Forecasting the Distribution of Option Returns

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Some Important Questions Relating to Options Markets

1. What is the expected return of an option?
2. What is the riskiness of an option?

Economic importance of these questions:

- Option returns reveal compensation investors demand for taking on state-dependent exposures
- Needed for portfolio choice problems involving options

Motivating Example: No-arbitrage SVJ Model

Option Risk and Return in Affine No-Arbitrage Models

SVJ model under $\mathbb{P}$ measure

$$
\begin{aligned}
dS_t &= (r+\mu)S_t dt + S_t\sqrt{V_t}dW_t^s(\mathbb{P}) + J_t(\lambda^{\mathbb{P}}, \mu_J^{\mathbb{P}}, \sigma_J^{\mathbb{P}}) \\
dV_t &= \kappa(\theta^{\mathbb{P}} - V_t)dt + \sigma_v\sqrt{V_t}dW_t^v(\mathbb{P})
\end{aligned}
$$

$\mathbb{Q}$ measure shifts parameters $\theta^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \mu_J^{\mathbb{Q}}, \sigma_J^{\mathbb{Q}}$

Fix parameters (from Broadie et al. 2009). For each contract,

1. Retrieve latent state V_t (invert each t from ~ 90 day ATM call)
2. $\mathbb{P}$ spec: Simulate $S_{t+1}^b, V_{t+1}^b, b = 1 : B$
3. $\mathbb{Q}$ spec: Evaluate option price $(P_{i,t+1}^b)$ at each simulated S_{t+1}^b, V_{t+1}^b
4. Result: $\{P_{i,t+1}^b\}_{b=1}^B$

Model-based conditional distribution of next-period prices for <u>each</u> contract-day

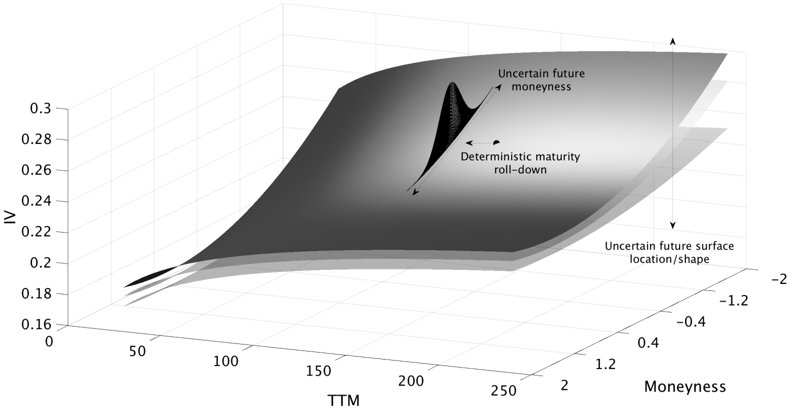
Option Risk and Return in Affine No-Arbitrage Models

- How frequently does realized option price fall below the x^{th} percentile of the simulated next-day price distribution?

Target	SVJ
1.0	21.8
5.0	26.6
10.0	29.5
25.0	35.1
50.0	41.9
75.0	48.3
90.0	53.4
95.0	56.2
99.0	60.8

- Model-forecasted distributions starkly inconsistent with data
  - Simulated distribution too narrow: Actuals frequently below low quantiles and above high quantiles

Overview of Method



Overview of Method

Uncertainty about future option price comes in two layers

1. Where on the surface will the contract migrate to at $t+1$?
    - Deterministic roll-down in time-to-maturity dimension
    - Stochastic change in moneyness dimension due to underlying return
    - Option return uncertainty due to "shocks to contract moneyness"

2. What will the shape of the IV surface look like at $t+1$?
    - Option return uncertainty due to "shocks to surface shape"
    - Time series factor models describe surface variation with R^2 approaching 100%

- Characterize distribution of future option returns by characterizing distribution of r_{t+1} (1) and IV_{t+1} (2)

Step 1: Define the IV Surface

- S&P 500 data from 1996-2015, OTM contracts only
- Define dimensions of the surface
  1. Time-to-maturity, τ, on grid $[30, 60, 91, 122, 152, 182, 273, 365]$
  2. Moneyness of contract, $m = \frac{\log(K/S)}{\text{VIX}\sqrt{\tau}}$, on grid $[-3 : 0.25 : 1.5]$
- Grid point interpolation of contract-level IV's each day

Step 2: Model of the Surface, Underlying

System backbone: Factor vector $X_t = (r_t, \log VIX_t, \text{PC's})$, evolves as

$$X_t = \mu + \rho X_{t-1} + \Sigma_{t-1}\epsilon_t \tag{1}$$

IV at grid point (m, τ) obeys factor model:

$$\log IV_{m,\tau,t} = \beta_{m,\tau}(1, X_t')' \; + \; \gamma_{m,\tau,t-1} u_{m,\tau,t} \tag{2}$$

Comments:

- Important to specify (2) in logs
- Panel R^2 in (2) is 99%, minimum R^2 at any grid point is 96%
- Co-variance matrices are GARCH models
- $\hat{\epsilon}_t = \hat{\Sigma}_t^{-\frac{1}{2}} e_t$, where e_t are VAR errors (likewise for $\hat{u}_t$)

Step 3: Bootstrap Conditional Price Distribution $(t+1|t)$

1. Fix day t conditioning information. This includes
   - State of the system: $S_t, X_t, \hat{\Sigma}_t, \hat{\gamma}_t$ and model parameters $(\hat{\mu}, \hat{\rho}, \hat{\beta}, \ldots)$
   - Fix contract: τ, K, m
2. Draw sample, $b = 1 : B$, from the empirical distribution of $\hat{\epsilon}_t$ and $\hat{u}_t$
3. For each draw b, feed shocks through estimated system
   - $X^b_{t+1} = \hat{\mu} + \hat{\rho} X_t + \hat{\Sigma}_t \epsilon^b_{t+1} \quad \Rightarrow \quad VIX^b_{t+1},\ S^b_{t+1} = S_t \exp(r^b_{t+1})$
   - $\tau^b = \tau - 1/365$ and $m^b = \frac{log(K/S^b_{t+1})}{\mathrm{VIX}^b_{t+1}\sqrt{\tau_{t+1}}}$
   - $\log IV_{m^b,\tau^b,t+1} = \hat{\beta}_{m^b,\tau^b} X^b_{t+1} + \hat{\gamma}_{m^b,\tau^b,t} \hat{u}^b_{t+1}$
   - Finally, $P^b_{t+1} = BS(S^b_{t+1}, IV_{m^b,\tau^b,t+1}, K, \tau^b)$

$\{P^b_{t+1}\}^B_{b=1}$ is conditional forecast of price distribution for *individual contract*

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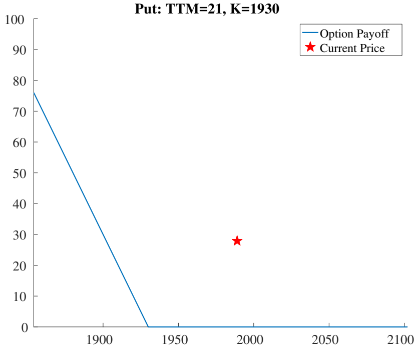
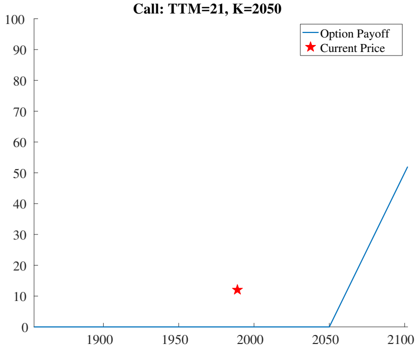
Case Study: Aug. 28, 2015

S&P at 1989, VIX at 26





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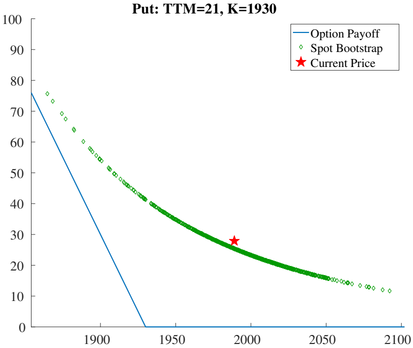

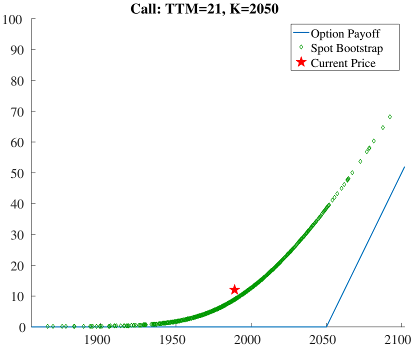

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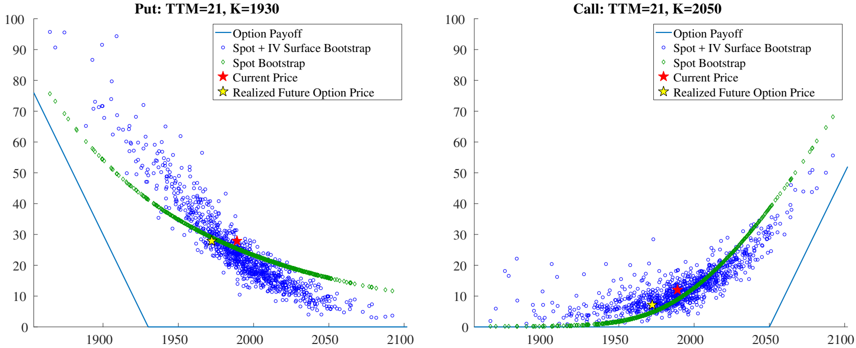

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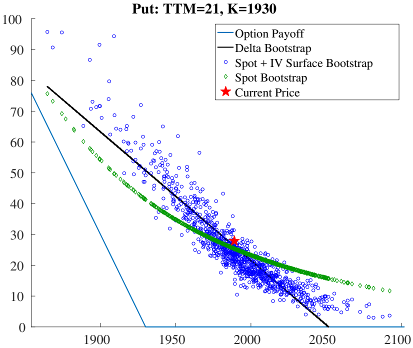

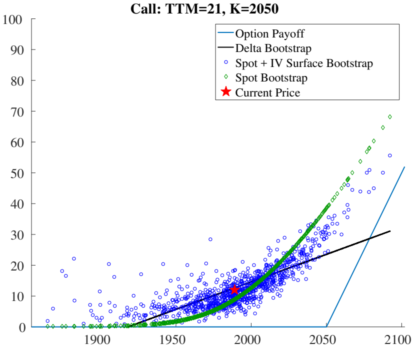

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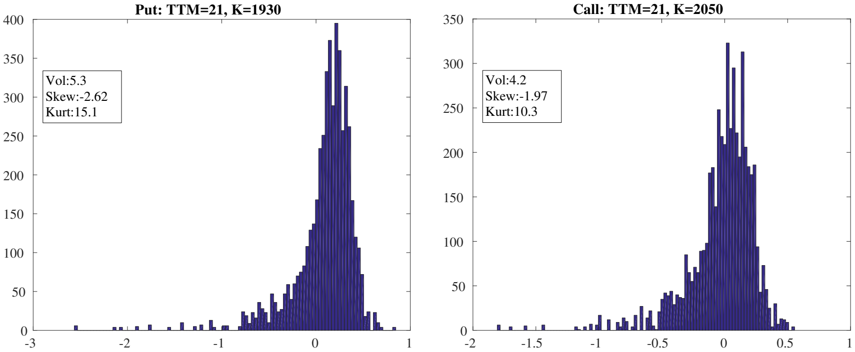
Δ-Hedged P&L to Short Option Position



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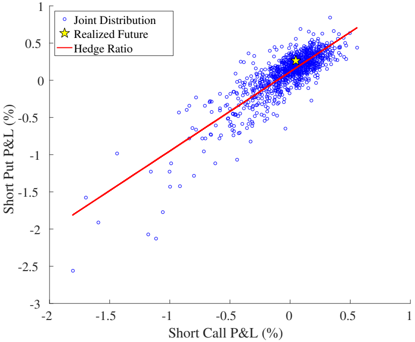

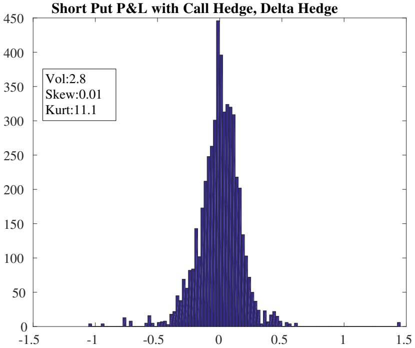

Portfolio Management: Bootstrap Hedge Ratios

Delta-Neutral Risk Reversal

- Evaluating accuracy of portfolio forecast distribution

Panel A: Mean Regressions		
	IS	OOS
Boot Mean	1.10	0.66
	(40.53)	(27.92)
R^2 (%)	24.9	16.5
T	4946	3949

Panel B: Percentiles		
Target	IS	OOS
1.0	1.0	0.9
5.0	5.3	4.5
10.0	10.5	9.2
25.0	26.4	23.9
50.0	51.6	49.6
75.0	76.8	75.0
90.0	91.3	90.0
95.0	95.8	94.9
99.0	99.4	98.9

- Dynamic bootstrap hedge ratio: Out-of-sample Sharpe 1.2 (p.a.)
- Static hedge ratio: Sharpe 0.6 (p.a.)

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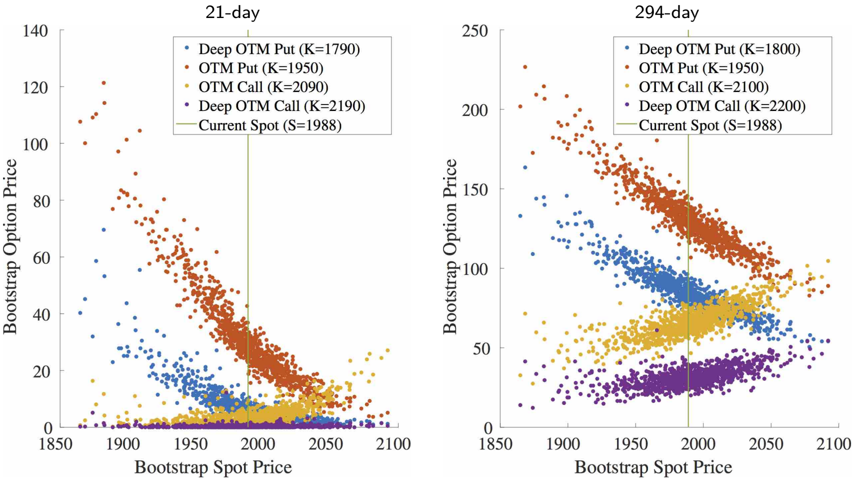


Empirical Analysis: S&P 500 Index Options

Assessing Accuracy of Distribution Forecasts

- Target distributions
  - Delta-hedged P&L of short option position

$$PL_{t+1}^{b} = \frac{1}{S_t}\left[P_t - P_{t+1}^{b} + \Delta_t (S_{t+1}^{b} - S_t)\right]$$

- Test accuracy of forecasted conditional distributions at the contract-level
  - Mean, variance, and quantile forecasts
  - Hedge ratios, optimized portfolios

- Baseline for comparison: SVJ model-based forecasts

Out-of-Sample Performance Evaluation

For each day t

- Use 1,000-day estimation sample ending at t
- Estimate model, form forecast $t + 1$ price distribution
- Form mean, std. dev., percentiles, etc., of bootstrap distribution

Forecasting P&L

	Dependent variable: Delta-hedged option P&L							
	1	2	3	4	5	6	7	8
Bootstrap		0.38		0.39		0.41		0.41
		(6.71)		(7.03)		(11.16)		(10.89)
SVJ			0.01	0.00			0.01	0.00
			(6.38)	(1.57)			(5.23)	(0.14)
Money	0.00			0.00	-0.01			-0.01
	(0.79)			-(0.28)	-(4.86)			-(2.53)
TTM	0.00			0.00	0.00			0.00
	-(1.00)			(2.57)	(0.53)			(3.84)
Gamma	0.00			0.02	-0.02			0.00
	-(0.12)			(1.78)	-(2.46)			-(0.37)
Vega	0.00			0.00	0.00			0.00
	(1.34)			-(0.51)	-(4.59)			-(2.62)
Theta	0.00			0.00	0.00			0.00
	-(0.73)			(0.60)	(0.55)			-(0.08)
IV	0.03			0.06	0.32			0.08
	(0.67)			(1.62)	(6.64)			(1.77)
Money*Put	0.00			0.00	0.03			0.01
	-(0.71)			(0.25)	(6.42)			(3.25)
TTM*Put	0.00			0.00	0.00			0.00
	(1.69)			-(2.38)	(0.41)			-(2.75)
Gamma*Put	0.00			-0.01	0.00			0.00
	(0.24)			-(0.66)	(0.40)			-(0.49)
Vega*Put	0.00			0.00	0.00			0.00
	-(1.05)			-(0.42)	(1.47)			-(0.32)
Theta*Put	0.00			0.00	0.00			0.00
	-(0.72)			-(1.78)	-(2.42)			-(0.85)
IV*Put	-0.02			-0.03	-0.15			-0.02
	-(0.60)			-(1.04)	-(5.03)			-(0.71)
Date FE	'No'	'No'	'No'	'No'	'Yes'	'Yes'	'Yes'	'Yes'
R^2	0.1	4.5	0.2	4.8	0.7	8.4	0.2	8.7
N	1157660	1157660	1157660	1157660	1157660	1157660	1157660	1157660

Note: Standard errors clustered by date. All Greek coefficients except those on Gamma are multiplied by 100

Forecasting Quantiles

- For each contract, we forecast quantiles of next-period price from the bootstrap distribution $\{P^b\}_{b=1}^B$
- $Q_x\left(\{P^b\}_{b=1}^B\right)$ is the price below which $x\%$ of the bootstrapped prices lie
- Assess accuracy of quantile forecast with

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\left[P_i \leq Q_x\left(\{P_i^b\}_{b=1}^B\right)\right]$$

  where i is a contract-day and N is number of observations
- The quantile exceedence frequency should be close to $x\%$ for a good forecast

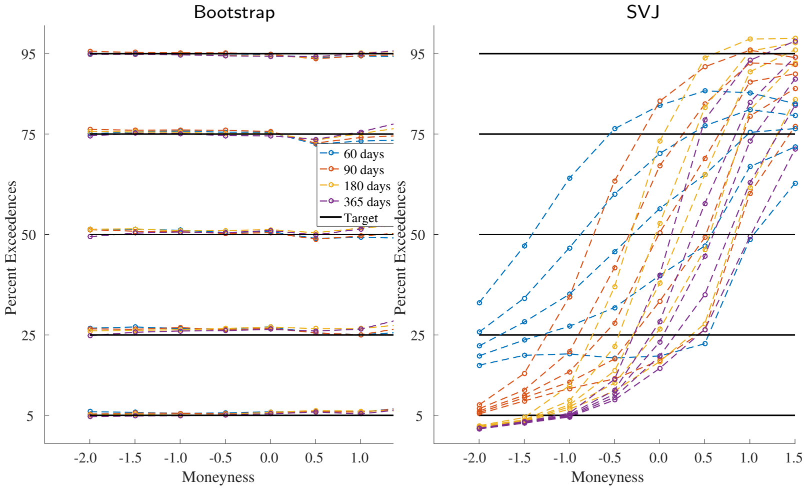
Forecasting Quantiles

- Exceedence frequency at various target quantiles
- Pooling all contract-days
- Bootstrapped statistical model, simulated SVJ model

	Out-of-sample	
Target	Bootstrap	SVJ
1.0	1.5	21.8
5.0	6.2	26.6
10.0	11.5	29.5
25.0	25.6	35.1
50.0	50.3	41.9
75.0	74.1	48.3
90.0	89.1	53.4
95.0	94.4	56.2
99.0	98.7	60.8

Forecasting Option Return Quantiles

- Exceedence frequency in moneyness/maturity bins



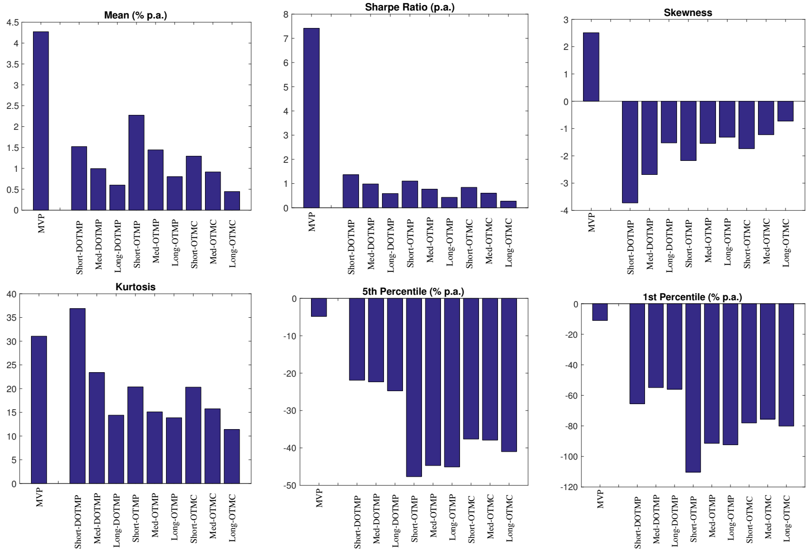
Hedging Application

Delta Hedge

- Regress $P_{i,t+1} - P_{i,t}$ on $\Delta_{i,t}(S_{t+1} - S_t)$, where Δ comes from
  1. Bootstrap
  2. Simulated SVJ
  3. Black-Scholes

	Dependent variable: Price change							
	1	2	3	4	5	6	7	8
Bootstrap	0.99			0.78	0.99			0.65
	(161.39)			(9.75)	(194.36)			(7.72)
SVJ		0.92		-0.09		0.96		0.01
		(149.87)		-(4.46)		(208.75)		(0.80)
Black-Scholes			0.98	0.31			0.98	0.33
			(164.29)	(3.69)			(201.07)	(3.93)
Date FE	'No'	'No'	'No'	'No'	'Yes'	'Yes'	'Yes'	'Yes'
R^2	91.8	82.4	89.1	91.9	95.0	92.3	94.1	95.1
N	1157660	1312973	1312973	1157660	1157660	1312973	1312973	1157660

Portfolio Choice Application



Conclusions

We propose a simple, statistically-driven means of measuring risk and return to options positions

I showed you

- Risk/return of state-dependent market exposures
- Hedging
- Portfolio optimization

We also study

- Empirical "Sharpe ratio surface"
    - Comparison with no-arb models
- Multi-horizon forecasts