VALUING THE QUALITY OPTION EMBEDDED IN FUTURES ON GERMAN GOVERNMENT BONDS

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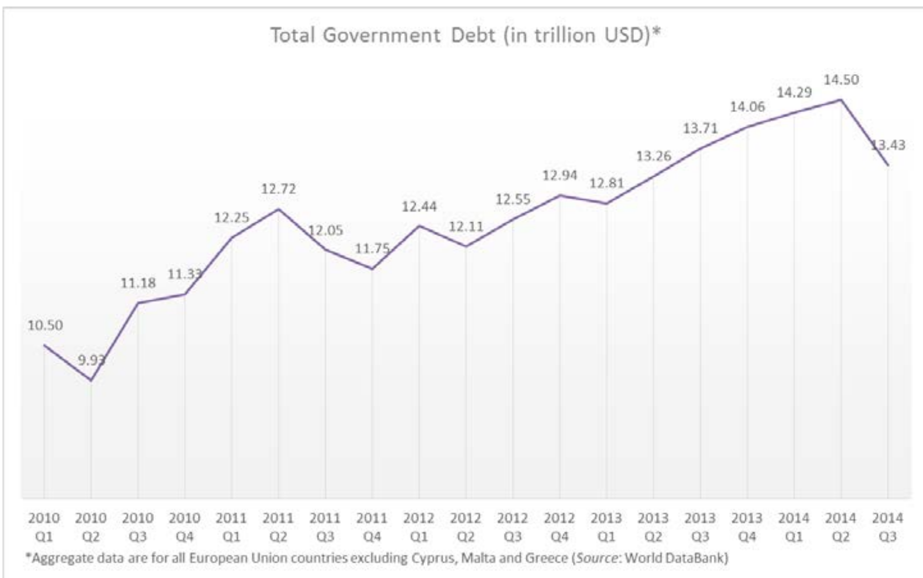
I. INTRODUCTION

The European sovereign bond market has been attracting the attention of investors around the world in the past several years due to rising debt levels in European economies. Countries such as Ireland, Portugal, Italy and, most notably, Greece have been struggling to meet their debt payments and to secure refinancing of their maturing tranches. That, however, hasn't stopped countries from using even more debt and as you can see from *Graph 1* the level of outstanding government debt has been steadily rising in past four years, reaching 13.43 trillion dollars in the third quarter of 2014.



Graph 1. Total government debt in European Union countries (in trillions USD)

The boom in sovereign bond issuance has been accompanied by an increase in the trading volume of exchange-traded interest rate derivatives. The increase has been very prominent in the years leading up to the financial crisis with the peak year being 2007, when more than 770 million futures and options contracts changed hands on the Eurex exchange[1]. The financial crisis led to an outflow of activity from the derivatives markets, but in the years after it ended we have seen some periods with interest in derivatives reaching levels close to the ones we had in 2005. The issuance of sovereign bonds also spurred new product offerings. For example, Eurex introduced futures on the Italian government bonds with short maturity in October 2010[2] and with medium maturity in September 2011[3]. The exchange also started offering futures on the long-term French bonds in April 2012[4]. Since the beginning of 2010 the total number of products offered on the Eurex exchange has doubled to a total of 18[5].

Although we have seen an increase in the volume and diversity of offered derivative contracts, to our knowledge, there hasn't been much work done on the analysis of the specifics of those contracts. In this paper we will take a look at one specific feature of the Eurex-offered futures contracts on German government bonds, namely the delivery option. The delivery option, as defined in this paper, is the option of the short side on the futures contract to deliver any bond from a pre-specified set of bonds at the time of the expiration of the futures contract. We will describe all details pertaining to the delivery option in a subsequent section of this paper.

The delivery option has been an area of academic interest ever since it was first introduced in commodity-linked futures. Different methodologies have been proposed for its pricing, which we will review in detail later on. Much of the research, however, has been focused on the option embedded in futures on U.S. Treasury bonds and almost none has been done on

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[1] *Source:* Eurex Monthly Statistics
[2] *Source*: https://www.eurexchange.com/exchange-en/resources/circulars/Short-Term-Euro-BTP-Futures--Introduction-of-Futures-Contracts-on-Short-Term-Italian-Government-Bonds/156948
[3] *Source:* http://www.futuresmag.com/2011/07/06/eurex-expanding-its-italian-bond-offering
[4] *Source:* http://www.bloomberg.com/news/articles/2012-03-21/eurex-to-offer-french-government-bond-futures-trading-next-month
[5] *Source:* Eurex Monthly Statistics

futures on sovereign bonds issued by European countries. That is an issue we believe should be addressed and we have tried to address it in the current paper.

Futures on German bonds in particular are an interesting area of study for several reasons. First, unlike futures on Treasury bonds, where there is only one contract expiration per quarter, futures on German bonds have four different contracts expiring each quarter. We believe that this fact might be materially affecting the value of the delivery option and thus we are expecting to see differences between the findings of this paper and those of previous authors, who focused their research on the option embedded in futures on U.S. Treasury bonds.

Second, in the futures on U.S. Treasury bonds there are multiple other embedded options alongside the delivery option such as a timing option, a wild-card option and an end-of-month option. The timing option refers to the ability of the short side on the futures contract to deliver any of the pre-defined set of bonds on any business day during the delivery month. The wild card option refers to the fact that during a certain period of time in each delivery month the spot market for U.S. Treasuries remains open after the futures market has closed for the day. The short side on the futures contract benefits from this additional time as the price changes of the underlying Treasury bonds are not reflected immediately in the price of the futures. The end-of-month option refers to the ability of the short side to choose the bond it wants to deliver in a seven-day period succeeding the last trading day for the futures contract. None of these options is relevant for the futures contracts on German bonds as will become clear when we review the specifications of those contracts in a section further in our analysis[6].

[6] Options other than the delivery option do exist for the futures on German bonds, but they are likely to be of very small value for reasons explained in the section outlining the specifications of those futures contracts

Third, we decided to focus our attention on futures contracts on German bonds instead of similar contracts on other European countries' sovereign debt because of the large trading volume of the former. Three of the four futures contracts on German bonds have been the most actively traded fixed income derivatives on Eurex for 2014[7]. In January of 2015 the trading volume in the active futures on the four futures contracts on German bonds represented approximately 73% of the total trading volume in all fixed income derivatives[8]. We believe that, given the heightened interest in the futures on German bonds, it is appropriate to conduct a proper analysis of the delivery option embedded in those contracts.

Lastly our decision was motivated by some developments in the investment industry. In the light of the recent European sovereign debt crisis, demand for hedging has increased among investors trying to reduce their exposure to the increased default risk of the affected European countries. Some of those investors have chosen to use futures contracts to construct their hedging positions. This increased demand for hedging has prompted us to ask the question whether futures hedging positions are effective enough in insuring against default risk, if they are ignoring the value of the delivery option. So, we embarked on a research project to estimate the value of the delivery option embedded in German bond futures contracts to see whether its value is large enough to be considered in the construction of hedging portfolios.

We have decided to focus our valuation efforts on the delivery option embedded in futures on German bonds for reasons mentioned earlier, but also because of lack of sufficient historical data to conduct a thorough analysis of futures contracts on sovereign bonds of troubled European countries. Italy is an example of a country, which has been hit hard by the sovereign

[7] *Source:* Eurex Monthly Statistics
[8] *Source:* Eurex Monthly Statistics, January 2015

debt crisis and which has actively traded government bond futures on the Eurex exchange. Unfortunately, those futures contracts were offered initially in 2010, which gives us only several years of relevant data. We do not believe that this is enough in order for a thorough analysis to be performed. But we do believed that the findings of our analysis of the delivery option embedded in futures contracts on German bonds could be extended to the futures contracts on Italian bonds. For that reason we have decided to focus our attention on the futures on German government bonds with the potential to apply our findings and extend our research to the futures on Italian bonds at a later stage.

II. GERMAN GOVERNMENT BOND CONTRACT SPECIFICATION[9]

Futures on German government bonds are trading on the Eurex exchange. As we mentioned earlier, there are four different contracts expiring each quarter. The first one in the Euro-Schatz Futures (Schatz futures), which allows for delivery of bonds with time until maturity of between 1.75 and 2.25 years. The second is the Euro-Bobl Futures (Bobl futures), which allows for delivery of bonds with time until maturity of between 4.5 and 5.5 years. The third is the Euro-Bund Futures (Bund futures) with deliverable securities having between 8.5 and 10.5 years until maturity. The fourth is the Euro-Buxl® Futures (Buxl futures) and the deliverable securities under the contract are all bonds with time until maturity between 24 and 35 years. The allowed maturities for the deliverable bonds under each of these contracts is one of the few things that distinguishes those contracts from one another. The only other is the minimum price change in value and percent terms.

[9] All information contained in this section is available on http://www.eurexchange.com/exchange-en/

The value of each futures contract is €100,000. The basket of deliverable securities is defined as the set of all securities that could be used to satisfy the obligation of the short side on the futures contract. The futures contract length is up to 9 months – the three nearest quarterly months of March, June, September and December. The delivery day on the contract is set to be the tenth day of each of the delivery months (March, June, September and December) if it is a business day. Otherwise, the first business day after that. The last trading day of the futures contract is two days prior to the delivery day and the final settlement price is set to equal the average price from a brief period of time before the futures trading ends, which is 12:30 CET.

The short side on the futures contract is obligated to notify Eurex on the last trading day which instrument it will deliver. The deadline, however, spans outside of the last trading hours on the contract, which gives the short an end-of-month option, similar to the one the short side on futures of U.S. Treasury bonds has. Here, however, the duration of the option is limited to several hours, while in the case of U.S. Treasury futures it has a duration of seven days.

The daily settlement price of the contract is determined by an average of all the prices in the last trading minute of the day. Trading in the futures contracts ends at 17:15 CET. The trading in the underlying securities, though, extends until 17:30 CET. This gives the short side on the futures contract an option similar to the wild card option embedded in futures on Treasury bonds. Here, again, the option is probably of no material value since it expires in only 15 minutes.

To make each of the deliverable bonds comparable to one another at expiration their prices have to be adjusted using a conversion factor. This conversion factor is set by Eurex and is not updated for the duration of the contract, which gives rise to the delivery option, known also as the quality option. As part of the process to make bonds comparable to one another, the price of each deliverable bond is also adjusted for all accrued interest for the period until the futures expires.

Before going on with our analysis, we believed it is appropriate to define one more term – the cheapest-to-deliver bond. This is the bond that, given its conversion factor and accrued interest, is the best deliverable option for the short side at some particular moment.

III. PREVIOUS WORK ON THE QUALITY OPTION

As we mentioned earlier, the delivery option has been the subject of extensive research in the past, ever since it was first introduced as part of the commodity-linked futures. Garbade and Silber (1983), for example, conclude that the multiple grades that the short side on the futures could potentially deliver, leave the parties involved in the contract subject to basis risk. The approach that the authors use for their analysis is a Monte Carlo simulation. Other authors have also tried using the Monte Carlo simulation method to estimate the value of the delivery option embedded in futures on Treasury bonds. One such notable example is the work of Kane and Marcus (1986a).The authors calculated the value of the option at the expiration of the futures contract. Using simulated payoffs and estimated yield curve from data for the period 1978-1982, the authors run a simulation assuming 10,000 realizations per month and conclude that the value of the delivery option is somewhere between 1.39% and 4.60% of par[10]. Kane and Marcus also state that a hedging strategy, which ignores the effects of the delivery option will result in a suboptimal behavior. Chance and Hemler (1993) later offer some critique to Kane and Marcus (1986a) results, claiming that the relatively high value for the delivery option that the latter have

[10] Going forward in this paper, the value of the delivery (quality) option will be expressed in percentage points of par in order to make results across different studies comparable

estimated is a result of the simulated payoff approach and the assumed 10,000 realizations per expiry. Also, Chance and Hemler point out that the high value of the quality option can be partially attributed to the yield curve data, which is obtained from a period characterized by high level of interest rate volatility. In one of his earlier papers Hemler (1990) also lists several reasons for the high estimates of Kane and Marcus (1986a), including the selected data sources by the authors and the estimation process, which prices the option at expiration of the futures contract. Hemler (1990) concludes that the most likely reason for the results of Kane and Marcus (1986a) is the authors' assumptions about the term structure of interest rates. LaBarge (1988) later replicates the authors' study and comes up with values for the option that are lower than those in their original paper.

One of the first purely theoretical framework developed for valuing the delivery option was introduced by Margrabe (1978). The author developed a model to price an option to exchange one asset for another, which operates under the assumption of market efficiency. The model is a generalized version of the Black-Scholes option pricing formula, in which the riskless asset is substituted with a risky one. As the Black-Scholes model does, Margrabe (1978)'s model assumes a diffusion process for the underlying asset. Also, the model assumes that the option function is linear homogenous in the prices of both assets. Margrabe (1978)'s work, however, was not focused on valuing the delivery option, but rather on valuing common financial agreements such as performance incentive fees, general margin accounts, exchange offers and standby commitments. His study, however, was applied and extended by other authors, who did research in the delivery option domain. Margrabe (1978)'s original research ended up being generalized to accommodate more than two assets, making it more suitable for pricing the delivery option embedded in futures contracts on government bonds.

One application of Margrabe (1978)'s model is proposed by Gay and Manaster (1984), whose work was focused on estimating the delivery option embedded in wheat futures. The model they proposed offered a closed-form solution for the value of the option and assumed it followed a Weiner process. The authors concluded that the value of the delivery option is somewhere between 2.2% and 5.2% of par, depending on whether the equation is used to value an out-of-the-money or in-the-money option. Since the estimated values are so high, some of the assumptions that the model relies on are worth mentioning. First, the model does not allow for convenience yields and thus cannot explain inverted markets. Second, it assumes non-stochastic process for the interest rates. Third, the location option (i.e. the ability of the short position to choose among a set of permissible locations to deliver the underlying wheat) is assumed to have no value. Fourth, it relies only on bid prices and assumes no transaction costs. The data that the authors use might also be the reason for the high estimates of the delivery option as the spot and futures prices used as inputs in the pricing formula are nonsynchronous.

Another application of Margrabe (1978)'s model was proposed by Hemler (1990), who generalized it for three assets. As noted by the author, however, the further extension of the model to accommodate more than three assets involves multi-dimensional integrals and calculation of covariance matrices, which makes the computation of the delivery option quite challenging. Using Margrabe (1978)'s approach, Hemler (1990) estimated values of the delivery option of between 0.7% and 1.2%. In his paper Hemler (1990) also used two other approaches of estimating the delivery option. The first one of these approaches represents a replication strategy that involves the investor switching between the bond, which was cheapest-to-deliver three months prior to the expiry of the futures contract and the bond, which was cheapest-to-deliver at the time of the actual delivery on the futures contract. This approach for pricing the delivery option results in values of around 0.3% of par. There is one shortcoming of this approach, however – it sometimes produces negative values for the delivery option. The author explains the negative numbers with the noisy market data, but it is difficult to test whether that is the reason for the results. The second approach Hemler (1990) uses is based on the T-bond futures pricing formula and assumes that futures prices and forward prices are equal. In addition, the approach assumes that financial markets are efficient and that all futures contracts expire on the same business day of the month. Under this approach Hemler (1990) concludes that the value of the quality option is 0.2% three months prior to delivery. Because in reality future and forward prices differ, the last estimate might be considered an upward bound for the delivery option.

Hemler (1990) is not the only author to use a replication-based approach for pricing the delivery option embedded in futures on Treasury bonds. Hegde (1989) uses a similar method and comes up with a value for the delivery option of 2.1%. Hegde (1989)'s approach is based on a continuously rolled over forward position in the cheapest-to-deliver bond at any point during the life of the futures contract. The value of the option is calculated as the sum of the profits from such strategy. The author also calculates the average number of rollovers between the different deliverable bonds and it is seven. That goes to show that the cheapest-to-deliver bond changes relatively frequently before the futures finally expires. In his paper Hegde (1989) also uses a buy-and-hold strategy for pricing the quality option. The result of that strategy is an ex-post valuation of the quality option, which makes the approach comparable to the one employed by Kane and Marcus. The buy-and-hold strategy yields much lower values than the ones Hegde (1989) gets from the previously described method – around 0.5%. A third pricing approach, proposed by the author is an ex ante valuation, which estimates the option from the difference between the price of a forward on the current cheapest-to-deliver bond and the conversion factor times the futures price. This approach yields results similar to the buy-and-hold strategy.

Here, again, it is appropriate to mention some of Hegde (1989)'s assumptions underpinning his proposed pricing methods. The author assumes marker efficiency with no transaction costs, no taxes and no margin requirements. He also assumes non-stochastic evolution of interest rates and thus equates the price of the forward contracts to that of the futures contracts. Both futures and forwards require instantaneous payment at the last trading day. Hegde (1989), much like Gay and Manaster (1984), admits to the limitations of the data he is using in his analysis. His spot and futures prices are nonsynchronous. The spot prices are also of bad quality and the author is forced to approximate the switching strategy with weekly data. Some of Hegde (1989)'s estimates for the delivery option turn out to be negative, much like those of Hemler (1990). The author is blaming the noisy market data for the results, but, this remains difficult to test.

We mentioned earlier the critique of Chance and Hemler (1993) on the results of Kane and Marcus (1986a), but we didn't mention that the authors also offered an overview of other work done on the valuation of the quality option. Particularly interesting is their overview of the research conducted by Barnhill and Seale (1988), who, like Hemler (1990) and Hegde (1989), are using a dynamic trading strategy to price the quality option. The strategy involves holding a long cash and short futures position with continuous switching of the cash position into the cheapest-to-deliver bond issue. The authors go one step further than just examining the distribution of profits from that strategy. They try to establish any relationship between those profits on one hand and the transaction costs and hurdle rates on the other. In this case hurdle rates are defined as predefined values at which the holders of the long cash/short futures position decide to switch between the different bonds. Barnhill and Seale (1988) find out that a strategy, which allows for more frequent switches between the bonds has a higher expected value than a strategy designed to make less frequent switches. The authors also find out that the quality option is positively related to the interest rate volatility and transaction costs.

Chance and Hemler (1993) also summarize the findings of Livingston (1987), who claims that in a perfect market with infinite deliverable securities and no marking-to-market, a hedge exists, which will drive the value of the quality option towards zero. The findings have been met with criticism from authors such as Barnhill (1988) and Kane and Marcus (1988), who identified some problems with Livingston's reasoning. Barnhill (1988) noted that under Livingston's assumptions the futures prices will never reach an equilibrium and Kane and Marcus (1988) claimed that the author's hedging strategy ignores a fundamental fact related to diffusion processes.

So far we mentioned research, focused primarily on the pricing of the quality option embedded in futures contracts. Some of the previously mentioned authors, however, have also tried to account for the various other options, which the short side on the contract has. Hegde (1989), for example, pays a special attention to the wild-card option and the end-of-month option in his research. He notes that the timing and delivery options are embedded in the wild card as well as end-of-month options and that the wild card and end-of-month options exist outside of the futures trading hours. Kane and Marcus (1986) focus their attention on the wild card and come up with a value of 0.2% for it at the beginning of the delivery month, which decays to zero by the end of the month. Gay and Manaster (1986) examine both the end-of-month and wild card options and claim that an optimal delivery strategy exists, such that when exercised it can result in positive economic profits. They also find out that the behavior of market participants differs significantly from the proposed optimal strategy.

The work of Boyle (1989) on the timing option is particularly interesting as he investigates how the latter interacts with the quality option. The author uses order statistics and looks for the expected value of the lowest order statistics. He starts by assuming that a forward contract is equivalent to holding a long call position and a short put position. He further assumes that the short side on the futures contract can choose any date in an interval of time to exercise its timing option. As some of the authors we mentioned earlier, Boyle (1989) uses a non-stochastic approach to interest rates, assuming equality between forward and futures prices. The author also assumes lognormal distribution for asset prices and same standard deviation for all of them. Using this methodology Boyle (1989) performs simulations on hypothetical instruments and reaches to some interesting conclusions. He finds out that the value of the quality option is significant even when the correlation between the deliverable assets is close to one. He also concludes that the value of the quality option increases as the number of deliverable securities increases. Boyle (1989) comes up with a separate value for the timing option and concludes that it is very small, but, at the same time, he claims that the quality option and the timing option interact with each other. The value of the timing option is enhanced when the correlation between the deliverable assets declines.

Although the findings of Boyle (1989) suggest that the value of the timing option is small, a recent paper by Hranaiova, Jarrow and Tomek (2005) seems to confirm Boyles (1989) observations that there is an interaction between the timing and the quality option. According to Hranaiova, Jarrow and Tomek (2005) the interaction diminishes the options' combined effect. These findings prompted us to focus our attention on the quality option embedded in futures on

German bonds. Since the other options embedded in futures contract are of negligible value, we can avoid any problems related to separating the value of the quality option from those other options or worrying about the possible interaction among them and the quality option[11]. Our study, however, is not the first one to try to isolate the value of the quality option from the influence of other options. Similar research has been performed by Lin, Chen and Chou (1999) on the futures on Japanese government bonds. Futures on Japanese bonds have only one option embedded in them – the quality option. The authors of the paper use a single-factor Hull White model to price that quality option. Their model evaluates the quality option starting from 13 weeks before the expiration of the futures contract to one week before expiry. The average value of the option the authors come up with is 0.021%, much lower than the previous estimates we mentioned. We will take a closer look at this study later on in the paper.

Our study is also not original in focusing on the futures on German bonds. For example, Balbas and Reichardt (2006) used a static replication method to price the quality option embedded in German Bund futures. The authors come up with a closed-form solution for the quality option and manage to incorporate transaction costs and coupon payments in their equation. Balbas and Reichardt (2006) conclude that the value of the quality option three months prior to expiry of the futures contract is between 1.9% and 2.8%. We will review the authors' results in a later section of this paper. We will also compare those results to ours

The current paper will try to build on the aforementioned research in several directions. First, we will focus our study not just on the futures on the German Bund, but also on the German Schatz, German Bobl and German Buxl. This will allow us to see whether the time to

[11] Also there is no timing option in the futures contracts on German bonds. There are only two other options – the wild card option and the end-of-month option

maturity of the basket of deliverable securities affects the value of the option. It will also allow us to see whether the option values embedded in the futures on the different German bonds exhibit different sensitivity to changes in the parameters of the model we have selected to use as well as to the macroeconomic environment. Our research will also span a period of almost eleven years (March 1999 – December 2009), which is much longer than the periods in both Balbas and Reichardt (2006) and Lin, Chen and Chou (1999). We have selected a longer period to analyze because we wanted to see how the changing macroeconomic environment is affecting the value of the quality option. We were particularly interested in the shifts of the yield curve, the general level of interest rates and the interest rate volatility. We also wanted to see how the general decline in the number of deliverable securities for each futures contract, which is observed during the period is reflected in the value of the quality option in order to relate our findings to those of Boyle (1989).

Our reasons for the selection of a model to perform the option valuation were three-fold: first, we wanted to choose a model different from the one selected by Balbas and Reichardt (2006), so we can see if we can reach to similar values for the quality option; second, we wanted to choose a model, which reflects accurately the reality and has proven to be accurate in the pricing of derivative instruments; third, we wanted to select a model, which we can relate to the Hull-White model used by Lin, Chen and Chou (1999), so we can see whether we can get similar values for the quality option for futures with similar contract specifications. We will describe the model we thought best fitted these three criteria in the next section.

With our research we wanted to achieve another goal as well – build the foundation for further analysis of the futures on European government bonds. An area of further study might be the estimation of the quality option embedded in French, Italian and Swiss bonds. The analysis could also be extended to cover the embedded options in futures of U.K. government bonds. Those futures, like U.S. Treasuries, have multiple options embedded in them and we can use our estimates of the quality option in futures on German bonds to isolate the values of those other options. Not only that, we can also try to look for any interactions between the different embedded options like the ones Boyle (1989) and Hranaiova, Jarrow and Tomek (2005) suggested.

IV. METHODOLOGY

We wanted the methodology for our analysis to fulfill the aforementioned three criteria and we also wanted to avoid some limitations of models previous authors have used. For example, most replication models that we mentioned assume implicitly that markets are efficient. They also do not allow for stochastic evolution of interest rates, which, we believe, do not correspond to the observed reality. The method originally proposed by Margrabe (1978) looks quite appealing, but it becomes quite complicated to apply in reality when the deliverable assets are more than three, as noted by Hemler (1990). Some of the described methods, such as the ones proposed by Kane and Marcus (1986a) and Hegde (1989) value the option only at the expiration of the futures contract, which makes its pricing at an earlier period problematic. To avoid these limitations and to achieve the three goals we set for ourselves earlier on, we have decided to use the Hull-White model to price the quality option embedded in futures on German government bonds.

The model was originally proposed by Hull and White (1990a) and represents a one-factor Markov model. In its original version the model was designed to price any security dependent on a single-state variable. Later on, the authors in Hull and White (1990b) adopted their approach for pricing interest rate derivatives. The authors claim that sometimes market expectations of future interest rates are time-dependent, which can be a result of economic cyclicality, expectation of monetary policy or expected trends in other macroeconomic variables. As a result, Hull and White proposed the incorporation of time-dependent drift to be included in the interest rate process with the reversion rate and volatility also made dependent on time. Market price of risk is defined as a function of time as well. The resulting model is essentially an extension to the model proposed by Vasicek (1977).

As noted by Lin, Chen and Chou (1999) the model with time-dependent parameters provides flexibility, but it gives no closed-form solutions for bond prices and relies on numerical procedures, because it involves integrals of unknown function. One version of the model that the authors consider more analytically tractable is the one they use in their analysis of the quality option embedded in futures of Japanese bonds. It assumes that the mean reversion factor and the volatility parameter are constant and has the form:

$$dr = (\theta(t) - ar)dt + \sigma dz \qquad (1)$$

where r is the short-term rate, z is a Weiner process, $\theta(t)$ is a time-dependent drift, a is a mean-reversion parameter and σ is a volatility parameter. This simplified version of the model was also considered by Hull and White (1990b). The authors mention that a constant volatility parameter might be more appropriate as choosing a time-dependent one results in a better fit to the current term structure, but may not give reasonable values for future short-rate volatility.

The Hull-White model has a lot of positives, including the fact that it can be used to compare results to other models by fitting the current structure of interest rates, the instantaneous short rate and (if we choose a time-dependent volatility parameter) the current structure of interest rate volatilities. The model is also a version of the so called explicit finite difference approaches of pricing derivative securities, which Hull and White (1990a) consider to be easier to implement and conceptually simpler than the implicit approaches. The approach, however, has one major disadvantage – like all single-factor models, it assumes perfect correlation between the returns of the deliverable assets. As noted by Ritchken and Sankarasubramanian (1995) this leads to results, which underestimate the true value of the quality option. Lin, Chen and Chou (1999) also admit that the model they use has this shortcoming and also note that single-factor models fail to capture the curvature of the yield curve. Naturally, some authors, such as Ritchken and Sankarasubramanian (1995), have proposed two-factor models in order to resolve these issues. Hull and White (1990b) have compared their single-factor model to multi-factor models and have reached to some interesting conclusions. First, the authors reveal that two-factor models have the flaw that they can't price options on coupon-bearing bonds. They also compare their model to the two-factor model proposed by Cox, Ingersoll and Ross (1985) and conclude that both models are producing results, which are not dramatically different from one another. Hull and White (1990b) also defend their model as computationally more efficient, which Lin, Chen and Chou (1999) also agree with.

One other disadvantage of the model, noted by Lin, Chen and Chou (1999) is that it produces negative interest rates. Indeed, some of the trinomial trees we constructed for the analyzed period result in probable negative interest rate environments. This disadvantage, however, might actually turn out to be an advantage of the model. The reason is that in recent periods we have actually observed negative interest rates for the German government bonds, which means that the Hull-White model is capturing better the reality than other models that do not allow for negative interest rates.

Given the analysis of benefits and shortcomings above, we believe that the Hull-White is not stretching too far away from reality, at the same time it is computationally efficient and, more importantly, it fulfills our initial criteria, so we have decided it is the best fit for the analysis we are planning to conduct.

For the construction of our model we will follow the approach in Lin, Chen and Chou (1999) and use the numerical procedure developed by Hull and White (1993). The methodology involves the construction of a trinomial tree in order to approximate the single-factor model proposed by Hull and White. The short rate *r* is defined as "the continuously compounded yield on a discount bond maturing in time Δt"[12] with Δt defined as the time period between two nodes of the trinomial tree. The values of r at each node of the tree are generated by the formula:

$$r = r_0 + j\Delta r \qquad (2)$$

where r_0 is the current instantaneous short-term rate and *j* is an integer from a pre-defined range of values, which we will discuss later. As Hull and White (1994) point out Δr is the size of the interest rate step, which is determined by the variance of $r(t + \Delta t) - r(t)$, denoted here as *V*. In a later paper[13] the authors reach to the following equation for *V*:

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[12] Hull, J. & White, A. (1993). One-factor interest rate models and valuation of interest-rate derivative securities. *Journal of Financial and Quantitative Analysis*, 28, 235-254

[13] Hull, J. & White, A. (1996). Using Hull-White interest rate trees, *Journal of Derivatives*, 3, 26-36

$$V = \frac{\sigma^2(1-e^{-2a\Delta t})}{2a} \quad (3)$$

Hull and White propose an interest rate step Δr satisfying the equation:

$$\Delta r = \sqrt{3V} \quad (4)$$

Next we define the expected change in the change of r as $E(dr) = Mr$. Here M is given by the equation:

$$M = e^{-a\Delta t} - 1 \quad (5)$$

The parameter M we can use for the calculation of the range for j and for estimating the probabilities of going up, down or straight in the trinomial tree. The maximum value for j is given by the equation (rounding up to the nearest integer)[14]:

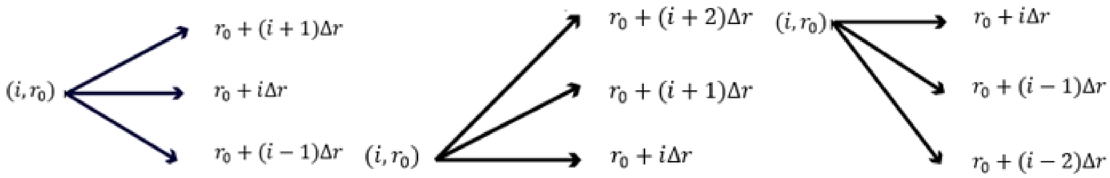
$$j_{max} = -0.184/M \quad (6)$$

The minimum value of j is just the maximum, taken with a negative sign.

The maximum and minimum values for j are important, because they show where the upward and downward branching process of the trinomial tree stop. This bounds are set to reflect the mean-reversion essence of the Hull-White model. Let's assume that we are situated at a node

[14] Hull and White have used this equation to find appropriate bounds for j in Hull, J. & White, A. (1996). Using Hull-White interest rate trees, *Journal of Derivatives*, 3, 26-36

of the trinomial tree where $j = i$. Going forward the branching process of the trinomial tree assumes three distinct shapes, depending on the value of i. If i is between j_{min} and j_{max} then the branching process is the same at the one presented in the first panel of *Graph 2*. If i is equal to j_{min} then the branching process is equivalent to the second panel of *Graph 2* and, finally, if i is equal to j_{max} the branching process is equivalent to the branching process in the third panel of *Graph 2*.



Graph 2. Branching process for Hull-White trinomial trees

The probabilities of going to the upper node, middle node or the lower node also depend on the branching process. If the branching process is like the one in the first panel of *Graph 2,* the equations are:

$$p_u = \frac{1}{6} + \frac{i^2M^2 + iM}{2} \quad (7)$$

$$p_m = \frac{2}{3} - i^2M^2 \quad (8)$$

$$p_d = \frac{1}{6} + \frac{i^2M^2 - iM}{2} \quad (9)$$

where p_u is the probability of going to the upper node, p_m is the probability of going to the middle node and p_d is the probability of going to the lower node. If the branching process is like the one in the second panel of *Graph 2* then the equations for the probabilities are:

$$p_u = \frac{1}{6} + \frac{i^2M^2 - iM}{2} \quad (10)$$

$$p_m = -\frac{1}{3} - i^2M^2 + 2iM \quad (11)$$

$$p_d = \frac{7}{6} + \frac{i^2M^2 - 3iM}{2} \quad (12)$$

And finally, if the branching process is like the one in the third panel of *Graph 2*, then the equations are:

$$p_u = \frac{7}{6} + \frac{i^2M^2 + 3iM}{2} \quad (13)$$

$$p_m = -\frac{1}{3} - i^2M^2 - 2iM \quad (14)$$

$$p_d = \frac{1}{6} + \frac{i^2M^2 + iM}{2} \quad (15)$$

To set up the trinomial tree Hull and White propose first to set $\theta(t)$ equal to 0, so that our initial equation becomes:

$$dr = -ardt + \sigma dz \quad (16)$$

Using this equation and the estimated values for Δr we can construct the first approximation of our tree. Then the values of r at each node are adjusted by a certain value, so that they are consistent with the current term structures of interest rates.

Once the tree has been constructed, the prices of all deliverable bonds have to be estimated at each of the terminal nodes of the tree. These values are obtained using the equation[15]:

$$P(t,T) = A(t,T)e^{-B(t,T)r_t} \quad (17)$$

where $P(t,T)$ is the price at some time t of a bond maturing at time T and r_t is the short-term rate at time t. The functions $A(t,T)$ and $B(t,T)$ are given by the equations:

$$A(t,T) = \frac{P(0,T)}{P(0,t)} e^{[B(t,T)F(0,t) - \frac{\sigma^2 B(t,T)(1-e^{-2at})}{4a}]} \quad (18)$$

$$B(t,T) = \frac{1-e^{-a(T-t)}}{a} \quad (19)$$

$F(0,t)$ is the instantaneous forward rate that applies to time t as observed from time 0. It can be calculated using the formula:

$$F(0,t) = -\frac{\delta \log[P(0,t)]}{\delta t} \quad (20)$$

The short rate r_t used to calculate the price of a bond at the terminal node of the tree is different from the interest rates estimated at each node of the Hull-White trees. Those rates are relating to periods with length Δt, while r_t is the instantaneous short-term rate. If we denote the

[15] The equation is taken from Hull, J. & White, A. (1996). Using Hull-White interest rate trees, *Journal of Derivatives*, 3, 26-36

rates at each of the nodes by R, then Hull and White (1996) link R and r_t by the following equation:

$$r_t = \frac{R\Delta t + logA(t,t+\Delta t)}{B(t,t+\Delta t)} \qquad (21)$$

Using equations (17) – (21) we can calculate the value of a discount bond at each of the terminal nodes[16] of the previously constructed tree. The value of a coupon-bearing bond can be obtained in a similar way by treating it as a series of cash flows. The total value of the bond is the sum of the values of all these cash flows at the terminal nodes.

Now that we have the values of the each of the deliverable bonds at the terminal nodes of the tree, we can adjust those prices for accrued interest and use their respective conversion factors to make the bonds comparable to one another. By doing these adjustments we come up with hypothetical values of futures contracts that allow for only one deliverable asset at the terminal nodes of the trinomial tree. Lin, Chen and Chou (1999) call these equivalent futures price and we will use their terminology here. This equivalent futures price at each node of the tree can be calculated using the equation:

$$EF_{i,k}(t) = \frac{P_{i,k}(t,T) - AI_i(t)}{CF_i} \qquad (22)$$

where $EF_{i,k}(t)$ is the equivalent futures price for bond k at a certain node of the tree i at time t. $P_{i,k}(t,T)$ is the price we have obtained for that bond using equations (17) – (21) at node i of the tree. $AI_i(t)$ is the accrued interest on that bond until time t and CF_i is the bond's conversion

[16] The terminal nodes are defined as the last nodes on the trinomial tree. In our case those terminal nodes correspond to the expiration time of the futures on German bonds.

factor. Using the equivalent futures prices of all the deliverable bonds at each of the terminal nodes of the tree, we can come with an estimate of that bond's price at the current moment zero. Comparing all these current prices for the bonds[17], we can find out which bond is the cheapest-to-deliver at the current moment. Let's assume that bond x is the current cheapest-to-deliver bond and its equivalent futures price today is $EF_x(0)$[18]. Let's now assume that a hypothetical bond y exists, which has equivalent futures prices at the terminal nodes of the tree satisfying the following condition:

$$EF_{i,y}(t) = \min[EF_{i,k}(t)] \quad (23)$$

We can obtain the equivalent futures price of such a bond today. Let's denote it by $EF_y(0)$. Given our construction of the equivalent futures price at each terminal node of the tree for our hypothetical bond y, the latter will always have a current price lower than or equal to the one of the cheapest-to-deliver bond at time zero. Its price today will be fully reflecting the ability of the short side to deliver the cheapest-to-deliver bond at expiration. We, therefore, will define the quality option as the difference at time zero between the equivalent futures price of that hypothetical bond and the equivalent futures price of the current cheapest-to-deliver bond. The following equation gives us that relationship:

$$QO(0) = EF_x(0) - EF_y(0) \quad (24)$$

---

[17] Those prices are not the bonds' actual current prices, because we are using values adjusted for accrued interest and conversion factors at the terminal nodes of the tree to come up with those current prices.

[18] We don't need to specify a node for the equivalent futures price, because at the current moment we will have only one node for the trinomial tree.

where $QO(0)$ is the value of the quality option at time zero[19].

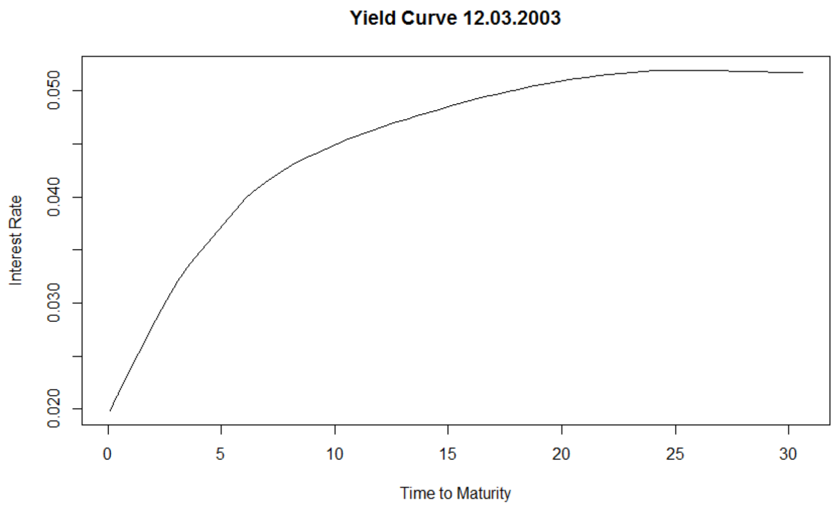
V. DATA

We have collected daily data for futures prices on Schatz futures, Bobl futures, Bund futures and Buxl futures starting from 1999 to 2009. We have also collected daily price data for all relevant deliverable bonds and their respective conversion factors for the same period. To estimate the term structure of interest rate, we have also collected daily price data on coupon and principal strips of German government bonds for the same period. All the price data we have collected come from daily closing prices. Lastly, we collected daily data for inter-bank lending rates among major banks in Europe. The data included weekly lending rates and was used for estimation of the volatility and mean reversion parameters on the Hull-White model. All data we collected for the purposes of our analysis was obtained from Bloomberg data services.

For the period under consideration we have a total of 43 quarterly expiries with four futures contract expirations per quarter. Like Lin, Chen and Chou (1999) we have decided to estimate the value of the quality option starting from 13 weeks to 1 week prior to the maturity of the futures contract. We have done this for two reasons – avoid periods during which the futures contracts are not actively traded and make our results comparable to those of Lin, Chen and Chou (1999). To construct the yield curve we have used the smooth cubic spline technique with twenty knots in almost all the cases. For one special case fewer than twenty knots have been used[20], because of lack of enough data points. The yield curve was estimated weekly, at any time

[19] The used definition of the quality option is consistent with the definition of Lin, Chen and Chou (1999)
[20] Fifteen knots have been used in that specific case

the quality option was priced. *Graph 3* shows one such estimation of the yield curve from December, 3rd 2003.



Graph 3. Estimated yield curve for 12.03.2003

The data points for the yield curve estimation were obtained from the prices of the coupon and principal strips. The number of data points varied from period to period depending on the outstanding strip securities at the moment of the yield curve estimation. In the construction of the yield curve preference was given to interest rates calculated from coupon strips as those instruments proved to be more liquid than the principal strips. For the analyzed period the yield curve exhibited all possible shapes. We had periods with upward-sloping, inverted and humped yield curves. In that sense, the period around the financial crisis was quite instructive, because we saw dramatic shifts in the yield curve on almost weekly basis. In general, the yield curve approach we used proved to be quite resilient as we didn't see sudden changes in the yield curve from week to week apart from the period we already mentioned. During that period the estimated yield curves were quite jumpy, which we attribute more so to the market conditions at the time and less so to the specifics of the model we have decided to use.

VI. INPUTS ESTIMATION

One of the inputs for the models we have chosen to use is the term-structure of interest rates. We have already described the methodology we have chosen for our estimation of the term-structure. Here, we just want to mention that our interest rate inputs for each period of the trinomial tree are directly inferred from our estimated term structure. These interest rates are usually for periods up to 13 weeks and should be viewed with some caution for two reasons: 1) they are usually extrapolated from yield curves constructed with no data points for periods shorter than three months 2) the yield curve tends to be quite volatile at the short end, which might introduce significant noise in our estimates.

The other two inputs we are using in our analysis are the mean reversion parameter and the volatility parameter. For their estimation we, again, are following the guidance offered by Lin, Chen and Chou (1999). We used an autoregressive process of order one on the data we have collected regarding the one-week inter-bank lending rates. We run a regression that gives us results of the form:

$$r_t = \alpha + \beta r_{t-\Delta t} + e_t \qquad (25)$$

where r_t[21] is the one-week inter-bank lending rate in period t and $r_{t-\Delta t}$ is the rate in the previous period. Using the mean square errors from the regression we come up with an estimate for the volatility parameter of our model. The mean reversion parameter a can be estimated using the equation:

[21] r_t here does not refer to the instantaneous short-term rate

$$\beta = e^{-a\Delta t} \quad (26)$$

Due to the long period we are analyzing, which is characterized by different macroeconomic environments, we have decided to divide it into sub-periods and estimate separate mean reversion and volatility parameters for those periods. When dividing our timeframe we used the 10-year U.S. Treasury note volatility index to identify periods when volatility remained relatively constant. Based on the index's performance, we decided that the best way to divide the period (1999-2009) is in the following way: (March 1999-June 2002); (July 2002 – December 2005); (January 2006 – December 2009). The volatility parameters we have estimated for each of these periods are 0.01292, 0.00604, and 0.00971 respectfully. The associated mean reversion parameters are 0.27815, 0.29789, and 0.14575.

VII. RESULTS

Like Lin, Chen and Chou (1999) we decided to begin our analysis with an estimation of the accuracy of our model. For this purpose we calculated the difference between the estimated equivalent futures price of the cheapest-to-deliver bond and the actual futures price at time zero. Time zero is defined as any point at which we have decided to price the quality option. The difference could be used as a proxy of our model's accuracy. But part of that difference can be attributed to the value of the quality option itself. We calculated the average deviation from the futures price for each contract under consideration and found out that he discrepancy varied from contract to contract. For the Schatz futures it was – 0.456%, for the Bobl it was – 0.485% and for the Bund futures it was – 0.670%. For the Buxl futures there were several abnormalities in the data we had for the contract, which were significantly skewing our calculations. Excluding those few occasions we come up an average difference across the period of – 1.293%. For comparison, Lin, Chen and Chou (1999) estimate an average difference for their analyzed period of – 0.49%, which is consistent with our results with the notable exception of the Buxl futures. In *Tables 1-4* we have presented the calculated differences for each of the contracts during the period (2006-2009)[22].We have also estimated an average difference between equivalent futures prices for the cheapest-to-deliver bonds and actual futures prices depending on how long before the expiry that difference was calculated[23]. We calculated those averages for all contracts in our study – the Schatz futures, the Bobl futures, the Bund futures and the Buxl futures. The results are presented in *Table 5*. As expected there is some convergence between the price of the cheapest-to-deliver bond and the price of the futures as time to maturity of the futures contract declines. We can safely assume that one week prior to expiration of the futures contract the cheapest-to-deliver contract will be known[24] and so the difference between the equivalent futures price of the cheapest-to-deliver asset at that point and the actual futures contract should be close to zero. A quick look at *Table 5* reveals that for our model those differences are – 0.053% for the Schatz, – 0.071% for the Bobl, – 0.2580% for the Bund futures and – 0.8320% for the Buxl futures. This, again, confirms that our model produces results, which are a good approximation of the surrounding reality.

[22] Results from previous periods are not presented for the sake of brevity, but are available upon request
[23] From 13 weeks to 1 week prior to expiry
[24] This is confirmed also by our valuation of the quality option one week prior to expiry as will become clear from our results later on.

Weeks prior to expiry	Mar-06	Jun-06	Sep-06	Dec-06	Mar-07	Jun-07	Sep-07	Dec-07	Mar-08	Jun-08	Sep-08	Dec-08	Mar-09	Jun-09	Sep-09	Dec-09
13	-0.5483%	-0.6508%	-0.7996%	-0.9096%	-0.8061%	-1.0968%	-1.1408%	-1.5445%	-1.2190%	-1.2728%	-1.1223%	-1.1829%	-0.3533%	-0.2180%	-0.1171%	-0.0218%
12	-0.4965%	-0.6294%	-0.7690%	-1.1033%	-0.8993%	-1.3536%	-1.0809%	-1.3635%	-1.1128%	-1.1523%	-0.9622%	-1.0358%	-0.6238%	-0.2405%	-0.0854%	-0.0628%
11	-0.4859%	-0.5543%	-0.7298%	-0.6414%	-0.4781%	-1.3897%	-1.0525%	-1.2364%	-0.8710%	-1.1445%	-0.8682%	-0.8347%	-0.4561%	-0.3802%	-0.0524%	-0.0634%
10	-0.4521%	-0.5666%	-0.7723%	-0.6150%	-0.7716%	-1.2775%	-1.0081%	-1.2221%	-1.1934%	-1.0142%	-0.7831%	-0.4472%	-0.7330%	-0.4455%	-0.0421%	-0.0454%
9	-0.3450%	-0.6539%	-0.5866%	-0.5332%	-0.9862%	-1.2333%	-0.8437%	-1.0323%	-0.9069%	-0.8606%	-0.7323%	-0.4812%	-0.5237%	-0.2740%	-0.0195%	-0.0351%
8	-0.2771%	-0.4637%	-0.6938%	-0.4540%	-0.7759%	-1.3264%	-0.7699%	-0.8662%	-0.8333%	-0.7989%	-0.6311%	-0.1394%	-0.4511%	-0.2574%	-0.0675%	0.0415%
7	-0.2436%	-0.3649%	-0.3800%	-0.4076%	-0.8572%	-0.9060%	-0.7349%	-0.8763%	-0.8111%	-0.6347%	-0.5838%	-0.1513%	-0.3524%	-0.1528%	0.0934%	-0.0091%
6	-0.2140%	-0.3176%	-0.3260%	-0.3560%	-0.8589%	-0.8675%	-0.7280%	-0.8079%	-0.6965%	-0.5584%	-0.4367%	-0.0617%	-0.2584%	-0.0639%	0.1509%	0.0588%
5	-0.1662%	-0.2857%	-0.2659%	-0.2850%	-0.8646%	-0.6621%	-0.5601%	-0.5944%	-0.6346%	-0.4930%	-0.3554%	0.0504%	-0.1515%	-0.1169%	0.1320%	0.0234%
4	-0.1462%	-0.2392%	-0.2009%	-0.2347%	-0.5089%	-0.4862%	-0.4513%	-0.5483%	-0.6217%	-0.3087%	-0.2536%	0.0740%	-0.0659%	-0.0801%	0.0701%	-0.0260%
3	-0.0589%	-0.2013%	-0.1370%	-0.1657%	-0.4034%	-0.1857%	-0.8094%	-0.4942%	-0.5071%	0.0658%	-0.1582%	-0.0186%	0.5336%	-0.1349%	0.0446%	-0.0466%
2	-0.0450%	-0.1661%	-0.0902%	-0.1141%	-0.3141%	-0.1253%	-0.7138%	-0.4024%	-0.3448%	-0.1482%	-0.0777%	0.2865%	0.4100%	0.0089%	-0.0712%	-0.0659%
1	0.0856%	-0.1097%	-0.0241%	-0.0542%	-0.2369%	-0.1554%	-0.6660%	-0.3569%	-0.2810%	0.0561%	0.0538%	0.1819%	0.3392%	0.1009%	-0.0982%	0.0254%
Median	***-0.2436%***	***-0.3649%***	***-0.3800%***	***-0.4076%***	***-0.7759%***	***-0.9060%***	***-0.7699%***	***-0.8662%***	***-0.8111%***	***-0.6347%***	***-0.5838%***	***-0.1394%***	***-0.3524%***	***-0.1528%***	***-0.0421%***	***-0.0260%***
Average	***-0.2610%***	***-0.4003%***	***-0.4442%***	***-0.4518%***	***-0.6739%***	***-0.8512%***	***-0.8123%***	***-0.8727%***	***-0.7718%***	***-0.6357%***	***-0.5316%***	***-0.2892%***	***-0.2067%***	***-0.1734%***	***-0.0048%***	***-0.0175%***

Table 1. Difference between equivalent futures price of the cheapest-to-deliver bond and the actual futures price for the Schatz contract (2006-2009).

Weeks prior to expiry	Mar-06	Jun-06	Sep-06	Dec-06	Mar-07	Jun-07	Sep-07	Dec-07	Mar-08	Jun-08	Sep-08	Dec-08	Mar-09	Jun-09	Sep-09	Dec-09
13	-0.8966%	-0.7354%	-0.8285%	-0.5979%	-1.0214%	-1.0135%	-1.1046%	-1.4072%	-1.3280%	-1.3068%	-1.3480%	-1.2144%	-0.9984%	-0.3227%	-0.3496%	-0.1496%
12	-0.8567%	-0.6875%	-0.7220%	-0.6139%	-0.9316%	-1.0103%	-1.0373%	-1.2242%	-1.1721%	-1.2430%	-1.1938%	-1.2124%	-0.9698%	-0.3139%	-0.3116%	0.0271%
11	-1.0234%	-0.6071%	-0.6256%	-0.5207%	-0.7515%	-1.0162%	-0.9155%	-1.1199%	-0.8776%	-1.2062%	-1.0720%	-1.0906%	-0.8453%	-0.3315%	-0.2693%	-0.0738%
10	-0.7500%	-0.6065%	-0.5285%	-0.4510%	-0.7874%	-0.9240%	-0.8666%	-1.0840%	-1.7355%	-1.0144%	-0.9359%	-0.9803%	-0.9149%	-0.4767%	-0.2042%	0.0851%
9	-0.6726%	-0.4951%	-0.5263%	-0.4369%	-0.7168%	-0.8372%	-0.8358%	-0.9219%	-0.9284%	-0.8617%	-0.8866%	-0.9340%	-0.8303%	-0.3345%	-0.2170%	0.0668%
8	-0.6063%	-0.3935%	-0.5540%	-0.3935%	-0.6127%	-0.7408%	-0.7243%	-0.7796%	-0.8234%	-0.8109%	-0.7845%	-0.5511%	-0.7064%	-0.3274%	-0.1612%	0.0069%
7	-0.5129%	-0.3557%	-0.5349%	-0.3420%	-0.5753%	-0.6111%	-0.7122%	-0.7743%	-0.7151%	-0.6706%	-0.6594%	-0.5405%	-0.6036%	-0.2403%	-0.2537%	-0.0251%
6	-0.5397%	-0.3653%	-0.4934%	-0.2116%	-0.5057%	-0.5240%	-0.6537%	-0.6713%	-0.6321%	-0.6122%	-0.5211%	-0.5621%	-0.5609%	-0.1660%	-0.1973%	0.0296%
5	-0.5410%	-0.3105%	-0.4243%	-0.1243%	-0.4849%	-0.4577%	-0.5764%	-0.5010%	-0.5022%	-0.5681%	-0.4477%	-0.4752%	-0.4679%	-0.2604%	-0.0577%	0.1327%
4	-0.4897%	-0.2588%	-0.3231%	-0.0609%	-0.2992%	-0.3271%	-0.4540%	-0.4661%	-0.4547%	-0.4536%	-0.3726%	-0.3600%	-0.3766%	-0.2185%	-0.0805%	-0.0393%
3	-0.4601%	-0.2126%	-0.2318%	0.0315%	-0.2197%	-0.2115%	-0.4071%	-0.3892%	-0.3412%	0.2122%	-0.2480%	-0.3279%	0.0083%	-0.2666%	-0.1930%	0.2491%
2	-0.4181%	-0.1490%	-0.2147%	0.1052%	-0.1418%	-0.0387%	-0.4522%	-0.2937%	-0.1979%	-0.2367%	-0.1630%	-0.1360%	0.0336%	-0.0649%	-0.2504%	0.1121%
1	-0.3886%	-0.0638%	-0.1676%	0.1370%	-0.0567%	-0.0540%	-0.2903%	-0.2548%	-0.1546%	0.1039%	-0.0020%	-0.1232%	0.1251%	-0.0539%	-0.2060%	0.1662%
Median	***-0.5410%***	***-0.3653%***	***-0.5263%***	***-0.3420%***	***-0.5753%***	***-0.6111%***	***-0.7122%***	***-0.7743%***	***-0.7151%***	***-0.6706%***	***-0.6594%***	***-0.5511%***	***-0.6036%***	***-0.2666%***	***-0.2060%***	***0.0296%***
Average	***-0.6274%***	***-0.4031%***	***-0.4750%***	***-0.2676%***	***-0.5465%***	***-0.5974%***	***-0.6946%***	***-0.7605%***	***-0.7587%***	***-0.6668%***	***-0.6642%***	***-0.6544%***	***-0.5467%***	***-0.2598%***	***-0.2117%***	***0.0452%***

Table 2. Difference between equivalent futures price of the cheapest-to-deliver bond and the actual futures price for the Bobl contract (2006-2009).

Weeks prior to expiry	Mar-06	Jun-06	Sep-06	Dec-06	Mar-07	Jun-07	Sep-07	Dec-07	Mar-08	Jun-08	Sep-08	Dec-08	Mar-09	Jun-09	Sep-09	Dec-09
13	-0.6766%	-0.8848%	-0.9001%	-0.9432%	-1.1081%	-1.0777%	-1.1511%	-1.4269%	-1.2149%	-0.9024%	-1.4907%	-1.4669%	-1.2530%	-0.8253%	-0.0700%	-0.0145%
12	-0.6376%	-0.7870%	-0.7384%	-0.8758%	-1.0126%	-1.0569%	-1.0522%	-1.1642%	-1.0439%	-0.7235%	-1.3491%	-1.4006%	-0.9625%	-0.7261%	-0.0810%	-0.1588%
11	-1.0327%	-0.5937%	-0.6078%	-0.8798%	-0.8261%	-1.1094%	-0.9822%	-1.0911%	-0.7247%	-0.7179%	-1.2214%	-1.2555%	-0.6972%	-0.8684%	-0.0595%	-0.0654%
10	-0.5733%	-0.6480%	-0.6905%	-0.8215%	-0.7972%	-0.9874%	-0.9445%	-1.0291%	-2.0151%	-0.4556%	-1.1353%	-1.0485%	-0.5700%	-0.7336%	-0.2501%	-0.1574%
9	-0.5175%	-0.5872%	-0.6644%	-0.7947%	-3.8651%	-0.8980%	-0.8640%	-0.8418%	-0.9147%	-0.4248%	-1.1458%	-0.9170%	-0.8367%	-0.6691%	-0.3334%	-0.1255%
8	-0.4013%	-0.3535%	-0.6708%	-0.6829%	-0.6945%	-0.7907%	-0.6974%	-0.7645%	-0.8016%	-0.3597%	-1.1246%	-0.6158%	-0.8420%	-0.5510%	-0.1244%	-0.1560%
7	-0.3486%	-0.3533%	-0.6488%	-0.6234%	-0.6084%	-0.6473%	-0.7155%	-0.7662%	-0.7117%	-0.2367%	-0.8499%	-0.4574%	-0.9355%	-0.3100%	0.0149%	0.0351%
6	-0.2888%	-0.3567%	-0.6180%	-0.4919%	-0.5034%	-0.5666%	-0.6415%	-0.6390%	-0.6580%	-0.2138%	-0.7403%	-0.7029%	-0.9346%	-0.2036%	-0.2188%	-0.0509%
5	-0.2951%	-0.3238%	-0.5317%	-0.4338%	-0.4933%	-0.5243%	-0.5784%	-0.4593%	-0.5091%	-0.2107%	-0.6755%	-0.4583%	-0.6874%	-0.4023%	-0.4261%	0.0180%
4	-0.3619%	-0.2732%	-0.4689%	-0.4260%	-0.3824%	-0.4231%	-0.4696%	-0.4469%	-0.4928%	-0.1561%	-0.6051%	-0.5910%	-0.8731%	-0.3342%	-0.1191%	0.0580%
3	-0.2039%	-0.3265%	-0.3511%	-0.3498%	-0.3095%	-0.2419%	-0.3403%	-0.3815%	-0.2980%	0.7089%	-0.4607%	-0.5139%	-0.5986%	-0.4573%	-0.0423%	0.1710%
2	-0.1914%	-0.2310%	-0.2531%	-0.2577%	-0.1929%	0.0077%	-0.4177%	-0.2785%	-0.1701%	-0.0371%	-0.4783%	-0.4331%	-0.6282%	-0.1882%	-0.1936%	-0.0523%
1	-0.0525%	-0.2163%	-0.1798%	-0.2731%	-0.0801%	-0.1374%	-0.2225%	-0.1387%	-0.2115%	0.5243%	-0.4120%	-0.4786%	-0.5894%	0.0479%	-0.1436%	0.2115%
Median	***-0.3619%***	***-0.3535%***	***-0.6180%***	***-0.6234%***	***-0.6084%***	***-0.6473%***	***-0.6974%***	***-0.7645%***	***-0.7117%***	***-0.2367%***	***-0.8499%***	***-0.6158%***	***-0.8367%***	***-0.4573%***	***-0.1244%***	***-0.0509%***
Average	**-0.4293%**	**-0.4565%**	**-0.5633%**	**-0.6041%**	**-0.8364%**	**-0.6502%**	**-0.6982%**	**-0.7252%**	**-0.7512%**	**-0.2465%**	**-0.8991%**	**-0.7953%**	**-0.8006%**	**-0.4786%**	**-0.1575%**	**-0.0221%**

Table 3. Difference between equivalent futures price of the cheapest-to-deliver bond and the actual futures price for the Bund contract (2006-2009).

Weeks prior to expiry	Mar-06	Jun-06	Sep-06	Dec-06	Mar-07	Jun-07	Sep-07	Dec-07	Mar-08	Jun-08	Sep-08	Dec-08	Mar-09	Jun-09	Sep-09	Dec-09
13	-2.7325%	-2.0696%	-1.9147%	-1.6650%	-3.9258%	-3.9772%	-3.7815%	-3.1125%	-3.0072%	-3.1635%	-4.0995%	-0.4531%	-0.1299%	-0.1078%	-0.2919%	-1.4098%
12	-3.1611%	-1.6698%	-1.5483%	-1.5531%	-4.1144%	-3.9962%	-3.6063%	-2.9278%	-2.7955%	-2.8249%	-3.0561%	-2.0341%	-0.9578%	-0.2279%	-0.5559%	-0.3113%
11	-4.1500%	-1.1660%	-1.3522%	-1.1447%	-3.6169%	-3.9129%	-3.5722%	-2.6607%	-2.4507%	-3.2641%	-2.9455%	-2.0048%	-0.5396%	-0.3794%	-0.3674%	-0.2983%
10	-2.5881%	-1.4260%	-1.8420%	-1.0899%	-3.3156%	-3.7193%	-3.5446%	-2.7336%	-3.7689%	-2.6840%	-2.7552%	-2.0680%	0.5229%	-0.3581%	-0.5559%	-0.4148%
9	-2.1328%	-1.7912%	-1.5819%	-1.2612%	-4.1030%	-3.6751%	-3.3929%	-2.6122%	-2.8045%	-2.9282%	-2.7624%	-1.2108%	0.4446%	-0.3557%	-0.5697%	-0.4119%
8	-0.5367%	-1.4363%	-1.6424%	-1.3573%	-3.6546%	-3.5860%	-2.3247%	-2.4999%	-2.6531%	-2.7366%	-1.8415%	-1.9991%	-0.2154%	-0.3571%	-0.1137%	-0.4930%
7	-0.3696%	-1.4828%	-1.5099%	-1.5036%	-3.4659%	-3.5426%	-2.4055%	-2.7188%	-2.7886%	-2.6178%	-0.3480%	-1.0628%	-0.2251%	-0.2283%	-0.4043%	-0.5318%
6	-0.7987%	-1.5190%	-1.5812%	-0.8230%	-4.0295%	-3.4872%	-2.3722%	-2.5012%	-2.5318%	-2.6691%	-1.3483%	-1.1105%	-0.8342%	-0.1711%	-0.2383%	-0.2555%
5	-0.5997%	-1.8656%	-1.6498%	-0.8878%	-2.7900%	-3.4690%	-2.1504%	-2.3607%	-2.5577%	-2.6918%	-1.5572%	-0.8654%	-0.0166%	-0.3371%	-0.3527%	-0.2830%
4	-0.9072%	-1.5999%	-1.5537%	-0.7974%	-3.7613%	-3.2181%	-2.2327%	-2.4781%	-2.6219%	-2.5821%	-1.4975%	-1.1505%	-0.0313%	-0.3156%	-0.2664%	-0.0344%
3	-0.2864%	-1.3565%	-1.2087%	-0.5570%	-3.2914%	-3.0516%	-2.0135%	-2.3672%	-2.3329%	-1.8300%	-1.2791%	-0.7137%	-0.3407%	-0.3363%	-0.3612%	-0.0238%
2	-0.8919%	-1.2699%	-1.1417%	-0.6626%	-3.2502%	-2.9666%	-2.1919%	-2.2693%	-2.1616%	-2.2577%	-1.2330%	-0.1472%	-0.3393%	-0.1775%	-0.5760%	-0.1646%
1	0.0892%	-1.4272%	-1.1863%	-0.5689%	-3.1984%	-3.0050%	-2.0733%	-2.1793%	-2.3617%	-1.8517%	-1.2142%	-0.8812%	-0.0263%	-0.4540%	-0.6619%	-0.1528%
Median	**-0.8919%**	**-1.4828%**	**-1.5537%**	**-1.0899%**	**-3.6169%**	**-3.5426%**	**-2.3722%**	**-2.5012%**	**-2.6219%**	**-2.6840%**	**-1.5572%**	**-1.1105%**	**-0.2154%**	**-0.3363%**	**-0.3674%**	**-0.2983%**
Average	***-1.4666%***	***-1.5446%***	***-1.5164%***	***-1.0670%***	***-3.5782%***	***-3.5082%***	***-2.7432%***	***-2.5709%***	***-2.6797%***	***-2.6232%***	***-1.9952%***	***-1.2078%***	***-0.2068%***	***-0.2928%***	***-0.4089%***	***-0.3681%***

Table 4. Difference between equivalent futures price of the cheapest-to-deliver bond and the actual futures price for the Buxl contract (2006-2009).

	Schatz		Bobl		Bund		Buxl	
Weeks prior to expiry	***Median***	***Average***	***Median***	***Average***	***Median***	***Average***	***Median***	***Average***
13	-0.7890%	-0.7881%	-0.8285%	-0.7970%	-1.0067%	-1.0416%	-1.5043%	-1.7904%
12	-0.6775%	-0.7576%	-0.8092%	-0.7768%	-0.8679%	-0.9509%	-1.3060%	-1.6517%
11	-0.6065%	-0.6696%	-0.6960%	-0.7160%	-0.7733%	-0.8554%	-1.2079%	-1.6293%
10	-0.5723%	-0.6392%	-0.6950%	-0.6747%	-0.7411%	-0.8450%	-1.2017%	-1.5225%
9	-0.5237%	-0.5712%	-0.6023%	-0.6001%	-0.7947%	-0.8634%	-1.2360%	-1.4517%
8	-0.4551%	-0.5030%	-0.5548%	-0.5237%	-0.6593%	-0.6987%	-1.0048%	-1.3002%
7	-0.3810%	-0.4694%	-0.5393%	-0.6597%	-0.6234%	-0.6986%	-0.8675%	-1.2709%
6	-0.3176%	-0.3731%	-0.4615%	-0.4335%	-0.5502%	-0.6014%	-0.8286%	-1.2256%
5	-0.2850%	-0.3137%	-0.4228%	-0.3710%	-0.4593%	-0.5466%	-0.8043%	-1.1252%
4	-0.2392%	-0.4031%	-0.2992%	-0.3231%	-0.4260%	-0.5225%	-0.7009%	-1.1265%
3	-0.1645%	-0.1874%	-0.2233%	-0.2019%	-0.3511%	-0.4872%	-0.6354%	-0.9634%
2	-0.1162%	-0.2040%	-0.1630%	-0.1499%	-0.2785%	-0.3462%	-0.3928%	-0.9160%
1	-0.0542%	-0.0530%	-0.1248%	-0.0710%	-0.2163%	-0.2580%	-0.4862%	-0.8302%

Table 5. Average difference between equivalent futures price of the cheapest-to-deliver bond and the actual futures contract per period (1999-2009).

The estimated values for the quality option for the period 2007-2009 are presented in *Tables 6-9*[25]. The average values for the quality option at each estimation point (1 week to 13 weeks before expiry) for the whole analyzed period are calculated separately and presented in *Table 10*. The average option values for the four contracts across the whole period, irrespective of the estimation point are 0.000179% for the Schatz futures, 0.012046% for the Bobl futures, 0.000195% for the Bund futures and 0.000057% for the Buxl futures. Those values are much lower than the reported by Balbas and Reichardt (2006), which were between 1.9% and 2.8%. Balbas and Reichardt (2006) results, however, relate to the option value 13 weeks prior to expiry and thus it is appropriate compare them to the values we calculated for the quality option 13 weeks prior to expiry. The average prices we have 13 weeks prior to expiry for the Bund futures contract (which is the same contract Balbas and Reichardt (2006) analyzed) are still quite low,

[25] The option values for the remainder of the analyzed period are not included for the sake of brevity, but are available upon request.

compared to the results of Balbas and Reichardt (2006). We believe the difference might be attributed to the selected methodology on one hand, and the definition of the quality option on the other. The methodology Balbas and Reichardt (2006) use is replication-based and as any other replication-based methodologies it is assuming market efficiency. The authors are trying to incorporate transaction costs to introduce frictions in their model, but even with the inclusion of frictions the authors' estimated value for the quality option is too high. Previous research reports, which used replication-based models are also reporting relatively high values for the quality option, so the results of Balbas and Reichardt (2006) are not surprising. Balbas and Reichardt (2006) are also not defining a single quality option for the futures contract, but multiple ones depending on the way they construct their replication portfolio. The authors explicitly mention that they are pricing the most expensive of those quality options, so the discrepancy between our findings and theirs might be partially explained by their choice. Also, as we mentioned earlier, our model has a tendency to undervalue the derivatives it is used to price due to the assumed perfect correlation among the returns of the underlying assets.

Weeks prior to expiry	Mar-07	Jun-07	Sep-07	Dec-07	Mar-08	Jun-08	Sep-08	Dec-08	Mar-09	Jun-09	Sep-09	Dec-09
13	0.000000	0.000178	0.000000	0.000000	0.000000	0.000097	0.000000	0.000001	0.000000	0.000000	0.000000	0.000000
12	0.000000	0.000194	0.000000	0.000000	0.000000	0.000247	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
11	0.000000	0.000092	0.000000	0.000000	0.000000	0.000093	0.000002	0.000000	0.000000	0.000000	0.000000	0.000000
10	0.000000	0.000033	0.000000	0.000000	0.000000	0.000117	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
9	0.000000	0.000029	0.000000	0.000000	0.000000	0.000003	0.000000	0.000000	0.000000	0.017054	0.000000	0.000000
8	0.000000	0.000003	0.000000	0.000000	0.000000	0.000001	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
7	0.000000	0.000000	0.000000	0.000000	0.000000	0.000004	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
6	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 6. Quality option values for the Schatz futures (2007-2009).

Weeks prior to expiry	Mar-07	Jun-07	Sep-07	Dec-07	Mar-08	Jun-08	Sep-08	Dec-08	Mar-09	Jun-09	Sep-09	Dec-09
13	0.000000	0.000002	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.003476	0.000000	0.000000
12	0.000000	0.000006	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000172	0.000000	0.000000
11	0.000000	0.000006	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
10	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000570	0.000000	0.000000
9	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000454	0.000000	0.000000
8	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001	0.000000	0.000000	0.003101	0.000000	0.000000
7	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000020	0.000000	0.000000
6	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.001591	0.000000	0.000000
4	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 7. Quality option values for the Bobl futures (2007-2009).

Weeks prior to expiry	Mar-07	Jun-07	Sep-07	Dec-07	Mar-08	Jun-08	Sep-08	Dec-08	Mar-09	Jun-09	Sep-09	Dec-09
13	0.000000	0.000000	0.000000	0.000000	0.000000	0.000056	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
12	0.000000	0.000000	0.000000	0.000000	0.000000	0.003021	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
11	0.000000	0.000000	0.000000	0.000000	0.000000	0.000991	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
10	0.000000	0.000000	0.000000	0.000000	0.000000	0.004631	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
9	0.000000	0.000000	0.000000	0.000000	0.000000	0.005005	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
8	0.000000	0.000000	0.000000	0.000000	0.000000	0.000996	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
7	0.000000	0.000000	0.000000	0.000000	0.000000	0.010785	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
6	0.000000	0.000000	0.000000	0.000000	0.000000	0.007166	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000000	0.000000	0.000000	0.000000	0.005777	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0.000000	0.000000	0.000000	0.000000	0.000000	0.000953	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.007269	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000161	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 8. Quality option values for the Bund futures (2007-2009).

Weeks prior to expiry	Mar-07	Jun-07	Sep-07	Dec-07	Mar-08	Jun-08	Sep-08	Dec-08	Mar-09	Jun-09	Sep-09	Dec-09
13	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000005	0.000000	0.000000
12	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000011	0.000001	0.000000
11	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.009841	0.000000	0.000000
10	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000091	0.000000	0.000000
9	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
8	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
7	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
6	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.004114	0.000000	0.000000
4	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 9. Quality option values for the Buxl futures (2007-2009).

Comparing our finding to those of Lin, Chen and Chou (1999) we do not see any stark differences. Lin, Chen and Chou (1999) came up with an average value for the quality option 13 weeks prior to expiry of 0.021%. The contract the authors are using to conduct their research is comparable to the Bund futures contract we used in our analysis, so we compared their findings with ours. Our average value for the quality option embedded in the Bund futures 13 weeks prior to expiry is only 0.000042%. We will discuss the potential reasons for this discrepancy further below.

From the summary data in *Table 10* it is visible that generally the value of the option declines as time to maturity approaches. This is relatively clear for the quality option estimates for the Schatz futures and the Bobl futures, but not so clear for the quality option embedded in Bund futures and Buxl futures. Also, when we relate the summary data to the one represented in *Tables 6-9*, it is visible that the averages are significantly higher for the summary than those for the period (2007-2009). The reason is that the largest concentration of non-zero values for the quality option were found in the period 1999-2002. We will take a look at the reasons why further below.

Weeks prior to expiry	**Schatz**	**Bobl**	**Bund**	**Buxl**
13	0.000331%	0.015866%	0.000042%	0.000000%
12	0.000358%	0.011348%	0.000284%	0.000000%
11	0.000351%	0.011286%	0.000319%	0.000286%
10	0.000205%	0.011555%	0.000231%	0.000167%
9	0.000558%	0.012891%	0.000236%	0.000000%
8	0.000135%	0.012029%	0.000057%	0.000037%
7	0.000106%	0.013526%	0.000306%	0.000004%
6	0.000076%	0.013188%	0.000413%	0.000000%
5	0.000067%	0.011861%	0.000290%	0.000131%
4	0.000046%	0.012553%	0.000094%	0.000000%
3	0.000044%	0.010729%	0.000184%	0.000000%
2	0.000023%	0.010635%	0.000017%	0.000000%
1	0.000023%	0.009131%	0.000066%	0.000117%

Table 10. Average quality option values (1999-2009)

The highest values we had for each of the contracts were 0.01705% for the Schatz futures, 0.53911% for the Bobl futures, 0.01079% for the Bund futures and 0.00984% for the Buxl future. Those values were not registered in the same sub-period, which indicates that the quality option might assume high values in all macroeconomic environments.

Next we go on to analyze what are the main reasons for our results. First, as is visible from the data, there is no clear connection between the time to maturity of the deliverable bonds and the quality option. From the summary of the average values for the quality options in *Table 10* we can see that the quality option embedded in the Bobl futures seem to have the highest value. Also, the quality options in the Schatz futures and Bund futures does not seem to differ by much when compared at each estimation point.

Second, the difference in the estimated values for the quality options in the four contracts might be a result of the differing number of deliverable securities. As we already pointed out, Boyle (1989) reached to such conclusion in his work. In our results we are finding clear correlation between the number of deliverable securities and the value of the quality option for the Bund futures and the Schatz futures. For periods when we are using the same volatility and mean reversion parameters we are getting higher option values when higher number of bonds are deliverable. Of course, we cannot attribute that variation solely to the differing number of deliverable securities. Other reasons might be the shape of the yield curve and the general level of interest rates. But results seem to indicate that such a relationship exists. With the Buxl futures, the link is not as clear. In the periods with higher number of deliverable bonds the estimated, values for the quality option do not differ significantly from those in periods with fewer deliverable securities.

The data for the Bobl futures maybe provides the best evidence for the connection between the value of the quality option and the number of deliverable bonds. The basket of securities for the contract varies in quite wide range – from 2 to 17. For comparison, the number of deliverable bonds on any of the other contracts does not exceed 8 at any point during the analyzed period. For the Bobl futures we find a very strong relationship between the basket of securities and the value of the quality option. In fact, the maximum value for the quality option that we get for the period (1999-2009) for any of the contracts was for an option embedded in the Bobl futures and was for a period when the number of deliverable bonds was 17. Our findings for the connection between the value of the quality option and the number of securities in the basket, however, seems to be at odds with the finding of Lin, Chen and Chou (1999), who find no such link.

As we mentioned earlier, Lin, Chen and Chou (1999) also estimate a somewhat higher value for the quality option in general. We believe that the discrepancy between our results and theirs might be partially explained by the periods, which were analyzed. The authors studied a period that was characterized by relatively high level of interest rates, while our analyzed period included a time interval when interest rates dropped close to zero. Even if we exclude this period, however, our estimates remain lower, so we do not believe that the level of interest rates is a major driver for the difference. Both studies include sub-periods characterized by different shapes of the yield curve, so the yield curve is also not very likely to be responsible for the gap.

We find a more serious candidate for the difference in the face of the two input parameters in the Hull-White model. Lin, Chen and Chou (1999) have conducted a sensitivity analysis in their paper, which establishes that a negative relationship exists between the value of the quality option and the mean reversion parameter and a positive one exists between the volatility parameter and the value of the quality option. These findings give us a good basis to compare the two models. Lin, Chen and Chou (1999) are using constant volatility and mean reversion parameters in their paper which are $a = 0.13235$ and $\sigma = 0.01233$. The mean reversion parameter of the authors is lower than all our estimates for the sub-periods in our time frame, while the volatility parameter is lower for just one of those sub-periods and only by a small amount. In fact, our mean reversion parameters are twice as large as the one Lin, Chen and Chou (1999) use in two out of the three sub-periods we studied. And the authors show in their research that just a 50% increase in their mean-reversion parameter leads to a decline of more than 100% in the value of the quality option. Given these estimates, the difference between our results and those of the authors should not be viewed as a surprise.

A third candidate for explaining the difference between our results and those of Lin, Chen and Chou (1999) is the underlying security and its characteristics. We believe this factor might be contributing for the discrepancies, but not with the same magnitude as the volatility and mean reversion parameters.

Lastly, the reason for the difference might hide in the number of deliverable securities under the Japanese contract and the contracts we are considering. For the period Lin, Chen and Chou (1999) covered, the basket of bonds never included less than 20 securities. By contrast, in our case the maximum number of deliverable bonds was 17 with the median number for the whole period being between 3 and 5, depending on the contract. Even though Lin, Chen and Chou (1999) did not find any relationship between the number of deliverable bonds and the value of the quality option, our findings suggest otherwise. We believe that the number of bonds in the basket might have at least moderate explanation power for the difference between our average value for the quality option and the one estimated by Lin, Chen and Chou (1999).

We would also like to take a look at the potential reasons for the variation of the value of the quality option in our results. Those variations could be explained by several factors, which coincide with the ones we already discussed in the previous several paragraphs. First, the general level of interest rates seems to be positively related to the value of the quality option; that is, the option seems to have higher values in high interest rate environments. Second, no clear connection has been established between the shape of the yield curve and the value of the quality option. Third, as we mentioned, the number of the bonds in the basket seems to be positively correlated with the option value. Fourth, the quality option seems to exhibit high sensitivity to our assumptions regarding the volatility parameter and the mean reversion parameter. The period between the middle of 2002 and the end of 2005 is instructive in that regard as it has notably lower estimates for the quality option than the other two sub-periods. These results might be at least partially due to the low assumed volatility, accompanied with high mean reversion parameter.

In the end of the discussion of our results we would like to remind the reader about some limitations of our analysis. First, we are using a single-factor model, which assumes perfect return correlation between the deliverable bonds. Second, the trinomial tree we are using consistently assumes future realizations with negative interest rates. And third, the proposed methodology does not explicitly deal with the wild card option and the end-of-month option embedded in the futures on German government bonds[26].

[26] As we mentioned earlier in our analysis, we believe that those options are of negligible value.

VIII. CONCLUSION

In this paper we tried to estimate the value of the quality option embedded in four futures contracts on German bonds – the Schatz futures, the Bobl futures, the Bund futures and the Buxl futures. Our analysis spanned the period between March 1999 and December 2009. We used a one-factor Markov model to estimate the values of the quality option at thirteen discrete points in time, one week apart. Our results show that the quality option has a very small value, which is highest of the Bobl futures – around 0.012046%. The option's value, however, varies in wide ranges and is dependent on several factors. Of those factors the most important seem to the mean reversion and volatility parameter estimates of our model. Other factors worth mentioning are the level of interest rates and the number of deliverable bonds under the futures contract. Our results seem to differ from those obtained from previous research on the quality option, which uses similar methodology or focuses on similar contracts. But those dissimilarities seem to be relatively easy to explain, given the differences between the current study and those previous studies. The results of this study could be used by hedgers, which are trying to construct their hedging portfolios. It looks like that the quality option does not seem to have a big impact on the construction of those positions, but under certain conditions, its value can reach as high as 0.5% of par. In such instances the quality option might start playing a larger role in the usage of futures as a hedging instrument; therefore hedgers should be aware of the factors, which might be driving the quality option's value higher.

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