Asset Pricing for the Shortfall Averse*

Paolo Guasoni † Gur Huberman ‡

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Abstract

Compare the utility from consuming a certain amount under two circumstances which vary by the levels of past historical consumptions. This papers premise is that the higher past historical consumption, the lower is the utility from current consumption. The paper and its companion, GHR, extend standard models of utility from dynamic consumption to have this property. Here and in GHR, the behavior of the optimizing consumer/investor takes into consideration the effect of current choices on the utility of future consumption. Assuming a representative agent framework this paper derives implications for the term structure of real interest rates and for equity pricing which generally are consistent with US data.

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†Boston University, Department of Mathematics and Statistics, 111 Cummington Street, Boston, MA 02215, USA, and Dublin City University, School of Mathematical Sciences, Glasnevin, Dublin 9, Ireland, email guasoni@bu.edu.
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‡Columbia Business School, 3022 Broadway Uris Hall 807, New York, NY 10027, USA

1 Introduction

Prospect Theory (Kahneman and Tversky, 1979) captures central features of decision making which go unremarked upon in expected utility theory. In particular, decision makers often derive utility (disutility) from outcomes which are better (worse) than a reference outcome and the marginal utility of a the better outcomes is discontinuously lower than that of the worse outcomes. Kahneman and Tversky (1979) leave open the identification of the reference point, and no consensus has emerged regarding its location. Moreover, as originally formulated (and in much of the subsequent literature) Prospect Theory is about mapping wealth outcomes into utility or welfare levels, as if wealth were not primarily a tool to obtain consumption rather than an end in itself.

Following a long tradition in the finance literature, the focus in the dynamic model developed here is on the dependence of welfare (or utility) on consumption, rather than wealth, with the added twist of a reference point at which marginal utility is discontinuous. Past peak consumption serves as the reference point. The marginal utility of increasing consumption above its historical peak is strictly lower than that of decreasing consumption from its peak to a level below it.

The choice of past peak consumption as the reference point has two apparently unrelated motivations. First, the peak-end rule (Kahneman et al., 1993) suggests that the welfare experienced over a period of time is affected primarily by the most extreme and by the most recent experience, whence the choice to identify the reference point with the strongest historical experience. Second, past peak consumption is a common reference to define a recession (According to the NBER (2010) definition, "A recession is a period between a peak and a trough, and an expansion is a period between a trough and a peak"). Moreover, the branch of explanations of the equity premium which invokes rare disasters measures them as "peak-to-trough fractional declines that exceed some threshold amount" (See Barro (2006) and Barro and Ursúa (2008).)

The utility function at the center of the model is a constant relative risk aversion utility of instantaneous consumption scaled by a power of the historical peak consumption rate. This scaling reflects *shortfall aversion* – the derivation of utility from consumption relative to the consumption's historical peak. Thus, the choice of historical peak consumption as a reference point is less radical than it seems. Moreover, it leads to a highly tractable model.

Formally, the utility function is

$$e^{-\beta t}U(c_t, h_t) = e^{-\beta t}\frac{(c_t h_t^{-\alpha})^{1-\gamma}}{1-\gamma}, \qquad \text{with} \qquad h_t = \max\left(h_0, \max_{0\le s\le t} c_s\right), \tag{1}$$

with h_0 some initial value of h, which for simplicity is assumed to be 0. The parameter α $(0 \le \alpha < 1)$ is the degree of shortfall aversion. The assumption $\alpha = 0$ reduces the model to familiar time-additive power utility.

The utility specification (1) supports unusual behavior in good times, i.e., when consumption is close to its historical peak. Reluctant to raise his reference point lest he lower the utility of future consumption, the model's representative agent has a stronger desire to save the closer his consumption rate is to its historical peak. In a market clearing environment this stronger desire to save translates into higher prices of the savings vehicles – the bonds and the stocks. Therefore interest rates and expected returns are counter-cyclical in the model. The effect is stronger for the interest rates.

The consumption rate c_t follows geometric Brownian motion in this paper's continuous time model,

$$\frac{dc_t}{c_t} = \mu_c dt + \sigma_c dW_t^c. \tag{2}$$

A:	Empirical Market Inputs	
	Average	S.D.
Consumption Growth	1.93	2.13
Dividend Growth	1.15	11.05
Correlation $\rho = 0.25$		

B:		Calibrated Preference Parameters
Discount Rate	β	<u>0</u>
Risk Aversion	γ	**4.220**
Loss Aversion	α	**0.498**

C:	Average		S.D.	
	Data	Model	Data	Model
Equity Premium	5.47	4.72	20.17	12.43
Price/Dividend	31.85	25.25	15.09	0.48
3-Month Real Rate	0.56	0.55	?	?
Long-Term Real Rate	?	4.83	0	0

Table 1: Directly estimated, calibrated and model-produced parameters. The model's input parameters in Panel A, from Beeler and Campbell (2012), govern the consumption and dividend processes (2) and (3), and their correlation which is assumed to be 0.25, as in Benzoni et al. (2011). The model's coefficients of risk aversion γ and shortfall aversion α (Panel B) are calibrated to minimize the sum of the squared differences between the directly estimated and the model-produced equity premium, the equity volatility, the three-month safe rate and the average price-dividend ratio. The time discount rate β is assumed, since calibration perfomance has low sensitivity to its value. Directly estimated and model-produced parameters are in Panel C.

Equations (1)-(2) jointly deliver the evolution of the pricing kernel which is used to derive the term structure of the default-free interest rate and the price of the claim on the stock market, represented by the claim on the stream of dividends D_t whose evolution is governed by the geometric Brownian motion (cf. Bansal and Yaron (2004); Campbell and Cochrane (1999),

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D(\rho dW_t^c + \sqrt{1-\rho^2}dW_t^D). \tag{3}$$

The model delivers also the unconditional and state-dependent first two moments of the stock market's returns at all horizons as well as its Sharpe ratios.

Table 1 Summarizes the model's input parameters (Panel A), its calibrated parameters (Panel B) and the first two moments of the equity premium, the 3-month real rate and price dividend ratio as estimated and as predicted by the model (Panel C).

The model appears to address well the equity premium and interest rate puzzles with an estimated coefficient of risk aversion of about 4. The model's glaring failure is in predicting that the volatility of the price dividend ratio is .48, about 30 times less than its historical average.

Figure 1 shows that an increase in the state variable c_t/h_t near 1 sharply decreases the three months rate. The rate of decrease in the equity's expected return appears milder. Therefore the Sharpe ratio increases as the state variable approaches 1.

The model studied here suggests that the welfare loss due to variation in consumption is substantial. This assessment is in sharp contrast with Lucas (2003), who argues that if the Lucas (1978)

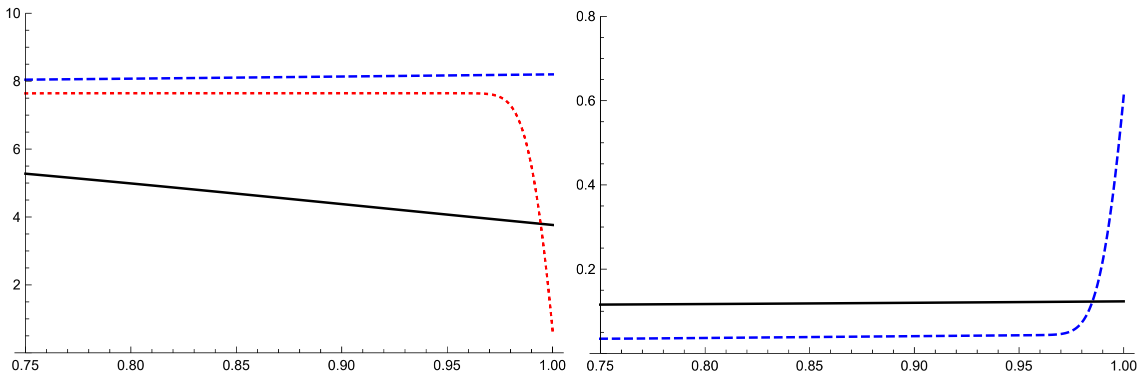

Figure 1: Left: Dividend yield (solid), stock return (dashes), and three-month rate (dotted), in percent per annum (vertical axis) against the state variable $x_t = c_t/h_t$ (horizontal axis). Right: Volatility (solid) and Sharpe ratio (dashed) per annum (vertical axis) against the state variable $x_t = c_t/h_t$ (horizontal axis).

environment is a good representation of the modern economy then the welfare loss associated with business cycle fluctuations is minimal.

2 Welfare Implications of the Business Cycle

Before exploring the asset pricing implications of (1)-(3) it is worthwhile to examine (1)-(2) in light of the Lucas (2003) claim that business cycle volatility (i.e., the volatility in consumption growth) entails only small welfare losses.

Lucas (2003) suggests a model-based, quantitative method to assess the cost of business cycle fluctuations which he associates with variability in consuumption growth. His starting point is a utility function that depends on consumption growth rate and its volatility, $U(\mu_c, \sigma_c)$. Then he looks for an equivalent growth rate, μ_e which satisfies $U(\mu_c, \sigma_c) = U(\mu_e, 0)$. The (relative) welfare loss due to consumption fluctuation is $1 - \mu_e/\mu_c$.

In words, using the estimated parameters of the consumption process Lucas calculates the expected utility of consumption under the model. He then considers a putative alternative consumption process with no variability which delivers the same expected utility. The elimination of consumption variability results in the consumption growth of the latter process being lower by a certain fraction of the consumption growth of the former process. That fraction is the equivalent loss in growth due to the business cycle. According to Lucas that fraction is very small.

The Lucas argument challenges a motivation underlying the study of business cycles: if they aren't so harmful as had been thought, why study them? Why devise policies to mitigate consumption fluctuations? Society may benefit from macro economists directing their efforts to study other issues, e.g., the drivers of economic growth.

Attempts to address Lucas' challenge include...

An application of the Lucas calculation to the model presented here delivers very different conclusions from those of Lucas, as illustrated in Figure 2.

In the Lucas setting ($\alpha = 0$), the expected utility of the representative agent is

$$u = E\left[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] = \frac{1}{\beta - (1-\gamma)\left(\mu_c - \frac{\gamma}{2}\sigma_c^2\right)} \tag{4}$$

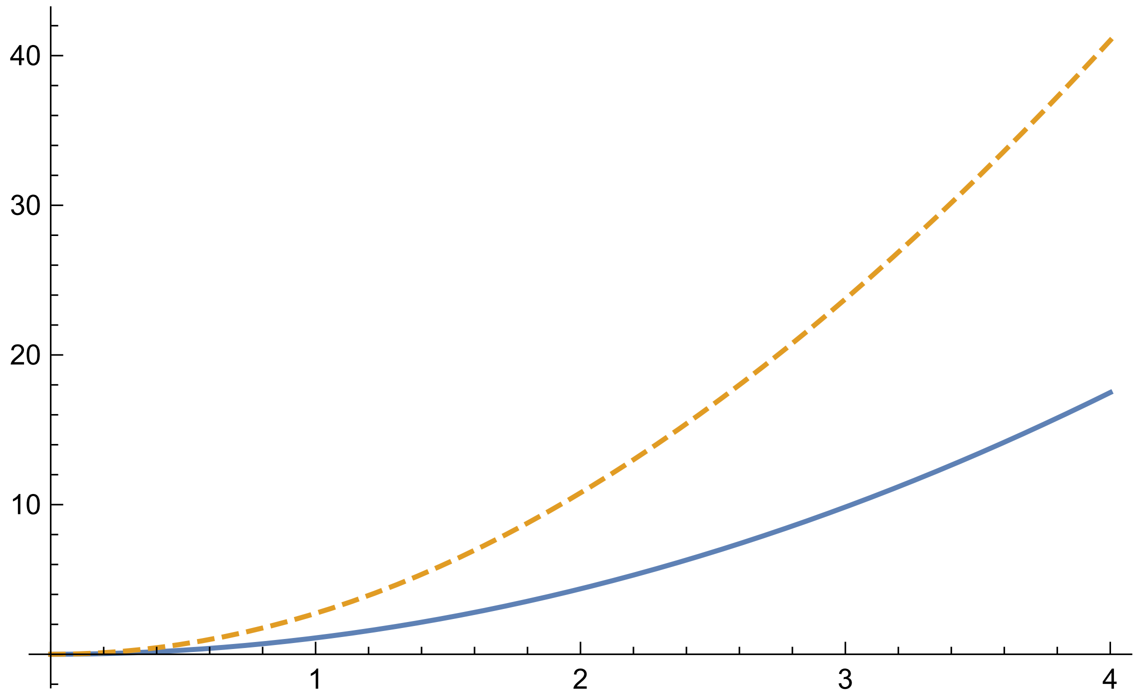

Figure 2: Fraction of mean consumption growth that a representative would forego (vertical axis, in percent) to remove its standard deviation (horizontal axis, in percent), in the benchmark model $\alpha = 0$ (solid) and with shortfall aversion $\alpha = 0.498$ (dashed).

implying that the representative agent is indifferent between a consumption stream that grows at rate μ_c with volatility σ_c and a steady consumption stream that grows at the constant rate

$$\mu_e(0) := \mu_c - \frac{\gamma}{2}\sigma_c^2. \tag{5}$$

Without shortfall aversion, consumption volatility has a relatively small effect; for example, with the parameters in Table 1, the certainty-equivalent growth rate is 1.83%, about 10 basis points below the natural growth rate of 1.93%, which amount to a reduction of 4.96% in consumption growth. In the classical model fluctuations in consumption growth have a relatively small welfare effect, as argued by Lucas (2003).

By contrast, with shortfall aversion, the certainty-equivalent growth rate is (see section (A.7))

$$\mu_e(\alpha) = \mu_c - \frac{\sigma_c^2}{2}\left(\gamma + \frac{\alpha}{1-\alpha}\left(2(\gamma-1) + \frac{\beta}{\mu_c}\right)\right) + O\left(\sigma_c^3\right) \tag{6}$$

This formula shows that shortfall aversion increases the effect of consumption volatility on welfare With the parameters in Table 1, the certainty-equivalent growth rate is now 1.69%, 24 basis points lower than the natural growth rate, which amounts to a 12.2% reduction in consumption growth.

In summary, the utility specification (1) is both consistent with the intuitive definitions of recessions and of severe economic disasters and leads one to conclude that with empirically sensible consumption fluctuations, recessions are costly.

3 Literature Review (VERY INCOMPLETE)

Guasoni, Huberman and Ren (2015) study a continuous time model of consumption and portfolio selection in which the utility function satisfies (1). In addition to the utility specification GHR assume a fixed safe interest rate and a single risky asset with price that follows a geometric Brownian motion. They derive the optimal consumption and portfolio rules. The optimizing agent internalizes the role of the reference point in that he has a region in which he increases his saving rate (and the weight of the risky asset in his portfolio) to avoid an increase in his reference consumption. Garleanu, Panageas and Yu (2012), generalizing Abel (1999), posit a similar utility function but in their model the reference is other people's peak consumption and the consumer-saver makes no attempt to influence it.) In contrast, this paper assumes that the consumption rate of the representative agent follows a geometric Brownian motion and derives implications for asset pricing.

There is a long and rich literature which studies the levels and term structure of interest rates (e.g., CIR, Vasacek), the equity risk premium (Mehra and Prescott, 1985) and the relation between the two (Weil, 1989). A common method of analysis consist of specification of preferences and of the consumption process of the representative agent followed by the derivation of the pricing kernel – the marginal rate of substitution – and applying the kernel to price different assets and derive the moments of their returns. A common objective is to match the model's predictions with the estimated moments.

A central issue is the reconciliation of the observed high equity premium (about 6% in the XX-XX sample) with the low correlation between equity returns and consumption growth rate (about XX in the XX-XX sample) and with the low rate of interest (XX in the XX-XX sample). Ideally, the reconciliation should be within a parsimonious and transparent model. Various modifications of either the representative agent's preferences or the process governing the consumption have been proposed.

Two strands of the literature study implications of a consumption process which is not governed by geometric Brownian motion. One deviation from the geometric Brownian motion environment is the possibility of a rare disaster. Rietz (1988) pioneers this approach, and Barro 2006; 2009 develops it further. Tsai and Wachter (2015) review the literature. A second modification of the geometric Brownian motion

To be continued, perhaps with more details & opinions.
Include Barberis, Huang, Santos in Lit Review

4 Overview

This section sketches the main results, most of which are derived analytically. To summarize them it is helpful to start as a benchmark with a version of the Lucas (1978) model in continuous time, i.e., the present model with $\alpha = 0$. The benchmark model, called here *the modified Lucas model*, is in continuous time (which is a deviation from the original model). Moreover, unlike in the original Lucas tree model, consumption and dividends are not identical but are correlated as in (2)-(3).

In the benchmark model the safe rate r_0, the dividend yield y_0, and the expected equity return e_0 are all constant at

$$r_0 = \beta + \gamma\mu_c - \frac{\sigma_c^2}{2}\gamma(\gamma + 1), \tag{7}$$

$$y_0 = r_0 - \mu_D + \gamma\rho\sigma_c\sigma_D, \tag{8}$$

$$e_0 = r_0 + \gamma\rho\sigma_c\sigma_D \tag{9}$$

respectively.

Probability	1%	5%	10%	50%
Theoretical	0.946	0.964	0.972	0.991
Empirical	0.987	0.994	0.999	1.000

Table 2: Empirical vs. theoretical quantiles of the state variables. The empirical quantiles, obtained from quarterly data for the period 1952:Q1-2015:Q1 (253 observations), are biased upwards because consumption (personal nondurable consumption expenditures plus services) is divided by the maximum of past consumption over quarterly rather than continuous observations.

These formulas summarize several classical results: the equilibrium interest rate includes the familiar time-preference, income, and precautionary savings terms. The dividend yield is determined by the discounted dividend growth (Gordon and Shapiro, 1956) in the first two terms, and by its risk exposure as in the consumption CAPM of Breeden (1979). The equity return equals the dividend yield plus expected dividend growth.

With the parameters Panels A and B of Table 1 the safe rate is an empirically unreasonably high 7.64%; the equity premium of .99% is far lower than the average historical equity premium. Risk aversion γ has to be much higher for the modified Lucas model to deliver an empirically reasonable value of the equity premium. But higher values of γ will result in even higher values of the safe rate r_0. Such high interest rate and low equity return are reminders of the inability of the classical model to reproduce the low real rates and high equity premium observed historically – the risk-free rate (Weil, 1989) and equity premium (Mehra and Prescott, 1985) puzzles.

The incorporation of shortfall aversion in the utility function appears to address both these puzzles.

4.1 The State Variable

With $\alpha > 0$, the single state variable is $x_t = c_t/h_t$, the ratio of current consumption rate to its historical maximum; by definition, it never exceeds 1. In fact, the model implies that this ratio fluctuates with consumption, reaching one whenever a new maximum is reached. The density distribution of the state variable x is

$$Prob(x_t \in dx) = \begin{cases} \lambda x^{\lambda-1} & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases} \text{where} \qquad \lambda = 2\mu_c/\sigma_c^2 - 1 \tag{10}$$

This distribution has mean $\lambda/(\lambda+1)$, variance $\lambda/((\lambda+1)^2(\lambda+2))$, and its lower p-quantile is $p^{1/\lambda}$ (see section A.1).

Empirically, λ is large (84, with the parameters in Table 1), consistent with the economy spending most of the time with x being very close to 1 which Table 2 suggests. In fact, the implications of x approaching 1 drive the model's results.

The distribution of the state variable x is highly skewed, and especially so because the estimated λ is so high. With the relatively short sample available it is not surprising that the empirical quintiles in Table 2 are above their theoretical counterparts.

As spending approaches the historical maximum, the agent is reluctant to increase it and tries to save more, which increases the prices of investment vehicles, and in particular of short-term safe bonds and of the risky asset. These price increases reduce the safe rate of interest and the expected return on the risky asset. The safe rate decreases faster than the expected return on the risky asset, and therefore the equity premium *increases* as spending approaches its historical maximum. Moreover, also the Sharpe ratio increases as the state variable x approaches 1 although the return volatility of the risky asset also decreases.

The marginal utility and the resulting pricing kernel are the starting points to the results that follow. When $c_t < h_t$, the marginal utility is

$$M_t = e^{-\beta t} U_c(c_t, h_t) = e^{-\beta t} c_t^{-\gamma} h_t^{-\alpha(1-\gamma)}. \tag{11}$$

Since $Prob(c_t = h_t) = 0$ a.s. for all t, this marginal utility can be used to price assets. The price at time t of an asset that generates a cumulative cash flow $(F_s)_{0 \le s \le \infty}$ is of the form

$$p_t(F) = E_t\left[\int_t^\infty \frac{M_s}{M_t} dF_s\right] \tag{12}$$

where E_t denotes the conditional expectation with respect to information available at time t. For securities which pay continuously, e.g., stocks which pay at the rate of D_t, $dF_t = D_t d_t$ The notation dF_t is needed for bonds which pay at discrete points in time.

4.2 Interest Rates

To apply (12) note that the cumulative cash flow of a zero-coupon bond with maturity t is $F_s = 0$ for $s < t$ and $F_s = 1$ for $s \ge t$. It follows that the price at time 0 of a zero-coupon bond with maturity t is (see section A.2)

$$B(x_t, T-t) = E_t\left[\frac{M_T}{M_t}\right] = e^{-(\beta+\phi(\alpha+\gamma-\alpha\gamma))(T-t)} E_t\left[e^{-\gamma\sigma_c(Y_T-Y_t)-\alpha(1-\gamma)\sigma_c(0\vee(Y_T^*-Y_t^*-\log(h_t/c_t)/\sigma_c))}\right] \tag{13}$$

where $Y_t = (\mu_c - \sigma_c^2/2)t/\sigma_c + W_t^c$ is a Brownian motion with drift, and $Y_t^* = \sup_{0 \le s \le t} Y_s$ denotes its running maximum. (In general the state variable $x_t = 1$ if and only if $Y_t = Y_t^*$.) The corresponding spot rate R_t^T and the term structure follow from the equality $e^{-(T-t)R^T(x_t)} = B(x_t, T-t)$.

Taking the limit of (13) as T goes to infinity delivers the long-term rate

$$R^\infty = \lim_{T\to\infty} R_t^T = \lim_{T\to\infty} -\frac{1}{T}\log B(x_t, T-t), \tag{14}$$

which is constant, i.e. independent of the state of the economy x_t (see section A.5)

$$R^\infty = \beta + \mu_c\gamma^* - \frac{\sigma_c^2}{2}\gamma^*(\gamma^*+1) \qquad \gamma^* = \alpha + (1-\alpha)\gamma \tag{15}$$

In other words, the model implies that the long term rate is still obtained from the Lucas formula (7), by replacing γ with γ^*, the α-weighted average of 1 and γ; i.e., $\gamma^* = \alpha + (1-\alpha)\gamma$. Usually, $\gamma > 1$ which implies that the long term rate is smaller than the interest rate in the benchmark modified Lucas model. For example, the parameter values estimated in Table 1 lead to a long-term rate equal to 4.83%. In the benchmark modified Lucas model the same parameters would lead to a long-term rate (which is equal to the short-term rate) of 7.64%.

Figure 3 displays the term structure of real interest rates at different states of the world x_t $(= c_t/h_t)$. In good times consumption is at or close to its historical maximum (bottom curve, $c_t/h_t = 99.9\%$) and the term structure is sloping upwards to the long-term rate. The model predicts low short rates in good times because when consumption is near its maximum, consumers are aware that increased consumption is likely to have low marginal value, hence they try to save more thereby depressing rates. As present consumption falls below its maximum, the term structure changes to humped (second curve from bottom) and inverted (top two curves) shapes, reflecting

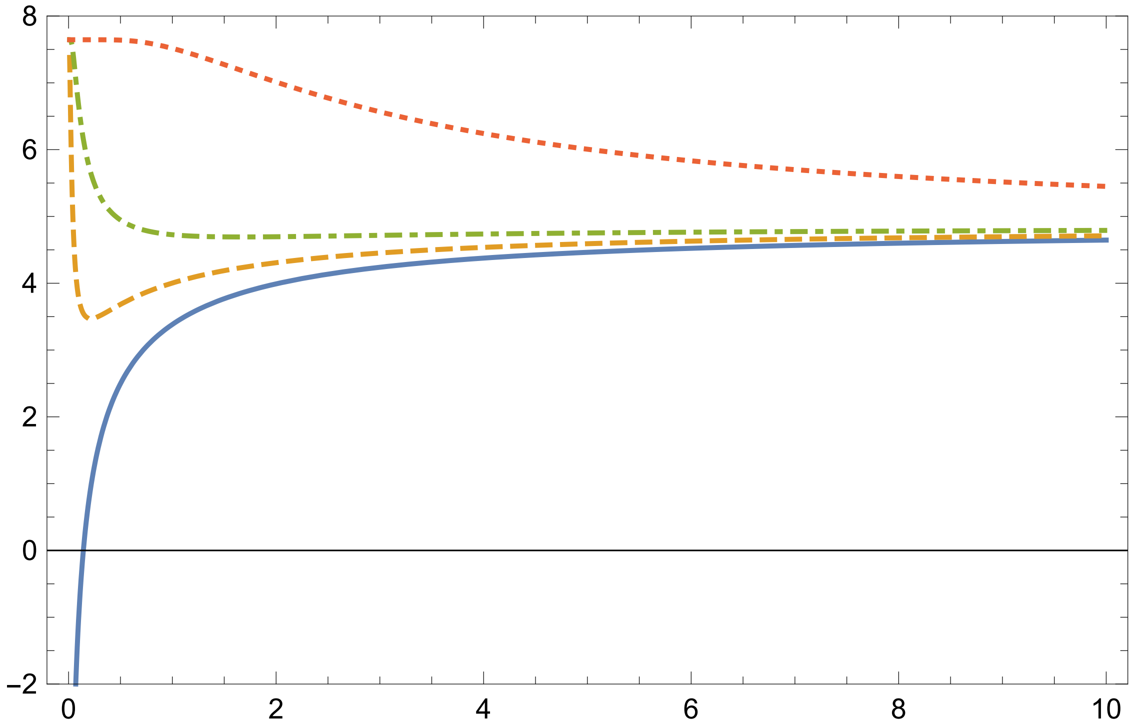

Figure 3: Term structure (horizontal axis, in years) or interest rates (vertical axis, in percent) at states of the economy ranging from $c_t/h_t = 99.9\%$ (bottom), 99.5%, 99%, to 95% (top). Market and preference parameters are as in Table 1.

the higher value of short-term consumption when it is away from its maximum. The value of consumption at medium terms depends on the probability of reaching maximum consumption at different points in the future, leading to a humped shape when a maximum is likely to be achieved soon, and an inverted shape when its reach is farther away.

Table 2 suggests that empirically, the state variable c_t/h_t exceeded .999 in about 90% of the 253 quarters between 1952 and 2015. Combining this observation with Figure 3 suggests that the average term structure is upward sloping.

The model's predictions for very short maturity bonds merit special attention. For states c_t/h_t which are sufficiently distant from 1 and for maturities sufficiently short, there is virtually no possibility that consumption will equal its historical peak within the life of the bond. Therefore the bond's yield is high at 7.64% which is the safe rate in the modified Lucas model. For states closer to 1 and for longer maturities it is possible that $c_t = h_t$ during the life of the bond. The set where $c_t = h_t$ is of measure zero, but on it saving is highly desired and the interest rate is minus infinity. The net result is low rates for very short maturities and for very high values of the state variable – see, e.g., the line corresponding to $c_t/h_t = 99.9\%$ in Figure 3. Holding the state fixed and shortening the bond's maturity results in positive rates (invisible in Figure 3 for $c_t/h_t = 99.9\%$.) Holding the bond maturity fixed and lowering the state x_t/h_t will also result in positive rates.

The average instantaneous interest rate in the model equals

$$E[R^0] = \beta + \mu_c\gamma - \frac{\sigma_c^2}{2}\gamma(\gamma+1) - \alpha(\gamma-1)(\mu_c - \sigma_c^2/2) \tag{16}$$

$$= r_0 - \alpha(\gamma-1)(\mu_c - \sigma_c^2/2) \tag{17}$$

which implies the average yield spread

$$R^{\infty} - E[R^0] = \alpha(\gamma - 1)\sigma_c^2 \left(\gamma(1 - \alpha/2) + \alpha/2\right), \tag{18}$$

which is positive for risk aversion γ greater than one, and vanishes either for logarithmic preferences ($\gamma = 1$) or without shortfall aversion ($\alpha = 0.$). Notably, the term spread is independent of the discount rate β. Rather, it arises because future cash flows are received at times when the state variable is more uncertain. Such uncertainty depends on the evolution of future consumption, thereby commanding a risk premium. With the parameters in Table 1, $E[R^0] = 4.59\%$ and $R^{\infty} = 4.85\%$, whence a term spread of 0.26%.

The unconditional average short-term interest rate $E[R^0]$ in the model is significantly higher than the typical short interest rate observed in usual good times, for two reasons. First, in rare bad times the short term rate reverts to the high Lucas rate, but such observation are not likely to be observed even in a relatively long sample. Second, the unconditional average $E[R^0]$ refers to instantaneous rates, while typical estimates of short term rates are based on one or three months rates, which tend to be lower.

On the whole, the continuous time shortfall aversion model is inappropriate to deliver realistic prediction about the very short-term rates because for very short horizons and for the average state shortfall aversion can be safely ignored, i.e., the predicted rate coincides with that of the modified Lucas model. On the other hand Table 1 suggests that it offers a realistic prediction for the average three month rate.

A consol is a promise to pay a constant unit rate into the open-ended future. Setting $F_s = 1$ in (12) yields the consol price (see section A.3)

$$p^C(x_t) = \frac{1 - \frac{\alpha(1-\gamma)}{\alpha(1-\gamma)+\delta_0} x_t^{\delta_0}}{r_0}, \tag{19}$$

where r_0 is the Lucas interest rate (7) and

$$\delta_0 = \gamma + \frac{1}{2} - \frac{\mu_c}{\sigma_c^2} + \sqrt{2\frac{\beta}{\sigma_c^2} + \left(\frac{\mu_c}{\sigma_c^2} - \frac{1}{2}\right)^2} \tag{20}$$

whence $\delta_0 = \gamma$ for $\beta = 0$. Since the consol price is the average of zero-coupon bond prices for all maturities and has a simple closed-form expression and β is close to zero, this case offers a simple benchmark to study the dependence of bond price on the model's parameters. In good times ($x = 1$) the consol rate equals

$$\frac{1}{p^C(1)} = r_0 \left(1 - \alpha \left(1 - 1/\gamma\right)\right) = r_0 \frac{\gamma^*}{\gamma}. \tag{21}$$

As γ is typically greater than one, this formula displays how shortfall aversion reduces interest rates from their classical level r_0 ($\alpha = 0$) down to r_0/γ ($\alpha = 1$). The effect on zero-coupon rates is complicated by the joint dependence on the state x_t and the maturity. Yet, the basic intuition remains that shortfall aversion depresses the motive to consume in excess of past maxima, thereby increasing the incentive to save and depressing interest rates.

The formula for the unconditional consol price enables the calculation of the average yield

$$\frac{1}{E[p^C(x_t)]} \approx R^{\infty} + o(\sigma_c^2) \tag{22}$$

which is well approximated by the long-term yield.

4.3 Price to Dividend Ratio

Specializing the general pricing formula (12) to the assumed dividend process (3) delivers the price to dividend ratio $P_D(x)$, which depends on the state x (see section A.3)

$$P_D(x) = \frac{1 - \frac{\alpha(1-\gamma)}{\alpha(1-\gamma)+\delta} x^{\delta}}{y_0}, \tag{23}$$

with

$$\delta = \gamma + \frac{1}{2} - \frac{\rho\sigma_D}{\sigma_c} - \frac{\mu_c}{\sigma_c^2} + \sqrt{2\frac{\beta - \mu_D}{\sigma_c^2} + \left(\frac{1}{2} - \frac{\rho\sigma_D}{\sigma_c} - \frac{\mu_c}{\sigma_c^2}\right)^2}. \tag{24}$$

This formula is better understood through the approximation (expanding for $\beta - \mu_D$ small)

$$\delta \approx \gamma + \frac{\beta - \mu_D}{\mu_c - \frac{\sigma_c^2}{2} + \rho\sigma_c\sigma_D} + o(\beta - \mu_D). \tag{25}$$

which is rather accurate for realistic parameters. For example, the figures in Table 1 lead to $\delta = 3.63$ with both the exact and approximate formulas. In turn, this approximation implies that in good times the dividend yield satisfies

$$1/P_D(1) \approx y_0 \left(1 - \alpha(1 - 1/\gamma)\left(1 + \frac{\beta - \mu_D}{\gamma(\mu_c - \frac{\sigma_c^2}{2} + \rho\sigma_c\sigma_D)}\right)\right). \tag{26}$$

Similar to the formula obtained for the consol rate, a higher shortfall aversion α reduces the dividend yield below the Lucas level y_0, significantly increasing the price to dividend ratio. (A consol is an asset paying a dividend that neither grows ($\mu_D = 0$) nor fluctuates ($\sigma_D = 0$), i.e., constant.) The figures in Table 1 imply stock prices of about 27 times dividends. By comparison, with $\alpha = 0$ the same parameters would imply a multiple of only 15. Shortfall aversion leads to higher stock prices than predicted by classical models, as the lower marginal utility from increasing consumption above its historical maximum strengthens the incentive to save in good times. In fact, comparing equation (21) to (26) shows that for typical parameter values shortfall aversion increases the consol multiple more than the stock multiple, as the term $(\beta - \mu_D)/\gamma(\mu_c - \frac{\sigma_c^2}{2} + \rho\sigma_c\sigma_D)$ is negative for realistic parameter values. (The numerator is negative, the denominator positive.)

In general, since the stationary density of the state variable x_t is $\lambda x^{\lambda-1}$ for $x \in (0, 1)$, the (unconditional) average price-dividend ratio is

$$E[P_D(x_t)] = \frac{1}{y_0}\left(1 - \alpha(1 - 1/\gamma)\frac{1}{\alpha(1 - 1/\gamma) - \delta/\gamma} \cdot \frac{\lambda}{\delta + \lambda}\right) \tag{27}$$

With the parameter values in Tables 1 and 2, this expression yields the average value of 26.07, close to its value in good times, reflecting the accuracy of the above approximation and the persistence of the state variable near its running maximum.

In the same fashion, the (unconditional) standard deviation of the price to dividend ratio equals

$$\text{StDev}(P_D(x_t)) = \frac{1}{y_0} \cdot \frac{1}{\frac{\delta}{\alpha(\gamma-1)} - 1} \cdot \frac{\delta}{\lambda + \delta} \cdot \left(\frac{\lambda}{\lambda + 2\delta}\right)^{1/2} \tag{28}$$

For the estimated parameter values, this formula yields the value of 0.47, which is well below the empirical figure of 15.09. This discrepancy is the main limitation of the present model: it can account for the low interest rates and high price multiples observed in the data, but it predicts that price multiples should be relatively stable over time compared to the variations observed empirically.

4.4 Equity and Consol Returns

The (unconditional) average expected equity return is (see section A.6)

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T\frac{dp_t+D_tdt}{p_t} = r_0+\gamma\rho\sigma_c\sigma_D+\alpha(1-\gamma)\sigma_c^2\left(\frac{\lambda}{2}-\gamma\int_0^1\frac{x^\delta}{1+\frac{\alpha(1-\gamma)}{\delta}(1-x^\delta)}\lambda x^{\lambda-1}dx\right) \tag{29}$$

$$\approx\beta+\gamma^*\mu_c-\frac{\sigma_c^2}{2}(\gamma(\gamma+1)-\alpha(\gamma-1)(2\gamma+1))+\gamma\rho\sigma_c\sigma_D+o(\sigma_c^2) \tag{30}$$

In particular, for $\alpha=0$ this expression reduces to the familiar formula $r_0+\gamma\rho\sigma_D\sigma_c$, a version of the consumption CAPM, whereby the expected excess return on a risky asset is proportional to the covariance of its cash flow with the consumption stream.

To understand the effect of shortfall aversion, consider first the case of a consol bond ($\mu_D=\rho=0$). Its average return is

$$\beta+\gamma^*\mu_c-\frac{\sigma_c^2}{2}(\gamma(\gamma+1)-\alpha(\gamma-1)(2\gamma+1))=R^\infty+\alpha^2(\gamma-1)^2\frac{\sigma_c^2}{2} \tag{31}$$

The consol's *average return* is higher than its yield by a Jensen's inequality term: the latter is the inverse of the unconditional expected price, while the former is the unconditional expectation of return.

For equity, the additional return is the familiar $\gamma\rho\sigma_c\sigma_D$, as in the Lucas benchmark. Thus, average equity returns are higher in the model only because of a term premium that also affects consols.

The impact of shortfall aversion on average expected return is negative for typical parameter values. For example, the parameters calibrated in Table 1 lead to an average real return of 5.13%, compared to a value of 7.89% obtained with $\alpha=0$ but otherwise the same parameters.

Likewise, the average standard deviation of stock returns equals to:

$$\sqrt{\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T\frac{d\langle p\rangle_t}{p_t^2}}=\sqrt{\int_0^1\left(\sigma_D^2(1-\rho^2)+\left(\frac{\alpha(\gamma-1)\delta x^\delta}{\delta+\alpha(\gamma-1)(x^\delta-1)}\sigma_c+\rho\sigma_D\right)^2\right)\lambda x^{\lambda-1}dx} \tag{32}$$

where $\langle p\rangle_t$ denotes the quadratic variation of p. For $\alpha=0$ this expression reduces to σ_D: as in the classical model the price to dividend ratio is constant, dividends and prices share the same volatility. Shortfall aversion leads to slightly higher volatility in stock prices: in the calibration of Table 2, the stock volatility rises to 12.43%, above the 11.05% observed for dividends, although it does not reach the 20.17% observed historically.

5 The Pricing Kernel and the State Variable

The fundamental pricing equation (12) can be written as

$$M_tp_t+\int_0^t M_sdF_s=E_t\left[\int_0^\infty M_sdF_s\right] \tag{33}$$

Being the conditional expectation at different times t of the same random variable, the right hand side is a martingale. Thus, also the left-hand side is a martingale, hence its expected conditional increments are zero. Decomposing this conditional increment is key to solving for p_t.

The first step in this direction is to calculate $d(M_t p_t) + M_t D_t dt$ or, equivalently, its growth rate (See section A.3)

$$\frac{d(M_t p_t)}{M_t p_t} + \frac{M_t dF_t}{M_t p_t} = \frac{dM_t}{M_t} + \frac{dp_t}{p_t} + \frac{d\langle M, p\rangle_t}{M_t p_t} + \frac{dF_t}{p_t} \tag{34}$$

The growth rate of the stochastic discount factor $M_t = e^{-\beta t} U_c(c_t, h_t) = e^{-\beta t} c_t^{-\gamma} h_t^{-\alpha(1-\gamma)}$ is

$$\frac{dM_t}{M_t} = -\beta dt - \gamma \frac{dc_t}{c_t} + \frac{\gamma}{2}(\gamma+1)\frac{d\langle c\rangle_t}{c_t^2} - \alpha(1-\gamma)\frac{dh_t}{h_t} \tag{35}$$

$$= -\left(\beta + \mu_c \gamma - \gamma(\gamma+1)\frac{\sigma_c^2}{2}\right) dt - \alpha(1-\gamma)\frac{dh_t}{h_t} - \gamma\sigma_c dW_t^c \tag{36}$$

where $\langle c\rangle_t$ denotes the quadratic variation of the consumption process.

The distinguishing feature of the present model is that $0 < \alpha$ and therefore the term associated with dh_t/h_t affects the solution.

In the classical case $\alpha = 0$, equation (36) reduces to the expression of the stochastic discount factor in the classical model, whereby the drift is minus the constant interest rate $r_0 = \beta + \mu_c\gamma - \gamma(\gamma+1)\frac{\sigma_c^2}{2}$ and the consumption risk premium is $\gamma\sigma_c$.

With $\alpha = 0$ the stochastic discount factor is governed by a process of the type

$$\frac{dM_t}{M_t} = f_t dt + g_t dW_t^c \tag{37}$$

where f_t and g_t are identified by the conditions that M_t, multiplied by any asset price, yields a martingale – which has drift zero. For the the money-market account $B_t = e^{\int_0^t r_s ds}$, it follows that

$$d(M_t B_t) = M_t dB_t + B_t dM_t = M_t r_t B_t dt + B_t dM_t = B_t M_t (r_t + f_t) dt + B_t M_t g_t dW_t^c$$

whence it follows that $f_t = -r_t$. Likewise, if the stock price satisfies

$$\frac{dS_t}{S_t} = (e_t + r_t)dt + \sigma_t^S dW_t^S$$

then the martingale condition yields (denoting by ρ the correlation between W^S and W^c)

$$d(M_t S_t) = M_t dS_t + S_t dM_t + d\langle M, S\rangle_t = \tag{38}$$

$$= M_t S_t (e_t + r_t) dt + M_t S_t \sigma_t^S dW_t^S + S_t M_t(-r_t dt + g_t dW_t^c) + \sigma_t^S g_t \rho dt \tag{39}$$

$$= M_t S_t (e_t + \sigma_t^S g_t \rho) dt + M_t S_t \sigma_t^S dW_t^S + S_t M_t g_t dW_t^c \tag{40}$$

Again, setting the total drift equal to zero yields the condition

$$g_t = -\frac{e_t}{\sigma_t^S \rho}$$

which identifies the market price of risk as the ratio between the equity premium and its exposure to consumption risk. Vice versa, given g_t, this equation yields a generic expected return-beta representation.

The novelty of equation (36) is in the term $-\alpha(1-\gamma)\frac{dh_t}{h_t}$, which is nonzero only at times when consumption increases above its past maximum, i.e., on the set $\{c_t = h_t\}$. Because c_t is a geometric Brownian motion, its supremum h_t is a continuous, increasing function with null derivative for almost all t, and h_t increases continuously precisely on the set of t where it is not differentiable.

Thus, the term dh_t is, unlike the martingale term dW_t^c, conditionally increasing and, unlike the drift term dt, it is not absolutely continuous. Informally, it behaves like a time-varying drift without jumps, that is zero most of the time and infinitely high in an infinitely small set, while resulting in a nonzero contribution.

The effect of the new term is best understood in conjunction with the classical consumption effect, i.e., by regrouping the terms in (35) and considering the sum

$$-\gamma\frac{dc_t}{c_t} - \alpha(1-\gamma)\frac{dh_t}{h_t} \tag{41}$$

When consumption is below its past maximum ($c_t < h_t$), the second term is null. As it establishes a new maximum ($c_t = h_t$ and $dc_t > 0$), then temporarily $dh_t/h_t = dc_t/c_t$ and (41) reduces to

$$-(\alpha + \gamma(1-\alpha))\frac{dc_t}{c_t} \tag{42}$$

which implies that the agent contemplates an increase above maximum past consumption *as if* preferences were classic, but risk aversion were $\gamma^* = \alpha + (1-\alpha)\gamma$, a value that is closer to one than γ. As risk aversion is typically greater than one, γ^* is typically lower than γ, which means that the agent views increases above the maximum with less concern than changes in consumption while below the maximum. This asymmetry vanishes in the special case of a logarithmic representative agent ($\gamma = 1$), who behaves myopically, and therefore is insensitive to the effect of shortfall aversion α on future marginal utility.

Any representative agent other than myopic evaluates future cash flows not only for their marginal utility at the time they are paid, but also for their covariation with the state of the economy – the consumption to historical peak ratio x_t. As a representative agent anticipates consumption to establish a new peak in the future, he realizes that it will reduce future marginal utility. To hedge against such future changes, the agent wishes to purchase today contracts that will increase in value when such a peak is reached – the asset prices themselves. As a result, hedging demand drives asset prices higher today.

Thus, the state variable $x_t = c_t/h_t$ fluctuates in the interval $(0, 1]$: when $c_t < h_t$, (i.e., $x_t \in (0, 1)$), it varies in lockstep with c_t. But when $c_t = h_t$, any further increase in c_t is offset in $x_t = c_t/h_t$ by a equal increase in h_t, which results in x_t following a diffusion reflected at the boundary 1. Indeed, the normalized ratio $z_t = -\log(x_t)/\sigma_c = \frac{\log(h_t/c_t)}{\sigma_c}$ satisfies

$$dz_t = -\frac{\mu_c - \sigma_c^2/2}{\sigma_c}dt - dW_t^c + d\eta_t \tag{43}$$

where the process η_t increases only on the set $\{c_t = h_t\} = \{z_t = 0\}$ as to keep $z_t \leq 0$ and hence $x_t \leq 1$. (Normalizing by σ_c is inessential, but it helps concentrate the parameter dependence on the drift alone.)

6 Concluding Remarks

The model identifies the state of the economy as current consumption relative to its past peak, in analogy with the literature on rare disasters, where such events are defined as drops from past peak consumption of 15% or more. In contrast to this approach, the present model does not make a sharp separation between normal times and disasters, but leaves a continuum of states x that vary between current peak consumption 1 and absolute ruin 0. In US postwar quarterly data, the ratio of consumption to its historical peak has remained above 0.987 for 99% of the time (Table 2), reflecting the absence of economic disasters. At the trough of the Great Depression, it reached the historical low of 80%.

Due to shortfall aversion the whole term structure of the default-free interest rate is lower than the (term-independent) rate delivered by the benchmark modified Lucas model. Shortfall aversion also reduces the expected return on equity but to a lesser extent. The suppression of the rates and of the expected returns is stronger the closer is current consumption to its historical peak.

A Appendix

A.1 Stationary density of the state variable (Eq. (10))

For the state variable $x_t = c_t/h_t$, i.e. current consumption as a fraction of its reference level h_t, consider the normalized log ratio $z_t = -\log(x_t)/\sigma_c = \frac{\log(h_t/c_t)}{\sigma_c}$, which satisfies

$$z_t = \frac{\log(h_t/c_t)}{\sigma_c} = -y_t + \sup_{0\leq s\leq t}\left(y_s \vee \frac{\log(h_0/c_0)}{\sigma_c}\right) \tag{44}$$

where $y_t = (\mu_c + \phi - \sigma_c^2/2)t/\sigma_c + W_t$ and $y_t^* = \sup_{0\leq s\leq t} y_s$. Therefore, by Skorohod's Lemma (Revuz and Yor, 1999, VI.2.1), z_t is a Brownian motion with negative drift, reflected at zero to remain positive, whence

$$dz_t = -\frac{\mu_c + \phi - \sigma_c^2/2}{\sigma_c}dt - dW_t^c + d\eta_t \tag{45}$$

where the process $\eta_t = \sup_{s\leq t}\left(y_s \vee \log(h_0/c_0)\right)/\sigma_c$ increases only on the set $\{c_t = h_t\} = \{z_t = 0\}$.

The long-term (stationary) distribution of a Brownian motion with negative drift reflected at zero is an exponential distribution with rate equal to twice the absolute value of the drift (Borodin and Salminen, p. 130), which means that the stationary density is exponential with rate $\frac{2(\mu_c+\phi)-\sigma_c^2}{\sigma_c}$. Since $x_t = e^{-\sigma_c z_t}$, setting $\lambda = 2\mu_c/\sigma_c^2 - 1$ it follows that the stationary density of x is a power law on $(0, 1)$

$$P(x_t \in dx) = m(x) = \begin{cases} \lambda x^{\lambda-1} & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}. \tag{46}$$

The curresponding cumulative distribution function is $P(x_t \leq x) = x^\lambda$ and its inverse, the quantile function, is $q(p) = p^{1/\lambda}$.

A.2 Zero Coupon Bond Price (Eq. (13))

Denote by $p(t, T)$ the price at time t of a zero-coupon bond with maturity T. Since

$$c_t = c_0 e^{(\mu_c-\sigma_c^2/2)t+\sigma_c W_t} \qquad h_t = e^{-\phi t}\left(h_0 \vee \sup_{0\leq s\leq t} c_0 e^{(\mu_c+\phi-\sigma_c^2/2)s+\sigma_c W_s}\right) \tag{47}$$

setting $y_t = (\mu_c + \phi - \sigma_c^2/2)t/\sigma_c + W_t$, and $y_t^* = \sup_{0\leq s\leq t} y_s$, it follows that

$$c_t = c_0 e^{-\phi t+\sigma_c y_t} \qquad h_t = e^{-\phi t}\left(h_0 \vee c_0 e^{\sigma_c y_t^*}\right) = c_0 e^{-\phi t+\log(h_0/c_0)\vee\sigma_c y_t^*} \tag{48}$$

whence

$$M_t = e^{-\beta t}U_c(c_t, h_t) = e^{-\beta t} c_t^{-\gamma} h_t^{-\alpha(1-\gamma)} \tag{49}$$

$$= c_0^{-(\alpha+\gamma-\alpha\gamma)} \exp\left(-\beta t + \phi(\alpha + \gamma - \alpha\gamma)t - \gamma\sigma_c y_t - \alpha(1-\gamma)(\log(h_0/c_0) \vee \sigma_c y_t^*)\right) \tag{50}$$

and

$$\frac{M_t}{M_0} = \exp\left(-\beta t + \phi(\alpha+\gamma-\alpha\gamma)t - \gamma\sigma_c y_t - \alpha(1-\gamma)\sigma_c(0 \vee (y_t^* - \log(h_0/c_0)/\sigma_c))\right) \tag{51}$$

Taking the expectation of this equation, it follows that, as claimed in (13),

$$p(t,T) = E\left[\frac{M_t}{M_0}\right] = e^{-(\beta+\phi(\alpha+\gamma-\alpha\gamma))t} E\left[e^{-\gamma\sigma_c y_t - \alpha(1-\gamma)\sigma_c(0\vee(y_t^* - \log(h_0/c_0)/\sigma_c))}\right] \tag{52}$$

This expression can be computed explicitly, as the expectation in the right-hand side involves a function of Brownian motion with drift y_t and its running maximum y_t^*. Recall that the joint law of these random variables is (Karatzas and Shreve, 1988, 2.8.1)

$$P(y_t \in d\zeta, y_t^* \in d\xi) = j(\zeta,\xi) = \frac{2(2\xi-\zeta)}{\sqrt{2\pi t^3}} e^{-\frac{(2\xi-\zeta)^2}{2t} - m\zeta - \frac{m^2}{2}t} \qquad 0 \vee \zeta < \xi \tag{53}$$

where $m = (\mu_c + \phi - \sigma_c^2/2)/\sigma_c$. Thus,

$$E\left[e^{-\gamma\sigma_c y_t - \alpha(1-\gamma)\sigma_c(0\vee(y_t^* - \log(h_0/c_0)/\sigma_c))}\right] = \int_{-\infty}^{+\infty}\int_{0\vee\zeta}^{+\infty} e^{-\gamma\sigma_c\zeta - \alpha(1-\gamma)\sigma_c(0\vee(\xi - \log(h_0/c_0)/\sigma_c))} j(\zeta,\xi)d\xi d\zeta \tag{54}$$

$$= \int_{0}^{\log(h_0/c_0)/\sigma_c}\int_{-\infty}^{\xi} e^{-\gamma\sigma_c\zeta} j(\zeta,\xi)d\zeta d\xi + \int_{\log(h_0/c_0)/\sigma_c}^{+\infty}\int_{-\infty}^{\xi} e^{-\gamma\sigma_c\zeta - \alpha(1-\gamma)\sigma_c(\xi - \log(h_0/c_0)/\sigma_c)} j(\zeta,\xi)d\zeta d\xi \tag{55}$$

$$= e^{\frac{1}{2}\gamma\sigma T(\gamma\sigma-2m)} + \frac{2(\gamma^*\sigma - m)x^{\gamma-\gamma^*} e^{\frac{1}{2}\gamma^*\sigma T(\gamma^*\sigma-2m)}\Phi\left(\frac{\sigma T(m-\gamma^*\sigma)+\log(x)}{\sigma\sqrt{T}}\right)}{\sigma(\gamma+\gamma^*)-2m} \tag{56}$$

$$+\frac{\sigma(\gamma-\gamma^*)e^{\frac{1}{2}\gamma\sigma T(\gamma\sigma-2m)}x^{2\gamma-\frac{2m}{\sigma}}\Phi\left(\frac{\sigma T(\gamma\sigma-m)+\log(x)}{\sigma\sqrt{T}}\right)}{\sigma(\gamma+\gamma^*)-2m} - e^{\frac{1}{2}\gamma\sigma T(\gamma\sigma-2m)}\Phi\left(\frac{\sigma T(m-\gamma\sigma)+\log(x)}{\sigma\sqrt{T}}\right) \tag{57}$$

where $\gamma^* = \alpha + (1-\alpha)\gamma$.

A.3 Stock and Consol Prices (Eqs (19) and (23))

Asset prices are of the form

$$p_t = E_t\left[\int_t^\infty \frac{M_s}{M_t} D_s ds\right] \tag{58}$$

where p_t is the price of a contract that generates the stream of cash flow $(D_s)_{t\le s\le\infty}$. The above equality is equivalent to

$$M_t p_t + \int_0^t M_s D_s ds = E_t\left[\int_0^\infty M_s D_s ds\right] \tag{59}$$

The right-hand side is the conditional expectation at different times t of the same random variable, hence a martingale. Thus, also the left-hand side is a martingale, hence its expected conditional increments are zero.

To identify the conditional increments $d(M_t p_t) + M_t D_t dt$ note that:

$$\frac{d(M_t p_t)}{M_t p_t} + \frac{M_t D_t}{M_t p_t} dt = \frac{dM_t}{M_t} + \frac{dp_t}{p_t} + \frac{d\langle M, p\rangle_t}{M_t p_t} + \frac{D_t dt}{p_t} \tag{60}$$

Since dividend growth is constant, the price should be of the form $p_t = D_t g(x_t)$ for some function g, i.e., linear in the current cash flow for a given state x_t. It follows that:

$$\frac{dp_t}{p_t} = \frac{dD_t}{D_t} + \frac{dg(x_t)}{g(x_t)} + \frac{\langle g(x), D\rangle_t}{g(x_t) D_t} \tag{61}$$

$$= \mu_D dt + \tfrac{x_t g'(x_t)}{g(x_t)} \left(\mu_c dt + \sigma_c dW_t - \tfrac{dh_t}{h_t}\right) + \tfrac{\sigma_c^2}{2} \tfrac{x_t^2 g''(x_t)}{g(x_t)} dt + \sigma_c \sigma_D \rho \tfrac{x_t g'(x_t)}{g(x_t)} dt \tag{62}$$

$$+ \sigma_D \rho dW_t^c + \sigma_D \sqrt{1-\rho^2} dW_t^D \tag{63}$$

$$= \left(\mu_D + \tfrac{x_t g'(x_t)}{g(x_t)} (\mu_c + \phi + \sigma_c \sigma_D \rho) + \tfrac{\sigma_c^2}{2} \tfrac{x_t^2 g''(x_t)}{g(x_t)}\right) dt \tag{64}$$

$$- \frac{x_t g'(x_t)}{g(x_t)} d\eta_t \tag{65}$$

$$+ \left(\rho \sigma_D + \sigma_c \frac{x_t g'(x_t)}{g(x_t)}\right) dW_t^c + \sigma_D \sqrt{1-\rho^2} dW_t^D \tag{66}$$

and the covariation rate between the growth rates of M and p equals

$$\frac{d\langle M, p\rangle_t}{M_t p_t} = -\gamma \sigma_c \left(\rho \sigma_D + \sigma_c \frac{x_t g'(x_t)}{g(x_t)}\right) dt \tag{67}$$

Because any local martingale of finite-variation is necessarily constant (Revuz and Yor, 1999, Proposition IV.1.2), the martingale condition for (59) requires that both (64) and (65) are zero, leading to the differential equation

$$-r_0 + \mu_D + \frac{x g'(x)}{g(x)} (\mu_c + \phi + \sigma_D \sigma_c \rho) + \frac{\sigma_c^2}{2} \frac{x^2 g''(x)}{g(x)} - \gamma \sigma_c \left(\rho \sigma_D + \sigma_c \frac{x g'(x)}{g(x)}\right) + \frac{1}{g(x)} = 0 \qquad x \in (0,1) \tag{68}$$

with the boundary condition

$$-\alpha(1-\gamma) - \frac{x g'(x)}{g(x)} = 0 \qquad \text{for} \quad x = 1 \tag{69}$$

which reflects that the process η_t increases only on the set $\{x_t = 1\}$, while remaining constant on the set $\{x_t \in (0,1)\}$

This is a inhomogeneous, second-order, linear ODE. One of its solutions is the constant $g(x) = 1/y_0$, and we assume that y_0 is positive. Thus, the general solution to the above ODE equals to this constant, plus a linear combination of the solutions of the corresponding homogeneous equation, i.e.,

$$g(x) = \frac{1}{y_0} + C_1 x^{\delta_-} + C_2 x^{\delta_+} \qquad \delta_\pm = \gamma + \frac{1}{2} - \frac{\rho \sigma_D}{\sigma_c} - \frac{\mu_c + \phi}{\sigma_c^2} \pm \sqrt{2 \frac{\beta - \gamma \phi - \mu_D}{\sigma_c^2} + \left(\frac{1}{2} - \frac{\rho \sigma_D}{\sigma_c} - \frac{\mu_c + \phi}{\sigma_c^2}\right)^2} \tag{70}$$

The constants C_1 and C_2 are identified by conditions on $g(x)$ at the boundaries $x = 0, 1$. Since $\delta_- < 0$, at $x = 0$ the classic solution recovers only if $C_1 = 0$. For the other boundary, the martingale property implies the Neumann condition (69). The resulting yield $D_t/p_t = 1/g(x_t)$ is then, denoting δ_+ as simply δ:

$$y(x_t) = \frac{y_0}{1 - \frac{\alpha(1-\gamma)}{\alpha(1-\gamma)+\delta} x_t^{\delta}} \tag{71}$$

In the special case $\mu_D = \sigma_D = 0$, the above formula gives the price of an asset that pays a constant cash flow – a consol bond. The resulting consol rate is therefore:

$$r(x_t) = \frac{r_0}{1 - \frac{\alpha(1-\gamma)}{\alpha(1-\gamma)+\delta_0} x_t^{\delta_0}} \tag{72}$$

where r_0 is the Lucas interest rate, while δ_0 is the value of δ obtained with $\mu_D = \sigma_D = 0$:

$$\delta_0 = \gamma + \frac{1}{2} - \frac{\mu_c + \phi}{\sigma_c^2} + \sqrt{2\frac{\beta - \gamma\phi}{\sigma_c^2} + \left(\frac{1}{2} - \frac{\mu_c + \phi}{\sigma_c^2}\right)^2} \tag{73}$$

A.4 Short-Term Rate

Consider the price of a bond that is repaid over time at constant rate, i.e. with the cash flow $D_t = e^{-t/T}/T$, where T denotes the average maturity, in that:

$$\int_0^\infty t/T e^{-t/T} dt = T, \tag{74}$$

and note that, with a constant interest rate r_0, the price of this asset is

$$\int_0^\infty e^{-r_0 t} e^{-t/T}/T dt = \frac{1}{1 + r_0 T} \tag{75}$$

i.e., the price of a zero-coupon bond with maturity T without compounding.

The price of this bond equals its current cash flow times a function of the state variable, i.e. $p_t = e^{-t/T}/T b(x_t)$ for some function b, whence

$$\frac{dp_t}{p_t} = \frac{b'(x_t)dx_t}{b(x_t)} + \frac{1}{2}\frac{b''(x_t)d\langle x\rangle_t}{b(x_t)} - \frac{dt}{T} \tag{76}$$

$$= \left(-\frac{1}{T} + \frac{x_t b'(x_t)}{b(x_t)}\mu + \frac{\sigma^2}{2}\frac{x_t^2 b''(x_t)}{b(x_t)}\right) dt - \frac{x_t b'(x_t)}{b(x_t)}(-\phi dt + d\eta_t) + \frac{x_t b'(x_t)}{b(x_t)}\sigma dW_t \tag{77}$$

Thus, collecting the drift terms arising from the martingale condition

$$\frac{d(M_t p_t)}{M_t p_t} + \frac{M_t e^{-t/T}/T}{M_t p_t} dt = \frac{dM_t}{M_t} + \frac{dp_t}{p_t} + \frac{d\langle M, p\rangle_t}{M_t p_t} + \frac{e^{-t/T}/T dt}{p_t} \tag{78}$$

the next ordinary differential equation follows

$$-r_0 - \frac{1}{T} + \frac{xb'(x)}{b(x)}(\mu + \phi) + \frac{\sigma^2}{2}\frac{x^2 b''(x)}{b(x)} - \gamma\sigma^2 \frac{xb'(x)}{b(x)} + \frac{1}{b(x)} = 0 \qquad x \in (0,1) \tag{79}$$

This ODE has the constant solution $b(x) = T/(1 + r_0 T)$, which corresponds to the familiar $p_t = e^{-t/T}\frac{1}{1+r_0 T}$ for $\alpha = 0$. The general solution is thus

$$b(x) = \frac{T}{1 + r_0 T} + C_1 x^{\delta_{T-}} + C_2 x^{\delta_{T+}} \qquad \delta_{T\pm} = \gamma - \frac{\mu + \phi}{\sigma^2} + \frac{1}{2} \pm \sqrt{2\frac{\beta - \gamma\phi + 1/T}{\sigma^2} + \left(\frac{\mu + \phi}{\sigma^2} - \frac{1}{2}\right)^2} \tag{80}$$

and the constants C_1, C_2 are identified by the boundary conditions at $x = 0, 1$. For $x = 0$ the classical setting recovers, whence $C_1 = 0$, while the Neumann condition

$$-\alpha(1-\gamma) - \frac{xb'(x)}{b(x)} = 0 \qquad x = 1 \tag{81}$$

leads to the bond price formula:

$$p_t = \frac{e^{-t/T}}{1 + r_0 T}\left(1 - \frac{\alpha(1-\gamma)}{\alpha(1-\gamma) + \delta_T} x^{\delta_T}\right) \tag{82}$$

To calculate the average term structure, consider the unconditional average at time 0 of the above expression, which is

$$E[p_0] = \frac{\delta_T\left(\sigma_c^2\left(-\left(\alpha(-\gamma) + \alpha + \delta_T - 1\right)\right) - 2\left(\mu_c + \phi\right)\right)}{\left(r_0 T + 1\right)\left(\alpha(\gamma - 1) - \delta_T\right)\left(2\left(\mu_c + \phi\right) + \sigma_c^2\left(\delta_T - 1\right)\right)} \tag{83}$$

A.5 Long Term Rate (Eq. (15))

The spot rate R_t^T at time t with maturity T is the solution to the equation

$$e^{-R_t^T(T-t)} = E_t\left[\frac{M_T}{M_t}\right] \tag{84}$$

and the long-term rate is defined as the limit, i.e.

$$R_t^\infty = \lim_{T\to\infty} R_t^T = \lim_{T\to\infty} -\frac{1}{T-t}\log E_t\left[\frac{M_T}{M_t}\right] \tag{85}$$

Recall that, by the Fenyman-Kac formula, the price $B(t, x_t)$ of a zero-coupon bond with maturity T satisfies the partial differential equation

$$B_t(t,x) - r_0 B(t,x) + xB_x(t,x)\left(\mu_c + \phi - \gamma\sigma_c^2\right) + \frac{\sigma_c^2}{2}x^2 B_{xx}(t,x) = 0 \qquad (t,x) \in [0,T) \times (0,1) \tag{86}$$

with boundary conditions

$$B(t,0) = 0 \qquad t \in [0,T) \tag{87}$$

$$B_x(t,1) = 0 \qquad t \in [0,T) \tag{88}$$

$$B(T,x) = 1 \qquad x \in (0,1) \tag{89}$$

Guessing that the long-term rate R_t^∞ is independent of t (in view of the positive recurrence of the state variable x_t), and thus equal to some constant $r_0 + \lambda$, for long horizons bond prices should be of the approximate form $p(t,x) \approx g(x)e^{-(r_0+\lambda)(T-t)}$. Replacing this guess into the previous equation leads to the ordinary differential equation

$$\lambda g(x) + xg'(x)(\mu_c + \phi - \gamma\sigma_c^2) + \frac{\sigma_c^2}{2}x^2 g''(x) = 0 \tag{90}$$

with boundary conditions

$$g(0) = 0 \qquad g'(1) = 0 \tag{91}$$

Note that the function $g(x)$ is determined up to a multiplicative constant, as the value of the maturity T is arbitrary. The general solution to equation (90) is $g(x) = C_1 x^{\nu_-(\lambda)} + C_2 x^{\nu_+(\lambda)}$, where

$$\nu_\pm(\lambda) = -\frac{\mu_c + \phi}{\sigma_c^2} + \gamma + \frac{1}{2} \pm \sqrt{\left(-\frac{\mu_c + \phi}{\sigma_c^2} + \gamma + \frac{1}{2}\right)^2 - \frac{2\lambda}{\sigma_c^2}} \tag{92}$$

Noting that $\nu_-(\lambda) < 0$, it follows that the boundary condition $g(0) = 0$ implies that $C_1 = 0$, while the value of C_2 is arbitrary, in view of the arbitrariness of the multiplicative constant. Indeed, the boundary condition on $g'(1)$ identifies

$$\lambda = \frac{1}{2}\alpha(\gamma - 1)\left((\alpha(1-\gamma) + 2\gamma + 1)\sigma_c^2 - 2(\mu_c + \phi)\right) \tag{93}$$

which in turn gives the following formula for the long-term rate

$$R^\infty = r_0 + \lambda = \beta - \alpha(\gamma - 1)\phi + \mu_c\gamma^* - \frac{\sigma_c^2}{2}\gamma^*(\gamma^* + 1) \qquad \gamma^* = \alpha + (1-\alpha)\gamma \tag{94}$$

Thus, the long term rate is obtained by replacing γ with $\gamma^* = \alpha + (1-\alpha)\gamma$ in the Lucas formula (7) and β with $\beta - \alpha(\gamma - 1)\phi$.

A.6 Expected Returns (Eq. (29))

The average, conditional expected return at time t for horizon T is defined as $E_t\left[\int_t^{t+T} \frac{dp_s + D_s ds}{p_s}\right]$. In view of (61) and (68),

$$\frac{dp_t + D_t dt}{p_t} = \left(\mu_D + \tfrac{x_t g'(x_t)}{g(x_t)}(\mu_c + \phi + \sigma_c\sigma_D\rho) + \tfrac{\sigma_c^2}{2}\tfrac{x_t^2 g''(x_t)}{g(x_t)} + \tfrac{1}{g(x_t)}\right)dt - \tfrac{x_t g'(x_t)}{g(x_t)} d\eta_t \tag{95}$$

$$+ \left(\rho\sigma_D + \sigma_c \frac{x_t g'(x_t)}{g(x_t)}\right) dW_t^c + \sigma_D\sqrt{1-\rho^2}\, dW_t^D \tag{96}$$

$$= \left(r_0 + \gamma\sigma_c\left(\sigma_D\rho + \sigma_c\frac{x_t g'(x_t)}{g(x_t)}\right)\right)dt - \frac{x_t g'(x_t)}{g(x_t)} d\eta_t \tag{97}$$

$$+ \left(\rho\sigma_D + \sigma_c \frac{x_t g'(x_t)}{g(x_t)}\right) dW_t^c + \sigma_D\sqrt{1-\rho^2}\, dW_t^D, \tag{98}$$

The long-term expected return follows from:

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T \frac{dp_t + D_t dt}{p_t} = \lim_{T\uparrow\infty}\frac{1}{T}\int_0^T \left(r_0 + \gamma\sigma_c\left(\sigma_D\rho + \sigma_c\frac{x_t g'(x_t)}{g(x_t)}\right)\right)dt - \lim_{T\uparrow\infty}\frac{1}{T}\int_0^T \frac{x_t g'(x_t)}{g(x_t)} d\eta_t \tag{99}$$

$$= r_0 + \gamma\sigma_c\left(\sigma_D\rho + \sigma_c\int_0^1 \frac{x g'(x)}{g(x)} m(x)dx\right) - \frac{g'(1)}{g(1)}\frac{\sigma_c^2}{2} m(1) \tag{100}$$

The last term, in view of the boundary condition $-g'(1)/g(1) = \alpha(1-\gamma)$ and the density $m(x) = \lambda x^{\lambda - 1}$, equals

$$-\frac{g'(1)}{g(1)}\frac{\sigma_c^2}{2} m(1) = \alpha(1-\gamma)\frac{\sigma_c^2}{2}\left(2\frac{\mu_c + \phi}{\sigma_c^2} - 1\right) = \alpha(1-\gamma)\left(\mu_c + \phi - \frac{\sigma_c^2}{2}\right) \tag{101}$$

whereas the above integral satisfies

$$\int_0^1 \frac{x g'(x)}{g(x)} m(x)dx = \int_0^1 \frac{\alpha(\gamma - 1)\delta_+ x^{\delta_+}}{\delta_+ + \alpha(\gamma - 1)\left(x^{\delta_+} - 1\right)} \lambda x^{\lambda - 1} dx \tag{102}$$

A.7 Welfare (6)

In the Lucas setting ($\alpha = 0$), the expected utility of the representative agent is

$$u = E\left[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right] = \frac{1}{\beta - (1-\gamma)\left(\mu_c - \frac{\gamma}{2}\sigma_c^2\right)} \tag{103}$$

This means that an economy that grows at rate μ_c with volatility σ_c is equivalent to a riskless economy that grows at rate μ_e defined by

$$\mu_e := \mu_c - \frac{\gamma}{2}\sigma_c^2 \tag{104}$$

For $\alpha > 0$, expected utility is more complex, because it depends also on the loss aversion α and on the state x. Suppose that at time t future expected utility is of the form $(c_t/h_t^\alpha)^{1-\gamma}/(1-\gamma)u(x_t)$, where the function $u(x)$ is to be determined, i.e.

$$\frac{(c_t/h_t^\alpha)^{1-\gamma}}{1-\gamma}u(x_t) = E_t\left[\int_t^\infty e^{-\beta(s-t)}\frac{(c_s/h_s^\alpha)^{1-\gamma}}{1-\gamma}ds\right] \tag{105}$$

It follows that

$$\int_0^t e^{-\beta s}\frac{(c_s/h_s^\alpha)^{1-\gamma}}{1-\gamma}ds + e^{-\beta t}\frac{(c_t/h_t^\alpha)^{1-\gamma}}{1-\gamma}u(x_t) = E_t\left[\int_0^\infty e^{-\beta s}\frac{(c_s/h_s^\alpha)^{1-\gamma}}{1-\gamma}ds\right] \tag{106}$$

Thus, since the right-hand side is a martingale, so is the left-hand side, hence its drift must be zero. Calculating the dynamics of the left-hand side with Itô's formula, and setting the drift to zero, implies that

$$\frac{\sigma_c^2}{2}x^2u''(x) + (\mu_c + \phi)xu'(x) + 1 - u(x)\left(\beta - (1-\gamma)\left(\mu_c - \frac{\gamma}{2}\sigma_c^2\right)\right) = 0 \qquad x \in (0,1) \tag{107}$$

One boundary condition is that at $x = 0$ the classical case obtains, i.e.

$$u(0) = \frac{1}{\beta - (1-\gamma)\left(\mu_c - \frac{\gamma}{2}\sigma_c^2\right)} \tag{108}$$

The second condition is that at $x = 1$, when a new target is established, the marginal utility of the present consumption increase equals the marginal decrease in future utility from a higher target, i.e.

$$-\alpha(1-\gamma) - \frac{xu'(x)}{u(x)} = 0 \qquad \text{for} \quad x = 1 \tag{109}$$

The solution to equation (107) is

$$u(x) = \frac{2}{2\beta + (\gamma-1)\left(2\mu_c - \gamma\sigma_c^2\right)} \tag{110}$$

$$+ \frac{4\alpha(\gamma-1)\sigma_c^2 x^{\frac{\sqrt{-4\sigma_c^2(-2\beta+(3-2\gamma)\mu_c+\phi)-(4(\gamma-1)\gamma-1)\sigma_c^4+4(\mu_c+\phi)^2}-2\mu_c+\sigma_c^2-2\phi}{2\sigma_c^2}}}{((\gamma-1)(\gamma\sigma_c^2-2\mu_c)-2\beta)\left((2\alpha(\gamma-1)-1)\sigma_c^2-\sqrt{-4\sigma_c^2(-2\beta+(3-2\gamma)\mu_c+\phi)-(4(\gamma-1)\gamma-1)\sigma_c^4+4(\mu_c+\phi)^2}+2\mu_c+2\phi\right)} \tag{111}$$

This formula is complicated because it describes the expected utility from future consumption as a function of the current state x. For $\sigma_c = 0$ the above expression reduces to

$$\frac{1}{\beta + (\gamma-1)\mu_e} \tag{112}$$

Matching the two expressions to solve for μ_e, and expanding the result for σ_c small, it follows that:

$$\mu_e = \mu_c + \frac{\sigma_c^2\left(\alpha(\beta-\phi) + (\alpha(\gamma-2)+\gamma)\mu_c + \gamma\phi\right)}{2(\alpha-1)\left(\mu_c+\phi\right)} + O\left(\sigma_c^3\right) \tag{113}$$

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