Discussion on Derivation of Risk Neutral Probabilities from Option Prices
References

Estimation of Moments of RND:


Application of Cubic Spline Directly to Relation Between Option Prices and Strike Prices:

Ana Monteiro Reha Tutuncu and Luis Vicente, “Recovering Risk Neutral Probability Density Functions from Options Prices using Cubic Splines” July 20, 2004 Carnegie Mellon
\[ g(z) = n(z) \left[ 1 + \frac{\mu_3}{3!} (z^3 - 3z) + \frac{\mu_4 - 3}{4!} (z^4 - 6z^2 + 3) \right] \]

where

\[ n(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2); \]

\[ z = \frac{\ln(S_t/S_0) - (r - \sigma^2/s)t}{\sigma\sqrt{t}}; \]

\[ S_0 = \text{current stock price}; \]
\[ S_t = \text{random stock price at time } t; \]
\[ r = \text{risk-free interest rate}; \]
\[ t = \text{time remaining until option maturity}; \text{ and} \]
\[ \sigma = \text{standard deviation of returns for the underlying stock}. \]
\[ C_{GC} = C_{BS} + \mu_3 Q_3 + (\mu_4 - 3)Q_4 \]  \hspace{1cm} (3)

where

\[ C_{BS} = S_0 N(d) - Ke^{-rt} N(d - \sigma \sqrt{t}) \] is the Black-Scholes option pricing formula;

\[ Q_3 = \frac{1}{3!} S_0 \sigma \sqrt{t} \left( (2\sigma \sqrt{t} - d) n(d) - \sigma^2 t \right) N(d) \];

\[ Q_4 = \frac{1}{4!} S_0 \sigma \sqrt{t} \left( (d^2 - 1 - 3\sigma \sqrt{t} (d - \sigma \sqrt{t})) n(d) + \sigma^3 t^{3/2} \right) N(d) \]; and

\[ d = \frac{\ln(S_0/K) + (r + \sigma^2/2)t}{\sigma \sqrt{t}}. \]
\[
\min_{\text{ISD, ISK, IKT}} \sum_{j=1}^{N} \left[ C_{\text{OBS},j} - (C_{\text{BS},j} (\text{ISD}) + \text{ISK} Q_3 + (\text{IKT} - 3)Q_4) \right]^2
\]
Definitions:
\[ \tilde{r}_i = \ln (1 + \tilde{r}_i) - E[\ln(1 + \tilde{r}_i)] \]
\[ \sigma_{ij} = E(\tilde{r}_i \cdot \tilde{r}_j) \]
\[ S_{ijk} = E(\tilde{r}_i \cdot \tilde{r}_j \cdot \tilde{r}_k) \]
\[ K_{ijkl} = E(\tilde{r}_i \cdot \tilde{r}_j \cdot \tilde{r}_k \cdot \tilde{r}_l) \]

Portfolio Moments:
\[ \sigma_p^2 = \sum_i \sum_j w_{ip} w_{jp} \sigma_{ij} = \sum_i w_{ip} \sigma_{ip} \]
\[ S_p^3 = \sum_i \sum_j \sum_k w_{ip} w_{jp} w_{kp} \cdot S_{ijk} = \sum_i w_{ip} S_{iip} \]
\[ K_p^4 = \sum_i \sum_j \sum_k \sum_l w_{ip} w_{jp} w_{kp} w_{lp} K_{ijkl} = \sum_i w_{ip} K_{iip} \]
Implied Correlations

\[ \sigma_p = \sigma_p \sum_i \omega_{ip} \beta_{ip} = \sum_i \omega_{ip} \beta_{ip} \sigma_i \]

\[ \beta_{ip} = \frac{\sigma_{ip}}{\sigma_p^2} \]

\[ \gamma_{ip} = \beta_{ip} \frac{\sigma_p}{\sigma_i} \quad \gamma = \sum_i \omega_{ip} \beta_{ip} = 1 \]

Under constant correlation \( \beta_{ip} = \beta_{ij} \neq \beta_{ij} \)

\[ \gamma_p = \frac{\sigma_p}{\sum_i \omega_{ip} \sigma_i} \quad \beta_{ip} = \frac{\sigma_p \sigma_{ij}}{\sigma_i} \]

Alternative assumptions: (1) Constant \( \beta_{ip} \)

[2] Adjust historical \( \beta_{ip} \)s, (3) Scaled historical correlation

\[ \sigma_p = \sum_i \omega_{ip} \delta_{ip} = \sum_i \omega_{ip} \gamma_{ipp} \frac{\sigma_p}{\sigma_i} \]

Under constant \( \gamma_{ipp} = \gamma_{ipp} \neq \gamma_{jj} \)

\[ \gamma_{ipp} = \frac{\sigma_p}{\sum_i \omega_{ij} \sigma_i} \quad \gamma_{ip} = \frac{\gamma_{ipp} \sigma_i}{\sigma_p} \]
Relation between $B_{ip}$ and $Y_{ip}$

Under Quadratic Characteristic Curve

$$\tilde{\phi}_i = b_{ip} \tilde{\phi}_p + c_{ip} (\tilde{\phi}_p^2 - \sigma^2_p) + \epsilon_i$$

$$E(\epsilon_i | \tilde{\phi}_p) = 0$$

$$\sum_i w_{ip} b_{ip} = 1 \quad \sum_i w_{ip} c_{ip} = 0$$

$$B_{ip} = b_{ip} + c_{ip} \left( \frac{S^3_p}{\sigma^3_p} \right)$$

$$Y_{ip} = b_{ip} + c_{ip} \left( \frac{\kappa^3_p - (\sigma^3_p)^2}{S^3_p} \right)$$

Concave characteristic curves $c_{ip} < 0$ must be offset by convex curves $c_{ip} > 0$. If $c_{ip} = c_{mp} = 0 \implies i$.

$$B_{ip} = Y_{ip} + \epsilon_i$$

Conditional distribution can be calculated using conditional mean and conditional moments.
Under zero linear and non-linear autocorrelations

\[ \sigma_{T,T}^2 = \sum_{t=1}^{T} \sigma_{t,1}^2 \]

\[ S_{T,T}^3 = \sum_{t=1}^{T} S_{t,1}^3 \]

\[ K_{T,T}^4 = \sum_{c=1}^{C_{T,1}} K_{c,1}^4 \]

The implied vol. surface becomes \( \sigma_{t}, S_{t}, \) \& \( K_t \) curves.