Discussion on Derivation of Risk Neutral Probabilities from Option Prices

References

Estimation of Moments of RND:

Robert Jarrow and Andrew Rudd (1982) "Approximation of Option Valuation for Arbitrary Stochastic Processes", J.F.E. 10 (1982) 347-369.

Charles Cornado and Tie Su (1996) "Skewness and Kurtosis in S&P Index Returns Implied by Option Prices", Journal of Financial Research Vol. XIX, No. 2 (Summer 1996)
pp. 175-192.

Application of Cubic Spline Directly to Relation Between Option Prices and Strike Prices:

Ana Monteiro Reha Tutuncu and Luis Vicente, "Recovering Risk Neutral Probability Density Functions from Options Prices using Cubic Splines" July 20, 2004 Carnegie Mellon

$$g(z) = n(z)\left[1 + \frac{\mu_3}{3!}(z^3 - 3z) + \frac{\mu_4 - 3}{4!}(z^4 - 6z^2 + 3)\right]$$

where

$$n(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2);$$

$$z = \frac{\ln(S_t/S_0) - (r - \sigma^2/s)t}{\sigma\sqrt{t}};$$

S_0 = current stock price;
S_t = random stock price at time t;
r = risk-free interest rate;
t = time remaining until option maturity; and
σ = standard deviation of returns for the underlying stock.

$$C_{GC} = C_{BS} + \mu_3 Q_3 + (\mu_4 - 3) Q_4 \tag{3}$$

where

$C_{BS} = S_0 N(d) - Ke^{-rt} N(d - \sigma\sqrt{t})$ is the Black-Scholes option pricing formula;

$$Q_3 = \frac{1}{3!} S_0 \sigma\sqrt{t}\,((2\sigma\sqrt{t} - d) n(d) - \sigma^2 t\, N(d));$$

$$Q_4 = \frac{1}{4!} S_0 \sigma\sqrt{t}\,((d^2 - 1 - 3\sigma\sqrt{t}\,(d - \sigma\sqrt{t}\,)) n(d) + \sigma^3 t^{3/2} N(d)); \text{ and}$$

$$d = \frac{\ln(S_0/K) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}.$$

$$\min_{\mathrm{ISD},\mathrm{ISK},\mathrm{IKT}} \sum_{j=1}^{N} \left[C_{\mathrm{OBS},j} - \left(C_{\mathrm{BS},j}(\mathrm{ISD}) + \mathrm{ISK}\,Q_3 + (\mathrm{IKT} - 3)Q_4\right)\right]^2$$

Definitions

$$\tilde{r}_i \equiv \ln(1+\tilde{R}_i) - E[\ln(1+\hat{R}_i)$$

$$\sigma_{ij} \equiv E(\hat{r}_i \cdot \hat{r}_j)$$

$$S_{ijk} \equiv E(\tilde{r}_i \cdot \tilde{r}_j \cdot \hat{r}_k)$$

$$K_{ijkl} \equiv E(\hat{r}_i \cdot \hat{r}_j \cdot \hat{r}_k)$$

Portfolio Moments

$$\sigma_p^2 = \sum_i \sum_j w_{ip} w_{jp} \sigma_{ij} = \sum_i w_{ip} \sigma_{ip}$$

$$S_p^3 = \sum_i \sum_j \sum_k w_{ip} w_{jp} w_{kp} S_{ijk} = \sum_i w_{ip} S_{ipp}$$

$$K_p^3 = \sum_i \sum_j \sum_k \sum_l w_{ip} w_{jp} w_{kp} w_{lp} K_{ijkl} = \sum_i w_{ip} K_{ippp}$$

Implied Correlations

$$\sigma_p = \sigma_p \sum_i w_{ip}\beta_{ip} = \sum_i w_{ip} r_{ip}\sigma_i$$

$$\beta_{ip} = \frac{\sigma_{ip}}{\sigma_p^2} \qquad r_{ip} = \beta_{ip}\frac{\sigma_p}{\sigma_i}, \quad \sum_i w_{ip}\beta_{ip} = 1$$

under constant correlation $r_{ip} = r_{jp} \ \forall i, j$

$$r_p = \frac{\sigma_p}{\sum_i w_{ip}\sigma_i} \qquad \beta_{ip} = r_p \frac{\sigma_i}{\sigma_p}$$

Alternative assumptions: (1) constant β_{ip}, (2) Adjust historical β_{ip}'s, (3) scaled historical correlation

$$S_p = S_p \sum_i w_{ip}\gamma_{ip} = \sum_i w_{ip} r_{ipp}\frac{S_p}{S_i}$$

under constant $r_{ipp} = r_{jpp} \ \forall i, j$

$$r_{pp} = \frac{S_p}{\sum_i w_i S_i}, \quad \gamma_{ip} = r_{pp}\frac{S_i}{S_p}$$

Relation between β_{ip} and γ_{ip}
Under Quadratic Characteristic Curve

$$\tilde{r}_i = b_{ip}\hat{r}_p + c_{ip}(\hat{r}_p^2 - \sigma_p^2) + v_i$$

$$E(\hat{v}_j \mid \tilde{r}_p) = 0$$

$$\sum_i w_{ip} b_{ip} = 1 \quad \sum_i w_{ip} c_{ip} = 0$$

$$\beta_{ip} = b_{ip} + c_{ip}(S_p^3/\sigma_p^2)$$

$$\gamma_{ip} = b_{ip} + c_{ip}(K_p^2 - (\sigma_p^2)^2)/S_p^3$$

concave charteristic curves $c_{ip} < 0$ must be offset by convex curves $c_{ip} > 0$. If $c_{ip} = c_{jp} = 0 \quad \forall i, j$

$$\beta_{ip} = \gamma_{ip} \quad \forall i$$

conditional distribution can be calculated using conditional mean and condition moments

Under zero Linear and non-linear autocorrelations

$$\sigma^2_{T,T} = \sum_{t=1}^{t=T} \sigma^2_{t,1}$$

$$S^3_{T,T} = \sum_{t=1}^{t=T} S^3_{t,1}$$

$$K^4_{T,T} = \sum_{t=1}^{t=T} K^4_{t,1}$$

The implied vol. surface becomes σ_t, S_t & K_t curves