A New Formula for the Expected Excess Return of the Market

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Abstract

Key to deriving the lower bound to the expected excess return of the market in Martin (2017) is the assumption of the negative correlation condition (NCC). We improve on the lower bound characterization by proposing an exact formula for the conditional expected excess return of the market. In our formula, each risk-neutral return central moment contributes to the expected excess return and is representable in terms of known option prices. To interpret theoretical and empirical distinctions between our formula and the lower bound, we develop and study the asset-pricing restrictions of the NCC.

Keywords: Expected excess return of the market, negative correlation condition, lower bound

Contents

I still remember the teasing we financial economists, Harry Markowitz, William Sharpe, and I, had to put up with from the physicists and chemists in Stockholm when we conceded that the basic unit of our research, the expected rate of return, was not actually observable. — Merton Miller. The History of Finance, Journal of Portfolio Management 1999 (Summer), 95–101.

1 The negative correlation condition in Martin (2017)

We employ the following notation:

S_{t+T} = price of the equity market index at some future date $t + T$;

$R_T \equiv \frac{S_{t+T}}{S_t}$ = gross return of the equity market index over t to $t+T$. Assume $R_T > 0$;

$\mathbb{E}_t^{\mathbb{P}}(.)$ = expectation under the real-world probability measure, $\mathbb{P}$;

$\mathbb{E}_t^{\mathbb{Q}}(.)$ = expectation under the risk-neutral probability measure, $\mathbb{Q}$;

M_T = stochastic discount factor, with $\mathbb{E}_t^{\mathbb{P}}(M_T R_T) = 1$ holding;

$R_{f,t} = \frac{1}{\mathbb{E}_t^{\mathbb{P}}(M_T)} = \mathbb{E}_t^{\mathbb{Q}}(R_T) = T$-period gross risk-free return;

$\mathbf{1}$ = conformable vector of ones;

$\mathbf{Z}_T[R_T]$ = gross return vector contingent on R_T and satisfies $\mathbb{E}_t^{\mathbb{P}}(M_T \mathbf{Z}_T[R_T]) = \mathbf{1}$;

$M_T[R_T]$ = projected stochastic discount factor, with $\mathbb{E}_t^{\mathbb{P}}(M_T[R_T] \mathbf{Z}_T[R_T]) = \mathbf{1}$ holding;

$\text{call}_{t,T}[K]$ = time t price of the call option on the market, with strike K, expiring in T-periods;

$\text{put}_{t,T}[K]$ = time t price of the put option on the market, with strike K, expiring in T-periods;

$\text{cov}_t^{\mathbb{P}}(\tilde{x}, \tilde{y})$ = conditional covariance between two random variables $\tilde{x}$ and $\tilde{y}$ under $\mathbb{P}$;

$\text{var}_t^{\mathbb{Q}}(R_T)$ = conditional variance of R_T under the risk-neutral measure $\mathbb{Q}$.

Definition 1 *(**Negative correlation condition, NCC**): The NCC is the assumption that* $\text{cov}_t^{\mathbb{P}}(M_T R_T, R_T) \leq 0$, *for all* M_T *satisfying* $\mathbb{E}_t^{\mathbb{P}}(M_T R_T) = 1$. ∎

Martin (2017) introduces the relation in equation (1) and derives the conditional lower bound, $R_{f,t}^{-1} \operatorname{var}_t^{\mathbb{Q}}(R_T)$, of the expected excess return of the market, as follows:

$$\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t} = \underbrace{\mathbb{E}_t^{\mathbb{P}}(M_T) \mathbb{E}_t^{\mathbb{P}}\left(\frac{M_T}{\mathbb{E}_t^{\mathbb{P}}(M_T)} R_T^2 - \frac{1}{\{\mathbb{E}_t^{\mathbb{P}}(M_T)\}^2}\right)}_{= R_{f,t}^{-1} \operatorname{var}_t^{\mathbb{Q}}(R_T)} - \operatorname{cov}_t^{\mathbb{P}}(M_T R_T, R_T) \tag{1}$$

$$\geq \underbrace{R_{f,t}^{-1} \operatorname{var}_t^{\mathbb{Q}}(R_T)}_{\text{Lower bound}}. \quad \text{(provided the NCC holds)} \tag{2}$$

The lower bound of the expected excess return of the market can be inferred from option prices as $\frac{2}{S_t^2} \int_{K < R_{f,t} S_t} \operatorname{put}_{t,T}[K] \, dK + \frac{2}{S_t^2} \int_{K > R_{f,t} S_t} \operatorname{call}_{t,T}[K] \, dK$.

The following remarks are central to the framework in Martin (2017) and guide our theoretical and empirical investigation. First, the inequality in equation (2) holds under all change-of-measure densities (i.e., for *every* M_T) such that $-\operatorname{cov}_t^{\mathbb{P}}(M_T R_T, R_T)$ is nonnegative for each t. Second, the derived lower bound is theoretically unsupportable if one could find an economically plausible M_T for which $\operatorname{cov}_t^{\mathbb{P}}(M_T R_T, R_T)$ is *positive*.

The lower bound in equation (2) touches a theme dating back to Merton (1980), Black (1993), and Elton (1999) on how elusive it is to estimate the expected return of the market. The topic is cutting-edge, and peer-reviewed academic evidence on the lower bound of the expected excess return remains relevant and influential, and is invigorating research complementarities in theory and practice (see, e.g., Chabi-Yo and Loudis (2019), Kadan and Tang (2019), Martin and Wagner (2019), and Schneider and Trojani (2019), among others).

Improving on the lower bound characterization in Martin (2017), our innovation is an exact theoretical representation for $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$, which can be synthesized from quantities inferred from option prices, specifically, the risk-neutral return central moments of order higher than one.

The big picture question is: How should one view our expression for $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ in light of the lower bound formula? First, we show empirically that there are sizable differences between our estimates of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ versus those from Martin (2017). Our steps to synthesizing $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ from option prices do not utilize the formalism of the NCC (i.e., whether $\operatorname{cov}_t^{\mathbb{P}}(M_T R_T, R_T) \leq 0$). Second, if the NCC is not a generic property *at each date t*, then having analytical formulations are of value. Addressing this issue, we identify the theoretical asset-pricing restrictions that underlie

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the NCC, and then propose empirical tests. Pivotal to our analysis, if $\text{cov}_t^{\mathbb{P}}(M_T R_T, R_T) > 0$, then $R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_T)$ is an *upper* bound and not a *lower* bound.

2 Decomposing the expected excess return of the market

We assume that there exists a *projected* SDF, $M_T[R_T]$, in the sense of Rosenberg and Engle (2002, Section 2.2), satisfying $M_T[R_T] > 0$, $\mathbb{E}_t^{\mathbb{P}}(M_T[R_T]) = \frac{1}{R_{f,t}} < \infty$, $\mathbb{E}_t^{\mathbb{P}}(M_T[R_T] R_T) = 1$, and $\mathbb{E}_t^{\mathbb{P}}(\{M_T[R_T]\}^2) < \infty$.

Why focus on SDFs that are a function of R_T? For one, Martin (2017) posits that if $M_T = \frac{1}{R_T}$, then $\text{cov}_t^{\mathbb{P}}(M_T R_T, R_T) = 0$, and $R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_T)$ is the tightest lower bound for $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ and can be extracted from option prices on the market index. For another, as shown in Rosenberg and Engle (2002), when it comes to considering claims written on the market return for which data can be exploited, it suffices to work with the projection of the SDF (even though the SDF itself may admit additional state dependencies) onto the space generated by the market return. Thus, we need only model $\mathbb{E}^{\mathbb{P}}(\mathcal{M} \mid R_T)$, for any $\mathcal{M}$ that represents the change of probability with the bond price as numeraire, without the necessity of exploring state-dependent specifications of the SDF (e.g., Campbell and Cochrane (1999), Epstein and Zin (1991), and Hansen and Renault (2009)).

Informed by theory and data from options markets, our novelty is to develop a theoretical representation for $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$, which can be compared with Martin (2017). Our characterization of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ is consistent with asset-pricing theory, does not require exploiting the workings of the NCC (i.e., $\text{cov}_t^{\mathbb{P}}(M_T R_T, R_T) \leq 0$), and can be employed in empirical applications.

Result 1 (Expected excess return of the market) *Suppose M_T is of the following form:*

$$M_T[R_T] = \exp\left(m_0 - 1 - \phi\left(R_T - R_{f,t}\right)\right), \qquad \text{for some constants } m_0 \text{ and } \phi > 0. \tag{3}$$

Then, the conditional expected excess return of the market can be decomposed, as follows:

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t} = \frac{1}{\phi} \Bigg(&\{\phi \, \text{SD}_t^{\mathbb{Q}}(R_T)\}^2 + \frac{1}{2}\{\phi \, \text{SD}_t^{\mathbb{Q}}(R_T)\}^3 \times \text{Skewness}_t^{\mathbb{Q}}(R_T) \\
&+ \frac{1}{6}\{\phi \, \text{SD}_t^{\mathbb{Q}}(R_T)\}^4 \times \{\text{Kurtosis}_t^{\mathbb{Q}}(R_T) - 3\} \\
&+ \frac{1}{24}\{\phi \, \text{SD}_t^{\mathbb{Q}}(R_T)\}^5 \{\text{Hskewness}_t^{\mathbb{Q}}(R_T) - 10\, \text{Skewness}_t^{\mathbb{Q}}(R_T)\} + \ldots \Bigg),
\end{aligned} \tag{4}$$

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where $\text{SD}_t^{\mathbb{Q}}(R_T) \equiv \sqrt{\text{var}_t^{\mathbb{Q}}(R_T)}$, $\text{Skewness}_t^{\mathbb{Q}}(R_T)$, $\text{Kurtosis}_t^{\mathbb{Q}}(R_T)$, $\text{Hskewness}_t^{\mathbb{Q}}(R_T) \equiv \frac{\mathbb{E}_t^{\mathbb{Q}}(\{R_T - R_{f,t}\}^5)}{\{\text{SD}_t^{\mathbb{Q}}(R_T)\}^5}$ *are, respectively, the conditional risk-neutral return volatility, skewness, kurtosis, and hyperskewness. Each required risk-neutral return central moment is algebraic in option prices (as displayed in equations (A36)–(A39) of Appendix A).*

Proof: See Appendix A. ∎

The formula in equation (4) deduces a *value* for $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ and not just a lower bound. This new estimate of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$, at each date t, is expressed as an infinite series, with each term related to risk-neutral return central moments of order higher than one. However, successively smaller weights are assigned to $\text{Skewness}_t^{\mathbb{Q}}(R_T)$, to $\{\text{Kurtosis}_t^{\mathbb{Q}}(R_T) - 3\}$, and to $\{\text{Hskewness}_t^{\mathbb{Q}}(R_T) - 10\,\text{Skewness}_t^{\mathbb{Q}}(R_T)\}$, provided that $\{\phi\,\text{SD}_t^{\mathbb{Q}}(R_T)\} < 1$.

Our approach is to consider the $\mathbb{P}$-measure return density as a weighted $\mathbb{Q}$-measure return density, as warranted by the form of M_T in equation (3). Then, we exploit the properties of the cumulant generating functions under $\mathbb{P}$ and $\mathbb{Q}$ (as detailed in Appendix A). Crucially, no parametric assumptions are made about the risk-neutral density of market returns.

Furthermore, equation (4) does not specialize to the tightest lower bound in Martin (2017) (i.e., $R_{f,t}^{-1}\{\text{SD}_t^{\mathbb{Q}}(R_T)\}^2$), as the specification $M_T = 1/R_T$ is not subsumed within equation (3).

The form of M_T in equation (3) is an assumption about the *projected* SDF and is complementary to Schneider and Trojani (2019). In particular, Schneider and Trojani (2019), in essence, project the SDF onto $(R_T, R_T^2, R_T^3, \ldots)$. Taking a Taylor series of $M_T[R_T]$ in equation (3), one obtains $M_T[R_T] = e^{m_0 - 1}\left(1 + \{-\phi(R_T - R_{f,t})\} + \frac{1}{2}\{-\phi(R_T - R_{f,t})\}^2 + \frac{1}{6}\{-\phi(R_T - R_{f,t})\}^3\right) + \frac{1}{24}\{-\phi(R_T - R_{f,t})\}^4 + \ldots)$, and, so, intuitively speaking, our equation (3) involves an infinite series of higher-order polynomials and is akin to Rosenberg and Engle (2002, equation (12)).

Our analytical representation of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ is a step forward, as Schneider and Trojani (2019) use assumptions about the sign of the covariance, under $\mathbb{P}$, between the SDF and various moments of the market return to obtain *bounds* on the $\mathbb{P}$ moments of the market return.

Our M_T in equation (3) is economically motivated, is informed by the data, and can arise in the context of minimum discrepancy problems considered in Borovička, Hansen, and Scheinkman (2016, Section VIII.B), Almeida and Garcia (2017), and Ghosh, Julliard, and Taylor (2017). Specifically,

one solves $\inf_{M \in \mathbb{M}} \mathbb{E}^{\mathbb{P}}(M \log M)$ with $\mathbb{M} \equiv \{M > 0$ such that $\mathbb{E}^{\mathbb{P}}(M(R - R_f)) = 0,\ \mathbb{E}^{\mathbb{P}}(M) = \mathbb{E}^{\mathbb{P}}(R_{f,t}^{-1}) \equiv \mu_M$, and $\mathbb{E}^{\mathbb{P}}(M \log M) < \infty\}$, where $\mathbb{E}^{\mathbb{P}}(.)$ is unconditional expectation. The optimal solution is $M^* = \exp(m_0^* - 1 - \phi^*(R - R_f))$, where (ϕ^*, m_0^*) solve $\arg \inf_{(\phi, m_0)}\{-m_0\, \mu_M + \mathbb{E}^{\mathbb{P}}(\exp(m_0 - 1 - \phi(R - R_f)))\}$. The essence of the estimation procedure is that M_T, parameterized by ϕ, enforces the correct unconditional pricing of the excess return of the market.

Our implementation indicates that the estimate of ϕ is 2.274, with a 90% bootstrap confidence interval of (1.30 3.29). See the note to Table 1. In our setup, ϕ reflects the sensitivity $\frac{d \log M_T[R_T]}{d R_T}$, and $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ is unaffected by m_0.

What is the value-added of our formula? It mitigates theoretical and empirical reliance on the NCC and on the conditional lower bound on the expected excess return of the market.

Consider Table 1, which presents features of our estimates of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$. The first question we ask is the following: *How reasonable are our estimates of the conditional expected excess return?* To answer this, we use monthly return data covering 348 nonoverlapping option expiration cycles (T=28 days) on the S&P 500 equity index, from January 1990 to December 2018 (29 years). The data on S&P 500 index options, used to construct each required risk-neutral return central moment, is described in the Internet Appendix (Section A).

The annualized average expected excess return is 8.97% (median is 5.8%) over the 29-year sample. There is considerable time variation, as depicted in Figure 1, with a 5th (95th) percentile value of 1.8% (26.2%). Our average $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ of 8.97% can be compared to the very long-run average market excess return of 7.81% over 1926:07–2018:12 (source: data library of Kenneth French). Furthermore, it is comparable to the estimate of 7.43% reported in Fama and French (2002) over 1951–2000. Fundamental to our methodology is the development of a portfolio of risk-neutral return central moments that synthesizes the conditional expected excess return of the market.

The next pertinent question is the following: How aligned is the lower bound, $R_{f,t}^{-1} \operatorname{var}_t^{\mathbb{Q}}(R_T)$, with our estimates of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$? We perform the following regression:

$$\underbrace{\mathbb{E}_t^{\mathbb{P}}(R_T)\ -\ R_{f,t}}_{\text{estimated from equation (4)}}\ =\ \Psi_0\ +\ \Psi_1\ \underbrace{\{R_{f,t}^{-1}\operatorname{var}_t^{\mathbb{Q}}(R_T)\}}_{\text{lower bound}}\ +\ \widetilde{\epsilon}_t, \tag{5}$$

$$\text{with null hypothesis of } \Psi_0 = 0 \text{ and } \Psi_1 = 1. \tag{6}$$

To compute p-values, we rely on the HAC estimator of Newey and West (1987), with automatically selected lags. The Wald test of $\Psi_0 = 0$ and $\Psi_1 = 1$, which reflects $\text{cov}_t^{\mathbb{P}}(M_T R_T, R_T) = 0$, is rejected with a p-value of 0.00. We obtain $\Psi_0 = 0.35$ (p-value of 0.00) and $\Psi_1 = 1.99$ (p-value of 0.00).

In Martin (2017, Table 1), the average lower bound, $R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_T)$, is 5% over the sample of 1/1996 to 1/2012. Our Table 1 (row (vi)) shows that the average $R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_T)$ is 4.32%, and is half the size of the average $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$. The difference $\{\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}\} - R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_T) = -\text{cov}_t^{\mathbb{P}}(M_T R_T, R_T)$, as plotted in Figure 2, manifests substantial dispersion through time.

Prompted by Barro (2006) and Wachter (2013), we investigate two additional questions using our method to estimate $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$.

How important is the perception of return extremes for $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$? This entails quantifying the role of the risk-neutral fourth cumulant in determining $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$. We find empirically that the contribution of $\frac{1}{24\phi}\{\phi\text{SD}_t^{\mathbb{Q}}(R_T)\}^5 \{\text{Hskewness}_t^{\mathbb{Q}}(R_T) - 10\,\text{Skewness}_t^{\mathbb{Q}}(R_T)\}$ to $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ is 0.006% annualized (on average; see row (v) of Table 1). Additionally, Table 1 (row (ii)) reveals that the average $\frac{1}{\phi}(\{\phi\,\text{SD}_t^{\mathbb{Q}}(R_T)\}^2 + \frac{1}{2}\{\phi\,\text{SD}_t^{\mathbb{Q}}(R_T)\}^3 \times \text{Skewness}_t^{\mathbb{Q}}(R_T))$ is 8.91%, implying an average contribution of $\frac{1}{6\phi}\{\phi\,\text{SD}_t^{\mathbb{Q}}(R_T)\}^4 \times \{\text{Kurtosis}_t^{\mathbb{Q}}(R_T) - 3\}$ of 0.049% (see row (iv)). Our estimates indicate that tail effects matter little quantitatively, on average.

How important is the perception of disasters, as reflected in the relative pricing of OTM puts versus OTM calls, for $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$? To answer this, we compute the average of $\frac{1}{\phi}\{\phi\,\text{SD}_t^{\mathbb{Q}}(R_T)\}^2$ and obtain a value of 9.84%. See Table 1 (row (iii)). Thus, the presence of $\frac{1}{2}\{\phi\,\text{SD}_t^{\mathbb{Q}}(R_T)\}^3 \times \text{Skewness}_t^{\mathbb{Q}}(R_T)$ *reduces* the conditional expected excess return of the market, on average, by -0.93% (the estimate of risk-neutral skewness is never positive). There is information for expected excess returns in the risk-neutral third central moment of market returns.

What are the other implications of formula (4) and Table 1? The cornerstone of Martin and Wagner (2019) is the expression for $\mathbb{E}_t^{\mathbb{P}}(R_T^j) - \mathbb{E}_t^{\mathbb{P}}(R_T)$ (their equation (14)), for individual stocks $j = 1, \ldots, J$, and for the expected excess return of an individual stock $\mathbb{E}_t^{\mathbb{P}}(R_T^j) - R_{f,t}$ (their equation (15)). The former expression does *not* depend upon the validity of the NCC, but the latter requires an estimate (and not only a lower bound) of the expected return of the market. To obtain this estimate, an assumption is made that $\text{cov}_t^{\mathbb{P}}(M_T R_T, R_T)$ *is identically zero*. Thus, given the documented disparities between $\{\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}\} - R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_T)$, a cen-

tral input to $\mathbb{E}_t^{\mathbb{P}}(R_T^j) - R_{f,t}$ is susceptible to misspecification. Our estimates of $\mathbb{E}_t^{\mathbb{P}}(R_T)$ and $-\text{cov}_t^{\mathbb{P}}(M_T R_T, R_T)) = \{\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}\} - R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_T)$, using formula (4), can improve implementation aspects in Martin and Wagner (2019).

Chabi-Yo and Loudis (2019, equations (27) and (31)) present a *lower bound* on $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$, derived from risk-neutral return variance, skewness, and kurtosis. Our departure in equation (4) is that it offers an *explicit* expression for the conditional expected excess return of the market.

Finally, what could be the reasons that the lower bound (i.e., $R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_T)$) is not adequately aligned with our estimates of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$? First, informed by theory, our formula unpacks the contribution of other risk-neutral return central moments. Second, if the NCC were not to be a generic property, what, at first sight, appears to be the lower bound, could, in fact, be an upper bound. We develop this formally in Result 3. Hence, knowledge of the *lower bound* on the expected excess return of the market is of limited value if the NCC were to be violated.

3 A general formula for the expected excess return of the market

The specification of M_T in equation (3), which is an exponential function of R_T, balances tractability and generality. The projected SDF is analytic (i.e., $M_T \in \mathcal{C}^\omega$) and can be viewed as a polynomial in R_T through its Taylor expansion.

One may inquire: Can one generalize the functional form for M_T for which one could obtain analytical expressions for $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$, and still use input variables constructed from option prices?

To explore this possibility, we maintain that $M_T[R_T] > 0$ almost surely, and $M_T[R_T]$, or equivalently, $M_T[1 + r_T]$ (recalling $r_T \equiv R_T - 1$ and $r_{f,t} \equiv R_{f,t} - 1$), is continuous and infinitely differentiable in r_T, that is, $M_T[1 + r_T] \in \mathcal{C}^\infty$. We define

$$H[r_T] \;=\; \frac{1}{M_T[R_T]} \;=\; \frac{1}{M_T[1 + r_T]} \in \mathcal{C}^\infty. \tag{7}$$

This class may encompass theoretically plausible and empirically interesting specifications of projected SDFs, allowing us to comment on the robustness of our estimates of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ in Table 1. Additionally, no assumptions are made about whether the NCC holds.

Denoting $H'[r_T] \equiv \frac{dH[r_T]}{dr_T}$, $H''[r_T] \equiv \frac{d^2 H[r_T]}{dr_T^2}$, $H'''[r_T] \equiv \frac{d^3 H[r_T]}{dr_T^3}$, and $H''''[r_T] \equiv \frac{d^4 H[r_T]}{dr_T^4}$, and $H[r_{f,t}] = H[r_T]\big|_{r_T=r_{f,t}}$, $H'[r_{f,t}] = H'[r_T]\big|_{r_T=r_{f,t}}$ and so on, we consider the Taylor expansion

$$
\begin{aligned}
H[r_T] &= H[r_{f,t}] + H'[r_{f,t}](r_T - r_{f,t}) + \frac{H''[r_{f,t}]}{2}(r_T - r_{f,t})^2 \\
&+ \frac{H'''[r_{f,t}]}{3!}(r_T - r_{f,t})^3 + \frac{H''''[r_{f,t}]}{4!}(r_T - r_{f,t})^4 + \cdots .
\end{aligned}
\tag{8}
$$

We assume that $\frac{H'[r_{f,t}]}{H[r_{f,t}]}$, $\frac{H''[r_{f,t}]}{H[r_{f,t}]}$, $\frac{H'''[r_{f,t}]}{H[r_{f,t}]}$, and $\frac{H''''[r_{f,t}]}{H[r_{f,t}]}$ exist and are well-defined.

Result 2 (Expected excess return of the market when $H[r_T] \in \mathcal{C}^\infty$) *With the function $H[r_T] \in \mathcal{C}^\infty$ defined in equation (7), the conditional expected excess return of the market is*

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t} &= \frac{H'[r_{f,t}]}{H[r_{f,t}]}\{\text{SD}_t^{\mathbb{Q}}(R_T)\}^2 + \frac{1}{2}\frac{H''[r_{f,t}]}{H[r_{f,t}]}\{\text{SD}_t^{\mathbb{Q}}(R_T)\}^3 \times \text{Skewness}_t^{\mathbb{Q}}(R_T) \\
&+ \frac{1}{6}\frac{H'''[r_{f,t}]}{H[r_{f,t}]}\{\text{SD}_t^{\mathbb{Q}}(R_T)\}^4 \times \{\text{Kurtosis}_t^{\mathbb{Q}}(R_T) - 3\} + \cdots .
\end{aligned}
\tag{9}
$$

Proof: See Appendix B. ∎

One may ask if Result 2 specializes to Result 1? When $M_T = \exp\left(m_0 - 1 - \phi(R_T - R_{f,t})\right)$ (as in equation (3)), then $H[r_T] = \exp\left(-m_0 + 1 + \phi(r_T - r_{f,t})\right)$. Hence, $\frac{H'[r_{f,t}]}{H[r_{f,t}]} = \phi$, $\frac{H''[r_{f,t}]}{H[r_{f,t}]} = \phi^2$, and $\frac{H'''[r_{f,t}]}{H[r_{f,t}]} = \phi^3$. In this case, equation (9) coincides with the exact expression for $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ in equation (4). Our motivation for featuring M_T in equation (3) was twofold. First, to show that risk-neutral return central moments higher than *two* may be important for $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$. This effort was guided by the bounds formula of Martin (2017), whose sole driver is the risk-neutral return variance. Second, to offer a parsimonious, easy to implement formula in which the unknown parameter could be estimated using established methods.

Result 2 inherits the convenience and generality of Result 1, but enhances flexibility. For one, it can allow for state-dependent sensitivity of M_T to R_T. For another, it can accommodate alternative forms of dependencies of M_T on R_T. The following parameterizations illustrate our point and may offer potential avenues for improvement.

Case 1 (State-dependent sensitivity) *Suppose M_T is of the following form:*

$$
M_T[R_T] = \exp\left(m_0 - 1 - \{\phi + \phi_z z_t\}(R_T - R_{f,t})\right), \quad \text{for constants } m_0, \ \phi > 0, \text{and } \phi_z,
\tag{10}
$$

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where z_t is some economically relevant variable known at time t (e.g., changes in variance uncertainty). We deduce that $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ satisfies equation (9), with

$$\frac{H'[r_{f,t}]}{H[r_{f,t}]} = \phi + \phi_z\, z_t, \qquad \frac{H''[r_{f,t}]}{H[r_{f,t}]} = (\phi + \phi_z\, z_t)^2, \qquad \text{and} \qquad \frac{H'''[r_{f,t}]}{H[r_{f,t}]} = (\phi + \phi_z\, z_t)^3. \tag{11}$$

Case 2 (Alternative form of dependencies) *Suppose*

$$M_T[R_T] = \exp\left(m_0 - 1 - \phi\,(R_T - R_{f,t}) + \psi\{\frac{\log(R_T)}{\wp_t} - R_{f,t}\}\right). \tag{12}$$

where $\wp_t \equiv \frac{1}{R_{f,t}}\mathbb{E}_t^{\mathbb{Q}}(\log(R_T)) = 1 - \frac{1}{R_{f,t}} + \int_{K<S_t} \frac{1}{K^2}\mathrm{put}_{t,T}[K]\,dK + \int_{K>S_t} \frac{1}{K^2}\mathrm{call}_{t,T}[K]\,dK$ *is the price of the* $\log(R_T)$ *payoff. We deduce that* $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ *satisfies equation (9), with*

$$\frac{H'[r_{f,t}]}{H[r_{f,t}]} = \phi - \frac{\psi}{\wp_t\, R_{f,t}}, \tag{13}$$

$$\frac{H''[r_{f,t}]}{H[r_{f,t}]} = \frac{\wp_t^2(R_{f,t})^2\,\phi^2 + \wp_t\,\{\psi - 2\,R_{f,t}\,\psi\,\phi\} + \psi^2}{\wp_t^2(R_{f,t})^2}, \qquad \text{and} \tag{14}$$

$$\frac{H'''[r_{f,t}]}{H[r_{f,t}]} = \frac{\wp_t^3(R_{f,t})^3\,\phi^3 - \wp_t^2\psi\,(3(R_{f,t})^2\phi^2 - 3R_{f,t}\phi + 2) + 3\,\wp_t\psi^2(R_{f,t}\phi - 1) - \psi^3}{\wp_t^3(R_{f,t})^3}. \tag{15}$$

What are the defining features of these formulae? In each case, $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ depends on all risk-neutral return central moments, and not just on the risk-neutral variance. Second, the sensitivity of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ to each risk-neutral central moment is time-varying.

Case 2 admits a general form of dependencies since $\log(1+r_T) = r_T - \frac{r_T^2}{2} + \frac{r_T^3}{3} - \frac{r_T^4}{4} + \frac{r_T^5}{5}\ldots$, and hence $\log M_T$ is a polynomial in the net market return r_T, with state-dependent (via $\wp_t$) parameterized sensitivities. While such generalizations are implementable, they appear more cumbersome.

Framing our issues in the context of Case 1 and equation (10), Table 2 reports the results when we take a stand on the choice of z_t. The question is whether data supports time-varying sensitivity, $\phi + \phi_z\, z_t$, of M_T to R_T, which is equivalent to testing whether $\phi_z = 0$. Our estimation procedure is analogous to that used for equation (3), except that M_T is parameterized by ϕ and ϕ_z (for a choice of z_t). See the note to Table 2.

Guided by empirical and theoretical considerations (e.g., see, among others, Cochrane (1996) and Menzly, Santos, and Veronesi (2004)), we consider z_t to be the change in prior month realized

market variance or the prior one month return on the HML factor. The HML factor is often associated with variation in business conditions (e.g., Fama and French (1993)). The theoretical framework of Case 1 accommodates the use of economically relevant z_t variables amenable to construction over option expiration cycles.

There are two takeaways from Table 2. First, the 90% confidence intervals of both ϕ_z estimates bracket zero, implying that the data does not support a time-varying effect of R_T on M_T beyond that reflected in ϕ (as ascertained in the context of Case 1 and for our choices of z_t). Second, the estimates of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$, using equation (10), align with those in Table 1. Thus, the parsimoniously specified M_T in equation (3) may be preferred over the more complicated counterparts.

4 Developing and testing the asset-pricing restrictions of the NCC

The theoretical lower bound (i.e., $R_{f,t}^{-1} \operatorname{var}_t^{\mathbb{Q}}(R_T)$) — which is a consequence of the NCC — is the estimate of the conditional expected excess return of the market. This section develops and identifies theoretical restrictions on M_T to ascertain the generality of the NCC. Additionally, we construct theoretical (but empirically plausible) economies in which the NCC fails to hold. The empirical evidence on testing the NCC helps to reconcile our perspective on considering exact theoretical representations of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$.

4.1 Theoretical restrictions of the NCC

We first present a theoretical result that gives a sufficient condition on the sign of the conditional covariance between a random variable X and a function $g[X]$ (suppressing subscript T on X_T).

Lemma 1 (Sign of conditional covariance) *Let X be a random variable with a finite second moment, $\mathcal{D}$ be a subset of the real line $\mathbb{R}$ with $\mathbb{E}_t^{\mathbb{P}}(X) \in \mathcal{D}$, and $g : \mathcal{D} \to \mathbb{R}$ be a function for which $g[X]$ has a finite second moment. The following statements are true:*

$$\text{If} \quad g[X] \text{ is a decreasing function on } \mathcal{D}, \text{ then } \operatorname{cov}_t^{\mathbb{P}}(g[X], X) \leq 0. \tag{16}$$

$$\text{If} \quad g[X] \text{ is an increasing function on } \mathcal{D}, \text{ then } \operatorname{cov}_t^{\mathbb{P}}(g[X], X) \geq 0. \tag{17}$$

Proof: See Appendix C. ∎

The essence of Lemma 1 is that if $\frac{dg[X]}{dX} \equiv g'[X] \leq (\geq) 0$, then $g[X]$ is decreasing (increasing), so Lemma 1 tells us that the conditional covariance $\text{cov}_t^{\mathbb{P}}(g[X], X)$ is nonpositive (nonnegative).

In view of Lemma 1, what are the theoretical restrictions on plausible M_T under which the NCC does not hold? We answer this question by developing a result on the sign of $\text{cov}_t^{\mathbb{P}}(M_T[R_T]R_T^n, R_T^n)$ for any positive integer n and then specializing the result to $n = 1$. Let $M_T'[R_T] \equiv \frac{dM_T[R_T]}{dR_T}$.

Result 3 (Restrictions of the NCC) *The following hold for any positive integer n:*

$$\text{If } \frac{M_T'[R_T]}{M_T[R_T]} + \frac{n}{R_T} \leq 0 \text{ almost surely, then } \text{cov}_t^{\mathbb{P}}(M_T[R_T]R_T^n, R_T^n) \leq 0. \tag{18}$$

$$\text{If } \frac{M_T'[R_T]}{M_T[R_T]} + \frac{n}{R_T} \geq 0 \text{ almost surely, then } \text{cov}_t^{\mathbb{P}}(M_T[R_T]R_T^n, R_T^n) \geq 0. \tag{19}$$

Furthermore,

$$\text{If } \frac{M_T'[R_T]}{M_T[R_T]} + \frac{n}{R_T} \leq 0, \quad \text{then } \mathbb{E}_t^{\mathbb{P}}(R_T^n) \geq \overbrace{\frac{\mathbb{E}_t^{\mathbb{Q}}(R_T^{2n})}{\mathbb{E}_t^{\mathbb{Q}}(R_T^n)}}^{\text{lower bound}} \quad \text{and} \tag{20}$$

$$\text{If } \frac{M_T'[R_T]}{M_T[R_T]} + \frac{n}{R_T} \geq 0, \quad \text{then } \mathbb{E}_t^{\mathbb{P}}(R_T^n) \leq \underbrace{\frac{\mathbb{E}_t^{\mathbb{Q}}(R_T^{2n})}{\mathbb{E}_t^{\mathbb{Q}}(R_T^n)}}_{\text{upper bound}} . \tag{21}$$

Proof: The result follows from Lemma 1 and the steps in Appendix D. ∎

Consider equation (20) of Result 3 corresponding to $n = 1$:

$$\text{If } \frac{M_T'[R_T]}{M_T[R_T]} + \frac{1}{R_T} \leq 0, \quad \text{then } \mathbb{E}_t^{\mathbb{P}}(R_T) \geq \frac{\mathbb{E}_t^{\mathbb{Q}}(R_T^2)}{\mathbb{E}_t^{\mathbb{Q}}(R_T)}. \tag{22}$$

This restriction implies a conditional lower bound $\mathbb{E}_t^{\mathbb{P}}(R_T) \geq \frac{\mathbb{E}_t^{\mathbb{Q}}(R_T^2) - \{\mathbb{E}_t^{\mathbb{Q}}(R_T)\}^2 + \{\mathbb{E}_t^{\mathbb{Q}}(R_T)\}^2}{\mathbb{E}_t^{\mathbb{Q}}(R_T)}$ (observed at time t), and, hence, $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t} \geq R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_T)$, in view of $\mathbb{E}_t^{\mathbb{Q}}(R_T) = R_{f,t}$.

What do we do here that is different from Martin (2017)? Specifically, equation (21) is a new observation and shows that $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t} \leq R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_T)$, so $R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_T)$ represents an *upper* bound, when $\frac{M_T'[R_T]}{M_T[R_T]} + \frac{1}{R_T} \geq 0$ for some theoretically and empirically supportable $M_T[R_T]$. Since the sign of $\frac{M_T'[R_T]}{M_T[R_T]} + \frac{1}{R_T}$ could potentially alternate through (calendar) time, our characterizations do not preclude the possibility that $R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_T)$ represents an *upper* bound on the expected

excess return at some time t, while it could represent a *lower* bound at some other time $t^\bullet$. Such a possibility is at the crux of our theoretical treatment and empirical approach.

The conditions in Result 3 are *sufficient*, but not necessary. This is because derivatives are local properties. Result 3 is free of distributional assumptions.

4.2   Economic rationale for the counterexamples where the NCC fails

In light of equations (20)–(21) evaluated at $n = 1$, we consider $\frac{M_T'[R_T]}{M_T[R_T]} + \frac{1}{R_T}$ and develop example economies in which $\frac{1}{R_T}$ is sufficiently positive to counteract the possible negative value of $\frac{M_T'[R_T]}{M_T[R_T]}$. We will argue that the class of SDFs violating the NCC are theoretically and empirically tenable.

Counterexample 1 Suppose the gross return vector, $\mathbf{Z}_T[R_T]$, is given by

$$\underbrace{\mathbf{Z}_T[R_T]}_{2\times 1} = [R_T \quad R_T^{\text{straddle}}]^\top, \quad \text{where} \quad R_T^{\text{straddle}} \equiv \frac{S_t \max(R_T - 1, 0) + S_t \max(1 - R_T, 0)}{\text{call}_{t,T}[S_t] + \text{put}_{t,T}[S_t]}. \quad (23)$$

Consider the following candidate for the projected SDF that enforces the correct pricing of $\mathbf{Z}_T[R_T]$:

$$M_T[R_T] = \mathbf{Z}_T^\top[R_T]\,\boldsymbol{\alpha} \quad \text{with} \quad \boldsymbol{\alpha} = \{\mathbb{E}^{\mathbb{P}}(\mathbf{Z}_T^\top[R_T]\,\mathbf{Z}_T[R_T])\}^{-1}\,\mathbf{1}. \quad (\text{since } \mathbb{E}^{\mathbb{P}}(M_T[R_T]\,\mathbf{Z}_T[R_T]) = \mathbf{1}) \quad (24)$$

Here $M_T[R_T]$ depends on the gross return of (i) the market and (ii) at-the-money straddle. ♣

The model of M_T in equation (24) is market volatility sensitive. Employing data from the S&P 500 index options market and computing the matrix of unconditional second-moments $\mathbb{E}^{\mathbb{P}}(\mathbf{Z}_T^\top[R_T]\,\mathbf{Z}_T[R_T])$, as in Cochrane (2005, pages 65–66), we determine $\boldsymbol{\alpha} = [0.73 \ 0.30]^\top$, with 90% bootstrap confidence intervals of (0.61 0.83) and (0.17 0.48), respectively. For this model,

$$\frac{M_T'[R_T]}{M_T[R_T]} + \frac{1}{R_T} = \frac{1}{M_T[R_T]} \begin{cases} \alpha_{\text{market}} + \frac{\alpha_{\text{straddle}}\,S_t}{\text{call}_{t,T}[S_t] + \text{put}_{t,T}[S_t]} + \frac{M_T[R_T]}{R_T} > 0 & \text{if } R_T > 1, \\[2mm] \alpha_{\text{market}} - \frac{\alpha_{\text{straddle}}\,S_t}{\text{call}_{t,T}[S_t] + \text{put}_{t,T}[S_t]} + \frac{M_T[R_T]}{R_T} & \text{if } R_T < 1. \end{cases}$$

The NCC fails to hold when $R_T > 1$.

Addressing plausibility, this $M_T[R_T]$ has a decreasing region and an increasing region, and is convex (i.e., $M_T''[R_T] \equiv \frac{d^2 M_T[R_T]}{dR_T^2} > 0$) in R_T. $M_T[R_T]$ with the said properties has empirical

support,[1] and is consistent with the empirical observation of *negative* average returns of call options on the market index (e.g., Bakshi, Madan, and Panayotov (2010)).

Next, as an illustration, suppose returns can take values along a grid (confining to a 7% down or up move over the option expiration cycle) with *equal* probabilities. Then

$$
\begin{aligned}
R_T[\omega] &= \frac{S_{t+T}[\omega]}{S_t} = [e^{-0.07} \cdots, e^{-0.050}, \ldots, e^{-0.01}, e^{-0.005}, 1, e^{0.005}, e^{0.01} \ldots, e^{0.05}, \ldots, e^{0.07}]^\top, \\
M_T[\omega] &= \mathbf{Z}_T^\top[\omega]\boldsymbol{\alpha} \quad \text{and} \quad R_T^{\text{straddle}}[\omega] \equiv \frac{S_t \max(R_T[\omega]-1,0) + S_t \max(1-R_T[\omega],0)}{\text{call}_{t,T}[S_t] + \text{put}_{t,T}[S_t]}.
\end{aligned} \tag{25}
$$

On 11/23/2015, $S_t = \$2086.59$, $R_{f,t} = 1$, and interpolated $\text{call}_{t,T}[S_t] = \text{put}_{t,T}[S_t] = \$29.07$.

The key point is that the correlation between $M_T R_T$ and R_T is computable and equal to 0.35 (i.e., positive). Thus, under our assumption about $M_T[R_T]$, the validity of the NCC is questioned for data from 11/23/2015.

We test the model in equation (24) in an another way in Section 4.3.

Counterexample 2 Suppose

$$
M_T[R_T] = \exp(m_0 - 1 + \Lambda_{\text{straddle}} \underbrace{(R_T^{\text{straddle}} - R_{f,t})}_{\text{excess return}}), \quad \text{with} \quad \Lambda_{\text{straddle}} > 0. \tag{26}
$$

This (projected) SDF depends on the excess return of a variable that is sensitive to volatility, has a decreasing region, has an increasing region, and is convex in R_T. Then

$$
\frac{M_T'[R_T]}{M_T[R_T]} + \frac{1}{R_T} = \begin{cases} \frac{\Lambda_{\text{straddle}} S_t}{\text{call}_{t,T}[S_t] + \text{put}_{t,T}[S_t]} + \frac{1}{R_T} > 0 & \text{if } R_T > 1, \\ -\frac{\Lambda_{\text{straddle}} S_t}{\text{call}_{t,T}[S_t] + \text{put}_{t,T}[S_t]} + \frac{1}{R_T} & \text{if } R_T < 1. \end{cases} \tag{27}
$$

Additionally, $\frac{M_T'[R_T]}{M_T[R_T]} + \frac{1}{R_T}$ is positive in the region $R_T < 1$, when $\Lambda_{\text{straddle}} < \frac{\text{call}_{t,T}[S_t] + \text{put}_{t,T}[S_t]}{S_t}$. ♣

---

[1] We refer the reader to the theoretical characterizations and empirical defence in, e.g., Aït-Sahalia and Lo (2000), Jackwerth (2000), Rosenberg and Engle (2002), Beare (2011), Christoffersen, Jacobs, and Heston (2013), Hens and Reichlin (2013), Chaudhuri and Schroder (2015), Wolfgang, Hardle, and Kratschmer (2017), and Schneider and Trojani (2019). The study of Linn, Shive, and Shumway (2018) offers a counterpoint.

Counterexample 3 Suppose, for constants m_0, $\upsilon > 0$, and $\delta > 0$,

$$M_T[R_T] \;=\; \exp\left(m_0 - 1 + \frac{\upsilon}{2}\,\Big(\frac{R_T^2}{\mathbf{q}_{t,\{R_T^2\}}} - R_{f,t}\Big) + \delta\Big(\frac{\frac{1}{R_T}}{\mathbf{q}_{t,\{\frac{1}{R_T}\}}} - R_{f,t}\Big)\right), \quad \text{for } R_T \in [\underline{R}, \overline{R}], \qquad (28)$$

where $\underline{R} > 0$ and $\overline{R} < \infty$. Here $(\frac{R_T^2}{\mathbf{q}_{t,\{R_T^2\}}} - R_{f,t})$ and $(\frac{\frac{1}{R_T}}{\mathbf{q}_{t,\{\frac{1}{R_T}\}}} - R_{f,t})$ are excess returns, where $\mathbf{q}_{t,\{R_T^2\}} = R_{f,t}^{-1}\,\mathbb{E}_t^{\mathbb{Q}}(R_T^2)$ (respectively, $\mathbf{q}_{t,\{\frac{1}{R_T}\}} = R_{f,t}^{-1}\,\mathbb{E}_t^{\mathbb{Q}}(R_T^{-1})$) is the price of the payoff R_T^2 (respectively, $\frac{1}{R_T}$) and can be synthesized via a static positioning in options. Then

$$\frac{M_T'[R_T]}{M_T[R_T]} \;+\; \frac{1}{R_T} \;=\; \frac{\frac{\upsilon}{\mathbf{q}_{t,\{R_T^2\}}}R_T^3 \;+\; R_T \;-\; \frac{\delta}{\mathbf{q}_{t,\{\frac{1}{R_T}\}}}}{R_T^2}. \qquad (29)$$

With $\upsilon > 0$, $\upsilon R_T^3 + R_T$ is strictly increasing in R_T, which implies that the equation $\upsilon R_T^3 + R_T - \frac{\delta}{\mathbf{q}_{t,\{\frac{1}{R_T}\}}} = 0$ has a unique real solution given by (see Weisstein (2010, equation (80)))

$$R_T^{\text{critical}} = -2\sqrt{\frac{\mathbf{q}_{t,\{R_T^2\}}}{3\upsilon}}\,\sinh\left(\frac{1}{3}\text{arcsinh}\left(-\frac{3\delta}{2\mathbf{q}_{t,\{\frac{1}{R_T}\}}}\sqrt{\frac{3\upsilon}{\mathbf{q}_{t,\{R_T^2\}}}}\right)\right). \qquad (30)$$

R_T^{critical} is crucial for determining the sign of $\frac{M_T'[R_T]}{M_T[R_T]} + \frac{1}{R_T}$ and, thus, of $\text{cov}_t^{\mathbb{P}}\big(M_T[R_T]R_T, R_T\big)$.

Suppose $R_T^{\text{critical}} < \underline{R}$. Then, since $R_T \in [\underline{R}, \overline{R}]$, we have $R_T > R_T^{\text{critical}}$, and the numerator of equation (29) is positive. Therefore, $\text{cov}_t^{\mathbb{P}}(M_T[R_T]\,R_T, R_T) > 0$ and the NCC is violated. ♣

Counterexample 4 Suppose $M_T[R_T]$ is the *weighted* sum of a completely monotone function and an absolutely monotone function (motivated by Bakshi, Madan, and Panayotov (2010)) of the type

$$M_T[R_T] = w\,R_T^{-\theta} \;+\; (1-w)\,(\frac{1}{R_T})^{-\theta}, \quad \text{for some constants } \theta > 1 \text{ and } 0 < w < 1. \qquad (31)$$

This $M_T[R_T]$ is convex in R_T, and is negatively (positively) sloped at low (large) R_T, with $(\frac{1}{R_T})^{-\theta}$ reflecting the marginal utility of agents long the *inverse of market* (their wealth declines in the region $R_T > 1$). Also,

$$\frac{M_T'[R_T]}{M_T[R_T]} + \frac{1}{R_T} = \frac{\frac{\theta}{R}\{-w\,R_T^{-\theta} + (1-w)\,R_T^{\theta}\}}{w\,R_T^{-\theta} + (1-w)\,R_T^{\theta}} + \frac{1}{R} > 0, \quad \text{when } R_T > \left(\frac{w\,(\theta-1)}{(1-w)(\theta+1)}\right)^{\frac{1}{2\theta}}. \qquad (32)$$

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Thus, $\text{cov}_t^{\mathbb{P}}(M_T[R_T]\,R_T, R_T)$ can be positive, which poses a challenge to the notion of the NCC. ♣

We next build on specifications of M_T that are insightful in an empirical context.

4.3   Empirical evidence on the NCC

The hypothesis of the NCC is that the conditional covariance $\text{cov}_t^{\mathbb{P}}(M_T\,R_T, R_T) \leq 0$ for all M_T. Therefore,

we can reject the hypothesis of the NCC if $\text{cov}_t^{\mathbb{P}}(M_T[R_T]\,R_T, R_T) > 0$, for some $M_T[R_T]$. (33)

Denoting the conditional covariance by $\mathbb{E}_t^{\mathbb{P}}(\tilde{\mathbf{c}}_T)$, where the demeaned cross-product $\tilde{\mathbf{c}}_T$ is given by $\tilde{\mathbf{c}}_T \equiv \{M_T[R_T]\,R_T - \mathbb{E}_t^{\mathbb{P}}(M_T[R_T]\,R_T)\}\{R_T - \mathbb{E}_t(R_T)\}$, the following two comments are in order:

- First, suppose we wish to assess the hypothesis H: $\mathbb{E}_t^{\mathbb{P}}(\tilde{\mathbf{c}}_T) > 0$. Having infinitely many, but countable, observations of positive $\tilde{\mathbf{c}}_T$ does not violate the possibility that $\mathbb{E}_t^{\mathbb{P}}(\tilde{\mathbf{c}}_T) \leq 0$.

- Second, suppose we wish to assess the same hypothesis H: $\mathbb{E}_t^{\mathbb{P}}(\tilde{\mathbf{c}}_T) > 0$ by utilizing the sufficient condition in Result 3; then it seems that we need to test $H^{\bullet}$: $\frac{M_T'[R_T]}{M_T[R_T]} + \frac{1}{R_T} > 0$ almost surely. While straightforward, this condition is not easily testable because, even if we have infinitely many, but countable, observations of positive $\frac{M_T'[R_T]}{M_T[R_T]} + \frac{1}{R_T}$, it does not exclude the possibility that $\frac{M_T'[R_T]}{M_T[R_T]} + \frac{1}{R_T} < 0$ almost surely.

Motivated by the discussions above, the next result is at the center of our empirical exercises.

Result 4 *Let $\mathbb{E}^{\mathbb{P}}(.)$ and $\text{cov}^{\mathbb{P}}(.,.)$ denote the unconditional expectation and unconditional covariance, respectively. The following statement is true:*

$$\underbrace{\text{cov}^{\mathbb{P}}(M_T[R_T]\,R_T, R_T)}_{\text{unconditional covariance}} = \underbrace{\mathbb{E}^{\mathbb{P}}(\text{cov}_t^{\mathbb{P}}(M_T[R_T]\,R_T, R_T)).}_{\text{unconditional expectation of conditional covariance}} \qquad (34)$$

Proof: By the law of total covariance formula, $\text{cov}^{\mathbb{P}}(M_T[R_T]\,R_T, R_T) = \mathbb{E}^{\mathbb{P}}(\text{cov}_t^{\mathbb{P}}(M_T[R_T]\,R_T, R_T)) + \text{cov}^{\mathbb{P}}(\mathbb{E}_t^{\mathbb{P}}(M_T[R_T]\,R_T), \mathbb{E}_t^{\mathbb{P}}(R_T))$. Since $\mathbb{E}_t^{\mathbb{P}}(M_T[R_T]\,R_T) = 1$, the result in equation (34) follows. ∎

If the NCC holds almost surely, that is, if, $\mathrm{cov}^{\mathbb{P}}_t(M_T[R_T]\,R_T, R_T) \leq 0$ almost surely, then the unconditional covariance $\mathrm{cov}^{\mathbb{P}}(M_T[R_T]\,R_T, R_T) \leq 0$. For ease of comparability of magnitudes across models of $M_T[R_T]$, we test the hypothesis that

$$H_0: \quad \mathrm{NCC}_T \leq 0, \text{ where } \mathrm{NCC}_T \text{ is the unconditional } correlation \text{ between } M_T[R_T]R_T \text{ and } R_T. \quad (35)$$

Specifically, if we reject H_0, then we reject the NCC, meaning that $\mathrm{cov}^{\mathbb{P}}_t(M_T[R_T]\,R_T, R_T) > 0$ holds with a strictly positive probability.

We next present empirical evidence (using data from options on the market index) from three models. While one could feature more models of M_T, we highlight settings that are theoretically and empirically revealing and amenable to implementation and validation.

Model A: The projected SDF, $M_T[R_T]$, depends on the gross return of the market (R_T) and the gross return of an at-the-money straddle $(R_T^{\mathrm{straddle}}$, as defined in equation (23)) as follows:

$$M_T[R_T] = \alpha_{\mathrm{market}} R_T + \alpha_{\mathrm{straddle}} \Big\{ \frac{S_t \max(R_T - 1, 0) + S_t \max(1 - R_T, 0)}{\mathrm{call}_{t,T}[S_t] + \mathrm{put}_{t,T}[S_t]} \Big\}. \quad (36)$$

Model B: The projected SDF, $M_T[R_T]$, depends on the gross return of the market and the gross return of a 2% out-of-the-money strangle, defined as $R_T^{\mathrm{strangle}} \equiv \frac{S_t \max(R_T - e^{0.02}, 0) + S_t \max(e^{-0.02} - R_T, 0)}{\mathrm{call}_{t,T}[S_t e^{0.02}] + \mathrm{put}_{t,T}[S_t e^{-0.02}]})$ as follows:

$$M_T[R_T] = \alpha_{\mathrm{market}} R_T + \alpha_{\mathrm{strangle}} R_T^{\mathrm{strangle}}. \quad (37)$$

Models A and B are distinct since R_T^{straddle} and R_T^{strangle} are imperfectly correlated.

Model C: The projected SDF, $M_T[R_T]$, is *exponential* in the excess returns of the squared log contract, allowing for asymmetric effect of variance in down and up equity markets:

$$M_T[R_T] = \exp\left(m_0 - 1 + \{\eta_{\mathrm{variance}}^- \mathbb{1}_{\{R_T < 1\}} + \eta_{\mathrm{variance}}^+ \mathbb{1}_{\{R_T > 1\}}\}(R_T^{\mathrm{variance}} - R_{f,t})\right). \quad (38)$$

The gross return, R_T^{variance}, is based on the payoff of the following squared log contract:

$$R_T^{\mathrm{variance}} = \frac{\{\log R_T\}^2}{\mathbf{q}_{t,\{\log R_T\}^2}}, \quad (39)$$

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where the price of the squared log contract, $\mathbf{q}_{t,\{\log R_T\}^2} = R_{f,t}^{-1}\mathbb{E}_t^{\mathbb{Q}}(\{\log R_T\}^2)$, is synthesized as

$$\mathbf{q}_{t,\{\log R_T\}^2} = \int_{K<S_t} \frac{2(1-\log\frac{K}{S_t})}{K^2}\,\mathrm{put}_{t,T}[K]\,dK \;+\; \int_{K>S_t} \frac{2(1-\log\frac{K}{S_t})}{K^2}\,\mathrm{call}_{t,T}[K]\,dK. \qquad (40)$$

It suffices, from Result 4, to compute the unconditional covariance $\mathrm{cov}^{\mathbb{P}}(M_T[R_T]\,R_T, R_T)$.

In our exercises, we compute the parameters affecting $M_T[R_T]$ and $\mathrm{cov}^{\mathbb{P}}(M_T[R_T]\,R_T, R_T)$. Tables 3 and 4 present our findings from implementing Models A, B, and C, respectively. The takeaway is that the correlation between $M_T[R_T]R_T$ and R_T; that is, NCC_T is positive. In each model, $M_T[R_T]$ does not attain negative values (i.e., the minimum $M_T[R_T]$ is positive).

The methodological appeal of Result 4 is that the sign of the expected conditional covariance between $M_T[R_T]R_T$ and R_T can be ascertained by the sign of the unconditional covariance between $M_T[R_T]R_T$ and R_T. The conclusion to draw is that the NCC fails to hold *even on average*.

Elaborating on these findings, we pose $\mathrm{cov}^{\mathbb{P}}(M_T[R_T]\,R_T, R_T)$ being negative as an explicit hypothesis and consider a bootstrap procedure. We bootstrap (with replacement) the gross returns in $\mathbf{Z}_T[R_T]$ and reestimate $\boldsymbol{\alpha}$ in the context, for example, of Model A and Table 3. Then we reconstruct $M_T[R_T] = \mathbf{Z}_T^{\top}[R_T]\,\boldsymbol{\alpha}$ and R_T. The 5th and 95th bootstrap values for NCC_T are positive, and we can reject the hypothesis of the NCC.

Our evidence, thus, refutes the notion that the NCC holds point by point and that $R_{f,t}^{-1}\,\mathrm{var}_t^{\mathbb{Q}}(R_T)$ is a universal lower bound across all plausible M_T. In essence, there is evidence, from Tables 3 and 4, that the NCC is *not* a generic property, and, consequently, $R_{f,t}^{-1}\,\mathrm{var}_t^{\mathbb{Q}}(R_T)$ may sometimes be an *upper* bound and not a *lower* bound.

What is special about featured Models A, B, and C? The distinguishing attribute is the dependence of $M_T[R_T]$ on a specific convex function of R_T that manifests decreasing and increasing regions. For example, $M_T[R_T]$ tends to be high when R_T^{straddle} or R_T^{strangle} are high, which, intuitively, reflects sensitivity to market volatility. The considered model classes can be differentiated from the ones in Martin (2017) that he employs to convey the empirical relevance of the NCC.

5 Conclusions

What is the compensation that agents require for investing in the equity market? Martin (2017) shows that a lower bound to the expected excess return of the market (and, thus, the minimum compensation) is equal to the discounted risk-neutral variance of the market return. The critical assumption that delivers this lower bound formula is the negative correlation condition (NCC); that is, $\text{cov}_t^{\mathbb{P}}(M_T R_T, R_T) \leq 0$ for all SDFs M_T.

What are the differentiating elements of our paper? Most crucially, we go beyond a lower bound characterization and propose an analytical expression for the conditional expected excess return of the market. Our theoretical formula distills the manner in which each risk-neutral return central moment contributes to the expected excess return of the market. Each source of the conditional expected excess return can be tractably extracted from known option prices.

Using data on S&P 500 index options from January 1990 to December 2018 (29 years, 348 option expiration cycles), our empirical exercises emphasize two insights. First, the components related to risk-neutral return variance and risk-neutral return skewness are the essential drivers of the expected excess return of the market. Second, our estimates of the conditional expected excess return materially differ from Martin (2017), and, on average, are twice the size of the lower bound (i.e., the discounted risk-neutral variance).

We take two perspectives to reconcile our findings. First, addressing theoretical distinctions, our formula assigns a weight to each risk-neutral return central moment. Our methodology is new and we do not rely on the assumption that $\text{cov}_t^{\mathbb{P}}(M_T R_T, R_T) \leq 0$ (i.e., that the NCC holds) to derive our formula. Second, we develop the asset-pricing restrictions of the NCC and provide evidence that the NCC is neither a generic property of the models nor does it necessarily hold in the data. Our theoretical and empirical treatment implies that discounted risk-neutral variance, identified as the lower bound in Martin (2017), could be an upper bound when the NCC fails.

The central problem of determining the expected excess return of the market — as rooted in the tradition of Merton (1980) and Black (1993) — is still hungry for consensus and resolution. Our explicit expression, analytical in quantities inferred from option prices, is a step in that direction.

References

Y. Aït-Sahalia and A. Lo. Nonparametric risk management and implied risk aversion. *Journal of Econometrics*, 94:9–51, 2000.

C. Almeida and R. Garcia. Robust economic implications of nonlinear pricing kernels. *Management Science*, 63(10):3361–3380, 2017.

G. Bakshi and D. Madan. Spanning and derivative-security valuation. *Journal of Financial Economics*, 55(2):205–238, February 2000.

G. Bakshi, D. Madan, and G. Panayotov. Returns of claims on the upside and the viability of U-shaped pricing kernels. *Journal of Financial Economics*, 97(1):130–154, 2010.

R. Barro. Rare disasters and asset markets in the twentieth century. *Quarterly Journal of Economics*, 121(9):823–866, 2006.

B. Beare. Measure preserving derivatives and the pricing kernel puzzle. *Journal of Mathematical Economics*, 47:689–697, 2011.

F. Black. Estimating expected returns. *Financial Analyst Journal*, 49:36–38, 1993.

J. Borovička, L. Hansen, and J. Scheinkman. Misspecified recovery. *Journal of Finance*, 71(6): 2493–2544, 2016.

J. Campbell and J. Cochrane. Force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy*, 107:205–251, 1999.

P. Carr and D. Madan. Optimal positioning in derivative securities. *Quantitative Finance*, 1(1): 19–37, 2001.

F. Chabi-Yo and J. Loudis. The conditional expected market return. *Journal of Financial Economics*, (forthcoming), 2019.

R. Chaudhuri and M. Schroder. Monotonicity of the stochastic discount factor and expected option returns. *Review of Financial Studies*, 28(5):1462–1505, 2015.

P. Christoffersen, K. Jacobs, and S. Heston. Capturing option anomalies with a variance-dependent pricing kernel. *Review of Financial Studies*, 26:1963–2006, 2013.

J. Cochrane. A cross-sectional test of an investment-based asset pricing model. *Journal of Political Economy*, 104:572–621, 1996.

J. Cochrane. *Asset Pricing*. Princeton University Press, Princeton, NJ, 2005.

E. Elton. Expected return, realized return and asset pricing tests. *Journal of Finance*, 54(4): 1199–1220, 1999.

L. Epstein and S. Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis. *Journal of Political Economy*, 99(2):263–286, 1991.

E. Fama and K. French. Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics*, 33:3–56, 1993.

E. Fama and K. French. The equity premium. *Journal of Finance*, 57:637–659, 2002.

A. Ghosh, C. Julliard, and A. Taylor. What is the consumption-CAPM missing? An information-theoretic framework for the analysis of asset pricing models. *Review of Financial Studies*, 30: 442–504, 2017.

L. Hansen and E. Renault. Pricing kernels and stochastic discount factors. In *"The term structure and future changes in long and short rates in the G-7 countries," Journal of Monetary Economics*, pages 255–283, 2009.

M. Harrison and D. Kreps. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20(3):381–408, 1979.

T. Hens and C. Reichlin. Three solutions to the pricing kernel puzzle. *Review of Finance*, 17: 1065–1098, 2013.

J. Jackwerth. Recovering risk aversion from option prices and realized returns. *Review of Financial Studies*, 13(2):433–451, 2000.

O. Kadan and X. Tang. A bound on expected stock returns. *Review of Financial Studies*, (forthcoming), 2019.

M. Kendall and A. Stuart. *The Advanced Theory of Statistics, Volume I*. Macmillan Publishing, New York, 1963.

M. Linn, S. Shive, and T. Shumway. Pricing kernel monotonicity and conditional information. *Review of Financial Studies*, 31(2):493–531, 2018.

I. Martin. What is the expected return on the market? *Quarterly Journal of Economics*, 150: 367–433, 2017.

I. Martin and C. Wagner. What is the expected return on a stock? *Journal of Finance*, (forthcoming), 2019.

L. Menzly, T. Santos, and P. Veronesi. Understanding predictability. *Journal of Political Economy*, 112:1–47, 2004.

R. Merton. On estimating the expected return on the market: An exploratory investigation. *Journal of Financial Economics*, 8:323–361, 1980.

W. Newey and K. West. A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix. *Econometrica*, 55(3):703–708, 1987.

J. Rosenberg and R. Engle. Empirical pricing kernels. *Journal of Financial Economics*, 64(3): 341–372, 2002.

P. Schneider and F. Trojani. (Almost) model-free recovery. *Journal of Finance*, 74:323–370, 2019.

J. Wachter. Can time-varying risk of rare disasters explain aggregate stock market volatility? *Journal of Finance*, 68:987–1035, 2013.

E. Weisstein. Cubic formula. Unpublished paper, MathWorld–A Wolfram Web Resource, 2010.

M. Wolfgang, K. Hardle, and V. Kratschmer. Reference-dependent preferences and the empirical pricing kernel puzzle. *Review of Finance*, 21:269–298, 2017.

Appendix: Proof of Results

A   Appendix A: Proof of the expected excess return formula (4) in Result 1

For ease of reference, we collect the following notations to keep the steps of our proof self-contained:

$r_T \equiv R_T - 1 =$ net return of the equity market index;

$r_{f,t} \equiv R_{f,t} - 1 =$ net risk-free return;

$\Omega \equiv \{r_T > -1\} =$ set of return possibilities;

$p[r_T] =$ density of r_T under the real-world probability measure, $\mathbb{P}$;

$q[r_T] =$ density of r_T under the risk-neutral probability measure, $\mathbb{Q}$;

$\text{mgf}_t^{\mathbb{P}}[\lambda] = \int_\Omega e^{\lambda r_T} p[r_T] \, dr_T =$ moment-generating function of the real-world distribution;

$\text{mgf}_t^{\mathbb{Q}}[\lambda] = \int_\Omega e^{\lambda r_T} q[r_T] \, dr_T =$ moment-generating function of the risk-neutral distribution;

$\text{cmoment}_{n,t}^{\mathbb{Q}}(r_T) =$ conditional n-th return central moment under the risk-neutral measure $\mathbb{Q}$.

In light of equation (3), we represent $M_T[R_T] = M_T[1 + r_T] = N_T[r_T]$, that is,

$$N_T[r_T] = \exp\left(m_0 - 1 - \phi(r_T - r_{f,t})\right) \equiv a_0 \, e^{-\phi r_T}, \quad \text{where } a_0 \equiv \exp(m_0 - 1 + \phi \, r_{f,t}). \quad (A1)$$

The constant a_0 will turn out to be irrelevant in our calculations.

Next, to derive analytical results in the context of a projected $M_T[R_T]$, or $N_T[r_T]$, observe from Harrison and Kreps (1979), that one can hypothesize that

$$\frac{q[r_T]}{p[r_T]} = N_T[r_T], \quad \text{for some positive function.} \quad (A2)$$

We make the normalization such that $p[r_T]$ integrates to unity, that is,

$$p[r_T] = \frac{\frac{1}{N_T[r_T]} q[r_T]}{\int_\Omega \frac{1}{N_T[r_T]} q[r_T] \, dr_T}, \quad (A3)$$

which implies that $\int_\Omega p[r_T] \, dr_T = 1$.

The n-th order raw conditional return moments of the distribution, under $\mathbb{P}$ and $\mathbb{Q}$, are

$$\mu_n^{\mathbb{P}} \equiv \int_{\Omega} \{r_T\}^n \, p[r_T] \, dr_T \qquad \text{and} \qquad \mu_n^{\mathbb{Q}} \equiv \int_{\Omega} \{r_T\}^n \, q[r_T] \, dr_T. \qquad (A4)$$

We denote the moment-generating functions under $\mathbb{P}$ and $\mathbb{Q}$, as $\mathrm{mgf}_t^{\mathbb{P}}[\lambda]$ and $\mathrm{mgf}_t^{\mathbb{Q}}[\lambda]$, respectively. Assume $\mathrm{mgf}_t^{\mathbb{P}}[\lambda] < \infty$ and $\mathrm{mgf}_t^{\mathbb{Q}}[\lambda] < \infty$. Then,

$$\mathrm{mgf}_t^{\mathbb{P}}[\lambda] \equiv \int_{\Omega} e^{\lambda r_T} \, p[r_T] \, dr_T = 1 + \frac{\lambda}{1!} \mu_1^{\mathbb{P}} + \frac{\lambda^2}{2!} \mu_2^{\mathbb{P}} + \frac{\lambda^3}{3!} \mu_3^{\mathbb{P}} + \frac{\lambda^4}{4!} \mu_4^{\mathbb{P}} + \ldots, \quad \text{for } \lambda \in \Re, \qquad (A5)$$

$$\mathrm{mgf}_t^{\mathbb{Q}}[\lambda] \equiv \int_{\Omega} e^{\lambda r_T} \, q[r_T] \, dr_T = 1 + \frac{\lambda}{1!} \mu_1^{\mathbb{Q}} + \frac{\lambda^2}{2!} \mu_2^{\mathbb{Q}} + \frac{\lambda^3}{3!} \mu_3^{\mathbb{Q}} + \frac{\lambda^4}{4!} \mu_4^{\mathbb{Q}} + \ldots, \quad \text{for } \lambda \in \Re. \qquad (A6)$$

Now consider

$$\mathrm{mgf}_t^{\mathbb{P}}[\lambda] = \int_{\Omega} e^{\lambda r_T} \, p[r_T] \, dr_T \qquad (A7)$$

$$= \int_{\Omega} e^{\lambda r_T} \frac{\frac{1}{N_T[r_T]}}{\int_{\Omega} \frac{1}{N_T[r_T]} q[r_T] \, dr_T} \, q[r_T] \, dr_T \qquad \text{(using equation (A3))} \qquad (A8)$$

$$= \frac{\int_{\Omega} e^{\lambda r_T} e^{\phi r_T} \, q[r_T] \, dr_T}{\int_{\Omega} e^{\phi r_T} \, q[r_T] \, dr_T} \qquad \text{(using equation (A1))} \qquad (A9)$$

$$= \frac{\mathrm{mgf}_t^{\mathbb{Q}}[\lambda + \phi]}{\mathrm{mgf}_t^{\mathbb{Q}}[\phi]}. \qquad (A10)$$

Taking logs on both sides of equation (A10), we arrive at

$$\log \mathrm{mgf}_t^{\mathbb{P}}[\lambda] = \log \mathrm{mgf}_t^{\mathbb{Q}}[\lambda + \phi] - \log \mathrm{mgf}_t^{\mathbb{Q}}[\phi]. \qquad (A11)$$

Equation (A11) implies that the cumulant-generating functions are related by the following identity:

$$\underbrace{\mathbb{C}_t^{\mathbb{P}}[\lambda]}_{= \, \log \mathrm{mgf}^{\mathbb{P}}[\lambda]} = \mathbb{C}_t^{\mathbb{Q}}[\lambda + \phi] - \mathbb{C}_t^{\mathbb{Q}}[\phi], \quad \text{where from Kendall and Stuart (1963),} \qquad (A12)$$

$$\log \mathrm{mgf}_t^{\mathbb{P}}[\lambda] \equiv \mathbb{C}_t^{\mathbb{P}}[\lambda] = \kappa_1^{\mathbb{P}} \frac{\lambda}{1!} + \kappa_2^{\mathbb{P}} \frac{\lambda^2}{2!} + \kappa_3^{\mathbb{P}} \frac{\lambda^3}{3!} + \kappa_4^{\mathbb{P}} \frac{\lambda^4}{4!} + \ldots = \sum_{n=1}^{\infty} \kappa_n^{\mathbb{P}} \frac{\lambda^n}{n!} \qquad \text{and} \qquad (A13)$$

$$\log \mathrm{mgf}_t^{\mathbb{Q}}[\lambda] \equiv \mathbb{C}_t^{\mathbb{Q}}[\lambda] = \kappa_1^{\mathbb{Q}} \frac{\lambda}{1!} + \kappa_2^{\mathbb{Q}} \frac{\lambda^2}{2!} + \kappa_3^{\mathbb{Q}} \frac{\lambda^3}{3!} + \kappa_4^{\mathbb{Q}} \frac{\lambda^4}{4!} + \ldots = \sum_{n=1}^{\infty} \kappa_n^{\mathbb{Q}} \frac{\lambda^n}{n!}. \qquad (A14)$$

Following Kendall and Stuart (1963, page 73, eq. (3.43)), $\kappa_n^{\mathbb{P}}$ are cumulants under $\mathbb{P}$, defined, in relation to the raw moments, as

$$\kappa_1^{\mathbb{P}} \equiv \mu_1^{\mathbb{P}} = \mathbb{E}_t^{\mathbb{P}}(r_T), \tag{A15}$$

$$\kappa_2^{\mathbb{P}} \equiv \mu_2^{\mathbb{P}} - (\mu_1^{\mathbb{P}})^2, \tag{A16}$$

$$\kappa_3^{\mathbb{P}} \equiv \mathbb{E}_t^{\mathbb{P}}(\{r_T - \mu_1^{\mathbb{P}}\}^3), \tag{A17}$$

$$\kappa_4^{\mathbb{P}} \equiv \mathbb{E}_t^{\mathbb{P}}(\{r_T - \mu_1^{\mathbb{P}}\}^4) - 3(\kappa_2^{\mathbb{P}})^2, \quad \text{and} \tag{A18}$$

$$\kappa_5^{\mathbb{P}} \equiv \mathbb{E}_t^{\mathbb{P}}(\{r_T - \mu_1^{\mathbb{P}}\}^5) - 10\,\kappa_3^{\mathbb{P}}\,\kappa_2^{\mathbb{P}}. \tag{A19}$$

Under the risk-neutral probability measure $\mathbb{Q}$, the corresponding cumulants are

$$\kappa_1^{\mathbb{Q}} \equiv \mu_1^{\mathbb{Q}} = \mathbb{E}_t^{\mathbb{Q}}(r_T) = r_{f,t}, \tag{A20}$$

$$\kappa_2^{\mathbb{Q}} \equiv \mu_2^{\mathbb{Q}} - (\mu_1^{\mathbb{Q}})^2 = \mathbb{E}_t^{\mathbb{Q}}(\{r_T - \mu_1^{\mathbb{Q}}\}^2) = \text{var}_t^{\mathbb{Q}}(r_T), \tag{A21}$$

$$\kappa_3^{\mathbb{Q}} \equiv \mathbb{E}_t^{\mathbb{Q}}(\{r_T - \mu_1^{\mathbb{Q}}\}^3) = (\text{var}_t^{\mathbb{Q}}(r_T))^{3/2}\text{Skewness}_t^{\mathbb{Q}}(r_T), \tag{A22}$$

$$\kappa_4^{\mathbb{Q}} \equiv \mathbb{E}_t^{\mathbb{Q}}(\{r_T - \mu_1^{\mathbb{Q}}\}^4) - 3(\kappa_2^{\mathbb{Q}})^2 = (\text{var}_t^{\mathbb{Q}}(r_T))^2(\text{Kurtosis}_t^{\mathbb{Q}}(r_T) - 3), \quad \text{and} \tag{A23}$$

$$\kappa_5^{\mathbb{Q}} \equiv \mathbb{E}_t^{\mathbb{Q}}(\{r_T - \mu_1^{\mathbb{Q}}\}^5) - 10\kappa_3^{\mathbb{Q}}\kappa_2^{\mathbb{Q}} = (\text{var}_t^{\mathbb{Q}}(r_T))^{5/2}(\text{Hskewness}_t^{\mathbb{Q}}(r_T) - 10\,\text{Skewness}_t^{\mathbb{Q}}(r_T)). \tag{A24}$$

$\text{Hskewness}_t^{\mathbb{Q}}(r_T)$ is the risk-neutral hyperskewness, defined in equation (A39). Our analysis does not require a parametric assumption about the risk-neutral distribution.

We are ready to verify Result 1. The conditional expected return under the $\mathbb{P}$ measure is

$$\mathbb{E}_t^{\mathbb{P}}(r_T) = \left.\frac{d\mathbb{C}_t^{\mathbb{P}}[\lambda]}{d\lambda}\right|_{\lambda=0} \qquad \text{(now use equation (A12))} \tag{A25}$$

$$= \left.\frac{d\mathbb{C}_t^{\mathbb{Q}}[\lambda+\phi]}{d\lambda}\right|_{\lambda=0} - \underbrace{\left.\frac{d\mathbb{C}_t^{\mathbb{Q}}[\phi]}{d\lambda}\right|_{\lambda=0}}_{=0} \tag{A26}$$

$$= \frac{1}{1!}\kappa_1^{\mathbb{Q}} + 2\frac{\phi}{2!}\kappa_2^{\mathbb{Q}} + 3\frac{\phi^2}{3!}\kappa_3^{\mathbb{Q}} + 4\frac{\phi^3}{4!}\kappa_4^{\mathbb{Q}} + 5\frac{\phi^4}{5!}\kappa_5^{\mathbb{Q}} + \sum_{n=6}^{\infty}\frac{\phi^{n-1}}{(n-1)!}\kappa_n^{\mathbb{Q}}, \tag{A27}$$

where we have exploited the derivative $\frac{d\mathbb{C}_t^{\mathbb{Q}}[\lambda+\phi]}{d\lambda} = \kappa_1^{\mathbb{Q}}\frac{1}{1!} + 2\kappa_2^{\mathbb{Q}}\frac{(\lambda+\phi)}{2!} + 3\kappa_3^{\mathbb{Q}}\frac{(\lambda+\phi)^2}{3!} + 4\kappa_4^{\mathbb{Q}}\frac{(\lambda+\phi)^3}{4!} + \ldots.$

Noting that $\kappa_1^{\mathbb{Q}} = \mathbb{E}_t^{\mathbb{Q}}(r_T) = \mathbb{E}_t^{\mathbb{Q}}(R_T - 1) = R_{f,t} - 1 = r_{f,t}$, the net risk-free return, it follows from equation (A27) that

$$
\underbrace{\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}}_{\text{expected excess return}} = \phi\,\kappa_2^{\mathbb{Q}} + \frac{\phi^2}{2}\,\kappa_3^{\mathbb{Q}} + \frac{\phi^3}{6}\,\kappa_4^{\mathbb{Q}} + \frac{\phi^4}{24}\,\kappa_5^{\mathbb{Q}} + \sum_{n=6}^{\infty} \frac{\phi^{n-1}}{(n-1)!}\,\kappa_n^{\mathbb{Q}}. \tag{A28}
$$

The conditional expected excess return in equation (A28) can be equivalently written as

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t} = {} & \frac{1}{\phi}(\{\phi\,\mathrm{SD}_t^{\mathbb{Q}}(r_T)\}^2 + \frac{1}{2}\,\{\phi\mathrm{SD}^{\mathbb{Q}}(r_T)\}^3\,\mathrm{Skewness}_t^{\mathbb{Q}}(r_T) \\
& + \frac{1}{6}\{\phi\mathrm{SD}_t^{\mathbb{Q}}(r_T)\}^4\,(\mathrm{Kurtosis}_t^{\mathbb{Q}}(r_T) \;-\; 3) \\
& + \frac{1}{24}\{\phi\mathrm{SD}_t^{\mathbb{Q}}(r_T)\}^5\,(\mathrm{Hskewness}_t^{\mathbb{Q}}(r_T) - 10\,\mathrm{Skewness}_t^{\mathbb{Q}}(r_T)) + \ldots),
\end{aligned} \tag{A29}
$$

where $\mathrm{SD}_t^{\mathbb{Q}}(r_T) = \sqrt{\mathrm{var}_t^{\mathbb{Q}}(r_T)}$, $\mathrm{Skewness}_t^{\mathbb{Q}}(r_T)$, $\mathrm{Kurtosis}_t^{\mathbb{Q}}(r_T)$, and $\mathrm{Hskewness}_t^{\mathbb{Q}}(r_T)$ are the conditional risk-neutral volatility, skewness, kurtosis, and hyperskewness, respectively.

The final step is to infer the risk-neutral return central moments from option prices, known at time t. Specifically, let

$$
\begin{aligned}
\mathrm{cmoment}_{n,t}^{\mathbb{Q}} &\equiv \mathbb{E}_t^{\mathbb{Q}}(\{r_T - r_{f,t}\}^n) = \mathbb{E}_t^{\mathbb{Q}}(\{(R_T - 1) - (R_{f,t} - 1)\}^n) \tag{A30} \\
&= \mathbb{E}_t^{\mathbb{Q}}(\{\frac{S_{t+T}}{S_t} - R_{f,t}\}^n) \tag{A31} \\
&= \frac{n(n-1)R_{f,t}}{S_t^2} \int_{K<R_{f,t}S_t} (\frac{K}{S_t} - R_{f,t})^{n-2} \mathrm{put}_{t,T}[K]\,dK \\
&\quad + \frac{n(n-1)R_{f,t}}{S_t^2} \int_{K>R_{f,t}S_t} (\frac{K}{S_t} - R_{f,t})^{n-2} \mathrm{call}_{t,T}[K]\,dK. \tag{A32}
\end{aligned}
$$

We can move from equation (A31) to equation (A32) since, from Bakshi and Madan (2000, Appendix A.3) and Carr and Madan (2001, equation (2)), we have, for $n \geq 2$,

$$
\begin{aligned}
A[S_{t+T}] &\equiv \{\frac{S_{t+T}}{S_t} - R_{f,t}\}^n \tag{A33} \\
&= \underbrace{A[S_{t+T}]\big|_{S_{t+T}=R_{f,t}S_t}}_{=0} + \underbrace{A'[S_{t+T}]\big|_{S_{t+T}=R_{f,t}S_t}}_{=0}(S_{t+T} - S_t) \\
&\quad + \int_{K<R_{f,t}S_t} A''[K](K - S_{t+T})^+\,dK + \int_{K>R_{f,t}S_t} A''[K](S_{t+T} - K)^+\,dK, \tag{A34}
\end{aligned}
$$

where

$$A''[K] = \frac{d^2 A[S_{t+T}]}{dS_{t+T}^2}\Big|_{S_{t+T}=K} = \frac{n(n-1)}{S_t^2}(\frac{S_{t+T}}{S_t} - R_{f,t})^{n-2}\Big|_{S_{t+T}=K} = \frac{n(n-1)}{S_t^2}(\frac{K}{S_t} - R_{f,t})^{n-2}. \quad \text{(A35)}$$

We obtain

$$\text{SD}_t^{\mathbb{Q}}(r_T) = \sqrt{\text{cmoment}_{n,t}^{\mathbb{Q}}\big|_{n=2}}, \quad \text{(A36)}$$

$$\text{Skewness}_t^{\mathbb{Q}}(r_T) = \frac{\text{cmoment}_{n,t}^{\mathbb{Q}}\big|_{n=3}}{\{\text{SD}_t^{\mathbb{Q}}(r_T)\}^3}, \quad \text{(A37)}$$

$$\text{Kurtosis}_t^{\mathbb{Q}}(r_T) = \frac{\text{cmoment}_{n,t}^{\mathbb{Q}}\big|_{n=4}}{\{\text{SD}_t^{\mathbb{Q}}(r_T)\}^4}, \quad \text{and} \quad \text{(A38)}$$

$$\text{Hskewness}_t^{\mathbb{Q}}(r_T) = \frac{\text{cmoment}_{n,t}^{\mathbb{Q}}\big|_{n=5}}{\{\text{SD}_t^{\mathbb{Q}}(r_T)\}^5}. \quad \text{(A39)}$$

The right-hand side of equation (A29) are quantities that can be inferred from option prices. ∎

B Appendix B: Proof of the expected excess return formula (9) in Result 2

In analogy to equations (A7)–(A10) and with $H[r_T] = \frac{1}{M_T[R_T]} = \frac{1}{M_T[1+r_T]} \in \mathcal{C}^\infty$, we consider

$$\text{mgf}_t^{\mathbb{P}}[\lambda] = \int_\Omega e^{\lambda r_T} p[r_T] \, dr_T \quad \text{(B1)}$$

$$= \int_\Omega e^{\lambda r_T} \frac{H[r_T]}{\int_\Omega H[r_T] q[r_T] \, dr_T} q[r_T] \, dr_T \quad \text{(using equation (A3))} \quad \text{(B2)}$$

$$= \frac{\int_\Omega e^{\lambda r_T} H[r_T] q[r_T] \, dr_T}{\int_\Omega H[r_T] q[r_T] \, dr_T} = \frac{\mathbb{E}_t^{\mathbb{Q}}(e^{\lambda r_T} H[r_T])}{\mathbb{E}_t^{\mathbb{Q}}(H[r_T])}. \quad \text{(B3)}$$

To express $\mathbb{E}_t^{\mathbb{P}}(R_T - R_{f,t})$ in terms of risk-neutral cumulants, we write equation (B3) as

$$\text{mgf}_t^{\mathbb{P}}[\lambda] = \frac{\mathbb{E}_t^{\mathbb{Q}}(e^{\lambda(r_T - r_{f,t})} H[r_T])}{\mathbb{E}_t^{\mathbb{Q}}(e^{-\lambda r_f} H[r_T])}. \quad \text{(B4)}$$

Hence, we obtain

$$\log \text{mgf}_t^{\mathbb{P}}[\lambda] = \log \mathbb{E}_t^{\mathbb{Q}}(e^{\lambda(r_T - r_{f,t})} H[r_T]) - \log \mathbb{E}_t^{\mathbb{Q}}(H[r_T]) + \lambda r_{f,t}. \quad \text{(B5)}$$

It follows that

$$\mathbb{E}_t^{\mathbb{P}}(r_T) \;=\; \left.\frac{d\log \mathrm{mgf}_t^{\mathbb{P}}[\lambda]}{d\lambda}\right|_{\lambda=0} \tag{B6}$$

$$\;=\; \left.\frac{d\log \mathbb{E}_t^{\mathbb{Q}}(e^{\lambda(r_T - r_{f,t})}H[r_T])}{d\lambda}\right|_{\lambda=0} \;+\; r_{f,t}. \tag{B7}$$

Next, we employ the fact that $e^{\lambda(r_T - r_{f,t})} = 1 + \lambda(r_T - r_{f,t}) + \frac{\lambda^2}{2}(r_T - r_{f,t})^2 + \frac{\lambda^3}{3!}(r_T - r_{f,t})^3 + \frac{\lambda^4}{4!}(r_T - r_{f,t})^4 + \cdots$, and $H[r_T] = H[r_{f,t}] + H'[r_{f,t}](r_T - r_{f,t}) + \frac{H''[r_{f,t}]}{2}(r_T - r_{f,t})^2 + \frac{H'''[r_{f,t}]}{3!}(r_T - r_{f,t})^3 + \frac{H''''[r_{f,t}]}{4!}(r_T - r_{f,t})^4 + \cdots$. Furthermore, assuming that one can switch the order of taking expectations and the infinite summations of the terms in the Taylor series expansion, we note that

$$\mathbb{E}_t^{\mathbb{Q}}(e^{\lambda(r_T - r_{f,t})}H[r_T]) \;=\; \mathbb{E}_t^{\mathbb{Q}}(\{1 + \lambda(r_T - r_{f,t}) + \frac{\lambda^2}{2}(r_T - r_{f,t})^2 + \frac{\lambda^3}{3!}(r_T - r_{f,t})^3 +$$
$$\frac{\lambda^4}{4!}(r_T - r_{f,t})^4 + \cdots\}\{H[r_{f,t}] + H'[r_{f,t}](r_T - r_{f,t}) + \frac{H''[r_{f,t}]}{2}(r_T - r_{f,t})^2$$
$$+\frac{H'''[r_{f,t}]}{3!}(r_T - r_{f,t})^3 + \frac{H''''[r_{f,t}]}{4!}(r_T - r_{f,t})^4 + \cdots\}), \tag{B8}$$

$$=\; c_0 + c_1[\lambda]\,(\kappa_1^{\mathbb{Q}} - \kappa_1^{\mathbb{Q}}) + c_2[\lambda]\,\kappa_2^{\mathbb{Q}} + c_3[\lambda]\,\kappa_3^{\mathbb{Q}} + c_4[\lambda]\,\{\kappa_4^{\mathbb{Q}} + 3(\kappa_2^{\mathbb{Q}})^2\} + \cdots$$

$$=\; c_0(1 + \underbrace{\frac{1}{c_0}\{c_2[\lambda]\,\kappa_2^{\mathbb{Q}} + c_3[\lambda]\,\kappa_3^{\mathbb{Q}} + c_4[\lambda]\,\{\kappa_4^{\mathbb{Q}} + 3(\kappa_2^{\mathbb{Q}})^2\} + \cdots\}}_{\equiv\, x[\lambda]}) \tag{B9}$$

$$=\; c_0(1 + x[\lambda]). \tag{B10}$$

In deriving equation (B9), the cumulants under $\mathbb{Q}$ are defined as $\kappa_1^{\mathbb{Q}} = \mathbb{E}_t^{\mathbb{Q}}(r_T) = r_{f,t}$, $\kappa_2^{\mathbb{Q}} = \mathbb{E}_t^{\mathbb{Q}}(\{r_T - \mu_1^{\mathbb{Q}}\}^2)$, $\kappa_3^{\mathbb{Q}} = \mathbb{E}_t^{\mathbb{Q}}(\{r_T - \mu_1^{\mathbb{Q}}\}^3)$, and $\kappa_4^{\mathbb{Q}} \equiv \mathbb{E}_t^{\mathbb{Q}}(\{r_T - \mu_1^{\mathbb{Q}}\}^4) - 3(\kappa_2^{\mathbb{Q}})^2$.

Each $c_j[\lambda]$ (for $j = 1, \cdots$) in equation (B9) depends on λ, where for example,

$$c_0 \;=\; H[r_{f,t}], \tag{B11}$$

$$c_1[\lambda] \;=\; \lambda H[r_{f,t}] \;+\; H'[r_{f,t}], \tag{B12}$$

$$c_2[\lambda] \;=\; \frac{\lambda^2 H[r_{f,t}] \;+\; 2\,\lambda\,H'[r_{f,t}] \;+\; H''(r_{f,t})}{2}, \tag{B13}$$

$$c_3[\lambda] \;=\; \frac{\lambda^3 H[r_{f,t}]}{6} + \frac{1}{2}\lambda^2 H'[r_{f,t}] + \frac{\lambda H''[r_{f,t}]}{2} + \frac{H'''[r_{f,t}]}{6}, \quad\text{and} \tag{B14}$$

$$c_4[\lambda] \;=\; \frac{\lambda^4 H[r_{f,t}]}{24} + \frac{\lambda^3 H'[r_{f,t}]}{6} + \frac{1}{4}\lambda^2 H''[r_{f,t}] + \frac{\lambda H'''[r_{f,t}]}{6} + \frac{H''''[r_{f,t}]}{24}, \tag{B15}$$

...

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Substituting the formula $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \ldots$ into equation (B10), we have

$$\log \mathbb{E}_t^{\mathbb{Q}}(e^{\lambda(r_T - r_{f,t})} H[r_T]) = \log c_0 + \log(1 + x[\lambda]) \tag{B16}$$

$$= \log c_0 + x[\lambda] - \frac{x^2[\lambda]}{2} + \frac{x^3[\lambda]}{3} - \frac{x^4[\lambda]}{4} \ldots \tag{B17}$$

Then, using equation (B7), and rearranging, we have

$$\mathbb{E}_t^{\mathbb{P}}(r_T) - r_{f,t} = \left.\frac{dx[\lambda]}{d\lambda}\right|_{\lambda=0} - \frac{1}{2}\left.\frac{dx^2[\lambda]}{d\lambda}\right|_{\lambda=0} + \frac{1}{3}\left.\frac{dx^3[\lambda]}{d\lambda}\right|_{\lambda=0} - \frac{1}{4}\left.\frac{dx^4[\lambda]}{d\lambda}\right|_{\lambda=0} + \cdots \tag{B18}$$

$$= \frac{H'[r_{f,t}]}{H[r_{f,t}]}\kappa_2^{\mathbb{Q}} + \frac{H''[r_{f,t}]}{2\,H[r_{f,t}]}\kappa_3^{\mathbb{Q}} + \frac{H'''[r_{f,t}]}{6\,H[r_{f,t}]}\{\kappa_4^{\mathbb{Q}} + 3(\kappa_2^{\mathbb{Q}})^2\} -$$

$$\underbrace{\frac{\left(H'[r_{f,t}]\kappa_2^{\mathbb{Q}} + \frac{H''[r_{f,t}]\kappa_3^{\mathbb{Q}}}{2} + \frac{H'''[r_{f,t}]\{\kappa_4^{\mathbb{Q}}+3(\kappa_2^{\mathbb{Q}})^2\}}{6}\right)\left(\frac{H''[r_{f,t}]\kappa_2^{\mathbb{Q}}}{2} + \frac{H'''[r_{f,t}]\kappa_3^{\mathbb{Q}}}{6} + \frac{H''''[r_{f,t}]\{\kappa_4^{\mathbb{Q}}+3(\kappa_2^{\mathbb{Q}})^2\}}{24}\right)}{(H[r_{f,t}])^2}}$$

$$\cdots .$$

Assuming $(\kappa_2^{\mathbb{Q}})^2 \approx 0$ (i.e., ignoring the effect of square of the risk-neutral variance of market return) and some cross-product risk-neutral return moments (i.e., $\kappa_2^{\mathbb{Q}} \kappa_3^{\mathbb{Q}} \approx 0$, and so on), we see that

$$\mathbb{E}_t^{\mathbb{P}}(R_T - R_{f,t}) = \frac{H'[r_{f,t}]}{H[r_{f,t}]}\kappa_2^{\mathbb{Q}} + \frac{H''[r_{f,t}]}{2\,H[r_{f,t}]}\kappa_3^{\mathbb{Q}} + \frac{H'''[r_{f,t}]}{6\,H[r_{f,t}]}\kappa_4^{\mathbb{Q}} + \cdots . \tag{B19}$$

Equation (B19) is in agreement with equation (A28) when $H[r_T] = \frac{1}{M_T[1+r_T]} = \exp(-m_0 + 1 + \phi(r_T - r_{f,t}))$. ■

C   Appendix C: Proof of Lemma 1

In what follows, we suppress the subscript T on the random variable X_T.

Since $\mathbb{E}_t^{\mathbb{P}}(X)$ is in the domain of g, we have

$$\text{cov}_t^{\mathbb{P}}(g[X], X) = \mathbb{E}_t^{\mathbb{P}}(\{X - \mathbb{E}_t^{\mathbb{P}}(X)\}\{g[X] - \mathbb{E}_t^{\mathbb{P}}(g[X])\}) \tag{C1}$$

$$= \mathbb{E}_t^{\mathbb{P}}(\{X - \mathbb{E}_t^{\mathbb{P}}(X)\}\{g[X] - g[\mathbb{E}_t^{\mathbb{P}}(X)]\}) + \mathbb{E}_t^{\mathbb{P}}(\{X - \mathbb{E}_t^{\mathbb{P}}(X)\}\{g[\mathbb{E}_t^{\mathbb{P}}(X)] - \mathbb{E}_t^{\mathbb{P}}(g[X])\})$$

$$= \mathbb{E}_t^{\mathbb{P}}(\{X - \mathbb{E}_t^{\mathbb{P}}(X)\}\{g[X] - g[\mathbb{E}_t^{\mathbb{P}}(X)]\}) + \{g[\mathbb{E}_t^{\mathbb{P}}(X)] - \mathbb{E}_t^{\mathbb{P}}(g[X])\}\underbrace{\mathbb{E}_t^{\mathbb{P}}(X - \mathbb{E}_t^{\mathbb{P}}(X))}_{=0}$$

$$= \mathbb{E}_t^{\mathbb{P}}(\{X - \mathbb{E}_t^{\mathbb{P}}(X)\}\{g[X] - g[\mathbb{E}_t^{\mathbb{P}}(X)]\}). \tag{C2}$$

The lemma follows by noticing that if $g[X]$ is decreasing on $\mathcal{D}$, when $X \leq (\geq) \mathbb{E}_t^{\mathbb{P}}(X)$, we have $g[X] \geq (\leq) g[\mathbb{E}_t^{\mathbb{P}}(X)]$, which implies that $(X - \mathbb{E}_t^{\mathbb{P}}(X))(g[X] - g[\mathbb{E}_t^{\mathbb{P}}(X)]) \leq 0$ everywhere.

Furthermore, if $g[X]$ is increasing on $\mathcal{D}$, when $X \leq (\geq) \mathbb{E}_t^{\mathbb{P}}(X)$, we have $g[X] \leq (\geq) g[\mathbb{E}_t^{\mathbb{P}}(X)]$, which implies that $(X - \mathbb{E}_t^{\mathbb{P}}(X))(g[X] - g[\mathbb{E}_t^{\mathbb{P}}(X)]) \geq 0$ everywhere. $\blacksquare$

D   Appendix D: Proof of Result 3

For brevity, we again suppress the subscript T on the random variable X_T.

It suffices to notice that if $g'[X] \leq (\geq)0$, then $g[X]$ is decreasing (increasing).

Apply Lemma 1 by setting, for any positive integer n,

$$X \; = \; R_T^n \qquad \text{and} \qquad g[X] \; = \; M_T[X^{\frac{1}{n}}] \, X. \tag{D1}$$

Then

$$g'[X] \;\; = \;\; M_T'[X^{\frac{1}{n}}] \frac{1}{n} X^{\frac{1}{n}-1} X \;\; + \;\; M_T[X^{\frac{1}{n}}] \tag{D2}$$

$$= \;\; \frac{M_T'[R_T] \, R_T}{n} \;\; + \;\; M_T[R_T] \tag{D3}$$

$$= \;\; \frac{M_T[R_T] \, R_T}{n} \left(\frac{M_T'[R_T]}{M_T[R_T]} + \frac{n}{R_T} \right). \tag{D4}$$

Next, to obtain $\mathbb{E}_t^{\mathbb{P}}(R_T^n)$, notice that

$$\mathrm{cov}_t^{\mathbb{P}}(M_T[R_T]R_T^n, R_T^n) = \mathbb{E}_t^{\mathbb{P}}(M_T[R_T] \, R_T^{2n}) - \mathbb{E}_t^{\mathbb{P}}(M_T[R_T] \, R_T^n) \, \mathbb{E}_t^{\mathbb{P}}(R_T^n) \tag{D5}$$

$$= \mathbb{E}_t^{\mathbb{P}}(M_T[R_T]) \, \mathbb{E}_t^{\mathbb{Q}}(R_T^{2n}) \;\; - \;\; \mathbb{E}_t^{\mathbb{P}}(M_T[R_T]) \, \mathbb{E}_t^{\mathbb{Q}}(R_T^n) \, \mathbb{E}_t^{\mathbb{P}}(R_T^n) \tag{D6}$$

$$= \frac{\mathbb{E}_t^{\mathbb{Q}}(R_T^{2n})}{R_{f,t}} - \frac{\mathbb{E}_t^{\mathbb{Q}}(R_T^n)}{R_{f,t}} \mathbb{E}_t^{\mathbb{P}}(R_T^n). \tag{D7}$$

Rearranging,

$$\mathbb{E}_t^{\mathbb{P}}(R_T^n) = \frac{\mathbb{E}_t^{\mathbb{Q}}(R_T^{2n}) - R_{f,t} \, \mathrm{cov}_t^{\mathbb{P}}(M_T[R_T] \, R_T^n, R_T^n)}{\mathbb{E}_t^{\mathbb{Q}}(R_T^n)}. \tag{D8}$$

In particular, for $n = 1$,

$$\mathbb{E}_t^{\mathbb{P}}(R_T) = \frac{\mathbb{E}_t^{\mathbb{Q}}(R_T^2) - R_{f,t} \operatorname{cov}_t^{\mathbb{P}}(M_T[R_T]R_T, R_T)}{\mathbb{E}_t^{\mathbb{Q}}(R_T)} \tag{D9}$$

$$= R_{f,t} + \frac{\operatorname{var}_t^{\mathbb{Q}}(R_T)}{R_{f,t}} - \operatorname{cov}_t^{\mathbb{P}}(M_T[R_T]R_T, R_T). \tag{D10}$$

Furthermore, for $n = 2$,

$$\mathbb{E}_t^{\mathbb{P}}(R_T^2) = \frac{\mathbb{E}_t^{\mathbb{Q}}(R_T^4) - R_{f,t} \operatorname{cov}_t^{\mathbb{P}}(M_T[R_T]R_T^2, R_T^2)}{\mathbb{E}_t^{\mathbb{Q}}(R_T^2)} \tag{D11}$$

$$= \frac{\mathbb{E}_t^{\mathbb{Q}}(R_T^4)}{\mathbb{E}_t^{\mathbb{Q}}(R_T^2)} - \frac{R_{f,t}}{\mathbb{E}_t^{\mathbb{Q}}(R_T^2)} \operatorname{cov}_t^{\mathbb{P}}(M_T[R_T]R_T^2, R_T^2). \tag{D12}$$

We have provided the intermediate steps of the proof. ∎

Reported are (i) estimates of (ϕ, m_0) of the specification of M_T in equation (3) and (ii) features of the conditional expected excess return of the market (based on equation (4) of Result 1). We obtain the following estimates of ϕ and m_0 by solving a minimum discrepancy problem:

	ϕ	m_0
Estimate	2.274	1.007
Bootstrap 5th percentile value	1.30	1.002
Bootstrap 95th percentile value	3.29	1.015

Specifically, one solves $\inf_{M \in \mathbb{M}} \mathbb{E}^{\mathbb{P}}(M \log M)$ with $\mathbb{M} \equiv \{M > 0 \text{ such that } \mathbb{E}^{\mathbb{P}}(M(R - R_f)) = 0, \mathbb{E}^{\mathbb{P}}(M) = \mathbb{E}^{\mathbb{P}}(R_{f,t}^{-1}) \equiv \mu_M, \text{ and } \mathbb{E}^{\mathbb{P}}(M \log M) < \infty\}$, where $\mathbb{E}^{\mathbb{P}}(.)$ is unconditional expectation. The optimal solution is $M^* = \exp(m_0^* - 1 - \phi^*(R - R_f))$, where (ϕ^*, m_0^*) solve $\arg\inf_{(\phi, m_0)} \{-m_0 \mu_M + \mathbb{E}^{\mathbb{P}}(\exp(m_0 - 1 - \phi(R - R_f)))\}$ (e.g., Borovička, Hansen, and Scheinkman (2016, Section VIII.B)). The sample period for estimating ϕ is 1926:07 to 2018:12 (1,110 monthly observations), and the time series of $R_T - R_{f,t}$ is taken from the data library of Kenneth French. We adopt a bootstrap procedure and draw $(R_{f,t}, R_T)$ with replacement. Then, we reestimate (ϕ, m_0). The reported confidence intervals are based on 10,000 bootstrap samples.

For each date t, we compute the conditional expected excess return of the market as

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t} = \frac{1}{\phi} \Big(&\{\phi\, \mathrm{SD}_t^{\mathbb{Q}}(R_T)\}^2 + \frac{1}{2}\{\phi\, \mathrm{SD}_t^{\mathbb{Q}}(R_T)\}^3 \times \mathrm{Skewness}_t^{\mathbb{Q}}(R_T) \\
&+ \frac{1}{6}\{\phi\, \mathrm{SD}_t^{\mathbb{Q}}(R_T)\}^4 \times \{\mathrm{Kurtosis}_t^{\mathbb{Q}}(R_T) - 3\} \\
&+ \frac{1}{24}\{\phi\, \mathrm{SD}_t^{\mathbb{Q}}(R_T)\}^5 \underbrace{\{\mathrm{Hskewness}_t^{\mathbb{Q}}(R_T)}_{\text{hyperskewness}} - 10\, \mathrm{Skewness}_t^{\mathbb{Q}}(R_T)\} + \ldots \Big),
\end{aligned}
$$

where $\mathrm{SD}_t^{\mathbb{Q}}(R_T) \equiv \sqrt{\mathrm{var}_t^{\mathbb{Q}}(R_T)}$, $\mathrm{Skewness}_t^{\mathbb{Q}}(R_T)$, $\mathrm{Kurtosis}_t^{\mathbb{Q}}(R_T)$, and $\mathrm{Hskewness}_t^{\mathbb{Q}}(R_T)$ are, respectively, the conditional risk-neutral return volatility, skewness, kurtosis, and hyperskewness, respectively. The reported features of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ rely on data from the S&P 500 index options market from January 1990 to December 2018 (29 years, 348 option expiration cycles). *All reported numbers are in annualized percentage units.*

		Average	SD	Percentiles		
				5th	50th	95th
(i)	Expected excess return of the market (annualized, %)	8.97	9.99	1.8	5.8	26.2
(ii)	$\frac{1}{\phi}(\{\phi\, \mathrm{SD}_t^{\mathbb{Q}}(R_T)\}^2 + \frac{1}{2}\{\phi\, \mathrm{SD}_t^{\mathbb{Q}}(R_T)\}^3 \times \mathrm{Skewness}_t^{\mathbb{Q}}(R_T))$	8.91	9.92	1.8	5.8	25.8
(iii)	$\frac{1}{\phi}\{\phi\, \mathrm{SD}_t^{\mathbb{Q}}(R_T)\}^2$	9.84	11.3	1.9	6.4	30.6
(iv)	$\frac{1}{6\phi}\{\phi\, \mathrm{SD}_t^{\mathbb{Q}}(R_T)\}^4 \times \{\mathrm{Kurtosis}_t^{\mathbb{Q}}(R_T) - 3\}$	0.049	0.16	-0.1	0.00	0.20
(v)	$\frac{1}{24\phi}\{\phi \mathrm{SD}_t^{\mathbb{Q}}(R_T)\}^5 \{\mathrm{Hskewness}_t^{\mathbb{Q}}(R_T) - 10\, \mathrm{Skewness}_t^{\mathbb{Q}}(R_T)\}$	0.006	0.07	-0.02	-0.00	0.02
(vi)	Martin (2017) lower bound, $R_{f,t}^{-1} \mathrm{var}_t^{\mathbb{Q}}(R_T)$	4.32	4.96	0.8	2.8	13.4
(vii)	$\{\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}\} - \{R_{f,t}^{-1} \mathrm{var}_t^{\mathbb{Q}}(R_T)\}$	4.64	4.96	1.0	3.0	13.6

Table 2: **Estimates of conditional expected excess return of the market from alternative models of M_T**

Reported are estimates of (m_0, ϕ, ϕ_z) of the specification of M_T in equation (10); that is, $M_T[R_T] = \exp(m_0 - 1 - \{\phi + \phi_z z_t\}(R_T - R_{f,t}))$. We consider z_t to be either the prior month change in realized market variance or the prior one month return on the HML factor. The market variance is constructed as the sum of daily squared log returns. To estimate (m_0, ϕ, ϕ_z), one solves $\arg\inf_{(m_0,\phi,\phi_z)} \{-m_0 \mu_M + \mathbb{E}^{\mathbb{P}}(\exp(m_0 - 1 - (\phi + \phi_z z_t)(R - R_f)))\}$, where $\mathbb{E}^{\mathbb{P}}(.)$ is unconditional expectation. The sample period for estimation is 1926:07 to 2018:12 (1,110 monthly observations), and the time series of $R_T - R_{f,t}$ is taken from the data library of Kenneth French. We adopt a bootstrap procedure and draw $(R_{f,t}, R_T, z_t)$ with replacement. Then, we reestimate (m_0, ϕ, ϕ_z). The reported confidence intervals are based on 10,000 bootstrap samples.

	z_t is change in market variance			z_t is HML		
	ϕ	m_0	ϕ_z	ϕ	m_0	ϕ_z
Estimate	2.277	1.005	-50.29	2.202	1.005	8.69
Bootstrap 5th percentile value	1.372	1.000	-175.38	1.218	1.000	-11.01
Bootstrap 95th percentile value	3.339	1.013	82.20	3.230	1.013	29.79

For each date t, we compute the conditional expected excess return of the market using the formula in equation (9) of Result 2. The reported features of $\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ rely on data from the S&P 500 index options market from January 1990 to December 2018 (29 years, 348 option expiration cycles). *All reported numbers are in annualized percentage units.*

		$\mathbb{E}_t^{\mathbb{P}}(R_T) - R_{f,t}$ (annualized, %)				
					Percentiles	
		Average	SD	5th	50th	95th
(i)	z_t is change in market variance	8.84	9.96	1.81	5.7	24.8
(ii)	z_t is HML factor	8.53	8.64	1.8	5.6	25.9

Table 3: **Testing the NCC when $\mathbf{Z}_T[R_T]$ contains the (i) gross return of the market and (ii) the gross return of an at-the-money straddle, or a 2% out-of-the-money strangle**

In Model A, the gross returns in $\mathbf{Z}_T[R_T]$ are

$$\mathbf{Z}_T[R_T] = \begin{pmatrix} R_T \\ R_T^{\text{straddle}} \end{pmatrix}, \quad \text{where} \quad R_T^{\text{straddle}} = \frac{S_t \max(R_T - 1, 0) + S_t \max(1 - R_T, 0)}{\text{call}_{t,T}[S_t] + \text{put}_{t,T}[S_t]}.$$

In Model B, the gross returns in $\mathbf{Z}_T[R_T]$ are

$$\mathbf{Z}_T[R_T] = \begin{pmatrix} R_T \\ R_T^{\text{strangle}} \end{pmatrix}, \quad \text{where} \quad R_T^{\text{strangle}} = \frac{S_t \max(R_T - e^{0.02}, 0) + S_t \max(e^{-0.02} - R_T, 0)}{\text{call}_{t,T}[S_t e^{0.02}] + \text{put}_{t,T}[S_t e^{-0.02}]}.$$

The form of the projected SDF, $M_T[R_T]$, is

$$M_T[R_T] = \mathbf{Z}_T^\top[R_T]\,\boldsymbol{\alpha}, \quad \text{and we infer} \quad \boldsymbol{\alpha} = \{\mathbb{E}(\mathbf{Z}_T^\top[R_T]\,\mathbf{Z}_T[R_T])\}^{-1}\,\mathbf{1}.$$

Reported is the unconditional correlation between $M_T[R_T]R_T$ and R_T, denoted as NCC_T. SD is standard deviation of $M_T[R_T] = \mathbf{Z}_T^\top[R_T]\,\boldsymbol{\alpha}$. We adopt a bootstrap procedure and draw $\mathbf{Z}_T[R_T]$ with replacement. Then, we reestimate $\boldsymbol{\alpha}$. The reported confidence intervals are based on 10,000 bootstrap samples. All reported results rely on data from the S&P 500 index options market from January 1990 to December 2018 (29 years, 348 option expiration cycles).

Panel A: SDF depends on R_T and R_T^{straddle} (Model A)

| | α_{market} | α_{straddle} | NCC_T | Properties of $M_T[R_T] = \mathbf{Z}_T^\top[R_T]\,\boldsymbol{\alpha}$ | | |
				SD (annual,%)	Mean (monthly)	Minimum (annual, %)
Estimates	0.73	0.30	0.26	78	0.992	72
Bootstrap						
5th	0.61	0.17	0.13	45	0.987	61
95th	0.83	0.48	0.49	114	0.997	82

Panel B: SDF depends on R_T and R_T^{strangle} (Model B)

| | α_{market} | α_{strangle} | NCC_T | Properties of $M_T[R_T] = \mathbf{Z}_T^\top[R_T]\,\boldsymbol{\alpha}$ | | |
				SD (annual, %)	Mean (monthly)	Minimum (annual, %)
Estimates	0.83	0.21	0.23	80	0.992	81
Bootstrap						
5th	0.76	0.11	0.09	46	0.987	74
95th	0.89	0.35	0.47	120	0.997	87

Table 4: **Testing the NCC when $\log M_T[R_T]$ depends on excess returns of variance in down and up equity markets (Model C)**

In Model C, we allow for asymmetric effects of return variance on the SDF. Specifically, the form of the projected SDF, $M_T[R_T]$, is

$$M_T[R_T] = \exp\left(m_0 - 1 + \eta_{\text{variance}}^- \, 1_{R_T<1} \, (R_T^{\text{variance}} - R_{f,t}) + \eta_{\text{variance}}^+ \, 1_{R_T>1} \, (R_T^{\text{variance}} - R_{f,t})\right).$$

We compute the gross return, R_T^{variance}, based on the payoff of the squared log contract, and accordingly synthesize its price from options

$$R_T^{\text{variance}} = \frac{\{\log R_T\}^2}{\mathbf{q}_{t,\{\log R_T\}^2}}, \qquad \text{where}$$

$$\mathbf{q}_{t,\{\log R_T\}^2} = R_{f,t}^{-1}\mathbb{E}_t^{\mathbb{Q}}(\{\log R_T\}^2) = \int_{K<S_t} \frac{2(1 - \log\frac{K}{S_t})}{K^2} \, \text{put}_{t,T}[K]\, dK + \int_{K>S_t} \frac{2(1 - \log\frac{K}{S_t})}{K^2} \, \text{call}_{t,T}[K]\, dK.$$

Reported is the unconditional correlation between $M_T[R_T]R_T$ and R_T, denoted as NCC_T. SD is the standard deviation of $M_T[R_T]$. We obtain the estimates of $(m_0, \eta_{\text{variance}}^-, \eta_{\text{variance}}^+)$ by solving $\inf_{M\in\mathbb{M}} \mathbb{E}^{\mathbb{P}}(M \log M)$ with $\mathbb{M} \equiv \{M > 0$ such that $\mathbb{E}^{\mathbb{P}}(M\{1_{R_T<1}(R_T^{\text{variance}} - R_{f,t})\}) = 0$, $\mathbb{E}^{\mathbb{P}}(M\{1_{R_T>1}(R_T^{\text{variance}} - R_{f,t})\}) = 0$, $\mathbb{E}^{\mathbb{P}}(M) = \mathbb{E}^{\mathbb{P}}(R_{f,t}^{-1}) \equiv \mu_M$, and $\mathbb{E}^{\mathbb{P}}(M \log M) < \infty\}$, where $\mathbb{E}^{\mathbb{P}}(.)$ is unconditional expectation (e.g., Borovička, Hansen, and Scheinkman (2016, Section VIII.B)). The optimal solution is $M^* = \exp(m_0^* - 1 + (\eta_{\text{variance}}^-)^* 1_{R_T<1} \, (R_T^{\text{variance}} - R_{f,t}) + (\eta_{\text{variance}}^+)^* 1_{R_T>1} \, (R_T^{\text{variance}} - R_{f,t}))$, where $(m_0^*, (\eta_{\text{variance}}^-)^*, (\eta_{\text{variance}}^+)^*)$ solve $\arg\inf_{(m_0, \eta_{\text{variance}}^-, \eta_{\text{variance}}^+)} \{-m_0\,\mu_M + \mathbb{E}^{\mathbb{P}}(\exp(m_0 - 1 + \eta_{\text{variance}}^- 1_{R_T<1} \, (R_T^{\text{variance}} - R_{f,t}) + \eta_{\text{variance}}^+ 1_{R_T>1} \, (R_T^{\text{variance}} - R_{f,t})))\}$. We adopt a bootstrap procedure and draw $(R_{f,t}, R_T, R_T^{\text{variance}})$ with replacement. Then, we reestimate $(m_0, \eta_{\text{variance}}^-, \eta_{\text{variance}}^+)$. The reported confidence intervals are based on 10,000 bootstrap samples. All reported results rely on data from the S&P 500 index options market from January 1990 to December 2018 (29 years, 348 option expiration cycles).

| | m_0 | η_{variance}^- | η_{variance}^+ | NCC_T | Properties of $M_T[R_T]$ | | |
					SD (annual, %)	Mean (monthly)	Minimum (annual, %)
Estimate	1.132	0.118	0.647	0.12	250	0.998	60
Bootstrap							
5th	1.082	0.008	0.470	0.04	179	0.998	41
95th	1.233	0.267	1.111	0.21	343	0.998	68

A New Formula for the Expected Excess Return of the Market

Internet Appendix: Not for Publication

Abstract

Section A describes the data on S&P 500 equity index, interest rates, and options on the S&P 500 index.

I Internet Appendix

A Interest rates and options data on the S&P 500 index

S&P 500 index options (ticker, SPX): The data on S&P 500 index options are daily and constructed over the sample period of January 1990 to December 2018. The call and put option data are extracted from the Optsum historical database (Market Data Express), maintained by the Chicago Board Options Exchange (CBOE).

The S&P 500 index options are European style and expire on the third Friday of the expiration month. We focus on the nearest maturity options and construct the data by expiration cycle dates (of which there are 348 in total).

The start date for the first expiration cycle is January 22, 1990. The final expiration cycle starts on December 24, 2018, and ends on January 18, 2019.

Our calculation of the number of days during each expiration cycle takes into account the fact that these options expire at the market close prior to August 24, 1992, and expire at the market open afterward. Accordingly, the Friday on which the option expires is included in the expiration cycle prior to August 24, 1992, and is excluded afterward.

The option price in our calculations is the midpoint of the bid and ask quotes, and we only keep out-of-the-money options (i.e., $K/S_t < 1$ for puts and $S_t/K < 1$ for calls). The final sample contains 36,630 end-of-the-day option price observations.

Gross interest rate $(R_{f,t})$: Constructed from (daily) four-week interest rate series (Center for Research in Security Prices) and is scaled to match the number of days in the options expiration cycles. ∎
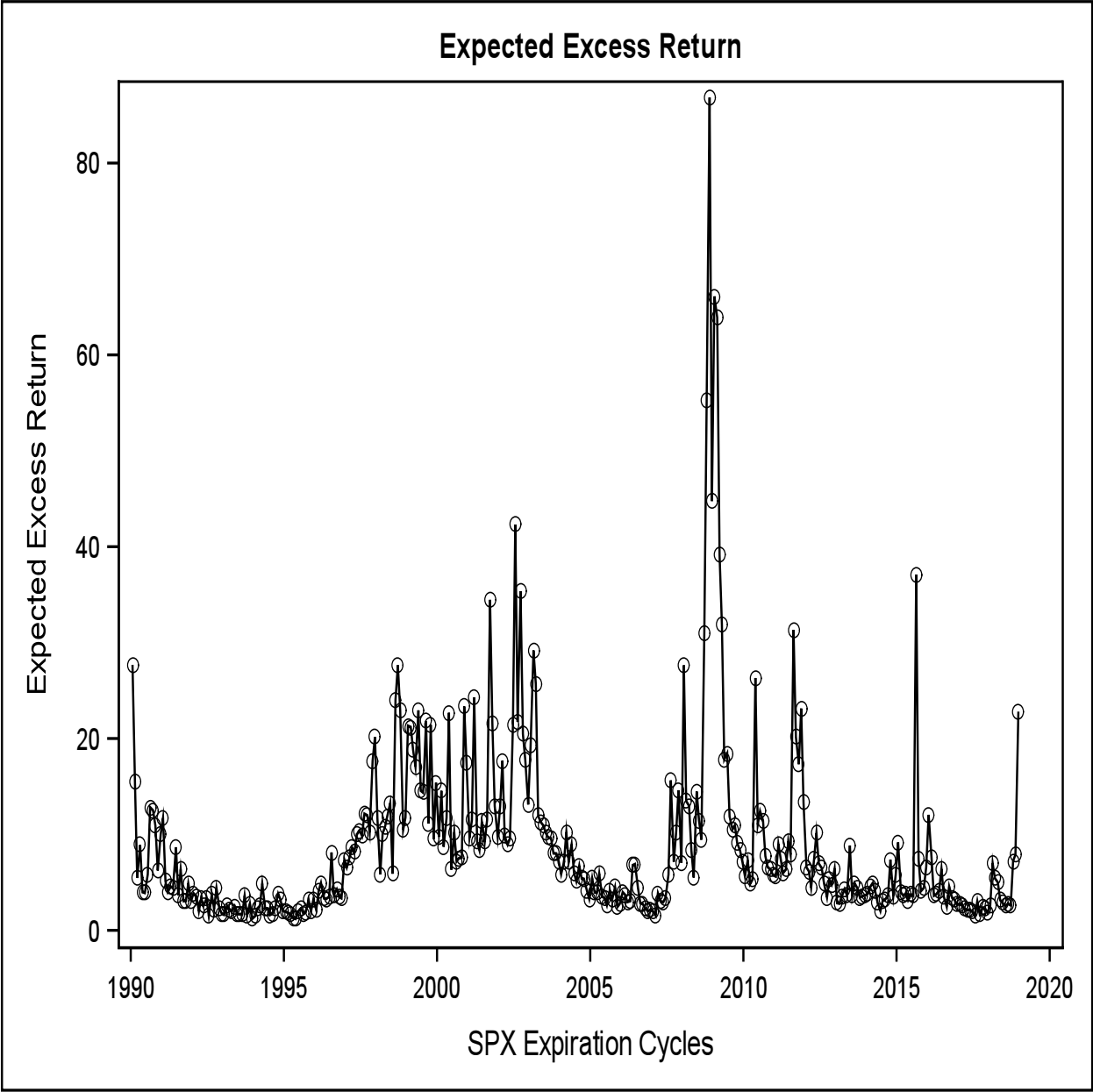
Figure 1: **Expected excess return of the market**

Plotted is the expected excess return of the market (expressed in *annualized percentage terms*) and is computed following equation (4). The study period is January 1990 to December 2018 (348 options expiration cycles).
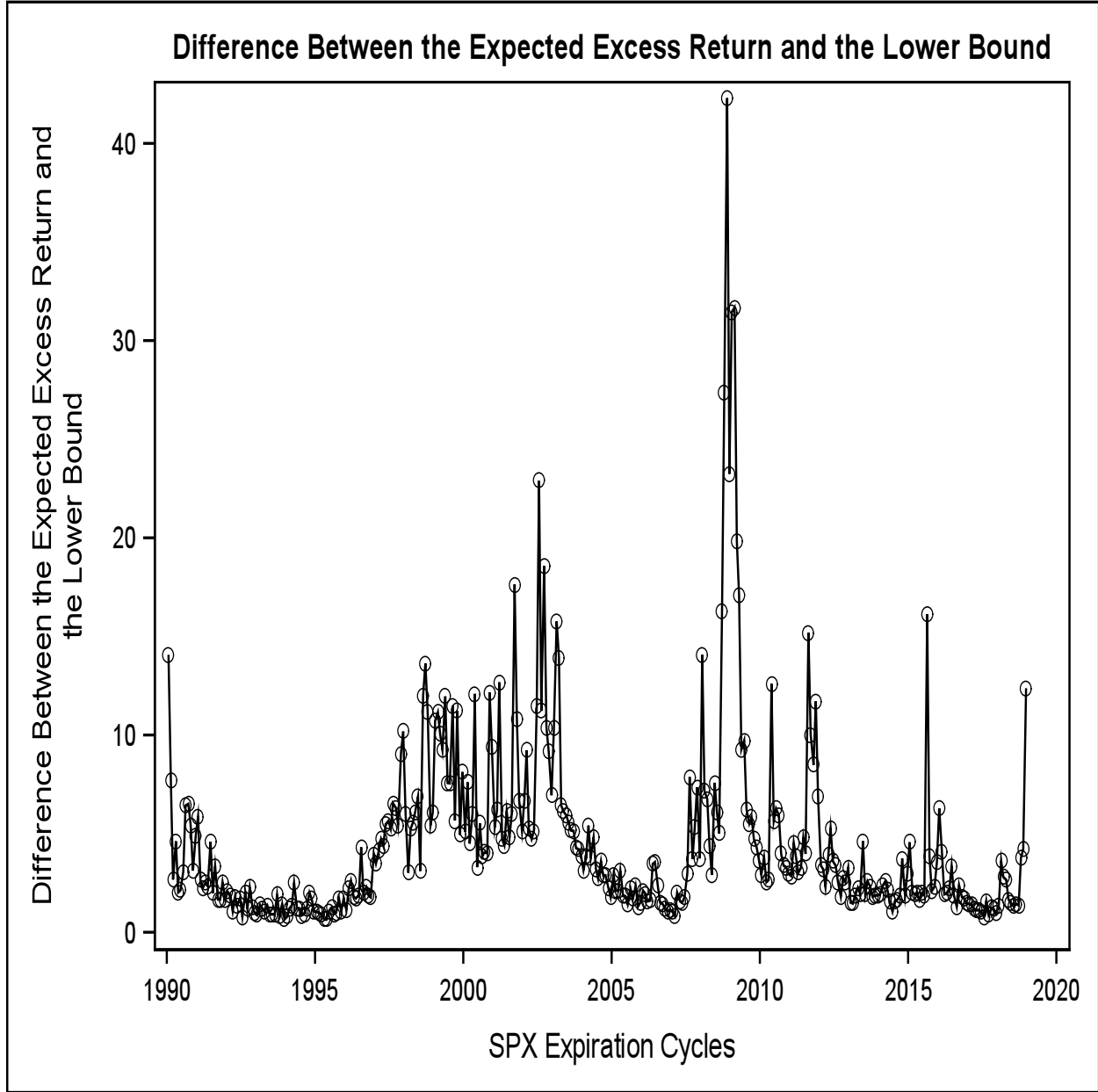
Figure 2: **Difference between the expected excess return of the market and the lower bound**

Plotted is the difference between the expected excess return of the market and the lower bound (expressed in *annualized percentage terms*):

$$\{\mathbb{E}_t^{\mathbb{P}}(R_T) \ - \ R_{f,t}\} \ - \ \underbrace{R_{f,t}^{-1} \operatorname{var}_t^{\mathbb{Q}}(R_T)}_{\text{lower bound}}.$$

The expected excess return of the market is computed following equation (4), whereas the lower bound, $R_{f,t}^{-1} \operatorname{var}_t^{\mathbb{Q}}(R_T)$, is based on Martin (2017, equation (5)). The study period is January 1990 to December 2018 (348 options expiration cycles).