Cournot Equilibrium with Free Entry

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Despite the fact that the assumptions underlying perfect competition never actually hold, the use of the competitive model, as an idealization, is justified if the predictions of the model approximate the outcomes of situations it is used to represent. In partial equilibrium analysis, this justification is embodied in the "Folk Theorem" which states that if firms are small relative to the market, then the market outcome is approximately competitive. This paper provides a precise statement and proof of the "Folk Theorem" for competitive markets with a single homogeneous good, and free entry and exit. It is shown that if firms are small relative to the market then there is a Cournot equilibrium with free entry; furthermore, any Cournot equilibrium with free entry is approximately competitive. More specifically, if we consider an appropriate sequence of markets in which firms become arbitrarily small relative to the market, then there is a Cournot equilibrium with free entry for all markets in the tail of the sequence, and aggregate equilibrium output converges to perfectly competitive output. If firms have strictly U-shaped average cost curves, then individual firm behaviour converges to competitive behaviour. The treatment of free entry distinguishes this paper from other papers dealing with the "Folk Theorem", where either the number of firms is exogenous, ruling out free entry, or free entry is treated as being equivalent to a zero profit condition, ignoring the integer problem that arises when the number of firms is finite but unspecified.

Firms may become small relative to the market in two ways: through changes in technology, absolute firm size (the smallest output at which minimum average cost is attained) may become small, or, through shifts in demand, the absolute size of the market (the market demand at competitive price) may become large. We allow both types of changes here, though shifts in demand, especially in the form of replication of the consumer sector, may be more familiar. In his conclusion, Ruffin (1971) presents a verbal argument for the "Folk Theorem" which is based on replication of demand and entry. Hart (1979), though not concerned with existence, shows that in a general equilibrium model with differentiated products and free entry, equilibria are approximately competitive (Pareto optimal) when consumers have been replicated a sufficient number of times.

The paper is organized as follows: Section 1 contains the perfectly competitive model and its assumptions, Section 2 contains the assumptions and definitions for the imperfectly competitive model, Section 3 contains an example contrasting the usual treatment of the "Folk Theorem" and the present approach, Section 4 contains the proofs of the main results, and Section 5 contains remarks on the results and indicates how some of the assumptions that are used can be weakened.

SECTION 1

The classical long run perfectly competitive model of a market for a single homogeneous good, where factor prices are constant, can be found in most textbooks which survey microeconomics at any level. All firms have identical technology, and in the long run firms
can choose plant size, and enter or leave the market. The long run perfectly competitive market result is aggregate output $X^*$ and price $F(X^*)$ ($F$ is inverse demand) such that $F(X^*) = \text{minimum long run average cost} = LRAC(y^*)$, with each active firm operating the optimally efficient size plant at output $y^*$, and earning zero profit. With constant input factor prices, long run supply is a horizontal line at price $= LRAC(y^*)$.

There are two reasons why firms must be infinitesimal in the perfectly competitive model: first, if firms produce significant output then they have an effect on price; second, if $y^*$ is significant, then long run supply at price $= LRAC(y^*)$ is the discrete set of points which are integer multiples of $y^*$, not a horizontal line, so if $X^*$ is not an integer multiple of $y^*$, long run perfectly competitive equilibrium does not exist. Both of these problems vanish when firms are infinitesimal.

The assumptions used in the long run perfectly competitive model are:

1. long run average cost is strictly U-shaped with minimum attained at $y^* \neq 0$ (non-zero in the scale of the firm);
2. there exists $X^* \in (0, \infty)$ such that inverse demand $F(X) \equiv LRAC(y^*)$ as $X \equiv X^*$;
3. (a) firms are identical and infinitesimal with respect to $X^*$,
   (b) firms maximize profit by choosing quantity viewing price as fixed,
   (c) firms are free to enter and leave the market.

Since firms are infinitesimal they have no effect on price. Thus viewing price as fixed is equivalent to viewing the aggregate output of other firms as fixed.

The long run average cost and inverse demand functions are also commonly assumed to satisfy differentiability conditions.

The long run perfectly competitive equilibrium is characterized by:

4. each active firm’s output is a profit maximizing response to price (and hence to the aggregate output of all other firms);
5. each active firm has non-negative profit;
6. no potential entrant can earn strictly positive profit by entry, assuming price (and hence the aggregate output of all other firms) is fixed.

**SECTION 2**

When firms are small but not infinitesimal ($y^*$ is small but significant) they have an effect on price, and we assume they recognize this effect, but no other substantial changes are made in the assumptions or the equilibrium properties. In this context, the “Folk Theorem” says that if $y^*$ is significant but small relative to $X^*$, then market equilibrium output exists and is approximately $X^*$. Henceforth, we deal exclusively with non-infinitesimal firms, and the terms “perfectly competitive price” and “perfectly competitive output” refer to minimum average cost and demand at price equal to minimum average cost respectively.

Let $AC$ and $C$ be long run average cost and long run cost functions respectively, and let $F$ be the inverse demand function. The first two assumptions for imperfectly competitive markets correspond to Assumptions (1) and (2) for the perfectly competitive model, with the addition of suitable differentiability conditions.

(C1) $AC$ is twice continuously differentiable on $(0, \infty)$ and strictly U-shaped, with minimum attained at $y^* \in (0, \infty)$ ($AC(y^*) \in (0, \infty)$). Also, $AC''(y^*) > 0$ and $C(0) = 0$.

(F1) There exists $X^* \in (0, \infty)$ such that $F(X) \equiv AC(y^*)$ as $X \equiv X^*$. $F$ is twice continuously differentiable on $(0, \infty)$ and downward sloping (i.e. $F'(X) < 0$ if $F(X) > 0$).
These assumptions are much stronger than necessary (see Section 5.4) but their use greatly simplifies the basic proofs of the theorems. The condition \( C(0) = 0 \) implies that firms can freely exit from the market by producing zero. Note that \( C \) need not be continuous at 0, so "set up" costs are allowed.

The measure of firm size is a natural one that is based on technology: the smallest output at which minimum average cost is attained, minimum efficient scale, \( MES \). In our partial equilibrium framework, each firm is completely described by its average cost function, so for convenience we identify the firm with its average cost function, and speak of the firm, \( A.C. \).

**Definition 1.** Let \( A.C. \) be an average cost function. Then the size of a firm, \( A.C. \), is
\[
MES(A.C.) = \inf \{ y \in (0, \infty) \mid A.C(y) \leq A.C(z), \quad \forall z \in (0, \infty) \}.
\]
If \( A.C. \) satisfies (C1) then \( MES(A.C.) = y^* > 0 \).

In order to generate a family of average cost functions (indexed by \( \alpha \)), a transformation which changes the scale of measurement of output is used. (More general families could be allowed. See Section 5.4.)

**Definition 2.** Let \( A.C. \) be an average cost function satisfying (C1). For each \( \alpha \in (0, \infty) \), the \( \alpha \)-size firm corresponding to \( A.C. \) is the firm \( A.C._\alpha \) defined by \( A.C._\alpha(y) = A.C(y/\alpha) \).

This transformation changes the scale of measurement of output by a factor of \( \alpha \)

\[
(AC(y) = A.C._\alpha(\alpha y)).
\]
With this transformation, we can assume that any basic average cost function under consideration has minimum efficient scale equal to one, and use \( \alpha \) to generate the other average cost functions with different minimum efficient scales. This assumption that minimum efficient scale equals one for basic cost functions entails no loss of generality.

(C2) \( MES(A.C.) = 1 \).

If \( A.C. \) satisfies (C1) and (C2), then an \( \alpha \)-size firm relative to \( A.C. \) has \( MES(A.C._\alpha) = \alpha \), and the use of size in Definitions 1 and 2 is consistent. Denote the cost function corresponding to \( A.C._\alpha \) by \( C_{\alpha} \), so \( C_{\alpha}(y) = \alpha C(y/\alpha) \).

The measure of market size is also natural: "perfectly competitive demand", \( D \) (which of course depends on the "perfectly competitive price" as well as the inverse demand function).

**Definition 3.** Let \( F \) be an inverse demand function and \( p \in (0, \infty) \). Then the size of a market with inverse demand \( F \) and "perfectly competitive price" \( p \) is
\[
D(F, p) = \sup \{ X \in (0, \infty) \mid F(X) \geq p \}.
\]
If \( A.C. \) satisfies (C1) and \( F \) satisfies (F1), then the "perfectly competitive price" is \( AC(y^*) \) and \( D(F, A.C(y^*)) = X^* \).

In order to generate a family of inverse demand functions, (indexed by \( \beta \)), a transformation which changes the scale of measurement of output is used. (More general families could be allowed. See Section 5.4.)

**Definition 4.** Let \( F \) be an inverse demand function satisfying (F1). For each \( \beta \in (0, \infty) \), the \( \beta \)-size market corresponding to \( F \) is the market with inverse demand function \( F_\beta \) defined by \( F_\beta(X) = F(X/\beta) \).

For \( \beta \) an integer, \( F_\beta \) corresponds to a \( \beta \)-times replication of the demand sector corresponding to \( F \). Without loss of generality, we assume the basic inverse demand function has competitive demand one and use \( \beta \) to generate other inverse demand functions.

(F2) \( D(F, A.C(y^*)) = 1 \).
If \( AC \) satisfies (C1) and \( F \) satisfies (F1) and (F2), then \( D(F_\beta, AC(y^*)) = \beta \), the "perfectly competitive demand", and the use of size in Definitions 3 and 4 is consistent.

The market under consideration is completely described by \( x, C, \beta, \) and \( F \), so we denote the market by \( (x, C, \beta, F) \). Note that in any \((x, C, \beta, F)\) market all firms, including potential entrants, have the same cost function \( C_a \).

The equilibrium concept used here has properties similar to those listed for the long run perfectly competitive equilibrium:

(i) The outputs of the active firms yield a Cournot equilibrium (without free entry), i.e. a Nash equilibrium with quantity as the strategic variable,

(ii) all firms make non-negative profit,

and

(iii) there is no profit incentive for additional firms to enter the market.

**Definition 5.** Given a cost function \( C \), an inverse demand function \( F \), and \( x, \beta \in (0, \infty) \), an \((x, C, \beta, F)\) market equilibrium with free entry is an integer \( n \) and a set \( \{y_1, ..., y_n\} \) of positive outputs such that, for

\[
X_{y_1} := \sum_{j=1}^n y_j \quad \text{and} \quad X_T := \sum_{t=1}^T y_t,
\]

(a) \( F_\beta(X_{y_1} + y_i) = C_a(y_i) \quad \forall y_i \in [0, \infty], \forall i \in \{1, 2, ..., n\} \);

and

(b) \( F_\beta(X_T + y) \leq C_a(y) \quad \forall y \in [0, \infty] \).

The set of all \((x, C, \beta, F)\) market equilibria with free entry is denoted by \( E(x, C, \beta, F) \).

Condition (a) is the Nash condition for producing firms. When firms are infinitesimal, \( y \) is infinitesimal with respect to (the integral) \( X_{y_1} \), so \( F(X_{y_1} + y) \) is a fixed price and (a) becomes the profit maximizing condition for a competitive firm. If \( C(0) = 0 \), condition (a) implies that all firms make non-negative profit since \( C_a(0) = 0 \). Condition (b), the free entry condition, requires that no potential entrant, acting alone, can achieve positive profit by entry. When firms are infinitesimal, this reduces to the competitive entry condition. Notice that the \( n \) used in the definition is endogenous, not prespecified, and given \( x, C, \beta, \) and \( F, E(x, C, \beta, F) \) may contain several equilibria, with different values of \( n \).

The main results of the paper are Theorems 1 and 2.

**Theorem 1.** Given a cost function satisfying (C1) and (C2), an inverse demand function \( F \) satisfying (F1) and (F2), and \( x, \beta \in (0, \infty) \), if \( n, \{y_1, ..., y_n\} \) is an element of \( E(x, C, \beta, F) \) then \( X_T := \sum_{t=1}^T y_t \in [\beta - \alpha, \beta] \).

Hence, if \( x \) is small relative to \( \beta \), then any equilibrium in \( E(x, C, \beta, F) \) (if one exists) yields a market output which is approximately the perfectly competitive output, \( \beta \).

**Theorem 2.** Given a cost function \( C \) satisfying (C1), (C2), and an inverse demand function \( F \) satisfying (F1) and (F2), there exists \( \kappa > 0 \) such that for all \( x, \beta \in (0, \infty) \), with \( x/\beta \leq \kappa \), \( E(x, C, \beta, F) \neq \emptyset \).

Theorems 1 and 2 provide a precise statement and proof of the "Folk Theorem": if firms are small relative to the market then there is a market equilibrium and the market output is approximately competitive.

**SECTION 3**

In this section, for a simple example with \( U \)-shaped average cost, we contrast the usual treatment of the "Folk Theorem" with the approach adopted in Section 2. The usual treatment fixes cost and demand functions, fixes the number of firms, \( n \), and finds an \( n \)-firm Cournot equilibrium. Then \( n \) is exogenously increased, so each firm's output (the
measure of size) becomes arbitrarily small. In this context, the "Folk Theorem" is valid if and only if the aggregate output of the n-firm Cournot equilibrium converges to perfectly competitive output as \( n \) becomes arbitrarily large. When average cost is U-shaped, if each n-firm equilibrium is viable (all \( n \) firms producing positive output earn non-negative profit) then the "Folk Theorem" must invariably fail, since price converges to \( AC(0+) \)—the limit of average cost as output converges to zero—in order to maintain non-negative profit for the firms whose output is becoming arbitrarily small. Treatment of the convergence of n-firm Cournot output to competitive output can be found in Frank (1965), Ruffin (1971) and Okuguchi (1973). Ruffin and Okuguchi recognize the importance of minimum average cost = \( C'(0+) \) (= \( AC(0+) \)) for \( C \) Continuous at 0 with \( C(0) = 0 \) for the validity of the "Folk Theorem" in this context.

Several extremely strong assumptions on the demand and cost functions are commonly made in order to ensure the existence of equilibrium for each \( n \). The assumption that profit functions are concave for all outputs of the individual firms, for all aggregate outputs of the other firms, is almost universal. Roberts and Sonnenschein (1977) have shown how unreasonably restrictive this assumption is.

In Example A, \( n \)-firm Cournot equilibrium fails to exist for all \( n \) greater than 1.

**Example A.**

\[
C(y) = \begin{cases} 
0 & \text{if } y = 0 \\
10 + y & \text{if } y \in (0, 1] \\
\infty & \text{if } y \in (1, \infty) 
\end{cases} \quad \text{and} \quad F(X) = \begin{cases} 
60 & \text{if } X \in [0, \frac{1}{2}] \\
14 - 2X & \text{if } X \in \left(\frac{1}{2}, 7\right] \\
0 & \text{if } X \in (7, \infty). 
\end{cases}
\]

The only Cournot equilibrium occurs when \( n \) equals 1.\(^1\) However, if entry is allowed, a second firm, assuming the output of the first firm is fixed, has profit incentive for entry. The discontinuity of \( F \), the fact that \( F' \) is not negative initially, the discontinuity of \( C \) at zero and the fact that output is bounded are not essential to the results, and the approach of Section 2 works for this example.

In contrast to the \( n \)-firm Cournot technique, the approach of Section 2 measures firm size by technology and treats the number of firms as an endogenous variable. In this context, the "Folk Theorem" is valid under very general assumptions.

Example B shows the failure of the "Folk Theorem" using the \( n \)-firm Cournot technique. Example B', using the same basic cost and demand functions, demonstrates the validity of the "Folk Theorem" using the approach of Section 2. For both examples, the non-differentiability of the cost function serves to simplify the reaction correspondences, and does not affect the nature of the results.

**Example B.** The cost function is

\[
C(y) = \begin{cases} 
\frac{3}{4}y - \frac{1}{2}y^2 & \text{if } y \in [0, 1] \\
\frac{1}{2}y + \frac{1}{4}y^2 & \text{if } y \in [1, \infty) 
\end{cases}
\]

and the inverse demand function is \( F(X) = 3 - 2X \). Let \( \{y_1, \ldots, y_n\} \) be an \( n \)-firm Cournot equilibrium, and let \( X_{ji} = \sum_{j=1, j \neq i}^n y_j \) for each \( i \). The reaction function for each firm is

\[
y_i(X_{ji}) = \begin{cases} 
\frac{3}{4}(3 - X_{ji}) & \text{if } X_{ji} \in [0, \frac{3}{4}] \\
0 & \text{if } X_{ji} \in \left(\frac{3}{4}, \infty\right) 
\end{cases}
\]

so summing over \( i \), recognizing that \( \sum_{i=1}^n X_{ji} = (n-1) \sum_{i=1}^n y_i \), and rearranging, we get

\[
\sum_{i=1}^n y_i = \frac{n}{n+\frac{3}{4}} < \frac{3}{4}.
\]

Then \( X_{ji} < \frac{3}{4} \) so \( y_i > 0 \) for all \( i \), and

\[
y_i = \frac{3}{4} \left( \frac{1}{n+\frac{1}{2}} \right) \quad \text{for all } i = 1, \ldots, n
\]
is an equilibrium, and it is the only $n$-firm equilibrium (to see that $y_i = y_j$ for all $i$, $j$, fix the output of any $n-2$ firms and notice that the only equilibrium for the two remaining firms, facing the residual demand as a Cournot duopoly, is $y_i = y_j$). As $n$ is exogenously increased, the output of each individual firm converges to 0, while aggregate output converges to $\frac{n}{2}$, not to competitive output 1. Price converges to $\frac{1}{2} = AC(0+) = C'(0+)$. Notice that for any finite $n$, price is greater than $\frac{1}{2}$, so if entry is allowed, it will take place; i.e. by producing an output sufficiently close to zero the entrant can maintain price above its average cost. Also, as $n$ increases the behavior of the firms does not converge to price taking behavior (the second order condition for a price taking profit maximizer is violated at $y_p$, and the price taking response to

$$X_H = \frac{1}{n+\frac{1}{2}} \left( \frac{n-1}{n+\frac{1}{2}} \right)$$

is always greater than 1, while $y_i$ converges to 0).

**Example B'.** The basic inverse demand and cost functions are the same as in Example B. For $\alpha, \beta \in (0, \infty)$,

$$C_\alpha(y) = \begin{cases} \frac{1}{2\alpha} y^2 & \text{if } y \in [0, \alpha] \\ \frac{1}{2\alpha} y^2 & \text{if } y \in (\alpha, \infty) \end{cases}, \quad F_\beta(X) = 3 - 2X/\beta.$$  

We show that for $\alpha/\beta \in (0, 1)$, $E(\alpha, C, \beta, F) \neq \emptyset$. In fact, $E(\alpha, C, \beta, F) \neq \emptyset$ for all $\alpha/\beta \in (0, \frac{1}{2})$, but for $\alpha/\beta \in [\frac{3}{2}, \frac{7}{3}]$ the properties of the reaction correspondence are different. This example is to serve as a preview to the proof of Theorem 2, so the proof for $\alpha \in [\frac{3}{2}, \frac{7}{3}]$ is extraneous. For $\alpha/\beta \in (\frac{3}{2}, \infty)$, $E(\alpha, C, \beta, F) = \emptyset$ since, as in example B, the free entry condition is not satisfied for any finite number of firms. Fix $\alpha, \beta$ with $\alpha/\beta < \frac{1}{2}$. The reaction correspondence of a firm with cost function $C_\alpha$, reacting to output $X$ by other firms, when inverse demand is $F_\beta$, is

$$y(X \mid \alpha, \beta) = \begin{cases} \frac{\alpha\beta}{4\alpha+\beta} (\frac{3}{2} - 2X/\beta) & \text{if } X \in \left[0, \frac{3\beta}{4} - 2\alpha\right) \\ \frac{3\beta}{4} - 2\alpha, \beta - \alpha & \text{if } X \in \left[\frac{3\beta}{4}, \beta - \alpha\right) \\ \{0, \alpha\} & \text{if } X = \beta - \alpha, \beta - \alpha \\ 0 & \text{if } X \in (\beta - \alpha, \infty). \end{cases}$$

The reaction correspondence is shown in Figure 1. There is a symmetric $n$-firm Cournot equilibrium (without free entry) if and only if the line

$$L(n) = \{(X, y) \mid (n-1)y = X\}$$

intersects the graph of the reaction correspondence. When $n$ is greater than $\beta/\alpha$, $L(n)$ goes through the jump at $X = \beta - \alpha$ and does not intersect the graph of the reaction correspondence. Let $n(\alpha, \beta) = \lfloor \beta/\alpha \rfloor$, the greatest integer less than or equal to $\beta/\alpha$. For all integers $n \leq n(\alpha, \beta)$, there is a symmetric $n$-firm Cournot equilibrium (without free entry), and in fact it is easy to show that all Cournot equilibria are symmetric.

It remains to show that at least one of these equilibria has aggregate output greater than or equal to $\beta - \alpha$, so that it is a Cournot equilibrium with free entry (from the reaction correspondence we see that if aggregate output is at least $\beta - \alpha$, then an optimal response for a potential entrant is to maintain zero output). In the $n(\alpha, \beta)$ firm equilibrium all firms produce $\alpha$, so aggregate output is $an(\alpha, \beta) = \alpha((\beta/\alpha) - 1) = \beta - \alpha$ where the inequality
follows from the definition of $n(\alpha, \beta)$. Thus, $n(\alpha, \beta), \{\alpha, \alpha, \ldots, \alpha\} \in E(\alpha, C, \beta, F)$. Notice that

(i) when $\alpha/\beta$ is small the aggregate output is near the perfectly competitive output $\beta$,

(ii) in equilibrium, each active firm's action is approximately equal to the action of a price taking firm faced with the same aggregate output by other firms,

(iii) all active firms earn strictly positive profit except when $\beta/\alpha$ is an integer. However, in that case, there is also another equilibrium with free entry, $(\beta/\alpha) - 1, \{\alpha, \alpha, \ldots, \alpha\}$ in which all active firms earn strictly positive profit.

SECTION 4

The results of Example B' are generalized in Theorems 1 and 2, the first of which shows that if $\alpha$ is small relative to $\beta$, then every element of $E(\alpha, C, \beta, F)$ yields a market output which is approximately the perfectly competitive output.

**Theorem 1.** Given a cost function $C$ satisfying (C1) and (C2), an inverse demand function $F$ satisfying (F1) and (F2), and $\alpha, \beta \in (0, \infty)$, if $n, \{y_1, \ldots, y_n\}$ is an element of $E(\alpha, C, \beta, F)$ then $X_T = \sum_{i=1}^{n} y_i \in [\beta - \alpha, \beta]$.

**Proof.** If $X_T > \beta$ then $y_i > 0$ for some $i$, and market price $F(X_T)$ is less than minimum average cost, so the firm producing $y_i > 0$ has strictly negative profit, contrary to the Nash condition and $C_a(0) = 0$.

If $X_T < \beta - \alpha$, by producing $\alpha$, an entrant changes price to $F_p(X_T + \alpha) > AC_a(\alpha)$ since $X_T + \alpha < \beta$, and earns profit $(F_p(X_T + \alpha) - AC_a(\alpha))\alpha > 0$, contrary to the free entry condition.

In order for this result to be meaningful, $E(\alpha, C, \beta, F)$ must be non-empty for $\alpha$ small relative to $\beta$. There are two ways in which $E(\alpha, C, \beta, F)$ can be empty. First, as in Example B' for $\alpha/\beta \in (4, \infty)$, no finite number of firms may be enough to remove the incentive for additional firms to enter, while $n$-firm Cournot equilibria without free entry exist for all $n$. Second, $n$-firm Cournot equilibrium may not exist for all but a finite number of $n$ values, with the free entry condition failing at those Cournot equilibria that do exist.
The second way in which \( E(\alpha, C, \beta, F) \) may be empty illustrates the integer problem that arises when free entry is allowed with non-infinitesimal firms. Most of the time, free entry has been treated as equivalent to a zero profit condition, and when firms are non-infinitesimal, the number of firms is treated as a continuous variable in order to get zero profit, after which some statement is made about rounding off the number of firms to an integer. Using that approach, equilibrium with free entry may fail to exist when the number of firms is rounded to an integer. In Example A of Section 3 the zero profit condition is satisfied with \( n = \frac{3}{4} \), but equilibrium with free entry does not exist for any \( n \), including \( n = 1 \) and \( n = 2 \).

Theorem 2 shows that if \( \alpha \) is small enough relative to \( \beta \), then both types of non-existence are overcome, and \( E(\alpha, C, \beta, F) \) is not empty.

**Theorem 2.** Given a cost function \( C \) satisfying (C1), (C2), and an inverse demand function \( F \) satisfying (F1) and (F2), there exists \( \kappa > 0 \) such that for all \( \alpha, \beta \in (0, \infty) \), with \( \alpha/\beta \leq \kappa \), \( E(\alpha, C, \beta, F) \neq \emptyset \).

In order to prove the Theorem, we show that for \( \alpha/\beta \) sufficiently small, the reaction correspondence is similar to that of Example B', at least in the interval \( [\beta - 2\alpha, \beta] \). In particular, we show that for all \( \alpha, \beta \in (0, \infty) \) with \( \alpha/\beta \) sufficiently small, there exists a unique \( X(\alpha, \beta) \in [\beta - \alpha, \beta] \) such that, for \( y(X | \alpha, \beta) \) the reaction correspondence for a firm of size \( \alpha \) in a market of size \( \beta \) when the aggregate action of other firms is \( X \),

(i) \( y(X | \alpha, \beta) = \{0\} \) for all \( X > X(\alpha, \beta) \),

(ii) \( y(X(\alpha, \beta) | \alpha, \beta) = \{0, \alpha(\beta)\} \) where \( \alpha(\beta) \) (which is defined by this condition) is non-zero and approximately equal to \( \alpha \),

(iii) \( y(X | \alpha, \beta) \) is single valued, greater than or equal to \( \alpha(\beta) \), and non-increasing in \( X \) for \( X \in [\beta - 2\alpha, X(\alpha, \beta)] \). Thus \( X(\alpha, \beta) \) and \( \alpha(\beta) \) correspond to \( \beta - \alpha \) and \( \alpha \) respectively in Example B', where for \( \alpha/\beta < \frac{1}{2} \),

\[
y(X | \alpha, \beta) = \begin{cases} 
\{0\} & X > \beta - \alpha \\
\{0, \alpha\} & X = \beta - \alpha \\
\{\alpha\} & X \in [\beta - 2\alpha, \beta - \alpha].
\end{cases}
\]

We then show that if the number of active firms, \( n(\alpha, \beta) \), is chosen in the same manner as in Example B' (i.e., as the greatest integer less than or equal to \( \{X(\alpha, \beta) + y(\alpha, \beta)\}/y(\alpha, \beta) \)) then there is an \( n(\alpha, \beta) \) firm equilibrium with free entry in \( E(\alpha, C, \beta, F) \).

**Proof of Theorem 2.**

Let \( y(X | \alpha, \beta) \) be the reaction correspondence and define \( X(\alpha, \beta) \) to be the largest \( X \) with a non-zero optimal response by the firm, and \( \alpha(\beta) \) to be the largest optimal response to \( X(\alpha, \beta) \):

\[
X(\alpha, \beta) := \sup \{X | y(X | \alpha, \beta) \neq \{0\}\},
\]

\[
\alpha(\beta) := \sup \{y | y \in y(X(\alpha, \beta) | \alpha, \beta)\}.
\]

A firm can just break even in response to \( X(\alpha, \beta) \). It is clear from the proof of Theorem 1 that \( X(\alpha, \beta) \in [\beta - \alpha, \beta] \). When \( \alpha/\beta \) is small, \( AC_\alpha \) is sharply U-shaped relative to the slope of \( F_\beta \) near \( \beta \), and if we consider a residual demand diagram as in Figure 2, we see that \( X(\alpha, \beta) \) and \( \alpha(\beta) \) are well defined and \( \alpha(\beta) \) is non-zero and approximately \( \alpha \).

By (C1) and (C2), for all \( \alpha \)

\[
C_\alpha(\alpha) = (1/\alpha)(2AC'(1) + AC''(1)) = (1/\alpha)AC''(1) > 0,
\]

so we can choose a \( \delta \in (0, 1] \) such that marginal cost is increasing \( (C''_\alpha(y) > 0) \) for all \( y \in [\alpha(1-\delta), \alpha(1+\delta)] \). Picking such a \( \delta \), and a \( Z \in (0, 1) \) such that

\[
AC(1)<F(Z)<\min \{AC(1-\delta), AC(1+\delta)\}, \text{ for any } \alpha,
\]
all optimal responses to any $X \geq Z$ when inverse demand is $F_\beta$ are either zero or in the interval $(\alpha(1-\delta), \alpha(1+\delta)) \cap [0, \beta - X]$. (If $y$ is greater than $\beta - X$ then

$$F_\beta(X+y) < AC(1) \leq AC_\alpha(y),$$

while if $0 \neq y \notin (\alpha(1-\delta), \alpha(1+\delta))$ then

$$AC_\alpha(y) > F_\beta(\beta Z) > F_\beta(X+y)$$

by downward sloping inverse demand and U-shaped average cost. In either case, $y$ cannot be an optimal response since zero profit is available for zero output.) For $\alpha/\beta < (1-Z)/2$, $\beta Z$ is less than $\beta - 2x$, and $X(\alpha, \beta)$ and $y(\alpha, \beta)$ are well defined. Clearly, by downward sloping inverse demand, zero is an optimal response to $X$ if and only if $X \geq X(\alpha, \beta)$, and zero is the unique optimal response if $X > X(\alpha, \beta)$. We are interested in the non-zero optimal responses when $X \in [\beta - 2x, X(\alpha, \beta)]$, and we have just seen that if $\alpha/\beta < (1-Z)/2$ and $X \in [\beta Z, X(\alpha, \beta)]$ and $y \in [0, \beta - X]$ then non-zero optimal responses exist and lie in an interval, $(\alpha(1-\delta), \alpha(1+\delta)) \cap [0, \beta - X]$, of increasing marginal cost.

We now show that when $\alpha/\beta$ is sufficiently small, marginal revenue is decreasing in both individual firm output, $y$, and aggregate output of other firms, $X$, when $X \in [\beta - 2x, \beta]$ and $y \in [0, \beta - X]$. Then, by the last paragraph, for each $X \in [\beta - 2x, X(\alpha, \beta)]$ there is a unique non-zero optimal response for which marginal revenue is equal to marginal cost. Let

$$\kappa = (4) \min \left\{ \min_{X \in [Z, 1]} \left\{ \frac{-F'(X)}{\max \{0, F''(X)\}} \right\}, 1 - Z \right\} > 0,$$
and let
\[ MR(y, X, \beta) := F_\beta(X + y) + yF_\beta'(X + y), \]
the marginal revenue function. For \( \alpha/\beta \leq \kappa \),
\[ X \in [\beta(1 - 2\kappa), \beta] \quad \text{and} \quad y \in [0, \beta - X] \subset [0, 2\beta \kappa] \]
so
\[ X + y \in [\beta(1 - 2\kappa), \beta] \]
\[ \partial MR/\partial y = 2F_\beta'(X + y) + yF_\beta''(X + y) \]
and
\[ \partial MR/\partial X = F_\beta'(X + y) + yF_\beta''(X + y). \]
Inverse demand is downward sloping so \( \partial MR/\partial y \leq \partial MR/\partial X \), and if \( F_\beta''(X + y) \leq 0 \), then marginal revenue is decreasing in both \( y \) and \( X \). However, if \( F_\beta''(X + y) > 0 \),
\[ \partial MR/\partial X = F_\beta'(X + y) + yF_\beta''(X + y) \leq F_\beta'(X + y) + 2\beta \kappa F_\beta''(X + y) \]
\[ = -F_\beta''(X + y) \left\{ \frac{-F_\beta'(X + y)}{F_\beta''(X + y)} - 2\beta \kappa \right\} \]
\[ = -F_\beta''(X + y) \left\{ \frac{-(1/\beta)F'(X + y)/\beta}{(1/\beta^2)F''(X + y)/\beta} - 2\beta \kappa \right\} \]
\[ = -\beta F_\beta''(X + y) \left\{ \frac{F''(X + y)/\beta}{F'(X + y)/\beta} - 2\kappa \right\} \]
\[ < 0, \]
since \( F_\beta''(X + y) > 0 \) by hypothesis and \( (X + y)/\beta \in [1 - 2\kappa, 1] \subset [Z, 1] \) so the term in brackets is strictly positive by the choice of \( \kappa \). Hence marginal revenue is decreasing in both \( y \) and \( X \) for \( \alpha/\beta \leq \kappa \), \( X \in [\beta(1 - 2\kappa), \beta] \) and \( y \in [0, \beta - X] \). Note that these \( X, y \) values include all those we are interested in since \( \beta(1 - 2\kappa) \leq \beta - 2\alpha, \kappa < (1 - Z)/2 \), and \( y \in [0, \beta - X] \) for all optimal \( y \). Marginal revenue is decreasing and marginal cost is increasing in the relevant regions, so for \( \alpha/\beta \leq \kappa \), for each \( X \in [\beta - 2\alpha, X(\alpha, \beta)] \) there is a unique non-zero optimal response, and that response satisfies the first order condition: marginal revenue equals marginal cost.

In order to show that the non-zero optimal response is non-increasing in \( X \) for \( X \in [\beta - 2\alpha, X(\alpha, \beta)] \), we can either use implicit differentiation of the first order condition to get
\[ \frac{\partial y(X \mid \alpha, \beta)}{\partial X} = -\left\{ \frac{F_\beta'(X + y) + yF_\beta''(X + y)}{2F_\beta'(X + y) + yF_\beta''(X + y) - C_\gamma(y)} \right\} \in (-1, 0), \]
or we can notice that whenever marginal revenue is decreasing in both \( y \) and \( X \), if \( X_1 < X_2 \) then the largest optimal response to \( X_2 \) is no larger than the smallest optimal response to \( X_1 \), regardless of the cost function.

To complete the proof of Theorem 2, let \( n(\alpha, \beta) := [\{X(\alpha, \beta) + y(\alpha, \beta)/y(\alpha, \beta)\}] \), the greatest integer less than or equal to \( \{X(\alpha, \beta) + y(\alpha, \beta)/y(\alpha, \beta)\} \). (Because of the transformations used to define \( AC_x \) and \( F_\beta \), \( X(\alpha, \beta) = \beta X(\alpha, \beta, 1) \) and \( y(\alpha, \beta) = \beta y(\alpha, \beta, 1) \), so \( n(\alpha, \beta) \) depends only on the ratio of \( \alpha \) to \( \beta \), as in Example B'.) We now show that with \( \kappa \) defined as above, if \( \alpha/\beta \leq \kappa \), then there is an \( n(\alpha, \beta) \) firm symmetric equilibrium in \( E(\alpha, C, \beta, F) \). This is easy to see from Figure 3. First
\[ X(\alpha, \beta) \geq \beta - \alpha \quad \text{and} \quad y(\alpha, \beta) \leq \beta - X(\alpha, \beta) \leq \alpha \]
so \( X(\alpha, \beta) - y(\alpha, \beta) \geq \beta - 2\alpha \). By definition of \( n(\alpha, \beta) \),
\[ X(\alpha, \beta) - y(\alpha, \beta) < (n(\alpha, \beta) - 1)y(\alpha, \beta) \leq X(\alpha, \beta) \]
so by the properties of the reaction correspondence for \( X \in [\beta - 2\alpha, X(\alpha, \beta)] \), the line \( \{(X, y) \mid (n(\alpha, \beta) - 1)y = X\} \) must intersect the graph of the reaction correspondence at a
5.1. It has been assumed throughout the paper that firms act non-cooperatively. Ignoring the possibility of threat strategies, any cartel must practice limit pricing because of the possibility of entry so the total industry gains from collusion are small, and converge to zero as \( \alpha \) converges to zero. Because of the problems and costs involved with collusion among a large number of firms, when \( \alpha \) is small, the gains from collusion will not justify the formation of large cartels. If a small coalition of firms acts collusively, other producing firms will generally be even better off than the coalition members, so with a large pool of producing firms, a “free rider problem” works against the formation of small coalitions. Finally, for \( \alpha \) sufficiently small, at an equilibrium, it is generally not profitable for a producing firm to act collusively with an entering firm. On the other hand, if threat strategies are allowed, cartel threats are only credible if the average cartel member has greater financial assets than an entrant, since the entrant can do at least as well as any active cartel member when the threat is carried out. Thus, the assumption of non-cooperative behavior seems justified when \( \alpha \) is small relative to \( \beta \).

5.2. If \((\alpha_j | \beta_j)_{j=1}^n\) is a sequence converging to 0 and \(n^j\), \(\{y_1^j, \ldots, y_{n^j}^j\} \in E(\alpha_j, C, \beta_j, F)\) for all \(j\), then \(n_j\) converges to \(\infty\) and \(\max_{1 \leq i \leq n_j} \{y_i^j / \beta_j\}\) converges to 0. This follows from the optimality of each firm’s response and the fact that aggregate output \(X_T \in [\beta_j - \alpha_j, \beta_j]\) for all \(j\). As the firms become technologically small with respect to the market, the endogenously determined number of operating firms becomes large, each firm’s actions converge to price taking actions, aggregate output “converges” to perfectly competitive output and price converges to perfectly competitive price. Compared to the \(n\)-firm Cournot technique, where the number of firms is exogenously increased and the output of each firm becomes small, the method used in this paper offers a much more natural interpretation of the “Folk Theorem”, and can be used to prove the “Folk Theorem” when average cost curves are U-shaped and \(n\)-firm Cournot equilibrium invariably fails to converge to the perfectly competitive equilibrium as \(n\) is increased.
Notice that in general firm profit is strictly positive at equilibrium even with free entry, because only integral numbers of firms can operate. All the equilibria constructed in the proof have strictly positive profit except in the case where

\[ n(\alpha, \beta) = \{X(\alpha, \beta) + y(\alpha, \beta)\} \mid y(\alpha, \beta) \]

(i.e. when it is not necessary to "round off" to obtain an integer), but even in that case there is another equilibrium with \(n(\alpha/\beta) - 1\) firms and positive profit for each firm.

5.3. It is clear that Cournot equilibrium with entry may exist for some large \(\alpha/\beta\). For example, if \(AC(0+)>F(0+)\) then for all \(\alpha/\beta\) sufficiently large, the average cost curve will always lie strictly above the inverse demand function, so no firm can profitably operate, and \(0, \emptyset \in E(\alpha, C, \beta, F)\). Also, for a sequence of \(\alpha/\beta\) values, the Chamberlinian type tangency will correspond to a Cournot equilibrium with entry with an integral number of firms and zero profit for each firm. This occurs when there is a non-zero

\[ y \in y(X(\alpha, \beta) \mid \alpha, \beta) \]

such that \(X(\alpha, \beta)y\) is an integer, \(n - 1\). Then

\[ n, \{y, y, \ldots, y\} \in E(\alpha, C, \beta, F) \]

(and \(n, \{s, s, \ldots, s\} \in E(\alpha, C, \beta, F)\) for all \(s > 0\)).

5.4. The assumptions used are stronger than necessary in some obvious ways: differentiability is only needed locally near \(F(1)\) and \(AC(1)\); non-differentiable kinks as in Example B' could be introduced; and capacity constraints on firm production could be allowed. If we define \(AC(0) = \lim_{y \to 0^+} AC(y)\) then we need not require that average cost be strictly U-shaped but only that \(AC(1) \leq AC(y)\) for all \(y \in [0, \infty)\), with strict inequality for \(0 \leq y < 1\). This allows flat bottomed average cost curves and multiple distinct minima (an extra step is necessary in the proof for certain of these cases). In (C1) we assumed \(AC''(1) > 0\), though for the method of proof used we need only assume that \(AC''(y)\) is non-negative for all \(y\) in some neighbourhood of one, which is violated only by non-economic curiosa.

It should also be noted that Theorem 2 is true even when average cost is always decreasing, so long as marginal cost is non-decreasing for all sufficiently large outputs (e.g. with a fixed cost plus constant marginal cost). Assumption (C2) must be dropped, and Theorem 1 must be modified to be of interest in this case since minimum average cost is not attained at any finite output.

Finally, it should be clear from the proofs that the families of average cost and inverse demand functions need not be formed in the manner used in Definitions 2 and 4. If we restrict our attention to \(\alpha \leq 1\) and \(\beta \geq 1\) then the results will hold as long as each individual function \(AC^\alpha \) for each \(\alpha, F^\beta\) for each \(\beta\) satisfies the appropriate properties, and there are "standardized limit functions" (such as \(AC^s\) and \(F^s\) where \(AC^s(y) := \lim_{x \to 0} AC(x)\) and \(F^s(x) := \lim_{\beta \to \infty} F(\beta x)\)) which also satisfy the appropriate properties.

5.5. When several cost functions are simultaneously available, results similar to Theorems 1 and 2 still hold. Given \(m\) cost functions \(C^1, C^2, \ldots, C^m\) each of which satisfies (C1), with \(MES(C^j) = y^*_j\) and \(AC^j(y^*_j) = p_j\) assume there is free entry for cost functions \(1, 2, \ldots, j\), but an upper bound \(n^*_j < \infty\) on the number of possible firms using cost function \(C^i\) for \(j < i \leq m\) by (C1) free exit applies to all cost functions). Also assume, without loss of generality, that cost functions are labelled and output is normalized so that \(p_1 \leq p_i, i = 2, 3, \ldots, j\) and \(1 = y^*_1 \leq y^*_i\) for all \(i \in \{2, 3, \ldots, j\}\) with \(p_1 = p_j\). Note that for \(i > j\) (the cost functions with restricted entry) \(p_i < p_1\) and \(y^*_i < y^*_1\) are possible.

With the obvious definitions of the market

\[(\alpha, C^1, C^2, \ldots, C^j, C^{j+1}, n^*_j, n^*_j, \ldots, C^m, n^*_m, \beta, F)\]
and Cournot equilibrium with free entry (the entry condition only applies to cost function $i > j$ if $n_i < n^*_j$), if $X^* = F^{-1}(\min_{1 \leq i \leq m} p_i)$ and $X^{**} = F^{-1}(p_1)$ then we obtain an analogue of Theorem 1: for all $\alpha/\beta$, aggregate output in any Cournot equilibrium with free entry lies in $[\beta X^{**} - \alpha, \beta X^*]$, and for $\alpha/\beta$ sufficiently small, the aggregate output lies in $[\beta X^{**} - \alpha, \beta X^{**}]$. When $p_1$ is the lowest minimum average cost, $\beta X^{**} = \beta X^*$ and we have convergence to the "perfectly competitive output", $\beta X^*$. When $p_1$ is not the lowest minimum average cost then $\beta X^{**} < \beta X^*$, and the restriction on entry for the most efficient technologies prevents convergence to the "perfectly competitive output", $\beta X^*$, as $\alpha/\beta$ becomes arbitrarily small.

If we assume $C_{ii}(y) \geq 0$ whenever $AC_i(y) < p_1$ (i.e., whenever the restricted entry technologies have average cost less than the lowest minimum average cost of the free entry technologies, the restricted technologies have non-decreasing marginal cost) then we obtain an existence theorem similar to Theorem 2. The equilibrium is constructed so that the only free entry technology used is $C_i$, and for $i > j$ all $n_i^*$ available firms are used if $p_1 < p_1^*$ or $p_1 = p_1^*$ and $y^*_j < y^*_j$ (otherwise no firms of type $i$ are used). A bit of computation shows that when $\alpha/\beta$ is sufficiently small, a Cournot equilibrium with free entry can be found, avoiding all the discontinuities in the various reaction correspondences.

6. CONCLUSION

In the partial equilibrium analysis of a market for a single homogeneous good, with constant factor prices, the "Folk Theorem" is valid under quite general assumptions. With firm size measured by technology, market size measured by perfectly competitive demand, and the number of firms endogenous, if firms are small relative to the market then Cournot equilibrium with free entry does exist, and the aggregate output is approximately perfectly competitive. The treatment of free entry recognizes that free entry is not equivalent to a zero profit condition when firms are significant and handles the integer problem that does arise with free entry.

Theorems 1 and 2 show that it is not necessary to mix the significant and infinitesimal cases in the discussion of a single perfectly competitive market in the long run. When firms are significant but small, they can be assumed to recognize their effect on price, and a Cournot equilibrium with free entry still exists. The market outcome in this equilibrium is approximately perfectly competitive, with aggregate output, price and individual firm output near perfectly competitive demand, perfectly competitive price, and efficient output respectively. This provides a justification for use of the long run perfectly competitive model, with infinitesimal firms, as an idealization of markets with free entry where firms are technologically small relative to the market.

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Conversations with Wayne Shafer led to the use of the long run perfectly competitive model in the presentation of the results, and also raised the question of possible incentives for collusion. Questions concerning the properties of equilibrium with free entry evolved out of conversations with Hugo Sonnenschein, whose comments on earlier drafts, along with the comments of Peter Hammond and a referee, greatly improved the presentation. Of course, all errors remain my own.

NOTES
1. The optimal responses are $1 - X$ for $X \in [0, 15/228]$, 1 for $X \in [15/228, 1]$, and 0 for $X \in [1, \infty)$.
2. The double tangency shown in Figure 2 is a special case. Later in the proof we show that when $\alpha/\beta$ is sufficiently small there is a unique tangency point as claimed in (ii).

REFERENCES
