Asset Pricing with Heterogeneous Agents
(Preliminary)

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Puzzles

- Asset pricing
  - High, volatile and counter-cyclical risk premia
  - Low and stable risk-free rate
  - Hard to account for with standard macro models

- Asset markets
  - Rich asset markets
  - Poor consumption smoothing
  - Skewed wealth distribution

- Can portfolio behavior help explain all this?
Heterogeneity of Portfolio Behavior

- No participation
  - Many households do not participate in equity markets, 50% in US, 70% in Europe.
  - Many who hold equities only do so in a small way

- Among participants: many still deviate from optimal portfolios
  - Make only very infrequent adjustments - inertia
  - Adjust but based on past returns (miss market timing)

- “Sophisticated” investors earn higher return by increasing risk

- Want to include these different investors in our model.
Heterogeneity of Preferences and Beliefs

Heterogeneity in portfolio behavior could result from

- **Heterogeneity in preferences**
  - Survey data: a wide dispersion in attitudes towards risk
  - Risk attitudes can be influenced by wealth, education, gender, experiences, and personality.

- **Heterogeneity in beliefs**
  - Survey data: a significant heterogeneity in beliefs on asset return and volatility
  - Different forecasts also reflect heterogeneity in beliefs

- Can this explain why those who trade, trade differently?
What We Do

- Extend our methodology to compute the equilibria of economies with **more heterogeneities**
  - In addition to heterogeneity in trading technologies
  - Now includes heterogeneity in preferences and beliefs

- One quantitative experiment
  - Focus on heterogeneity in beliefs
  - Introduce recency bias: volatile beliefs
Environment

- Aggregate endowment $Y_t = \exp(z_t) Y_{t-1}$ comes in two forms
  - *tradeable output* $(1 - \gamma) Y_t$ depends on $z_t$
  - *non-tradeable output* $\gamma Y_t \eta_t$ depends on $\eta_t$ too

- Idiosyncratic shocks
  - $\eta$ are i.i.d. across households and $E\{\eta_t | z^t\} = 1$
  - $\pi(z^t, \eta^t)$ is probability of observing event $(z^t, \eta^t)$

- For now, assume a continuum of ex ante identical households with CRRA utility

- Our methodology builds on Arrow-Debreu environment
Arrow-Debreu Economy

With standard Arrow-Debreu economy, household $i$ chooses consumption sequence $c_t(z^t, \eta^t)$ to

$$\max_{\{c_t^i\}} \sum_t \sum_{(z^t, \eta^t)} \beta_t c_t^i(z^t, \eta^t)^{1-\alpha} \pi(z^t, \eta^t)$$

s.t. $\sum_t \sum_{(z^t, \eta^t)} \{ \gamma Y(z^t)\eta_t - c_t^i(z^t, \eta^t) \} \tilde{P}(z^t, \eta^t) + \omega_0 \geq 0.$

- subject to present-value budget constraint
- $\tilde{P}(z^t, \eta^t) = P_t(z^t) \pi(z^t, \eta^t)$ is the time zero state price.
- $\omega_0$ is the initial wealth.
- $\gamma Y(z^t)\eta_t$ is risky "labor" income
Add Trading Technologies

- Trading Technologies: restrictions on asset holdings over time
- How to impose restrictions on asset holding in time zero trading problem?
  - Remember that "Arrow" = "Arrow-Debreu"
  - Net assets \( a_t(z^t, \eta^t) \) = net savings
  - \( a_t(z^t, \eta^t) \) must be consistent with their consumption plan

\[
S^i(z^t, \eta^t) = \sum_{\tau \geq t} \sum (z^\tau, \eta^\tau) \tilde{P}(z^\tau, \eta^\tau) (\gamma Y(z^\tau)\eta^\tau - c(z^\tau, \eta^\tau))
\]

\[
= -a_t(z^t, \eta^t)\tilde{P}(z^t, \eta^t)
\]

- Hence, any restriction on \( a_t(z^t, \eta^t) \) will also limit [·].
Examples on Trading Technologies

- **Debt bounds**
  \[ a_t(z^t, \eta^t) \geq D(z^t), \]

- **No contingent claim on idiosyncratic shocks (Z-com traders)**
  \[ a_t(z^t, \hat{\eta}^t) = a_t(z^t, \eta^t), \text{ for all } \hat{\eta}^t \text{ and } \eta^t \]

- **Fix portfolio**
  \[ \frac{a_t(z^t, \eta^t)}{R_P(z^t)} = \frac{a_t(\hat{z}^t, \hat{\eta}^t)}{R_P(\hat{z}^t)} = \hat{a}_{t-1}(z^{t-1}, \eta^{t-1}) \]

  where \( R_P \) is the return on the fix portfolio
Lagrangian Example

Take a household $i$ with debt bounds and subject to fix portfolio restrictions as an example:

$$L = \max_{\{c, \hat{\alpha}\}} \min_{\{\chi, \nu, \varphi\}} \sum_{t=1}^{\infty} \beta^t \sum_{(z^t, \eta^t)} \frac{c^i_t(z^t, \eta^t)^{1-\alpha}}{1-\alpha} \pi(z^t, \eta^t)$$

$$+ \chi^i \left\{ \sum_{t \geq 1} \sum_{(z^t, \eta^t)} \tilde{P}(z^t, \eta^t) \left[ \gamma Y(z^t) \eta_t - c^i(z^t, \eta^t) \right] + \omega(z^0) \right\}$$

$$+ \sum_{t \geq 1} \sum_{(z^t, \eta^t)} \nu^i(z^t, \eta^t) \left\{ -\tilde{P}(z^t, \eta^t) \hat{a}(z^{t-1}, \eta^{t-1}) R^P(z^t) \right\}$$

$$+ \sum_{t \geq 1} \sum_{(z^t, \eta^t)} \varphi^i(z^t, \eta^t) \left\{ D^i_t(z^t) \tilde{P}(z^t, \eta^t) - S^i(z^t, \eta^t) \right\}.$$

where

$$S^i(z^t, \eta^t) = \sum_{\tau \geq t} \sum_{(z^\tau, \eta^\tau)} \tilde{P}(z^\tau, \eta^\tau) \left( \gamma Y(z^\tau) \eta_\tau - c(z^\tau, \eta^\tau) \right).$$
Time and State Varied Multiplier

- Recursive multiplier (Marcet and Marimon(1998))

\[
\zeta^i(z^t, \eta^t) = \zeta^i(z^{t-1}, \eta^{t-1}) + \nu^i(z^t, \eta^t) - \varphi^i(z^t, \eta^t)
\]

- All traders have first-order conditions

\[
\beta^t u'(c^i_t(z^t, \eta^t)) = \zeta^i(z^t, \eta^t) P_t(z^t).
\]

where \(\zeta^i(z^t, \eta^t)\) varies to satisfy constraints on net savings and \(P_t(z^t)\) is the state price.

- Law of motion on multiplier + FOC in consumption \(\implies\) Euler equations
Consumption Allocations

- The FOCs with respect to consumption for all traders:

\[ \beta^t c^i_t(z^t, \eta^t)^{-\alpha} = \zeta^i_t(z^t, \eta^t) p_t(z^t) \]

- together with resource constraints \( C_t(z^t) = \sum_i c^i_t(z^t, \eta^t) \mu_i \)

- imply the consumption share rules

\[ c^i_t(z^t, \eta^t) = \left( \frac{\zeta^i_t(z^t, \eta^t)^{-1/\alpha}}{\sum_i \zeta^i_t(z^t, \eta^t)^{-1/\alpha} \mu_i} \right) C(z^t). \]

- \( h_t(z^t) \equiv \sum_i \zeta^i_t(z^t, \eta^t)^{-1/\alpha} \mu_i \) is the one key moment
Price Aggregation

- FOCs and consumption share rules imply that
  \[ P_t(z^t) = \beta^t C(z^t)^{-\alpha} h_t(z^t)^{+\alpha}. \]

- Perturbed version of Breeden-Lucas stochastic discount factor
  \[ m_{t+1} \equiv \frac{P_{t+1}(z^{t+1})}{P_t(z^t)} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha} \left( \frac{h_{t+1}}{h_t} \right)^{+\alpha}. \]

  - standard part from a representative CRRA agent
  - this is the new part

- Depends on common homogeneous preferences
Computation

- Stochastic discount factor

\[ m_{t+1} \equiv \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha} \left( \frac{h_{t+1}}{h_t} \right)^{+\alpha}. \]

- How to compute?
  - Conjecture on state contingent prices, \( h_{t+1}/h_t \)
  - Use individual multiplier \( \zeta^i \) as state variable
  - Compute the law of motion on \( \zeta^i \) for each individual
  - Update \( h_{t+1}/h_t \): moments of multiplier distribution
  - Equilibrium is fixed point \( F \left[ h_{t+1}/h_t \right] = [h_{t+1}/h_t] \).
Heterogeneity in Preferences and Beliefs

Previously, all households had same CRRA preferences, discount rates and beliefs. Now agent of type $i$ has preferences

$$\sum_{t \geq 1, (z^t, \eta^t)} \infty (\beta_i)^t u^i(c^i_t(z^t, \eta^t)) \tilde{\pi}^i(z^t, \eta^t),$$

- $u^i(c^i_t(z^t, \eta^t))$ is strictly concave
- own discount rate $\beta_i$
- $\tilde{\pi}^i(z^t, \eta^t)$ subjective probability of agent $i$ on event $(z^t, \eta^t)$. 
New Trick to Get Aggregation

How can we apply the price aggregation result without common homogeneous preferences?

**Answer:** Create reference economy

**Reference economy** has

- fraternal twin for each trader with nice preferences+beliefs
- has given (same) state prices
- uses social planning weights to allocate consumptions
- choose weights so consumptions are the same between twins.

\[
\sum_i \left\{ \beta^t \sum_{(z^t, \eta^t)} \frac{1}{\zeta^i (z^t, \eta^t)} \bar{u}(\bar{c}(z^t, \eta^t)) - P(z^t)\bar{c}(z^t, \eta^t) \right\} \mu_i.
\]
Mapping to Reference Trader

- The FOC for a type $i$ household

$$\beta_i^t u'(c_t(z^t, \eta^t)) \tilde{\pi}^i(z^t, \eta^t) = \zeta^i(z^t, \eta^t) P_t(z^t) \pi(z^t, \eta^t)$$

- Consider a reference trader, who has standard CRRA preference and correct belief.

$$\beta^t \bar{c}^i(z^t, \eta^t)^{-\bar{\alpha}} = \bar{\zeta}^i(z^t, \eta^t) P_t(z^t),$$

- Define an adjusted multiplier, $\bar{\zeta}^i(z^t, \eta^t)$ such that $c^i = \bar{c}^i$

$$\left( \frac{\bar{\zeta}^i(z^t, \eta^t) P_t(z^t)}{\beta^t} \right)^{-1/\bar{\alpha}} \equiv u'^{-1} \left( \frac{\zeta^i(z^t, \eta^t) \pi(z^t, \eta^t) P_t(z^t)}{\beta_i^t \tilde{\pi}^i(z^t, \eta^t)} \right)$$
Mapping to Reference Trader

- With these **adjusted multiplier** for the reference traders:
  - If markets clear in the economy with reference traders, they do in the original one too.
  - The same price aggregation applies in reference economy
  - Need law of motion for individual multipliers from original economy
  - Need (change of variable) reference trader multipliers to compute new $h$

- Also works on recursive utility or robust utility
- Simple mapping trick with a wide range of applications
Computation

- Stochastic discount factor

\[ m_{t+1} \equiv \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha} \left( \frac{\tilde{h}_{t+1}}{\tilde{h}_t} \right)^{+\alpha}. \]

- How to compute?

  - Conjecture on state contingent prices, \( \tilde{h}_{t+1}/\tilde{h}_t \)
  - Use \( \zeta^i \) as state variable
  - Compute the law of motion on \( \zeta^i \) for each individual
  - Map \( \zeta^i \) into \( \tilde{\zeta}^i \)
  - Update \( \tilde{h}_{t+1}/\tilde{h}_t \): moments of adjusted multiplier distribution
  - Equilibrium is fixed point \( F [\tilde{h}_{t+1}/\tilde{h}_t] = [\tilde{h}_{t+1}/\tilde{h}_t] \).
Some Remarks

- Key to our methodology: price aggregation
- Compute via simple iterative method
- No need to find prices to each market
- Allow rich asset markets
- Price aggregation applies to finite agent case
Quantitative Exercise: Heterogeneity in Beliefs

Why beliefs?

- Counter-cyclical market price of risk
- Survey evidence: Greenwood and Shleifer (2012)
  - expectations of returns are
  - positively correlated with past stock market returns and
  - negatively correlated with future returns
- Adjust portfolio based on past returns (miss market timing)
Evidence on missed market timing: US equity mutual funds
Volatile beliefs

- **Volatile beliefs**: traders form their belief \( \tilde{\pi}(z^t, \eta^t) \)
  - with probability \( \kappa \) on the ergodic transition \( \pi(z_{t+1}|z_t) \) and
  - with probability \( 1 - \kappa \) by the observed transition frequencies during the past 4 periods.

- Consistent with forecasting in a non-stationary world
  - Agent who thinks that the transition matrix might have changed a fixed number of periods ago.
  - Similar strategies are followed by many forecasting models which truncate the data or overweight recent observations.

- (Standard trader has \( \kappa = 1 \)).
Bayesian Regime Switching

- Bayesian **regime-switching belief** traders believe that
  - There is high and a low regime
  - In high regime more likely to get high growth rate.
  - Regime switching governed by Markov transition matrix
  - Regime cannot be observed so infer from past history
- Recursive relationship for probability of high regime.
  - Use past history to update beliefs that regime is high, $\omega$
  - High growth rates raise this probability.
- (Standard trader has $\pi^h = 1$, $\pi^l = 0$, and no persistence).
Calibration

- Preferences: CRRA with $\alpha = 5$ and $\beta = .95$

- Endowments:
  - Aggregate shock
    - iid version of Merha-Prescott
    - Two state Markov process
  - Idiosyncratic risk calibrated to Storeslatten et al (no CCV)
    - Two state Markov process

- Assets: equity, bond and aggregate-state contingent claim

- Non-negative net saving constraints for all agents
Calibration

- Fraction of traders
  - 50% does not participate equity market. (bond only)
  - 40% holds the market
  - 10% aggregate-complete (Z-com traders)

- Among the 10% of Z-com traders
  - Case 1: all of them have correct belief, $\kappa = 1$
  - Case 2: 1/2 have \textbf{volatile beliefs} with $\kappa = 0.75$
  - Case 3: 1/2 have Bayesian \textbf{regime-switching beliefs}. 
## Variation in Beliefs Results

<table>
<thead>
<tr>
<th></th>
<th>Baseline</th>
<th>Volatile</th>
<th>R-S</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\sigma(m)}{E(m)}$</td>
<td>0.41</td>
<td>0.41</td>
<td>0.42</td>
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<tr>
<td>$\text{Std} \left{ \frac{\sigma(m)}{E(m)} \right}$</td>
<td>2.78</td>
<td>8.07</td>
<td>9.28</td>
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<tr>
<td>$E(R_f)$</td>
<td>1.93</td>
<td>2.04</td>
<td>2.04</td>
</tr>
<tr>
<td>$E(\omega_z)$</td>
<td>0.79</td>
<td>0.89</td>
<td>0.88</td>
</tr>
<tr>
<td>$E(\omega_{\tilde{z}})$</td>
<td>-</td>
<td>0.69</td>
<td>0.70</td>
</tr>
<tr>
<td>$\text{Corr} (\omega_z, SR)$</td>
<td>0.93</td>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td>$\text{Corr} (\omega_{\tilde{z}}, SR)$</td>
<td>-</td>
<td>-0.93</td>
<td>-0.87</td>
</tr>
<tr>
<td>$E(W_z/W)$</td>
<td>2.15</td>
<td>2.33</td>
<td>2.36</td>
</tr>
<tr>
<td>$E(W_{\tilde{z}}/W)$</td>
<td>-</td>
<td>1.91</td>
<td>1.86</td>
</tr>
</tbody>
</table>

$\omega$: equity share of portfolio, $W$: value of wealth
### Variation in Beliefs Results

<table>
<thead>
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<th>Baseline</th>
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<th>R-S</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(\Delta \log(c_z))$</td>
<td>2.99</td>
<td>2.92</td>
<td>2.92</td>
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<tr>
<td>$\sigma(\Delta \log(C))$</td>
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<tr>
<td>$\sigma(\Delta \log(c_z))$</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$\sigma(\Delta \log(c_{cap}))$</td>
<td>3.38</td>
<td>3.37</td>
<td>3.37</td>
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<tr>
<td>$\sigma(\Delta \log(C))$</td>
<td></td>
<td></td>
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<tr>
<td>$\sigma(\Delta \log(c_{bond}))$</td>
<td>3.59</td>
<td>3.57</td>
<td>3.57</td>
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<tr>
<td>$\sigma(\Delta \log(C))$</td>
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<tr>
<td>$\sigma(\Delta \log(C_z))$</td>
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<tr>
<td>$\sigma(\Delta \log(C))$</td>
<td>6.98</td>
<td>7.08</td>
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<td>$\sigma(\Delta \log(C_z))$</td>
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<td>$\sigma(\Delta \log(C))$</td>
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<td>1.00</td>
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<tr>
<td>$\sigma(\Delta \log(C))$</td>
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<tr>
<td>$\sigma(\Delta \log(c_{bond}))$</td>
<td>0.91</td>
<td>0.92</td>
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</tr>
</tbody>
</table>

$c_i$ is individual consumption and $C_i$ is group average
Volatile Beliefs

Subjective Probability of Expansions

Sharpe Ratio

Equity Share of Portfolio Choices

Ratio of Group Wealth to Average Wealth

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Asset Pricing with Heterogeneous Agents
Heterogeneity in Beliefs

Regime-Switching Beliefs

Subjective Probability of Expansions

Sharpe Ratio

Equity Share of Portfolio Choices

Ratio of Group Wealth to Average Wealth

Notes:
The shaded areas indicate recessions.

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Asset Pricing with Heterogeneous Agents
Tentative Conclusion

► New methodology
  ▶ Compute a G.E. model with a rich degree of heterogeneity
  ▶ Heterogeneity in trading technologies, preferences and beliefs

► Volatile and R-S belief active traders
  ▶ Volatility of market price of risk up a lot
  ▶ Missed market timing

► Disciplined by trading behavior + belief survey data.

Can also report on risk aversion and discount differences.