Large Dynamic Covariance Matrices

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Outline

1. Introduction
2. The Model
3. Estimation in Large Dimensions
4. Nonlinear Shrinkage
5. Empirical Study
6. Conclusion
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Problem & Aim of the Paper

Problem:

- **Multivariate GARCH models** are popular tools for risk management and portfolio selection.
- However, the number of assets in the investment universe generally poses a challenge to such models.
- In other words, many multivariate GARCH models suffer from the **curse of dimensionality**.
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- **Multivariate GARCH models** are popular tools for risk management and portfolio selection
- However, the number of assets in the investment universe generally poses a challenge to such models
- In other words, many multivariate GARCH models suffer from the **curse of dimensionality**

Aim of the paper:

- **Robustify** the DCC model of Engle (2002, JBES) against large dimensions
- Comparison to all kinds of other multivariate GARCH models is left to future research
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Notation

Subscripts:
- \( i = 1, \ldots, N \) indexes assets
- \( t = 1, \ldots, T \) indexes time

Ingredients:
- \( r_{i,t} \): observed return, stacked into \( \mathbf{r}_t := (r_{1,t}, \ldots, r_{N,t})' \)
- \( d^2_{i,t} := \text{Var}(r_{i,t} | \mathcal{F}_{t-1}) \): conditional variance
- \( D_t \): diagonal matrix with generic entry \( d_{i,t} \)
- \( H_t := \text{Cov}(\mathbf{r}_t | \mathcal{F}_{t-1}) \): conditional covariance matrix; \( \text{Diag}(H_t) = D^2_t \)
- \( s_{i,t} := r_{i,t}/d_{i,t} \): devolatilized return, stacked into \( \mathbf{s}_t := (s_{1,t}, \ldots, s_{N,t})' \)
- \( R_t := \text{Corr}(\mathbf{r}_t | \mathcal{F}_{t-1}) = \text{Cov}(\mathbf{s}_t | \mathcal{F}_{t-1}) \): conditional correlation matrix
- \( \sigma^2_i := \mathbb{E}(d^2_{i,t}) = \text{Var}(r_{i,t}) \): unconditional variance
- \( C := \mathbb{E}(R_t) = \text{Corr}(\mathbf{r}_t) = \text{Cov}(\mathbf{s}_t) \): unconditional correlation matrix
Model Definition

Univariate volatilities governed by a GARCH(1,1) process:

\[ d_{i,t}^2 = \omega_i + a_i r_{i,t-1}^2 + b_i d_{i,t-1}^2 \]
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DCC model of Engle(2002, JBES) with correlation targeting:

\[ Q_t = (1 - \alpha - \beta) C + \alpha s_{t-1} s'_{t-1} + \beta Q_{t-1} \]  \hspace{1cm} (1)

where \( Q_t \) is a pseudo conditional correlation matrix.
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Conditional correlation and covariance matrices then:

\[ R_t = \text{Diag}(Q_t)^{-1/2} Q_t \text{Diag}(Q_t)^{-1/2} \]

\[ H_t = D_t R_t D_t \]
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Data generating process:

\[ r_t|\mathcal{F}_{t-1} \sim \mathcal{N}(0, H_t) \]
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Making Estimation Feasible

Estimating the model with a large number of assets is challenging.

Major difficulty:

- Inverting the conditional covariance matrix $H_t$ for the likelihood
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Solution by Pakel et al. (2014, WP):
- Instead of using the full conditional covariance matrix, use a selection of two-by-two blocks
- The composite likelihood is obtained by combining the likelihoods of (contiguous) pairs of assets
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Three-stage estimation scheme:

1. Fit a GARCH(1,1) model to each asset
2. **Estimate the unconditional correlation matrix** $C$ of the devolatilized returns for correlation targeting
3. Maximize the composite likelihood to estimate $(\alpha, \beta)$
Nonlinear Shrinkage to Counter Large Dimensions

Main contribution:

- Improved estimation of the unconditional correlation matrix \( C \), which serves as the correlation target in equation (1)
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Naïve approach:

- Use the sample correlation matrix of the devolatilized returns $\hat{s}_t$
- Corresponds to the original proposal of Engle (2002, JBES)
- This approach does not work well in large dimensions, and cannot even be used when $N > T$
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Superior approach:

- Apply nonlinear shrinkage to the devolatilized returns $\hat{s}_t$
- This approach works well in large dimensions, even when $N > T$
Generic setting:

- I.i.d. data $y_t \in \mathbb{R}^N$ with covariance matrix $\Sigma$
- Stacked into $T \times N$ matrix $Y$
Nonlinear Shrinkage: Starting Point

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- I.i.d. data $y_t \in \mathbb{R}^N$ with covariance matrix $\Sigma$
- Stacked into $T \times N$ matrix $Y$

The sample covariance matrix $S$ admits a spectral decomposition

$$S = U \Lambda U'$$

Here:
- $U$ is an orthogonal matrix whose columns are the sample eigenvectors $(u_1, \ldots, u_N)$
- $\Lambda$ is a diagonal matrix whose diagonal entries are the sample eigenvalues $(\lambda_1, \ldots, \lambda_N)$
Nonlinear Shrinkage: Class of Estimators

Rotation Equivariance

- Observed $T \times N$ data matrix: $Y$
- $W$ is an $N$-dimensional orthogonal / rotation matrix
- $\hat{\Sigma} := \hat{\Sigma}(Y)$ is a generic estimator of $\Sigma$
- It is rotation-equivariant if $\hat{\Sigma}(YW) = W'\hat{\Sigma}(Y)W$
## Nonlinear Shrinkage: Class of Estimators

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Without specific knowledge about $\Sigma$, rotation equivariance is a **desirable property** of an estimator.
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Without specific knowledge about $\Sigma$, rotation equivariance is a desirable property of an estimator.

We use the following class of rotation-equivariant estimators going back to Stein (1975, 1986):

$$\hat{\Sigma} := UDU' \quad \text{where} \quad D := \text{Diag}(d_1, \ldots, d_N) \text{ is diagonal}$$
Nonlinear Shrinkage In Action

Generic estimator in the class $\hat{\Sigma} := UDU'$.

Keep the sample eigenvectors.

Shrink the sample eigenvalues:
- $D := \text{Diag}(d(\lambda_1), \ldots, d(\lambda_N))$
- Based on nonlinear shrinkage function $d : \mathbb{R} \rightarrow \mathbb{R}$
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**Approach of Ledoit and Wolf (2012, AOS; 2015, JMVA):**

- Use large-dimensional asymptotics where $N/T \to c > 0$
- Consistently estimate optimal limiting shrinkage function $d^*$
- Feasible estimator: $\widetilde{\Sigma} := U \times \text{Diag}(\widetilde{d}(\lambda_1), \ldots, \widetilde{d}(\lambda_N)) \times U'$
Nonlinear Shrinkage In Action

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Proposed Estimation of the DCC Model

Estimation of the correlation target $C$:

- Apply nonlinear shrinkage to the devolatilized returns $\hat{s}_t$
- The resulting estimator is not a proper correlation matrix
- Post-processing the estimator takes care of this problem, that is, convert covariance matrix into a correlation matrix
Proposed Estimation of the DCC Model

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Three-stage estimation scheme:
- Fit a GARCH(1,1) model to each asset
- Use nonlinear shrinkage to estimate $C$
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Simpler alternative:
Linear Shrinkage

Easiest way to think about it:

- Convex linear combination of the sample covariance matrix and (a multiple of) the identity matrix:

\[ \hat{\Sigma} = c(\bar{s}^2 I) + (1 - c)S \]

- \( \bar{s}^2 \) is the average of the \( N \) sample variances \( s_i^2 \)
- \( c \in [0, 1] \) is the shrinkage intensity
Linear Shrinkage

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  \]
  - \(\bar{s}^2\) is the average of the \(N\) sample variances \(s_i^2\)
  - \(c \in [0, 1]\) is the shrinkage intensity

Alternative way to think about it:

- This estimator is also of the form \(UDU'\), but \(d\) is restricted to be a certain linear function:
  \[
  d(\lambda_i) := c\bar{\lambda} + (1 - c)\lambda_i
  \]
  - \(\bar{\lambda}\) is the average of the \(N\) sample eigenvalues \(\lambda_i\)
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Goal:

- Examine out-of-sample properties of Markowitz portfolios via backtest exercises
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Two applications:

- Global minimum variance (GMV) portfolio
- Full Markowitz portfolio with a signal
Big Picture

Goal:
- Examine out-of-sample properties of Markowitz portfolios via backtest exercises

Two applications:
- Global minimum variance (GMV) portfolio
- Full Markowitz portfolio with a signal

(Out-of-sample) Performance measures:
- Standard deviation
- Information ratio
Data & Portfolio Rules

Data:
- Download daily return data from CRSP
- Period: 01/01/1980–12/31/2015
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Updating:
- 21 consecutive trading days constitute one ‘month’
- Update portfolios on ‘monthly’ basis
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- 21 consecutive trading days constitute one ‘month’
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Out-of-sample period:
- Start investing on 01/08/1986
- This results in 7560 daily returns (over 360 ‘months’)

Portfolio sizes:

- We consider $N \in \{100, 500, 1000\}$
Data & Portfolio Rules

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Portfolio constituents:

- Select new constituents at beginning of each ‘month’
- Find the $N$ largest stocks that have
  
  (i) a complete 1250-day return history
  (ii) a complete 21-day return future
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Covariance matrix estimation:
- Use previous $T = 1250$ days to estimate the covariance matrix
Global Minimum Variance Portfolio

**Problem Formulation**

\[
\min_{w} w'H_tw
\]

subject to \[w'1 = 1\]

(where \(1\) is a conformable vector of ones)
Global Minimum Variance Portfolio

**Problem Formulation**

\[
\min_{w} \quad w^\prime H_t w \\
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(where \(1\) is a conformable vector of ones)

**Analytical Solution**

\[
w^* = \frac{H_t^{-1} 1}{1^\prime H_t^{-1} 1}
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Global Minimum Variance Portfolio

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\[
\hat{w}^* = \frac{H_t^{-1} 1}{1' H_t^{-1} 1}
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Feasible Solution

\[
\tilde{w} := \frac{\hat{H}_t^{-1} 1}{1' \hat{H}_t^{-1} 1}
\]
Global Minimum Variance Portfolio

Competing portfolios:

- **1/N**: as a simple benchmark
- **DCC-S**: based on the sample correlation matrix
- **DCC-L**: based on linear shrinkage
- **DCC-NL**: based on nonlinear shrinkage
- **RM-2006**: RiskMetrics 2006
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- **Standard deviation** (primary)
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Assessing statistical significance:
- Test for significant difference between DCC-S and DCC-NL uses Ledoit and Wolf (2011, WM)
# Global Minimum Variance Portfolio

Annualized **standard deviations:**

<table>
<thead>
<tr>
<th>N</th>
<th>1/N</th>
<th>DCC-S</th>
<th>DCC-L</th>
<th>DCC-NL</th>
<th>RM-2006</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>21.56</td>
<td>13.36</td>
<td>13.33</td>
<td><strong>13.17</strong>*</td>
<td>14.69</td>
</tr>
<tr>
<td>500</td>
<td>19.53</td>
<td>10.57</td>
<td>10.40</td>
<td><strong>9.64</strong>*</td>
<td>12.60</td>
</tr>
<tr>
<td>1000</td>
<td>19.04</td>
<td>10.59</td>
<td>9.14</td>
<td><strong>8.02</strong>*</td>
<td>14.86</td>
</tr>
</tbody>
</table>

Remarks:

- In each row, the **best number** appears in blue
- Stars indicate significant outperformance (DCC-NL vs. DCC-S)
### Global Minimum Variance Portfolio

**Annualized information ratios:**

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<tr>
<td>100</td>
<td>0.56</td>
<td>0.74</td>
<td>0.74</td>
<td>0.76</td>
<td>0.57</td>
</tr>
<tr>
<td>500</td>
<td>0.69</td>
<td>1.32</td>
<td>1.33</td>
<td>1.39</td>
<td>0.89</td>
</tr>
<tr>
<td>1000</td>
<td>0.75</td>
<td>1.11</td>
<td>1.33</td>
<td>1.52</td>
<td>0.77</td>
</tr>
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Markowitz Portfolio with Signal

Problem Formulation

\[
\begin{align*}
\min_{w} & \quad w' H_t w \\
\text{subject to} & \quad w' m_t = b \quad \text{and} \\
& \quad w' 1 = 1
\end{align*}
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(where \( m_t \) is a signal and \( b \) is a target expected return)
## Markowitz Portfolio with Signal

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### Analytical Solution

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w^* &= c_1 H_t^{-1} 1 + c_2 H_t^{-1} m \\
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Feasible Solution \( \tilde{w} \) replaces \( H_t \) with an estimator \( \hat{H}_t \).
Markowitz Portfolio with Momentum Signal

For simplicity and reproducibility, we use \textit{momentum} as the signal.
Markowitz Portfolio with Momentum Signal

For simplicity and reproducibility, we use *momentum* as the signal.

Competing portfolios:

- **EW-TQ:** equal-weighted portfolio of top-quintiles stocks → yields target expected return $b$ for other portfolios
- **DCC-S:** based on the sample correlation matrix
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  uses Ledoit and Wolf (2008, JEF)
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### Annualized standard deviations:

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<tr>
<td>100</td>
<td>28.43</td>
<td>17.05</td>
<td>17.03</td>
<td><strong>16.90</strong>*</td>
<td>18.87</td>
</tr>
<tr>
<td>500</td>
<td>24.42</td>
<td>12.36</td>
<td>12.16</td>
<td><strong>11.31</strong>*</td>
<td>16.14</td>
</tr>
<tr>
<td>1000</td>
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<td>1.34</td>
<td>1.37</td>
<td>1.48***</td>
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Two keys for making DCC model robust against large dimensions:

1. **Composite likelihood** makes estimation feasible
2. **Nonlinear shrinkage** estimation of the correlation targeting matrix ensures good performance
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Resulting **DCC-NL** model:

- Outperforms the basic DCC-S model by a wide margin
- Should become the new DCC standard
Conclusion

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1. **Composite likelihood** makes estimation feasible
2. **Nonlinear shrinkage** estimation of the correlation targeting matrix ensures good performance

Resulting **DCC-NL** model:

- Outperforms the basic DCC-S model by a wide margin
- Should become the new DCC standard

Remark:

- Nonlinear shrinkage can also help in robustifying other multivariate GARCH models against large dimensions
- A short description for the scalar BEKK model is in the paper


Asymptotic Framework

Let $N := N(T)$ and assume $N/T \to c > 0$, as $T \to \infty$.

The following set of assumptions is maintained throughout.

A1 The population covariance matrix $\Sigma_T$ is a nonrandom $N$-dimensional positive definite matrix.

A2 Let $X_T$ be an $T \times N$ matrix of real i.i.d. random variables with zero mean, unit variance, and finite twelfth moment. One observes $Y_T := X_T \Sigma_T^{1/2}$.

A3 Let $((\tau_{T,1}, \ldots, \tau_{T,N}); (v_{T,1}, \ldots, v_{T,N}))$ denote the eigenvalues and eigenvectors of $\Sigma_T$. The e.d.f. of the population eigenvalues, denoted by $H_T$, converges weakly to some limiting e.d.f. $H$.

A4 Supp$(H)$, the support of $H$, is the union of a finite number of closed intervals, bounded away from zero and infinity. Furthermore, there exists a compact interval in $(0, +\infty)$ that contains Supp$(H_T)$ for all $T$ large enough.
The Stieltjes transform of a nondecreasing function $G$ is:

$$\forall z \in \mathbb{C}^+ \quad m_G(z) := \int_{-\infty}^{+\infty} \frac{1}{\lambda - z} dG(\lambda)$$

(It has an explicit inversion formula too.)

Denote the e.d.f. of the sample eigenvalues by $F_T$. Marčenko and Pastur (1967) showed that $F_T$ converges a.s. to some nonrandom limit $F$ at all points of continuity of $F$.

They also discovered how $m_F$ relates to $H$ and $c$:

$$\forall z \in \mathbb{C}^+ \quad m_F(z) = \int_{-\infty}^{+\infty} \frac{1}{\tau \left[ 1 - c - cz m_F(z) \right] - z} dH(\tau)$$  \hspace{1cm} (2)

This is the celebrated Marčenko-Pastur (MP) equation.
Moral: knowing $H$ and $c$, one can ‘solve’ for $F$.

The particular expression (2) of the MP equation is due to Silverstein (1995).

Silverstein and Choi (1995) showed that

$$\forall \lambda \in \mathbb{R} \quad \lim_{z \in \mathbb{C}^+ \to \lambda} m_F(z) =: \dot{m}_F(\lambda) \text{ exists}$$

The quantity $\dot{m}_F(\lambda)$ will be of crucial importance.
Illustration

$H$ is a point mass at one (as for identity covariance matrix).

Plot density of $F$ for various values of $c$: 

- $c = 0.01$
- $c = 0.25$
- $c = 0.5$
- $c = 0.75$
Optimization Problem

(Standardized) Frobenius norm:

\[ ||A|| := \frac{\sqrt{\text{Tr}(AA')}}{r} \quad \text{for any matrix } A \text{ of dimension } r \times m \]
Optimization Problem

(Standardized) Frobenius norm:

$$\|A\| := \frac{\sqrt{\text{Tr}(AA')}}{r}$$

for any matrix $A$ of dimension $r \times m$

Loss function:

$$\mathcal{L}(U_T D_T U_T, \Sigma_T) := \|U_T D_T U_T - \Sigma_T\|^2$$
Optimization Problem

(Standardized) Frobenius norm:

\[ ||A|| := \frac{\sqrt{\text{Tr}(AA')}}{r} \]

for any matrix \( A \) of dimension \( r \times m \)

Loss function:

\[ \mathcal{L}(U_T D_T U_T, \Sigma_T) := ||U_T D_T U_T - \Sigma_T||^2 \]

Line of attack:

- It turns out that there is nonstochastic limit of the loss function, which involves the shrinkage function \( d \)
- We minimize the limiting expression with respect to \( d \)
We illustrate the methodology for the case $c \leq 1$. 
We illustrate the methodology for the case $c \leq 1$.

**Optimal limiting shrinkage function**

$$d^*(\lambda) := \frac{\lambda}{|1 - c - c \lambda \hat{m}_F(\lambda)|^2}$$
Nonlinear Shrinkage Estimator

We illustrate the methodology for the case $c \leq 1$.

**Optimal limiting shrinkage function**

$$d^*(\lambda) := \frac{\lambda}{\left| 1 - c - c \lambda \tilde{m}_F(\lambda) \right|^2}$$

A **feasible estimator** is obtained by:

- Replacing $c$ with $N/T$
- Consistently estimating $\tilde{m}_F$, which is achieved by consistently estimating $H$ and putting it in the MP equation together with $N/T$

Resulting estimator: $\tilde{\Sigma}_T := U_T \times \text{Diag}(\tilde{d}_T(\lambda_{T,1}), \ldots, \tilde{d}_T(\lambda_{T,N})) \times U'_T$
Nonlinear Shrinkage Estimator

We illustrate the methodology for the case $c \leq 1$.

**Optimal limiting shrinkage function**

\[ d^*(\lambda) := \frac{\lambda}{\left|1 - c - c \lambda \bar{m}_F(\lambda)\right|^2} \]

A **feasible estimator** is obtained by:

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The methodology can be extended to the case $c > 1$. 
We illustrate the methodology for the case $c \leq 1$.

**Optimal limiting shrinkage function**

$$d^*(\lambda) := \frac{\lambda}{\left|1 - c - c \lambda \tilde{m}_F(\lambda)\right|^2}$$

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The methodology can be extended to the case $c > 1$. 

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