t$ positions of the fund in each market as a fraction of the respective market depth

Limits of diversification

- Fire sales increase fund volatility, exactly in scenarios where the fund experiences difficulty, reducing the benefit of diversification.
- Even without liquidity drying up (D_i constant), feedback effects can modify significantly fund volatility when large positions are exited.
- Spikes in correlation and fund volatility can be triggered by fire sales, even in the absence of predatory trading by short sellers.

Spillover effects

Proposition 4

In the presence of fire sales in a reference fund with positions α , the realized variance for a target fund with (small) positions $(\mu_t^i)_{1 \leq i \leq n}$ between t_1 and t_2 is equal to $\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \gamma_s \, ds$ where

$$\gamma_s M_s^2 = \pi_s^\mu \cdot \Sigma \pi_s^\mu + \frac{2f'(\frac{V_s}{V_0})}{V_0} (\pi_s^\mu \cdot \Sigma \pi_s^\alpha) (\Lambda \cdot \pi_s^\mu) + \frac{f'(\frac{V_s}{V_0})^2}{V_0^2} (\pi_s^\alpha \cdot \Sigma \pi_s^\alpha) (\Lambda \cdot \pi_s^\mu)^2$$

where $\pi_s^\alpha = \begin{pmatrix} \alpha_1 P_s^1 \\ \vdots \\ \alpha_n P_s^n \end{pmatrix}$ and $\pi_s^\mu = \begin{pmatrix} \mu_t^1 P_s^1 \\ \vdots \\ \mu_t^n P_s^n \end{pmatrix}$ denote respectively the (dollar) holdings of the reference fund and the target fund,

$M_s = \sum_{i=1}^n \mu_s^i P_s^i$ is the target fund's value, and $\Lambda = (\frac{\alpha_1}{D_1}, \dots, \frac{\alpha_n}{D_n})^t$.

Orthogonality condition

- spillover from fire sales in portfolio α to portfolio μ proportional to the liquidity-weighted overlap $\Lambda \cdot \pi_t^\mu = \sum_{1 \leq i \leq n} \frac{\alpha_i}{D_i} \mu_t^i P_t^i$
- \rightarrow overlaps in allocation count more in illiquid asset classes
- 'orthogonality condition':

$$\Lambda \cdot \pi_t^\mu = 0 \Rightarrow \gamma_s M_s^2 = \pi_s^\mu \cdot \Sigma \pi_s^\mu$$

\rightarrow fire sales in the reference fund *do not affect* target fund's variance

Risk management and regulation

- model which can be used to assess risks of a financial portfolio in a more complete way
- can be coupled with models for the 'exogenous' variability of correlations and volatilities
- allows to evaluate the endogenous risk generated by fire sales
- allows to compute the impact of the liquidation of a large fund on a given portfolio of assets
- useful for monitoring fire-sale contagion and anticipating the impact of the failure or liquidation of a large fund

Realized covariance with and without fire sales

- no fire sales between 0 and T : $C_{[0,T]} = \Sigma$
- possible fire sales between T and $T + \tau_{liq}$:

$$C_{[T, T+\tau_{liq}]} = \Sigma + LM_0\Pi\Sigma + \Sigma\Pi M_0L + O(\|\Lambda\|^2, \|f''\|)$$

with

$$M_0 = \sum_{j=1}^J \frac{\gamma_j}{V_0^j} \times \alpha^j (\alpha^j)^t$$

with $\alpha^j = (\alpha_1^j, \dots, \alpha_n^j)^t$ the vector of positions of fund j , γ_j average rate of liquidation for fund j and L and Π diagonal matrices with i -th diagonal term equal respectively to $\frac{1}{D_i}$ and $\frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} P_t^i dt$.

Inverse problem

- We now consider the inverse problem of explaining 'abnormal' patterns in realized covariance and volatility in the presence of fire sales and estimating the parameters of the liquidated portfolio from empirical observations.
- Mathematically speaking, given Σ , $C_{[T, T+\tau_{liq}]}$, L and Π , we want to find M such that $C_{[T, T+\tau_{liq}]} = \Sigma + LM\Pi\Sigma + \Sigma\Pi ML$.
- The knowledge of M allows to *estimate the volume of fire sales in asset class i between T and $T + \tau_{liq}$* up to an error term of order one in $\|\Lambda\|$.

Identifiability

Proposition 5

If $\Pi\Sigma L^{-1}$ is diagonalizable with eigenvalues ϕ_1, \dots, ϕ_n such that

$$\phi_i + \phi_j \neq 0 \quad 1 \leq i, j \leq n$$

then there exists a unique matrix M verifying

$$C_{[T, T+\tau_{liq}]} = \Sigma + LM\Pi\Sigma + \Sigma\Pi ML$$

and we can write

$$M = \Phi(L, \Pi, \Sigma, C_{[T, T+\tau_{liq}]})$$

where Φ can be computed explicitly.

Estimator of M

- In practice, we do not know exactly Σ and $C_{[T, T+\tau_{liq}]}$.
- We can build estimators for Σ and $C_{[T, T+\tau_{liq}]}$, from asset prices observed at time intervals τ , denoted respectively $\widehat{\Sigma}(\tau)$ and $\widehat{C}(\tau)$, which converge in probability (see for example [Theorem 3.3.1, Ch. 5, Jacod-Protter, 2012]) to Σ and $C_{[T, T+\tau_{liq}]}$ respectively.

■

$$\widehat{\Sigma}(\tau) = \frac{1}{T} [\ln(P), \ln(P)]_T^{(\tau)} \xrightarrow[\tau \rightarrow 0]{\mathbb{P}} \Sigma$$

$$\widehat{C}(\tau) = \frac{1}{\tau_{liq}} \left([\ln(P), \ln(P)]_{T+\tau_{liq}}^{(\tau)} - [\ln(P), \ln(P)]_T^{(\tau)} \right) \xrightarrow[\tau \rightarrow 0]{\mathbb{P}} C_{[T, T+\tau_{liq}]}$$

Estimator of M

- We can hence define an estimator $\hat{M}(\tau)$ for M :

$$\hat{M}(\tau) = \Phi(L, \Pi, \hat{\Sigma}(\tau), \hat{C}(\tau))$$

- $\hat{M}(\tau)$ estimator for liquidated fund flows during $[T, T + \tau_{liq}]$, computed from price series observed at time intervals τ
- \rightarrow study the consistency and the large sample properties of $\hat{M}(\tau)$ when $\tau \rightarrow 0$ i.e. when the estimator is computed with more and more refined price data

Consistency and large sample properties of $\widehat{M}(\tau)$

Proposition 6

The estimator $\widehat{M}(\tau)$ is consistent:

$$\widehat{M}(\tau) = \Phi(L, \Pi, \widehat{\Sigma}(\tau), \widehat{C}(\tau)) \xrightarrow[\tau \rightarrow 0]{\mathbb{P}} M$$

Furthermore, it is possible to derive a central limit theorem for $\widehat{M}(\tau)$, which allows to build a statistical test for testing the null hypothesis $M = 0$ (ie: no fire sales between T and $T + \tau_{liq}$).

Statistical test for fire sales

Proposition 7

Under the null hypothesis $M = 0$ (no fire sales between T and $T + \tau_{liq}$),

$$\frac{1}{\sqrt{T}} \left(P_T^t \hat{M}^{(\tau)} (P_T - P_{T+\tau_{liq}}) \right) \xrightarrow{\tau \rightarrow 0} \mathcal{N}(0, V)$$

where $V = \left(\frac{1}{T} + \frac{1}{\tau_{liq}} \right) \sum_{1 \leq i, j, k, l \leq n} m_{ij} m_{kl} (\Sigma_{ik} \Sigma_{jl} + \Sigma_{jk} \Sigma_{il})$, with

$$m_{ij} = \sum_{1 \leq p, q \leq n} \frac{[\Omega^{-1} P_T]_p [\Omega^{-1} (P_T - P_{T+\tau_{liq}})]_q}{\phi_p + \phi_q} \Omega_{ip} \Omega_{jq} D_i D_j \text{ where}$$

$\Omega^{-1} \Pi \Sigma L^{-1} \Omega = \text{diag}(\phi_p)$, P_t is the vector of prices at date t and $(D_i)_{1 \leq i \leq n}$ are the asset market depths.

Statistical test for fire sales

- The previous Proposition gives the asymptotic law of $\left(P_T^t \widehat{M}^{(\tau)}(P_T - P_{T+\tau_{liq}})\right)$ under the assumption that $M = 0$.
- We can then define a level l such that $\mathbb{P}\left(\left|P_T^t \widehat{M}^{(\tau)}(P_T - P_{T+\tau_{liq}})\right| > l\right) \leq 1 - p_l$ where p_l is typically equal to 95% or 99%. If we find that $\left|P_T^t \widehat{M}^{(\tau)}(P_T - P_{T+\tau_{liq}})\right| > l$, then the null hypothesis of no fire sales may be rejected at confidence level p_l .

Systematic investigation of abnormal market events

Methodology:

- identify a period $[0, T]$ without fire sales (and with a stable correlation structure) and a period $[T, T + \tau_{liq}]$ with possible fire sales (and with abnormal correlation regime)
- compute $\hat{\Sigma}(\tau)$ and $\hat{C}(\tau)$ from price series observed at time interval τ
- compute $\hat{M}(\tau)$, estimator of the liquidation matrix during $[T, T + \tau_{liq}]$
- test for the presence of fire sales during $[T, T + \tau_{liq}]$ (i.e. test for the hypothesis $M = 0$)
 - if $M = 0 \rightarrow$ no fire sales during $[T, T + \tau_{liq}]$
 - if $M \neq 0 \rightarrow$ fire sales during $[T, T + \tau_{liq}]$ and $\hat{M}(\tau)$ allows to reconstitute the volume of liquidation in each asset

The great deleveraging following the collapse of Lehman Brothers

- September, 15th, 2008: Lehman Brothers files for chapter 11 bankruptcy protection, citing bank debt of \$613 billion, \$155 billion in bond debt, and assets worth \$639 billion and becoming the largest bankruptcy filing in the US history.
- This market shock generated the "Great Deleveraging": stop loss strategies were triggered, measures of risk exploded, obliging fund managers to deleverage their portfolios in order to meet higher capital requirements in this high volatility environment.

Correlation spikes in Fall 2008

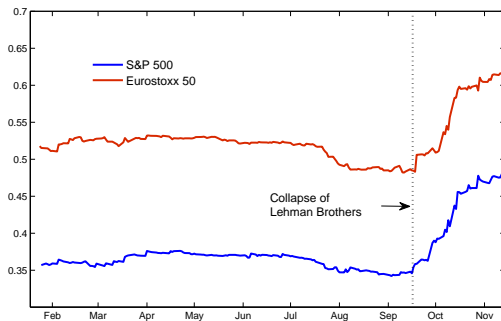


Figure : One-year average pairwise EWMA correlation in the S&P 500 (blue line) and in the Eurostoxx 50 (orange line)

Liquidations following the collapse of Lehman Brothers

We apply our estimation procedure and estimate the aggregate liquidated portfolio during the three months following the collapse of Lehman Brothers on the following universe of stocks:

- SPDRs: sector ETFs of the S&P500
- stocks belonging to the Eurostoxx 50, the main European equity index

Liquidations on SPDRs

Sector SPDR	daily \$ amount liquidated $\times 10^6 \$$	Weight
Financials	320	28%
Consumer Discretionary	55	5%
Consumer Staples	38	3.5%
Energy	300	26%
Health Care	63	5.5%
Industrials	90	8%
Materials	110	9.5%
Technology	65	5.5%
Utilities	100	9%

Table : Average liquidated portfolio on SPDRs during the 3 months following September, 15th, 2008

Liquidations on the Eurostoxx 50

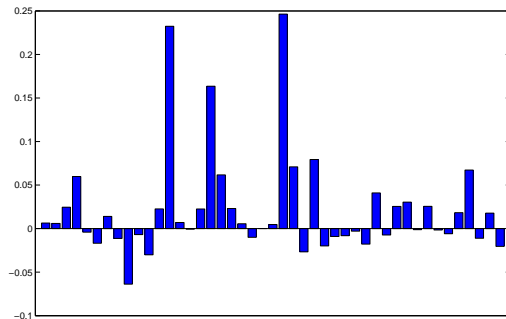


Figure : Composition of the aggregate portfolio on the Eurostoxx 50 liquidated during the 3 months following September, 15th, 2008

Main stocks liquidated on the Eurostoxx 50

Stock	Euros amount liquidated $\times 10^6 \text{ €}$	Weight
ING	1100	25%
Deutsche Bank	1000	23%
Eni	750	16%
Mittal	350	8%
Intesa San Paolo	320	7%
Unicredito	300	6.5%
EOAN	275	6%

Table : Most significantly liquidated stocks in the Eurostoxx 50 during the month following September, 15th, 2008

The quant event of August 2007

- first week of August 2007: huge daily losses for all market neutral hedge funds
- no major effect felt on major equity indices
- empirical studies suggest liquidation of large market neutral fund
- estimation procedure → reconstitute the average portfolio liquidated on the S&P 500

Aggregate portfolio liquidated during the quant event

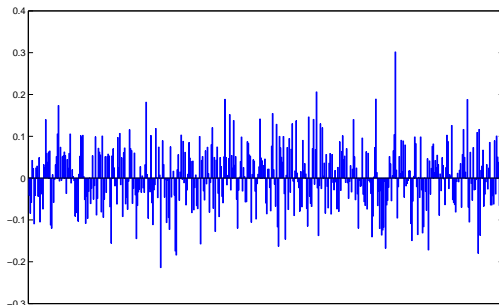


Figure : Composition of the aggregate portfolio on the S&P500 liquidated during the quant event

Strategy crowding: the example of August 2007

- investors exiting a large market-neutral long short fund \rightarrow large losses/excess volatility for *similar* long short funds
- *index funds* \rightarrow *orthogonal* to the reference fund \rightarrow unaffected

- $$\frac{\hat{\Lambda} \cdot \pi_t^{\hat{\mu}}}{\|\hat{\Lambda}\| \|\pi_t^{\hat{\mu}}\|} = \frac{\sum_{i=1}^n \frac{\alpha_i}{D_i} \mu_t^i P_t^i}{\|\hat{\Lambda}\| \|\pi_t^{\hat{\mu}}\|} = 0.0958 \rightarrow \text{angle of } 0.47\pi \text{ between the}$$
 vectors $\hat{\Lambda}$ and $\pi_t^{\hat{\mu}}$, i.e. very close to orthogonality

- can happen *without liquidity drying up*

Some concluding remarks

- Fire sales generate contagion effects and endogenous risk.
- The benefits of fund diversification is reduced and spillover effects can be observed.
- Our framework allows to explain large shifts in the realized covariance structure of asset returns in terms of supply and demand patterns across asset classes, which makes such events easier to analyze and understand.
- The estimation procedure that we propose may be useful for regulators in view of investigating unusual market events in a systematic way.

Research papers

- "Fire sales forensics: measuring endogenous risk", with Rama Cont, 2012, working paper
- "Running for the exit: distressed selling and endogenous correlation in financial markets", with Rama Cont, 2011, *Mathematical finance*, In press