

Safe Assets: A Dynamic Retrading Perspective*

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Abstract

The price of a safe asset reflects not only the expected discounted future cash flows but also future service flows, since retrading allows partial insurance of idiosyncratic risk in an incomplete markets setting. This lowers the issuers' interest burden. As idiosyncratic risk rises during recessions, so does the value of the service flows bestowing the safe asset with a negative β . The resulting exorbitant privilege resolves government debt valuation puzzles and allows the government to run a permanent (primary) deficit without ever paying back its debt, but the government faces a "Debt Laffer Curve".

Keywords: Safe Asset, Exorbitant Privilege, Government Debt, Debt Laffer Curve, Ponzi Scheme, Fiscal Capacity, I Theory of Money, r vs. g

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1 Introduction

What is a safe asset? What are its features? Why does it have a negative β ? Why is government debt of advanced countries like the US and Japan a safe asset? Why does it enjoy an exorbitant privilege? How much of it can the market absorb? Is there a debt valuation puzzle? When can governments run a permanent (primary) deficit without ever paying back its debt, like a Ponzi scheme, and is there a limit, a “Debt Laffer Curve”? When does one lose the safe asset status? How do we have to modify representative agent asset pricing and the government debt valuation equation to account for safe asset features? This paper presents a theory of safe assets that shed light on these questions.

We define a safe asset by its key characteristic, the *Good Friend Analogy*. A safe asset is like a good friend, it is around; that is, it is (i) valuable and (ii) liquid when one needs it. We illustrate this within a setting in which citizens face uninsurable idiosyncratic risks and save for precautionary reasons. Each citizen adjusts her portfolio consisting of risky physical capital and the safe asset, a government bond. Idiosyncratic shocks that cannot be diversified away (as well as aggregate shocks) make capital risky. This makes the safe asset attractive since it can be sold after an adverse shock. From an individual citizen’s perspective it is this ability to retrade which makes the government bond a desirable hedging instrument. Her planned dynamic trading strategy generates a payoff stream that is a good hedge.

Since it is the re trading after idiosyncratic shocks that creates this extra service flow, a safe asset should not be plagued by trading frictions. Hence, low risk debt that is informationally insensitive is better suited as a safe asset. Asymmetric information and other trading frictions are not conducive for re trading.¹

Since a safe asset generates this extra service flow in the form of self-insurance, it is attractive even at a low real interest rate, r , its cash flow return. It is instructive to consider a new asset pricing formula which nicely separates the two benefits of the safe asset: cash flows, possibly negative, and a service flow that results from the ability to

¹Hence, it makes sense for central banks to act as market maker of last resort to ensure that bid-ask spreads remain low. Viewed this way, John Law’s big achievement was to create a safe asset status for English and French government debt early in the 18th century.

self-insure through retrading. The real value of a safe asset (or any tradable asset) is

$$price_t = \mathbb{E}_t[PV_{r^{**}}[cash\ flows]] + \mathbb{E}_t[PV_{r^{**}}[service\ flows]].$$

While the traditional asset pricing formula prices the cash flow of a buy and hold strategy of the safe asset, the above stated pricing formula prices the cash flow of a dynamic (equilibrium) strategy whose cash flow is positive when the asset is sold (after a negative shock) and negative when additional safe assets are bought (after a positive shock). Valuing individual dynamic trading cash flow streams and aggregating them leads to the above pricing equation, where a different discount rate, r^{**} , arises naturally. Interestingly, r^{**} can be viewed as the “representative agent interest rate” in an incomplete market setting. It is the risk-free rate that excludes the component that is due to precautionary demand driven by the exposure to uninsurable idiosyncratic risk.

A safe asset is really like a good friend if it not only allows citizens to self-insure against adverse idiosyncratic shocks, but also serves as a safe haven after adverse aggregate shock. That is, if it appreciates in recessions due to flight-to-safety capital flows. To see why a safe asset has a non-positive β consider an economy in which idiosyncratic risk rises when entering a recession and aggregate output declines. A drop in output reduces payoffs and increases the marginal utility, leading to the traditional positive β in the asset pricing equation for the cash flow term. The second term, the service flow term, behaves very differently. As idiosyncratic risk rises in recessions, citizens prefer to shift their portfolio away from capital towards the safe asset, resulting in a force that pushes up the real value of safe assets. It is due to the second term capturing the discounted stream of service flows that the safe asset has a non-positive β .

Our model has also interesting stock market asset pricing implications due to "flight-to safety" phenomena. During recessions, idiosyncratic risk is assumed to rise. While for outside equity, idiosyncratic risk can be diversified away, the residual claimant to each firm is an insider who remains exposed to the idiosyncratic risk via her inside equity holdings. During recessions, these insiders demand a higher insider risk premium which depresses payouts to outside equity holders. As a consequence, the (outside) equity stock index depreciates relative to the safe asset. That is, outside equity even when held in a diversified mutual fund has a positive β .

For the safe asset the required cash flow return r is low due to its service flow and its negative β . Any entity that issues a safe asset, be it the government or private corpo-

rations, benefits from the low required cash flow return. Since similar service flows can be derived from all safe assets, whether issued by the government or by corporations, traditional measures of convenience yields, like the BAA-Treasury interest rate spread used in [Krishnamurthy and Vissing-Jorgensen \(2012\)](#), do not capture the full service flow from trading safe assets.

However, not any entity that issues a safe asset with service flows can run a Ponzi scheme. When precautionary savings due to idiosyncratic risk depresses the required real cash flow return r below the economic growth rate g , sustainable Ponzi schemes become feasible. One can pay off the maturing bonds with newly issued debt and issue more to fund additional expenditures. Viewed differently, in this case one can issue a “bubbly” safe asset.² In our model with uninsurable idiosyncratic risk, bubbles are possible even though individual citizens’ transversality conditions hold. Strictly speaking, which entity can run a Ponzi scheme depends on which equilibrium is selected. In other words, the selected “bubble equilibrium” determines who is subject to a no-Ponzi constraint. [Brunnermeier et al. \(2021a\)](#) argue that the government’s ability to tax and impose regulations on the private sector puts it in a unique position to defend a bubble on its debt. According to this view, the government enjoys an exorbitant privilege as a safe asset issuer that sets it apart from private entities. While the latter may also be able to issue a safe asset with service flows, they can – unlike governments – not run a Ponzi scheme.

Note that safe assets do not need to be bubbly. Safe assets can also arise in the absence of bubbles. However, a bubble component can make an asset “safer”, since the value of the service flow is proportional to the market value of the (bubbly) asset – and the service flow is highly priced, not least because it carries a negative β . Indeed, under certain circumstances, the same asset without a bubble can have a positive β (driven by discounted cash flows), while when the bubble is associated with the asset, its β becomes negative. In that case, the asset is only a safe asset if the bubble is attached to the asset. In other words, bubble and safe asset characteristics are complementary. This leads us to a second concept, the “safe asset tautology” that holds for bubbly safe assets: A safe asset is safe because it is perceived to be safe. In this case, the safe asset status can be easily lost, when the bubble pops. [Brunnermeier et al. \(2021a\)](#) describes policy measures and necessary fiscal space to defend the bubble and with it the safe asset status.

²A standard asset pricing equation carries then a bubble term.

Finally, the government can generate seigniorage revenues by “mining the bubble” (expanding the Ponzi scheme to fund deficits). One contribution of this paper is to document quantitatively that “bubble mining” only raises sizable government revenue if safe public debt has a negative β . A government can generate revenue by issuing bonds at a faster pace, create higher inflation, and thereby reduce the real return on holding the government bond. “Printing” bonds at a faster rate acts like a tax on bond holdings or, better said, on partial self-insurance through holding and retrading the safe asset. It is a form of “financial repression.” Increasing the tax rate increases the “tax revenue”, but erodes the “tax base”, the value of the bonds. A “*Debt Laffer Curve*” emerges. When the tax exceeds a certain level, overall tax revenue from bubble mining declines. Our calibration quantifies the Laffer Curve and establishes that the negative β is crucial to generate quantitatively significant revenue from “bubble mining”.

Literature. This paper touches upon many strands of classic and recent economic literature. [Dang et al. \(2015\)](#) emphasize the information insensitivity of safe assets. In [Gorton and Pennachi \(1990\)](#), [Dang et al. \(2017\)](#), and [Greenwood et al. \(2016\)](#) intermediaries create information insensitive assets. [He et al. \(2019\)](#) model a safe asset tautology within a generalized global games setting. [Caballero et al. \(2017\)](#) stress the importance of safe asset shortage. [Brunnermeier et al. \(2017, 2016\)](#) propose the creation of a safe asset via securitization and argue that the main problem is the asymmetric supply of safe assets leading to eruptive cross-border capital flows. [Brunnermeier et al. \(2021b\)](#) discuss the loss of safe asset status in the context of an international framework for emerging market economies. Emerging market government bonds’ safe asset status competes with advanced economies safe assets and hence are deeply affected by spillovers from US monetary policy.

This paper resolves the “Public Debt Valuation Puzzle” proposed in [Jiang et al. \(2019\)](#), which argues that government debt appears overvalued not least because primary surpluses, the total payments to all bond holders, are procyclical and should thus be discounted at a higher rate. In our setting, the price of debt is countercyclical since the value of service flows rises in bad times, resulting in a negative β asset. Second, it also resolves the “Government Debt Risk Premium Puzzle” ([Jiang et al., 2020](#)), the puzzle that government debt appears to insure simultaneously bond holders and taxpayers whereas in standard models, it can insure only one of the two groups. Our analysis shows that sufficiently large service flows can make the bond a negative β -asset, a good hedge for bond holders, while primary surpluses are procyclical at the

same time, thus providing insurance for taxpayers.

There is a large literature on the risk-sharing benefits of money or government debt in environments with uninsured idiosyncratic risk. The classic reference is [Bewley \(1980\)](#). [Aiyagari and McGrattan \(1998\)](#) calibrate the optimal debt level in an [Aiyagari \(1994\)](#)-type model without aggregate risk. In [Kiyotaki and Moore \(2008\)](#) citizens use money as liquidity to exploit investment opportunity shocks. Our model environment is closest to [Angeletos \(2007\)](#), who studies idiosyncratic investment risks, and [Brunnermeier and Sannikov \(2016a,b\)](#) who add a ‘bubbly’ safe asset in the form of government debt or money into that framework and allow for aggregate risk. [Di Tella \(2020\)](#) and [Merkel \(2020\)](#) study similar environments in which the safe asset is money and has a convenience yield instead of being bubbly. In this paper, we emphasize that government debt generates a service flow from retrading even in the absence of bubbles and convenience yields and provide a formal characterization its value. Government debt retains its safe asset role even if we add publicly traded equity that is also fully liquid and free of idiosyncratic risk.

Our discussion of public debt bubbles is related to a literature on rational bubbles and debt sustainability when the interest rate r is lower than the economic growth rate g . Papers generating $r < g$ with Overlapping Generations (OLG) include [Samuelson \(1958\)](#), [Diamond \(1965\)](#) with capital, [Tirole \(1985\)](#) with a bubble and, most recently, [Blanchard \(2019\)](#). In the above cited [Bewley \(1980\)](#)-type models, as well as ours, interest rates are depressed due to a precautionary savings motive. [Bassetto and Cui \(2018\)](#) and [Brunnermeier et al. \(2021a\)](#) study the fiscal theory of the price level (FTPL) when $r < g$. [Brunnermeier et al. \(2021a\)](#) show how the government can ensure that the bubble is associated with government debt and does not jump to other assets, e.g. crypto assets. They also point out that a bubble gives rise to a “Debt Laffer Curve”, but consider an environment without aggregate risk. In this paper, we quantify the Laffer curve and show that aggregate risk and the negative- β are quantitatively important. [Reis \(2021\)](#) studies fiscal debt capacity in a related framework with a bubble on government debt. To avoid an opposite infinity problem in the debt valuation equation when $r < g$, [Reis \(2021\)](#) discounts at the higher marginal product of capital $m > g$. In this paper, we propose instead r^{**} -discounting, which also avoids the opposite infinity problem but has a very simple economic interpretation. There is an extensive literature on rational bubbles outside of the government debt context. Survey papers include [Miao \(2014\)](#) and [Martin and Ventura \(2018\)](#).

Like us, [Constantinides and Duffie \(1996\)](#) show how variation in idiosyncratic risk exposures in an incomplete markets setting can resolve several asset pricing puzzles. Unlike our paper, they abstract from asset trading and focus exclusively on the no-bubble equilibrium, while in our theory re-trading of the safe asset is key and can generate bubbles. [Heaton and Lucas \(1996, 2000a,b\)](#) are prominent papers quantifying incomplete market models that include trading costs and the importance of entrepreneurial risk. In our analysis traded equity exhibits excess volatility and predictability as in recessions idiosyncratic risk rises the risk compensation for inside equity rises leaving less for outside (publicly traded) equity holders.

2 Model

2.1 Model Setup

The model is set in continuous time with an infinite horizon.

There is a continuum of households indexed by $i \in [0, 1]$. All households have identical logarithmic preferences

$$V_0^i := \mathbb{E} \left[\int_0^\infty e^{-\rho t} \log c_t^i dt \right]$$

with discount rate $\rho > 0$.³

Each agent operates one firm that produces an output flow $a_t k_t^i dt$, where k_t^i is the capital input chosen by the firm and a_t is an exogenous productivity process that is common for all agents. Capital of firm i evolves according to

$$\frac{dk_t^i}{k_t^i} = \left(\Phi \left(i_t^i \right) - \delta \right) dt + \tilde{\sigma}_t d\tilde{Z}_t^i + d\Delta_t^{k,i},$$

where Φ is an increasing concave function that captures adjustment costs in capital accumulation, i_t^i the investment rate (in output goods) per unit of capital, δ is the depreciation rate, $\tilde{\sigma}_t d\tilde{Z}_t^i$ represents idiosyncratic Brownian shocks, and $d\Delta_t^{k,i}$ represents firm i 's market transactions in physical capital. Brownian motions \tilde{Z}^i are agent-specific and

³In Appendix A.4, we present a generalization with [Duffie and Epstein \(1992\)](#) preferences (continuous time [Epstein and Zin \(1989\)](#) preferences).

i.i.d. across agents. The levels of idiosyncratic risk $\tilde{\sigma}_t$ and productivity a_t are exogenous processes.

To obtain simple closed-form expressions, we choose the functional form $\Phi(\iota) = \frac{1}{\phi} \log(1 + \phi\iota)$ with adjustment cost parameter $\phi \geq 0$ for the investment technology.

Each agent i can reduce idiosyncratic risk exposure by selling equity to other agents. Outside equity claims on i 's capital k^i have the same aggregate and idiosyncratic risk as capital itself, but may pay a lower expected return, reflecting an insider premium that i earns for managing the capital stock. Agents can hold a diversified equity portfolio and thereby eliminate idiosyncratic risk.

The key friction in the model is that agents are unable to share idiosyncratic risk perfectly. Specifically, we assume that agents face a skin-in-the-game constraint and must retain at least a fraction $\bar{\chi} \in (0, 1]$ of their capital in undiversified form. As a consequence, agents have to bear the residual idiosyncratic risk of at least $\bar{\chi}\tilde{\sigma}_t d\tilde{Z}_t^i$ per unit of capital in their firms.

Besides this limit on idiosyncratic risk sharing, there are no further financial frictions. Agents are allowed to trade physical capital and contingent claims on aggregate risk subject to standard no Ponzi conditions.

In addition to households, there is a government that issues nominal government bonds, funds government spending, and imposes taxes on firms. Outstanding nominal government debt has a face value of \mathcal{B}_t and pay nominal interest i_t . The face value follows a continuous process $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$ with growth rate $\check{\mu}^{\mathcal{B}}$. There is an exogenous need for real spending $\mathfrak{g}K_t dt$, where $K_t := \int k_t^i di$ is the aggregate capital stock and \mathfrak{g} is a model parameter. The government can finance this spending by setting a proportional tax τ_t (subsidy if negative) on firms' output and by adjusting the bond issuance rate $\mu_t^{\mathcal{B}}$ (repurchasing bonds if negative). We assume that $\mu_t^{\mathcal{B}}$ and i_t are exogenous processes while taxes τ_t adjust to satisfy the nominal government budget constraint

$$i_t \mathcal{B}_t + \mathcal{P}_t \mathfrak{g} K_t = \mu_t^{\mathcal{B}} \mathcal{B}_t + \mathcal{P}_t \tau_t a_t K_t, \quad (1)$$

where \mathcal{P}_t denotes the price level.

We assume that the exogenous processes a_t , $\tilde{\sigma}_t$, $\mu_t^{\mathcal{B}}$, and i_t follow a joint Markov diffusion process that is driven by some Brownian motion Z_t , which captures aggregate risk and is independent of all idiosyncratic Brownian motions \tilde{Z}_t^i .

Finally, the aggregate resource constraint is

$$C_t + \mathfrak{g}K_t + \iota_t K_t = a_t K_t, \quad (2)$$

where $C_t := \int c_t^i di$ is aggregate consumption and $\iota_t = \int \iota_t^i k_t^i / K_t di$ is the average investment rate.

2.2 Model Solution

Price Processes and Returns. Let q_t^K be the market price of a single unit of physical capital. Then, $q_t^K K_t$ is private capital wealth. Let further $q_t^B := \frac{\mathcal{B}_t / \mathcal{P}_t}{K_t}$ be the ratio of the real value of government debt to total capital in the economy.⁴ Then, the real value of the total stock of government bonds is $q_t^B K_t$ and the real value of a single government bond is $\frac{q_t^B K_t}{\mathcal{B}_t}$. It is convenient to define the share of total wealth in the economy that is due to bond wealth,

$$\vartheta_t := \frac{q_t^B K_t}{(q_t^B + q_t^K) K_t}.$$

We postulate that q_t^B and q_t^K have a generic Ito evolution

$$dq_t^B = \mu_t^{q,B} q_t^B dt + \sigma_t^{q,B} q_t^B dZ_t, \quad dq_t^K = \mu_t^{q,K} q_t^K dt + \sigma_t^{q,K} q_t^K dZ_t.$$

Whenever $q_t^B, q_t^K \neq 0$, the unknown (geometric) drifts $\mu_t^{q,B}, \mu_t^{q,K}$ and volatilities $\sigma_t^{q,B}, \sigma_t^{q,K}$ are uniquely determined by the local behavior of q_t^B and q_t^K , respectively. In the following, we also use the notation μ_t^ϑ and σ_t^ϑ for the (geometric) drift and volatility of ϑ_t .⁵

Households can trade two assets in positive net supply (if $q_t^B \neq 0$), bonds and capital. Assume that in equilibrium $\iota_t = \iota_t^i$ for all i (to be verified below) such that aggregate capital grows locally deterministically at rate $\Phi(\iota_t) - \delta$. Then, the return on bonds is

$$dr_t^B = i_t dt + \frac{d(q_t^B K_t / \mathcal{B}_t)}{q_t^B K_t / \mathcal{B}_t} = \frac{d(q_t^B K_t)}{q_t^B K_t} - \overbrace{\left(\mu_t^B - i_t \right)}{=: \tilde{\mu}_t^B} dt$$

⁴It is convenient to work with this normalized version of the inverse price level $1/\mathcal{P}_t$, because the latter depends on the scale of the economy and the nominal quantity of outstanding bonds in equilibrium, whereas q_t^B does not.

⁵This means, $d\vartheta_t = \mu_t^\vartheta \vartheta_t dt + \sigma_t^\vartheta \vartheta_t dZ_t$.

$$= \left(\Phi(\iota_t) - \delta + \mu_t^{q,B} - \check{\mu}_t^B \right) dt + \sigma_t^{q,B} dZ_t. \quad (3)$$

The return on agent i 's capital is

$$\begin{aligned} dr_t^{K,i} \left(\iota_t^i \right) &= \frac{(1 - \tau_t) a_t - \iota_t^i}{q_t^K} + \frac{d(q_t^K k_t^i)}{q_t^K k_t^i} \\ &= \left(\frac{(1 - \tau_t) a_t - \iota_t^i}{q_t^K} + \Phi \left(\iota_t^i \right) - \delta + \mu_t^{q,K} \right) dt + \sigma_t^{q,K} dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^i. \end{aligned}$$

Using the government budget constraint (1) to substitute out $\tau_t a$ yields

$$dr_t^{K,i} \left(\iota_t^i \right) = \left(\frac{a_t - \mathfrak{g} - \iota_t^i}{q_t^K} + \frac{q_t^B}{q_t^K} \check{\mu}_t^B + \Phi \left(\iota_t^i \right) - \delta + \mu_t^{q,K} \right) dt + \sigma_t^{q,K} dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^i.$$

In addition to bonds and capital, there are outside equity claims in zero net supply. Outside equity claims issued by household i have the same risk characteristics as the capital return $dr_t^{K,i}$ but may have a different expected return. The return on outside equity issued by agent i is therefore

$$dr_t^{E,i} = \mathbb{E}_t[dr_t^{E,i}] + \sigma_t^{q,K} dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^i,$$

where the expected return component $\mathbb{E}_t[dr_t^{E,i}]$ is determined in equilibrium. In equilibrium, all agents optimally hold a perfectly diversified equity portfolio. The return on that portfolio is

$$d\bar{r}_t^E = \int dr_t^{E,i} di = \mathbb{E}_t[d\bar{r}_t^E] + \sigma_t^{q,K} dZ_t.$$

Because all individual varieties of outside equity $dr_t^{E,i}$ generate the same aggregate risk contribution to the overall equity portfolio, it will be the case in equilibrium that $\mathbb{E}_t[dr_t^{E,i}] = \mathbb{E}_t[d\bar{r}_t^E]$ for all i .

Household Problem and Equilibrium. We formulate the household problem as a standard consumption-portfolio-choice problem that does not make explicit reference to the capital trading process $d\Delta_t^{k,i}$ as a choice variable. For this purpose, denote by n_t^i the net worth of household i and let $\theta_t^{k,i}$, $\theta_t^{E,i}$, $\theta_t^{\bar{E},i}$ be the fraction of net worth invested into capital, own outside equity, and the diversified portfolio of equity, respectively.⁶

⁶The own outside equity share $\theta_t^{E,i}$ is negative as this asset is issued by the household.

Net worth evolves according to

$$\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i} dt + \underbrace{dr_t^{\mathcal{B}} + \theta_t^{K,i} \left(dr_t^{K,i} (l_t^i) - dr_t^{\mathcal{B}} \right) + \theta_t^{E,i} \left(dr_t^{E,i} - dr_t^{\mathcal{B}} \right) + \theta_t^{\bar{E},i} \left(d\bar{r}_t^E - dr_t^{\mathcal{B}} \right)}_{=:dr_t^{n,i}}. \quad (4)$$

The household chooses consumption c_t^i , real investment l_t^i , and the portfolio shares $\theta_t^{K,i}$, $\theta_t^{E,i}$, and $\theta_t^{\bar{E},i}$ in capital, own outside equity and the diversified equity portfolio, respectively, to maximize utility V_0^i subject to (4), the return expressions stated previously, the skin-in-the-game constraint

$$-\theta_t^{E,i} \leq (1 - \bar{\chi})\theta_t^{K,i}, \quad (5)$$

and a constraint that rules out Ponzi schemes.⁷

We solve this problem in Appendix A.1 using the stochastic maximum principle. Here, we merely report the main conclusions.

The optimal consumption and investment choice are determined by the two conditions

$$\begin{aligned} c_t^i &= \rho n_t^i, \\ q_t^K &= \frac{1}{\Phi'(l_t^i)} = 1 + \phi l_t^i. \end{aligned}$$

The first line is the familiar permanent income consumption equation for log preferences. The second line is a Tobin's q condition for physical investment in the presence of capital adjustment costs. The second equality in that line follows from the functional form assumption $\Phi(l) = \frac{1}{\phi} \log(1 + \phi l)$. Because all households face the same capital price q_t^K , they all choose the same investment rate l_t^i , so that we drop the i superscript from now on.

Aggregating the first condition across all agents i and combining the two equations with goods market clearing (2) and the definition of ϑ_t implies the following Lemma

Lemma 1. *The investment rate, value of government bond per unit of K_t , and price of a physical*

⁷Formally, the no Ponzi constraint is $\liminf_{T \rightarrow \infty} \mathbb{E}[\bar{\zeta}_T^i n_T^i] \geq 0$, where $\bar{\zeta}_T^i$ is the stochastic discount factor defined below in Section 2.4.

capital unit is

$$\iota_t = \frac{(1 - \vartheta_t)(a_t - \mathfrak{g}) - \rho}{1 - \vartheta_t + \phi\rho}, \quad (6)$$

$$q_t^B = \vartheta_t \frac{1 + \phi(a_t - \mathfrak{g})}{1 - \vartheta_t + \phi\rho}, \quad (7)$$

$$q_t^K = (1 - \vartheta_t) \frac{1 + \phi(a_t - \mathfrak{g})}{1 - \vartheta_t + \phi\rho}. \quad (8)$$

These equations determine the equilibrium uniquely as a function of the exogenous process a_t and the (endogenous) bond wealth share ϑ_t . To fully characterize the equilibrium, we thus only need to determine ϑ_t .

ϑ_t can be thought of as a relative price between capital assets (including equity which is a claim to capital) and government bonds and is determined by households' portfolio choice and asset market clearing. Households' portfolio choice conditions for the three choices $\theta_t^{k,i}$, $\theta_t^{E,i}$, $\theta_t^{\bar{E},i}$ take the form of standard Merton-type portfolio conditions that ensure that for the optimal portfolio, each asset's expected excess return over $\frac{\mathbb{E}_t[dr_t^B]}{dt}$ is equalized to the risk premium that the agent requires to be willing to hold the aggregate and idiosyncratic risk associated with the asset.⁸ We present the formal equations in Appendix A.1.

There, we also show that by combining these portfolio choice conditions and using asset market clearing to eliminate the individual portfolio weights $\theta_t^{k,i}$, $\theta_t^{E,i}$, $\theta_t^{\bar{E},i}$, we can reduce them to a single equation for the expected change in ϑ_t :

$$\mathbb{E}_t [d\vartheta_t] = \left(\rho + \check{\mu}_t^B - (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2 \right) \vartheta_t dt \quad (9)$$

This is a backward stochastic differential equation (BSDE) for ϑ_t . It characterizes all possible stochastic processes for the bond wealth share ϑ_t (\approx relative price between bonds and capital) that are consistent with household portfolio choice and market clearing. Together with a specification for the evolution of the exogenous states $\tilde{\sigma}_t$ and a_t and for policy $\check{\mu}_t^B$, equation (9) determines the equilibrium process for ϑ_t . Equations (6), (7), (8) and goods market clearing (2) can then be used to back out the remaining quantities

⁸In the case of capital and outside equity, there is also a Lagrange multiplier term related to the skin-in-the-game constraint (5). In the problem considered here, the constraint is always binding, so that households issue the maximum amount of outside equity consistent with it.

of interest.

Remark: Equilibrium Selection and Bubbles. Depending on the precise assumptions on parameters and government policy, our model environment may allow for the emergence of rational bubbles. A competitive equilibrium is therefore not necessarily unique. Our model solution procedure incorporates some implicit assumptions that affect equilibrium selection if the equilibrium is non-unique. We highlight these assumptions here and maintain them throughout with the exception of Section 8, where we briefly discuss alternative equilibria.⁹

First, by assuming that private households face standard no Ponzi conditions and that government debt is the only asset in positive net supply that is free of idiosyncratic risk, we have effectively ruled out bubbles on any asset other than government debt.

Second, even under these assumptions, there may be multiple equilibria corresponding to multiple solutions to equation (9). However, one can show that there is a unique stationary equilibrium in which nominal bonds are valued (“stationary monetary equilibrium”). By considering a Markov equilibrium in which ϑ can be written as a function of the exogenous state variables, as we do from Section 4 onward, we effectively select this unique stationary monetary equilibrium. Note that under conditions that ensure equilibrium uniqueness, this is the only competitive equilibrium. Otherwise, this equilibrium implies the existence of a bubble on government debt.

2.3 Closed-Form Steady State

We provide a brief characterization of the model’s steady state.¹⁰ To do so, we assume that productivity a , idiosyncratic risk $\tilde{\sigma}$, and policy $\check{\mu}^B$ are constant. We restrict attention equilibria with constant q^B and q^K and a positive value of government bonds, $q^B > 0$ (in line with our equilibrium selection choice). These assumptions immediately imply that also ϑ and taxes τ must be constant in a steady state.

Any such equilibrium must solve equation (9) with $d\vartheta_t = 0$. The right-hand side is a third-order polynomial, so there are three solutions to this equation, $\vartheta = 0$, $\vartheta =$

⁹Our selected equilibrium can be made unique with the fiscal policy arrangements discussed in Brunnermeier et al. (2021a).

¹⁰The “steady state” is in fact a balanced growth path. In our AK -type model, there is always a growth trend in the capital stock K_t .

$\frac{\bar{\chi}\tilde{\sigma} + \sqrt{\rho + \check{\mu}^B}}{\bar{\chi}\tilde{\sigma}}$, and $\vartheta = \frac{\bar{\chi}\tilde{\sigma} - \sqrt{\rho + \check{\mu}^B}}{\bar{\chi}\tilde{\sigma}}$. Among these solutions, only the third can be consistent with $q^B, q^K > 0$ and thus a valid steady state equilibrium in which bonds have a positive value.¹¹ It is consistent with such an equilibrium if in addition the condition

$$\bar{\chi}\tilde{\sigma} \geq \sqrt{\rho + \check{\mu}^B}$$

is satisfied. Effectively, this inequality imposes a constraint on bond growth in excess of interest payments $\check{\mu}^B$, a measure of dilution of existing bond holds. Dilution $\check{\mu}^B$ cannot be too large for the private sector to remain willing to hold government bonds. The higher is the residual idiosyncratic risk $\bar{\chi}\tilde{\sigma}$ that agents have to bear after optimal outside equity issuance, the less restrictive is this constraint.

If this condition is satisfied, investment is

$$i = \frac{\sqrt{\rho + \check{\mu}^B} (a - g) - \rho \bar{\chi}\tilde{\sigma}}{\sqrt{\rho + \check{\mu}^B} + \phi \rho \bar{\chi}\tilde{\sigma}}$$

and the (scaled) real asset values are

$$q^B = \frac{(\bar{\chi}\tilde{\sigma} - \sqrt{\rho + \check{\mu}^B}) (1 + \phi (a - g))}{\sqrt{\rho + \check{\mu}^B} + \phi \rho \bar{\chi}\tilde{\sigma}}, \quad q^K = \frac{\sqrt{\rho + \check{\mu}^B} (1 + \phi (a - g))}{\sqrt{\rho + \check{\mu}^B} + \phi \rho \bar{\chi}\tilde{\sigma}}.$$

These closed-form solutions yield straightforward conclusions regarding the impact of parameter changes on equilibrium outcomes. We emphasize here explicitly that capital valuations and investment are strictly decreasing while bond valuation are strictly increasing in idiosyncratic risk $\tilde{\sigma}$. This is because an increase in idiosyncratic risk leads to a portfolio reallocation from capital assets to government bonds as can be readily seen from equation (9). This same force also plays an important role in the flight-to-safety dynamics that we emphasize in Sections 4 and 5.

2.4 Safe Asset Definition

Individuals hold a safe asset for precautionary reasons, which they can “liquify” at an above average return when they face an idiosyncratic and/or aggregate shock

¹¹The second solution never corresponds to a valid equilibrium, while the first is only consistent with equilibrium if government primary surpluses are zero, see Brunnermeier et al. (2021a) for details.

and their marginal of an extra dollar is high. The process of the individual stochastic discount factor (SDF) process of citizen i is denoted by ζ_t^i . This process satisfies $\zeta_0^i = 1$ and $d\zeta_t^i/\zeta_t^i = -r_t^f dt - \zeta_t dZ_t - \tilde{\zeta}_t^i d\tilde{Z}_t^i$, with a negative drift term equal to the risk-free rate and aggregate and idiosyncratic prices of risk, ζ_t , $\tilde{\zeta}_t^i$ respectively.¹² The return of citizen i 's net worth, $r^{n,i}$ is given by equation (4).

The following definition makes the used "good friend analogy" of a safe asset precise.

Definition 1. *An asset j is in equilibrium a safe asset for individual i at time t if the conditional covariance between her SDF and return of the asset in excess to its net worth return, $dr_t^j - dr_t^{n,i}$, is negative, i.e. $\text{Cov}_t[d\zeta_t^i/\zeta_t^i, dr_t^j - dr_t^{n,i}] < 0$.*

Note that the safe asset concept is an equilibrium concept. The same asset with the same cash flow payoffs can be a safe asset in one equilibrium and not a safe asset in another equilibrium. The returns r_t^j and $r_t^{n,i}$ as well as the SDF ζ_t^i depend on the equilibrium the economy is in.

Second, an asset is safe for individual i relative to its net worth n^i . The return which could result from cash flows or capital gains shocks reduce individual i 's consumption relative to her net worth.¹³

Third, an asset is safe at a specific point in time t and lose its "safe-asset status".

Fourth, in our model we implicitly assumed that the asset enjoys perfect market liquidity (all the time). That is, an can be traded easily without large trading frictions, e.g. due to asymmetric information. Given that in our setting safety comes from retrading, this is an important feature of a safe asset.

Proposition 1 applies the safe asset definition to our model with constant idiosyncratic risk $\tilde{\sigma}_t = \tilde{\sigma} > 0$, productivity $a_t = a$, and policy $\check{\mu}_t^B = \check{\mu}^B$.

Proposition 1. *If $\tilde{\sigma}_t > 0$, a_t , and $\check{\mu}_t^B$ are time-invariant, the government bond is a safe asset for all agents at all times.*

¹²In integral form the individual SDF is

$$\zeta_t^i = \underbrace{\exp\left(-\int_0^t r_\tau^f d\tau\right)}_{\text{time discounting}} \cdot \underbrace{\exp\left(-\int_0^t \zeta_\tau dZ_\tau - \frac{1}{2}\int_0^t \zeta_\tau^2 d\tau\right)}_{\text{aggregate risk}} \cdot \underbrace{\exp\left(-\int_0^t \tilde{\zeta}_\tau d\tilde{Z}_\tau^i - \frac{1}{2}\int_0^t \tilde{\zeta}_\tau^2 d\tau\right)}_{\text{idiosyncratic risk}},$$

where the second and third factors are martingales.

¹³Alternatively, one could define an asset as (absolutely) safe by $\text{Cov}_t[d\zeta_t^i/\zeta_t^i, dr_t^j] \leq 0$.

In equilibrium each citizen's net worth return is positively correlated with her consumption growth rate and hence negatively correlated with her SDF, as the incomplete markets frictions prevents them from hedging their idiosyncratic risk. Since the government bond return is risk-free, $\text{Cov}_t[d\bar{\zeta}_t^i/\bar{\zeta}_t^i, r^f - dr_t^{n,i}] < 0$ and consequently the government bond is a safe asset. Capital is not a safe asset as $\text{Cov}_t[d\bar{\zeta}_t^i/\bar{\zeta}_t^i, dr_t^K] \geq \text{Cov}_t[d\bar{\zeta}_t^i/\bar{\zeta}_t^i, dr_t^{n,i}]$. In the equilibrium in which there is no government, the economy has no safe asset.

In our setting with stochastic idiosyncratic risk $\tilde{\sigma}_t > 0$, the real return of the government bond is not risk-free. However, as we will see in Section 4, it depreciate in volatile times less than citizen's net worth, indeed it even even appreciates, and hence more than satisfies the safe asset criterion.

3 Safe Asset Debt Valuation Equation: Two Perspectives

The value of government debt has to satisfy a debt valuation equation that relates the real value of debt to the present value of future primary surpluses. There are two ways to derive such an equation: (1) by iterating the government's flow budget constraint forward in time and pricing the total stock of government bonds with any marginal agent's stochastic discount factor or (2) by valuing each individual households' bond portfolio and then aggregating over all households. Both procedures imply the same valuation formula with complete markets, but, with incomplete markets, lead to two distinct equations that differ in the effective discount rate applied to government surpluses. These equations provide two different perspectives for pricing government debt.

The first procedure leads to a "buy and hold perspective" of government debt pricing. The value of government debt must equal the marginal valuation of an individual agent that buys and holds a (small) constant fraction of the total stock of outstanding bonds.¹⁴ The cash flow stream associated with this strategy is proportional to the stream of primary surpluses. Hence, in a setting without aggregate risk, the bond is risk-free and future payoffs are discounted at the risk-free rate. In a setting with aggregate risk, only the aggregate component of the stochastic discount factor enters the

¹⁴This may require the agent to trade despite the label "buy and hold", but only directly with the issuer, the government, in order to absorb new debt issuance, not with other agents.

debt valuation equation.

The second procedure leads to a “dynamic trading perspective” of government debt pricing. It recognizes that individual citizens do not intend to buy and hold the government bond, but plan to retrade it whenever they face a shock. After a negative shock, they raise cash flow by selling the bond, while after a positive shock they buy additional bonds. The cash flow stream associated with this optimal trading strategy is (idiosyncratically) stochastic. When valuing this cash flow stream, the resulting equation contains a “service flow” term from retrading that is absent in the buy and hold perspective.

Dynamic programming implies that a transversality condition has to hold only from the dynamic trading perspective, for each individual agent. Optimality does not imply a transversality condition from the buy and hold perspective (where discounting happens at a lower effective rate). For that reason, a gap may appear between the value of debt and the present value of surpluses from the buy and hold perspective. This gap is closed by an additional bubble term.

Unfortunately, it can even happen that both the bubble term and the present value of primary surpluses are infinite with opposite sign, yet their sum still converges as the time horizon approaches infinity. In contrast, the terms in the dynamic trading perspective are always well-defined and finite.

Buy and Hold Perspective. We denote by $s_t := \tau_t a_t - g$ the primary surplus per unit of aggregate capital. Recall that ζ_t^i is agent i 's SDF process. From the buy and hold perspective, individual uninsurable risk does not enter the valuation equation directly, so that only the aggregate component $\bar{\zeta}_t$ of the processes ζ_t^i matters, i.e. $d\bar{\zeta}_t/\bar{\zeta}_t = -r_t^f dt - \zeta_t dZ_t$.¹⁵ Absent aggregate shocks (including inflation shocks), the government bond is a risk-free asset and the relevant discount factor is simply $\bar{\zeta}_t = \exp(-\int_0^t r_\tau^f d\tau)$.

Proposition 2 (Buy and Hold Perspective). *The value of government debt at $t = 0$ satisfies*

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T \rightarrow \infty} \left(\mathbb{E} \left[\int_0^T \bar{\zeta}_t s_t K_t dt \right] + \mathbb{E} \left[\bar{\zeta}_T \frac{\mathcal{B}_T}{\mathcal{P}_T} \right] \right). \quad (10)$$

¹⁵The aggregate discount factor is the projection of any individual citizen's SDF onto a common filtration generated by the aggregate Brownian $\{Z_t\}_{t=0}^\infty$. Put differently, $\bar{\zeta}_t := \mathbb{E} [\zeta_t^i | Z_\tau : \tau \leq t]$, takes conditional expectations with respect to the history of aggregate shocks dZ_τ up to time t but without any knowledge of idiosyncratic shocks. Equivalently, $\bar{\zeta}_t = \int \zeta_t^i di$ is the unweighted average of individual SDFs.

This equation consists of two terms: a discounted stream of primary surpluses plus (the limit of) a discounted terminal value. The latter can be positive even in the limit, giving rise to a possible bubble on government debt.¹⁶ The reason is that in our model no private citizen's transversality condition necessary implies $\mathbb{E} \left[\bar{\zeta}_T \frac{\mathcal{B}_T}{\mathcal{P}_T} \right] \rightarrow 0$ because agents do not buy and hold a fixed fraction of the government debt stock but constantly trade bonds. If the discount factor is small enough so that the terminal condition does converge to zero, we obtain the traditional debt valuation equation that says that the value of debt must equal the present value of primary surpluses.

To derive equation (10), we start by using $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$ to rewrite the government flow budget constraint (1) as

$$-(d\mathcal{B}_t - i_t \mathcal{B}_t dt) = \mathcal{P}_t \underbrace{(\tau a_t - \mathfrak{g})}_{=s_t} K_t dt.$$

We now multiply both sides by the nominal SDF $\zeta_t^i / \mathcal{P}_t$ of agent i and use Ito's product rule to replace $\zeta_t^i / \mathcal{P}_t d\mathcal{B}_t$ with $d \left(\zeta_t^i / \mathcal{P}_t \mathcal{B}_t \right) - \mathcal{B}_t d(\zeta_t^i / \mathcal{P}_t)$:¹⁷

$$-d \left(\zeta_t^i \mathcal{B}_t / \mathcal{P}_t \right) + \mathcal{B}_t \left(d \left(\zeta_t^i / \mathcal{P}_t \right) + i_t \zeta_t^i / \mathcal{P}_t dt \right) = \zeta_t^i s_t K_t dt.$$

Integrating this equation from $t = 0$ to $t = T$, taking expectations, and solving for $\zeta_0^i \mathcal{B}_0 / \mathcal{P}_0$ yields

$$\zeta_0^i \frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E} \left[\int_0^T \zeta_t^i s_t K_t dt \right] - \mathbb{E} \left[\int_0^T \mathcal{B}_t \left(d \left(\zeta_t^i / \mathcal{P}_t \right) + i_t \zeta_t^i / \mathcal{P}_t dt \right) \right] + \mathbb{E} \left[\zeta_T^i \frac{\mathcal{B}_T}{\mathcal{P}_T} \right]. \quad (11)$$

Equation (11) is simply an accounting identity, an integrated version of the government flow budget constraint (1). We add economic content by noting that the individual SDF ζ_t^i must price the bond because agent i is marginal in the bond market. This implies that the associated nominal SDF $\zeta_t^i / \mathcal{P}_t$ must decay on average at the nominal market interest rate, so that the second term in equation (11) vanishes. In addition, we can replace the individual SDF ζ_t^i with the average SDF $\bar{\zeta}_t$ because equation (11) holds for all individuals i and $s_t K_t$ and $\mathcal{B}_T / \mathcal{P}_T$ are free of idiosyncratic risk. When taking the limit $T \rightarrow \infty$, we obtain equation (10)

¹⁶The bubble term on government debt is discussed in detail in Brunnermeier et al. (2021a).

¹⁷There is no quadratic covariation term because $d\mathcal{B}_t$ is absolutely continuous.

Dynamic Trading Perspective. Let $\eta_t^i := n_t^i/N_t$ be citizen i 's net worth share and denote again i 's SDF process by ζ_t^i . Pricing individual bond portfolios and aggregating over agents i yields our main valuation equation from the dynamic trading perspective,

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \int \left(\mathbb{E} \left[\int_0^\infty \zeta_t^i \cdot \eta_t^i s_t K_t dt \right] + \mathbb{E} \left[\int_0^\infty \zeta_t^i \cdot \eta_t^i (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \right] \right) di. \quad (12)$$

The real value of all outstanding public debt $\mathcal{B}_0/\mathcal{P}_0$ is the integral of the valuations of individual debt holdings. Each of these valuations consists of two terms, the discounted value of the share of future primary surpluses, $\eta_t^i s_t K_t := \eta_t^i (\tau_t a - g) K_t$, paid out to agent i plus the discounted value of future service flows, $\eta_t^i (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t}$, that agent i derives from trading bonds. The safe asset service flow is due to partial insurance, which increases in the value of public debt, and the amount of idiosyncratic risk the citizen is exposed to, which in turn depends on his portfolio share on physical capital $(1 - \vartheta_t)$ and undiversified risk $\bar{\chi} \bar{\sigma}_t$. Government bonds provide a positive service flow because the agent sells bonds precisely when she experiences a negative idiosyncratic shock, so that the bond portfolio generates a positive payout in times of high marginal utility ζ_t^i .

Equation (12) emphasizes that the total value is obtained by aggregating individual portfolio valuations. Mathematically, it is more convenient to interchange the order of integration. This yields the following proposition.

Proposition 3 (Dynamic Trading Perspective). *The value of government debt at $t = 0$ satisfies*

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E} \left[\int_0^\infty \underbrace{\left(\int \zeta_t^i \eta_t^i di \right)}_{=: \zeta_t^{**}} s_t K_t dt \right] + \mathbb{E} \left[\int_0^\infty \underbrace{\left(\int \zeta_t^i \eta_t^i di \right)}_{=: \zeta_t^{**}} (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \right]. \quad (13)$$

This equation discounts aggregate cash flows (surpluses and service flows) free of idiosyncratic risk like equation (10) obtained from the buy and hold perspective. But importantly, the “stochastic discount factor” ζ_t^{**} in this equation is a net-worth-weighted average of individual stochastic discount factors. Since a single citizen's individual net worth weight η_t^i co-moves negatively with her SDF ζ_t^i , the discount factor is lower (discount rate is higher) than the usual unweighted average discount factor (used in the buy and hold perspective above). It turns out this weighted average SDF

is not a mere mathematical artifact from swapping integrals but has an economic interpretation. It is the correct marginal rate of intertemporal substitution of aggregate cash flows for a pseudo-representative agent who is forced to distribute aggregate consumption to individuals according to the equilibrium consumption shares c_t^i/C_t in our model. We discuss this interpretation in more detail below.

To derive valuation equations (12) and (13), we start by valuing citizen i 's bond portfolio at time $t = 0$. Denote by $b_t^i := (1 - \theta_t^{K,i} - \theta_t^{E,i} - \theta_t^{\bar{E},i})n_t^i$ the value of agent i 's bond portfolio at time t and let $b_t^i d\Delta_t^{b,i}$ be the stochastic bond trading process, where

$$d\Delta_t^{b,i} = \mu_t^{\Delta,i} dt + \sigma_t^{\Delta,i} dZ_t + \tilde{\sigma}_t^{\Delta,i} d\tilde{Z}_t^i$$

denotes the proportional appreciation of b_t^i due to trading between t and $t + dt$. Under the optimal trading policy, the initial bond wealth b_0^i must equal the discounted value of future payouts (=outflows) from the bond portfolio,¹⁸

$$b_0^i = -\mathbb{E} \left[\int_0^\infty \tilde{\zeta}_t^i b_t^i \left(\mu_t^{\Delta,i} - \varsigma_t \sigma_t^{\Delta,i} - \tilde{\zeta}_t^i \tilde{\sigma}_t^{\Delta,i} \right) dt \right]. \quad (14)$$

As all agents hold the same constant fraction of their net worth in bonds ($\theta_t^i = \vartheta_t$), the value of the individual bond portfolio is simply the product of the agent's net worth share and aggregate bond wealth, $b_t^i = \eta_t^i q_t^B K_t$. In Appendix A.3, we show that the bond trading process satisfies

$$\mu_t^{\Delta,i} = -s_t/q_t^B, \quad \sigma_t^{\Delta,i} = 0, \quad \tilde{\sigma}_t^{\Delta,i} = (1 - \vartheta_t)\tilde{\chi}\tilde{\sigma}_t. \quad (15)$$

The proportional reduction in the value of all agents' bond portfolios due to trading with the government equals the surplus-debt ratio s_t/q_t^B , agents do not trade in response to aggregate shocks as they are symmetrically affected, but agents do trade in response to idiosyncratic shocks: they sell capital and buy bonds when they receive a positive shock and vice versa. We also show in the appendix that the price of idiosyncratic risk satisfies $\tilde{\zeta}_t^i = (1 - \vartheta_t)\tilde{\chi}\tilde{\sigma}_t$, where the right-hand expression is the residual (proportional) idiosyncratic wealth risk that agents have to bear in equilibrium.

Combining all these equations and using $q_t^B K_t = \mathcal{B}_t/\mathcal{P}_t$ leads to the individual

¹⁸A transversality condition always ensures that there is no additional nonvanishing terminal wealth term. We provide a formal derivation of this equation in Appendix A.3.

valuation equation

$$\eta_0^i \frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E} \left[\int_0^\infty \bar{\zeta}_t^i \eta_t^i s_t K_t dt \right] + \mathbb{E} \left[\int_0^\infty \bar{\zeta}_t^i \eta_t^i (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \right]. \quad (16)$$

Finally, integrating over individuals i yields equation (12).

Comparison of the Two Approaches. The SDFs used in equations (10) and (13) are both free of idiosyncratic risk and imply the same aggregate risk premium, but they differ with respect to their average rate of decay, the “risk-free rate” they imply. The average SDF $\bar{\zeta}$ decays at the equilibrium risk-free rate r_t^f . It is thus a proper SDF in this model that prices all assets free of idiosyncratic risk. The same is not true for the weighted average SDF $\bar{\zeta}_t^{**}$. The latter decays at a rate $r_t^f + \bar{\zeta}_t \bar{\sigma}_t^n$, where $\bar{\sigma}_t^n$ is the idiosyncratic net worth volatility of agents (which is identical for all agents in equilibrium). The weighted average SDF $\bar{\zeta}_t^{**}$ therefore discounts safe cash flows at a higher rate than the risk-free rate that contains a risk premium for idiosyncratic wealth risk. The reason for this is apparent from equation (12) which inverts the order of integration: while aggregate cash flows from bonds are free of idiosyncratic risk, each agent holds a stochastic share η_t^i of the aggregate bond portfolio so that individual bond portfolios do contain priced idiosyncratic risk.

These considerations imply that only equation (10) is a standard asset pricing formula, a discounted present value formula using a SDF that prices all assets (at least those free of idiosyncratic risk). But equation (10) can have a bubble and infinities with opposite sign. It can be more informative to work with equation (13) instead, as this equation makes the source of trading gains transparent. However, we need to keep in mind that this equation uses a SDF that does (in general) not price the assets in the economy correctly without additional service flow terms.

Relating the Dynamic Trading Perspective to a Representative Agent. The weighted-average SDF may not be a proper SDF that prices assets in the competitive equilibrium of our incomplete markets economy. Yet, it turns out to be the correct SDF of a representative agent in a Lucas-type asset pricing economy that generates the same allocation as our competitive equilibrium. In addition, if we interpret aggregate capital and aggregate bonds as two “trees” in this representative agent economy, then equation (13) is precisely the valuation equation for the bond tree from the perspective of the representative agent. The dynamic trading perspective is therefore equivalent to the perspective

of a hypothetical representative agent.

More precisely, consider a representative agent that maximizes a weighted welfare function $\mathcal{W}_0 = \int \lambda^i V_0^i di$ with some (positive) welfare weights $(\lambda^i)_{i \in [0,1]}$. If we denote by $\eta_t^i := c_t^i / C_t$ the consumption share of agent i , we can write utility of this representative agent as

$$\mathcal{W}_0 = \mathbb{E} \left[\int_0^\infty e^{-\rho t} \int \lambda^i \log(\eta_t^i C_t) didt \right], \quad (17)$$

which resembles standard time-separable utility in *aggregate consumption* C_t with period utility function $C_t \mapsto \int \lambda^i u(\eta_t^i C_t) di$. The consumption shares η_t^i in this utility function evolve according to $d\eta_t^i = \tilde{\sigma}_t^\eta d\tilde{Z}_t^i$ with volatility process $\tilde{\sigma}_t^\eta$ specified below in equation (19). We show in Appendix A.5 that \mathcal{W}_0 can also be written as

$$\mathcal{W}_0 = w_0 + \mathbb{E} \left[\int_0^\infty e^{-\rho t} \left(\log C_t - \frac{1}{2\rho} (\tilde{\sigma}_t^\eta)^2 \right) dt \right] \quad (18)$$

with some constant w_0 . Equation (18) eliminates the direct dependence on i and gives us the alternative interpretation that two “goods” enter the representative agent’s utility function, the aggregate consumption good and a “volatility reduction good” which is captured by the term $-\frac{1}{2\rho} (\tilde{\sigma}_t^\eta)^2$.

We assume that the representative agent has access to two assets, capital K_t , which produces a certain bundle of the aggregate consumption good and volatility $\tilde{\sigma}_t^\eta$, and “derivatives” X_t , which mimic the cash flows to individuals i generated by bond trades in our incomplete markets model and thereby reduce volatility. Capital grows at rate $g_t := \Phi(\iota_t) - \delta$ over time and generates consumption goods at rate $((1 - \tau_t)a_t - \iota_t) K_t dt$.¹⁹ The face value X_t of derivatives evolves according to

$$dX_t / X_t = (g_t + \mu_t^{q,B}) dt + \sigma_t^{q,B} dZ_t,$$

where $\mu_t^{q,B}, \sigma_t^{q,B}$ are the drift and volatility processes of q_t^B implied by the competitive equilibrium of the incomplete markets model. Derivatives generate a cash flow $-\check{\mu}_t^B X_t$ and reduce fluctuations in consumption shares η_t^i . Specifically, the volatility loading $\tilde{\sigma}_t^\eta$

¹⁹For the purpose of this representative agent economy, $g_t, \tau_t, a_t, \iota_t$ are exogenous processes. But, of course, we choose for them the stochastic processes implied by the competitive equilibrium of our incomplete markets model. The same remark holds for other lower-case variables $q_t^B, q_t^K, \check{\mu}_t^B$ used below.

satisfies the equation

$$\left(q_t^K K_t + X_t\right) \tilde{\sigma}_t^\eta = q_t^K K_t \bar{\chi} \tilde{\sigma}_t, \quad (19)$$

where q_t^K is the capital price process from the incomplete markets economy. We can interpret the product $X_t \tilde{\sigma}_t^\eta$ as a measure of the aggregate gross trading cash flows from bond trades in response to idiosyncratic shocks in the incomplete markets economy.²⁰

Let Q_t^K be the capital price that the representative agent faces, P_t^X the price per unit (face value) of derivatives, and let $N_t := Q_t^K K_t + P_t^X X_t$ be the representative agent's total net worth. The budget constraint of the representative agent is

$$dN_t = -C_t dt + Q_t^K K_t dr_t^K + P_t^X X_t dr_t^X \quad (20)$$

with return processes

$$\begin{aligned} dr_t^K &= \left(\frac{(1 - \tau_t) a_t - l_t}{Q_t^K} + \mu_t^{Q,K} + g_t \right) dt + \sigma_t^{Q,K} dZ_t, \\ dr_t^X &= \left(\mu_t^{P,X} + g_t - \check{\mu}_t^B + \sigma_t^{q,B} \sigma_t^{P,X} \right) dt + \left(\sigma_t^{q,B} + \sigma_t^{P,X} \right) dZ_t. \end{aligned}$$

The representative agent chooses $C_t, \tilde{\sigma}_t^\eta, K_t, X_t$ to maximize utility \mathcal{W}_0 subject to the budget constraint (20) and the risk constraint (19) taking the prices Q_t^K, P_t^X and the return processes as given. The representative agent model is closed by time-zero supplies of capital (K_0) and derivatives (X_0). We impose the additional relationship $X_0 = q_0^B K_0$, where q_0^B is the initial value of q_t^B in the incomplete markets model. While this supply restriction for X_0 may appear ad hoc, it can be micro-founded in an environment with information frictions in which idiosyncratic shocks are private information and agents have access to hidden trade and savings.²¹ In such an environment, incentive compatibility requires that any insurance transfer to an agent must be precisely offset by a reduction in the present value of that agent's future consumption. Otherwise, the agent would have incentives to misreport the size of the shock and secretly trade capital. Incentive compatibility thus limits the amount of insurance that can be provided, i.e. the quantity X of derivatives.

²⁰ $q_t^K k_t^i \bar{\chi} \tilde{\sigma}_t$ is sensitivity of an agent i 's capital wealth to shocks $d\tilde{Z}_t^i$ before portfolio rebalancing and $q_t^K k_t^i \tilde{\sigma}_t^\eta$ is the shock sensitivity after rebalancing. The difference, $q_t^K k_t^i \left(\bar{\chi} \tilde{\sigma}_t - \tilde{\sigma}_t^\eta \right)$ measures trading cash flows per unit of $d\tilde{Z}_t^i$ and aggregating over all agents yields $X_t \tilde{\sigma}_t^\eta$.

²¹Details on this micro-foundation can be found in Brunnermeier et al. (2020). This information environment has also been employed by Di Tella (2020) in a closely related model.

We show in Appendix A.5 that the competitive equilibrium of this representative agent economy features prices $Q_t^K = q_t^K$ and $P_t^X = 1$ (and thus $P_t^X X_t = q_t^K K_t$), so that asset prices are the same as in the incomplete markets economy.²² Using the utility representation (17), we see immediately that the representative agent's SDF process is

$$\Xi_t = e^{-\rho t} \frac{\int \lambda^i \eta_t^i u'(\eta_t^i C_t) di}{\int \lambda^i \eta_0^i u'(\eta_0^i C_0) di} = \frac{\int \lambda^i u'(c_0^i) \zeta_t^i \eta_t^i di}{\int \lambda^i u'(c_0^i) \eta_0^i di}.$$

We also show in the appendix that Ξ_t is independent of welfare weights λ^i and thus we can assume w.l.o.g. that $\lambda^i u'(c_0^i)$ is a constant independent of i .²³ This implies the following proposition.

Proposition 4. *The representative agent's SDF equals the weighted-average SDF,*

$$\Xi_t = \int \zeta_t^i \eta_t^i di = \zeta_t^{**}.$$

The valuation equation for derivatives from the perspective of the representative agent is

$$P_0^X X_0 = \mathbb{E} \left[\int_0^\infty \Xi_t \cdot \left(-\check{\mu}_t^\beta X_t \right) dt \right] + \mathbb{E} \left[\int_0^\infty \Xi_t \cdot (1 - \vartheta_t)^2 \bar{\chi}^2 \check{\sigma}_t^2 X_t dt \right]. \quad (21)$$

Here, the first term represents the discounted present value of cash flows $-\check{\mu}_t^\beta X_t$ and the second term represents the discounted volatility reduction service flows that derivatives provide by lowering $\check{\sigma}^\eta$ in the utility function (18). As derivatives in the representative agent economy play the same role as bonds in the incomplete markets economy, we can make the identification $X_t = q_t^K K_t$ and $-\check{\mu}_t^\beta X_t = s_t K_t$. With these replacements (and $P_0^X = 1$), equation (21) becomes equation (13), the debt valuation equation from the dynamic trading perspective.

²²Also aggregate consumption C_t and the consumption shares η_t^i are as in the incomplete markets economy. The representative agent economy therefore leads to the same allocation.

²³ $\lambda^i u'(c_0^i)$ would also be independent of i if we allowed the representative agent to choose the initial consumption allocation.

4 Quantitative Model: Counter-cyclical Safe Asset and Two Betas

4.1 Setup with Stochastic Idiosyncratic Risk and Recursive Utility

In this section, we solve a calibrated version of our model with aggregate risk numerically. We introduce aggregate risk as shocks to idiosyncratic risk $\tilde{\sigma}_t$. We interpret periods of high idiosyncratic risk as recessions and want them to be associated with lower consumption and higher marginal utility. Rather than microfounding this relationship explicitly, we simply impose an exogenous relationship $a_t = a(\tilde{\sigma}_t)$ with $a'(\cdot) < 0$ that generates the desired correlation structure.²⁴

For idiosyncratic risk $\tilde{\sigma}_t$, we specify a [Heston \(1993\)](#) model of stochastic volatility, i.e. we assume that the idiosyncratic variance $\tilde{\sigma}_t^2$ follows a Cox–Ingersoll–Ross process ([Cox et al., 1985](#)) process,

$$d\tilde{\sigma}_t^2 = -\psi \left(\tilde{\sigma}_t^2 - (\tilde{\sigma}^0)^2 \right) dt - \sigma \tilde{\sigma}_t dZ_t \quad (22)$$

with parameters $\psi, \sigma, \tilde{\sigma}^0 > 0$.

For the functional relationship between $\tilde{\sigma}_t$ and a_t , we impose the parsimonious linear relationship

$$a(\tilde{\sigma}_t) = a^0 - \alpha^a (\tilde{\sigma}_t - \tilde{\sigma}^0).$$

A sufficiently large coefficient $\alpha^a > 0$ ensures that both output and consumption fall in equilibrium when idiosyncratic risk rises.

For government policy, summarized by debt growth net of interest payments, $\check{\mu}_t^B$,²⁵ we also impose a simple linear relationship

$$\check{\mu}_t^B = \check{\mu}^{B,0} + \alpha^B (\tilde{\sigma}_t - \tilde{\sigma}^0) \quad (23)$$

²⁴For models similar to ours in which output and consumption naturally react negatively to risk shocks, see [DiTella and Hall \(2020\)](#) and [Li and Merkel \(2020\)](#).

²⁵To be precise, the government also chooses the nominal interest rate i_t . However, in our flexible price model, this policy choice merely affects the equilibrium inflation rate but not real allocations or asset prices.

with parameters $\check{\mu}^{\mathcal{B},0}$ and $\alpha^{\mathcal{B}} > 0$. Provided $\alpha^{\mathcal{B}}$ is sufficiently large, this implies that surpluses $s_t = -\check{\mu}_t^{\mathcal{B}} q_t^{\mathcal{B}}$ are positive for low idiosyncratic risk (in expansions) and negative for high idiosyncratic risk (in recessions). Primary surpluses therefore correlate negatively with marginal utility and any agent in the economy would require a positive risk premium for holding a (hypothetical) claim to primary surpluses, a feature that is empirically plausible (see, e.g., [Jiang et al. \(2019\)](#) for the US).

Finally, in order to match certain aspects of the data better, we make two small modifications to the model itself. First, in order for our model to generate quantitatively realistic aggregate risk premia, we replace logarithmic preferences of households with stochastic differential utility ([Duffie and Epstein, 1992](#)) with unit elasticity of intertemporal substitution (EIS) and arbitrary relative risk aversion $\gamma > 0$: household i maximizes V_0^i , where V_t^i is recursively defined by

$$V_t^i = \mathbb{E}_t \left[\int_t^\infty (1 - \gamma) \rho V_s^i \left(\log(c_s^i) - \frac{1}{1 - \gamma} \log \left((1 - \gamma) V_s^i \right) \right) ds \right].$$

In the special case $\gamma = 1$, this specification collapses to our baseline specification with logarithmic utility discussed in Section 2.²⁶ Second, to separate the level of investment from the adjustment cost parameter ϕ , which governs fluctuations in investment and capital prices, we consider the slightly more general capital adjustment cost function

$$\hat{\Phi}(l) = l^0 + \Phi(l - l^0)$$

with the additional parameter l^0 . All solution formulas for the baseline model remain valid for this more general specification if we replace a_t with $a_t - l^0$ and l_t with $l_t - l^0$.

4.2 Calibration and Model Fit

We calibrate our model such that, when we feed in a quantitatively realistic process for idiosyncratic risk, the model generates variation in macro aggregates and aggregate risk premia that are broadly consistent with US data. For our mapping from the model to the data, one time period in the model corresponds to one year. We briefly summarize our calibration choices here. Additional details as well as a description of the

²⁶Qualitatively, the two models behave identically. However, with $\gamma = 1$, the model does not generate a sufficiently large aggregate price of risk to capture the empirically observable equity premium.

underlying data sources can be found in Appendix A.8.

With regard to the parameters $\tilde{\sigma}^0$, ψ , σ of the exogenous risk process $\tilde{\sigma}_t$, we tie our hands by estimating them externally. Specifically, we choose these parameters such that $\tilde{\sigma}_t^2$ closely matches, in a maximum likelihood sense, the common idiosyncratic volatility (CIV) factor proposed by [Herskovic et al. \(2016\)](#). As that paper shows, CIV is a priced risk factor that is correlated with idiosyncratic risk exposures of both firms and households. In Appendix A.8.2, we argue that CIV is a model-consistent data counterpart of $\tilde{\sigma}_t^2$.²⁷

For the share of undiversifiable idiosyncratic risk, $\bar{\chi}$, the literature provides some imperfect guidance. [Angeletos \(2007\)](#) reports that, in the aggregate, private and public equity firms in the US are of approximately equal importance for production, employment, and wealth and uses this as the basis to calibrate a model with idiosyncratic return risk similar to ours that features both private and public equity. His calibration suggests $\bar{\chi} \approx 0.5$ in our framework. [Heaton and Lucas \(2000a,b\)](#) provide evidence on the portfolio composition of the wealth of US stockholders. Broadly speaking, two fifths of wealth are composed of liquid assets, and three fifths are composed of real estate and business wealth. Depending on whether one treats real estate wealth, which has no direct counterpart in our model, as diversifiable or non-diversifiable capital, their evidence suggests values of $\bar{\chi}$ between 0.2 and 0.6.²⁸ Based on this broad evidence, we choose an intermediate value of $\bar{\chi} = 0.3$.

We choose the nine parameters γ , ρ , a^0 , g , $\check{\mu}^{B,0}$, α^a , α^B , ϕ , ι^0 such that the model generates values for a number of moments that are broadly in line with the empirical evidence.²⁹ These moments are the average ratios of consumption, government expenditures, primary surpluses, capital, and debt to output, the average investment rate, the volatilities of output, consumption, investment, and the surplus-output ratio, and the equity premium and equity sharpe ratio. Our empirical moments are based on a sample from 1970 to 2019 just prior to the start of the covid pandemic with two excep-

²⁷We have also considered alternative measures for idiosyncratic uncertainty ([Bloom et al., 2018](#); [Hassan et al., 2019](#)) but ultimately chosen CIV both due to the quality and length of the available data series and because of the theoretical link between $\tilde{\sigma}_t^2$ in the model and CIV.

²⁸More recent work based on administrative data from Sweden and Norway ([Bach et al., 2020](#); [Fagereng et al., 2020](#)) provides corroborating evidence that private business wealth of households in the top percentile of the wealth distribution – those most relevant for asset pricing – makes up well above 20% of their total portfolio holdings.

²⁹We do not employ a formal simulated method of moments estimation but merely adjust parameters manually to achieve a good visual fit.

Table 1: Parameter Choice

parameter	description	value
$\tilde{\sigma}^0$	$\tilde{\sigma}_t^2$ stoch. steady state	0.54
ψ	$\tilde{\sigma}_t^2$ mean reversion	0.67
σ	$\tilde{\sigma}_t^2$ volatility	0.4
$\bar{\chi}$	undiversifiable idio. risk	0.3
γ	risk aversion	6
ρ	time preference	0.138
a^0	productivity (a) stoch. steady state	0.63
g	government expenditures	0.138
$\check{\mu}^{B,0}$	policy ($\check{\mu}^B$) stoch. steady state	0.0023
α^a	productivity (a) slope	0.071
α^B	policy ($\check{\mu}^B$) slope	0.12
ϕ	capital adjustment cost	8.5
l^0	capital adjustment intercept	-0.015
δ	capital depreciation rate	0.055

tions. The first is the debt-output ratio. Over the largest part of our sample period, this ratio has exhibited a clear upward trend. For this reason, we target the average over the last decade in the sample (0.71) instead of the average over the full sample (0.37). The second exception is the average investment rate, $\mathbb{E}[I/K]$, which we do not compute ourselves but take directly from [Cooper and Haltiwanger \(2006\)](#), who report a value estimated from micro data.

As we explain in [Appendix A.8.3](#), matching the average ratios is directly informative for the average value of the endogenous variable ϑ_t and the parameters ρ , a^0 , g , $\check{\mu}^{B,0}$, and l^0 . Requiring the model to match the macro volatilities is standard in the business cycle literature and also ensures that the model generates a broadly realistic amount of aggregate macro risk.³⁰ While including the surplus volatility is less standard, this moment is important to discipline the (cyclical) variation in primary surpluses, a key ingredient into the valuation of government debt.³¹ Finally, requiring the model to match the equity premium and equity sharpe ratio ensures that this aggregate macro volatility is realistically priced in capital markets.

The remaining parameter δ does not affect anything of interest for the purpose of

³⁰These moments also discipline the model parameter α^a and ϕ .

³¹That moment also disciplines the parameter α^B .

this paper.³² We set it to 0.055, a value slightly smaller than but broadly in line with typical calibrations. With this choice, the average growth rate of our model economy is 2.2%, close to the empirical counterpart of 2.1% in our sample. Incidentally, because $r^f \approx g$ in this model when primary surpluses are close to zero, this choice also ensures that the levels of returns are realistic.

Table 2: Quantitative Model Fit

symbol	moment description	model	data
$\sigma(Y)$	output volatility	1.3%	1.3%
$\sigma(C)/\sigma(Y)$	relative consumption volatility	0.63	0.64
$\sigma(I)/\sigma(Y)$	relative investment volatility	3.31	3.38
$\sigma(S/Y)$	surplus volatility	1.1%	1.1%
$\rho(Y, C)$	correlation of output and consumption	0.98	0.92
$\rho(Y, I)$	correlation of output and investment	0.98	0.94
$\rho(Y, S/Y)$	correlation of output and surpluses	0.98	0.60
$\mathbb{E}[C/Y]$	average consumption-output ratio	0.58	0.56
$\mathbb{E}[G/Y]$	average government expenditures-output ratio	0.22	0.22
$\mathbb{E}[S/Y]$	average surplus-output ratio	-0.0004	-0.0005
$\mathbb{E}[I/K]$	average investment rate	0.12	0.12
$\mathbb{E}[q^K K/Y]$	average capital-output ratio	3.48	3.73
$\mathbb{E}[q^B K/Y]$	average debt-output ratio	0.74	0.71
$\mathbb{E}[d\bar{r}^E - dr^B]$	average (unlevered) equity premium	3.62%	3.40%
$\frac{\mathbb{E}[d\bar{r}^E - dr^B]}{\sigma(d\bar{r}^E - dr^B)}$	equity sharpe ratio	0.31	0.31

Notes: $\sigma(x)$ denotes the standard deviation of x and $\rho(x, y)$ denotes the correlation of x and y , both at a quarterly frequency. Inputs x and y are HP-filtered with smoothing parameter 1600. For $x, y \in \{Y, C, I, G\}$, we take logarithms before filtering. $\mathbb{E}[x]$ denotes expectations over the ergodic model distribution, inputs x are *not* HP-filtered. Y : (aggregate) output, C : consumption, I : investment, G : government expenditures, S : primary surplus; $K, q^K, q^B, dr^B, d\bar{r}^E$ are defined as in Section 2.

Table 1 summarizes our parameter choice and Table 2 summarizes the quantitative model fit. In addition to our target moments, we report in Table 2 also the correlations of consumption, investment, and primary surpluses with output.

Table 2 reveals that our model achieves overall an excellent fit to the targets. As we have varied only nine of our parameters to match twelve moments, this is by no means a trivial observation. Most importantly, our model is consistent with the observed large equity premium and price of risk (Sharpe ratio) while at the same time matching the volatility and comovement of macro aggregates. This verifies that our model is capable of generating realistic aggregate risk premia without requiring excessive real volatil-

³²This is due to the combination of the AK structure of our economy with a unit EIS. The former implies that δ merely affects the growth rate of the economy and the latter that income and substitution effects from permanent variations in growth rates cancel out.

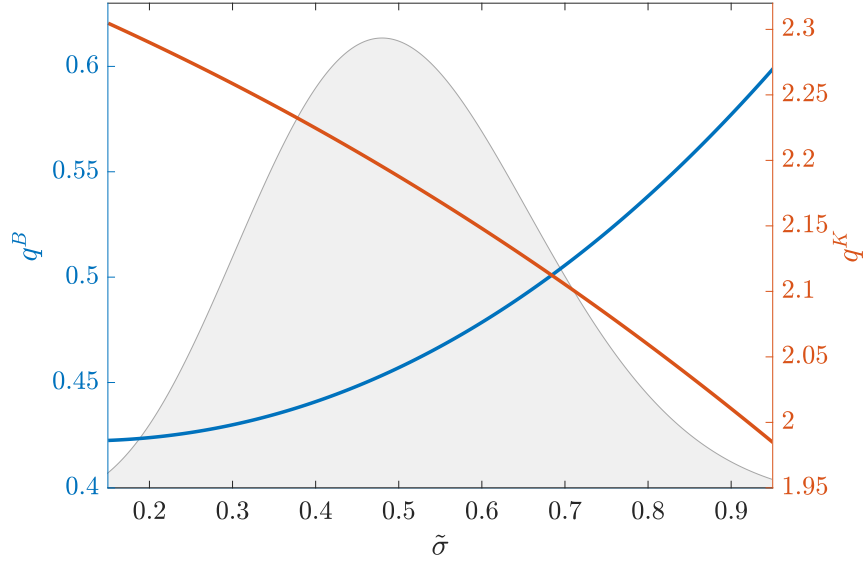


Figure 1: Equilibrium asset valuations q^B (blue line, left scale) and q^K (red line, right scale) as a function of idiosyncratic risk $\tilde{\sigma}$. The gray shaded area in the background depicts the (rescaled) ergodic density of the state variable $\tilde{\sigma}$.

ity. Notably, our model achieves quantitatively plausible aggregate risk pricing with a moderate risk aversion parameter (γ) of just 6.

4.3 Equilibrium Dynamics of Bond and Capital Values

Figure 1 illustrates the equilibrium dynamics of the value of the government bond stock q^B (blue line) and the value of the capital stock q^K (red line) per unit of capital in the economy by plotting these valuations as a function of the state variable $\tilde{\sigma}$. The gray shaded area depicts the stationary distribution of $\tilde{\sigma}$. q^B is strictly increasing in idiosyncratic risk whereas q^K is strictly decreasing. Because output comoves negatively with $\tilde{\sigma}$ by construction, these monotonicity patterns imply that bond valuations are counter-cyclical whereas capital valuations are pro-cyclical. It is this counter-cyclical valuation that makes government bonds a good safe asset. We analyze the source of the counter-cyclicity in the following subsection.

4.4 Analyzing the Two Bond Asset Pricing Terms Separately

We now consider the two terms in the government debt valuation equation derived from the dynamic trading perspective (equation (13)). Figure 2 plots the two present values³³

$$q^{B,CF}(\tilde{\sigma}) := \mathbb{E} \left[\int_0^\infty \left(\int \xi_t^i \eta_t^i di \right) s_t K_t dt \mid \tilde{\sigma}_0 = \tilde{\sigma}, K_0 \right] / K_0$$

$$q^{B,SF}(\tilde{\sigma}) := \mathbb{E} \left[\int_0^\infty \left(\int \xi_t^i \eta_t^i di \right) (1 - \vartheta_t)^2 \gamma \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \mid \tilde{\sigma}_0 = \tilde{\sigma}, K_0 \right] / K_0$$

for our calibrated model. The blue solid line shows the present value of future primary surpluses (cash flows) $q^{B,CF}$ as a function of the single state variable $\tilde{\sigma}$. This value is strictly decreasing in idiosyncratic risk and has a low – and in fact negative – value. Comparing the present value of surpluses $q^{B,CF}K$ in our model to the market value of government debt $q^B K$, which is represented by the black dashed line in Figure 2, reveals a large gap $(q^B - q^{B,CF})K$, a “debt valuation puzzle”. In addition, when compared with the present value of surpluses $q^{B,CF}K$, the total value of government debt $q^B K$ has also the opposite correlation with the aggregate state. Yet, there is no puzzle from the perspective of our model: government debt is a safe asset valued for its service flow from re-trading which is represented by the component $q^{B,SF}(\tilde{\sigma})$. As the red solid line in Figure 2 shows, this value is positive, large and positively correlated with $\tilde{\sigma}_t$. This additional component dominates the overall dynamics of the value of government debt and is the reason that q^B appreciates in bad times despite the simultaneous drop in $q^{B,CF}$. That $q^{B,SF}$ must be positively correlated with $\tilde{\sigma}$ can also be seen from the present value equation: for our policy specification, residual net worth risk $(1 - \vartheta_t)\bar{\chi}\tilde{\sigma}_t$ is increasing in $\tilde{\sigma}_t$, so that an increase in idiosyncratic risk increases the value of insurance service flows from re-trading.³⁴

The correlation structure apparent in Figure 2 implies that, if the two claims $q^{B,CF}$ and $q^{B,SF}$ could be traded separately, the cash flow claim would be a high- β asset,

³³Relative to equation (13), here an additional factor γ appears because we no longer assume logarithmic preferences.

³⁴This is not an entirely rigorous argument as it ignores changes in the discount rate. The effective discount rate in the weighted-average SDF $\int \xi_t^i \eta_t^i di$ can both increase or decrease with the aggregate state $\tilde{\sigma}_t$ depending on whether the *aggregate* risk premium increases or decreases. Note however, that the level of idiosyncratic risk does not directly matter for the effective discount rate because the risk premium on idiosyncratic risk exactly offsets the lower risk-free rate due to a precautionary motive.

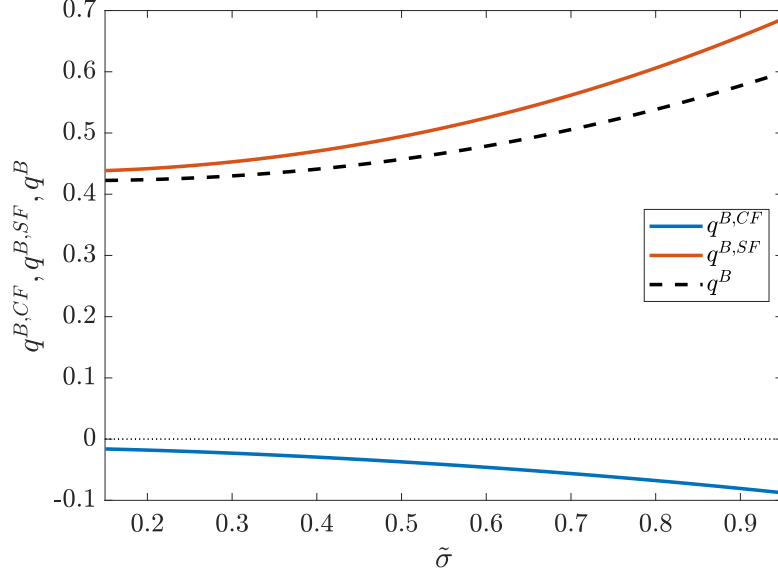


Figure 2: Decomposition of the value of government debt as a function of idiosyncratic risk $\tilde{\sigma}$. The blue solid line shows the present value of primary surpluses ($q^{B,CF}$), the red solid line the present value of service flows ($q^{B,SF}$) and the black dashed line the total value of government debt (q^B), all normalized by the capital stock.

while the service flow claim would be a negative- β asset. The presence of this second, negative- β component makes government debt as a whole a negative β asset. Government debt emerges as a “good friend” also with respect to aggregate shocks. Figure 3 depicts this explicitly by plotting (weighted) conditional betas for the two hypothetical assets.³⁵

4.5 The Possibility of Insuring Bond Holders and Tax Payers at the Same Time

In our simple setting, households are both capital owners and bond holders. In this section, we conceptually separate each household into two sub-units, a capital owner and a government debt holder. Surprisingly, it is possible to follow a government pol-

³⁵We define $\beta_t^j = \sigma_t^j / \zeta_t$, where $j \in \{CF, SF\}$ and dr^j is the return on the respective component and σ_t^j is the aggregate risk loading of that return. This definition can be interpreted as $\beta_t = -\frac{\text{Cov}_t(d\zeta_t / \zeta_t, dr_t^j)}{\text{Var}_t(d\zeta_t / \zeta_t)}$, where $d\zeta_t / \zeta_t$ is the SDF that discounts cash flows from $t + dt$ to time t . In addition, we weight β^j by its share $\omega^j := q^{B,j} / q^B$ of the total government debt claim.

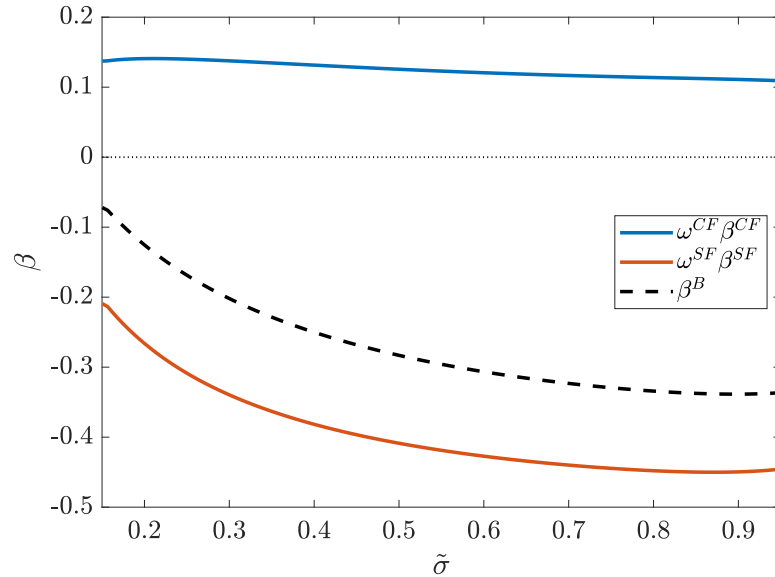


Figure 3: Conditional betas of hypothetical claims to the surplus and risk-sharing components of the government debt value

icy that provides insurance against negative aggregate shocks for both tax payers and bond holders at the same time. By cutting taxes (or even granting subsidies) for capital owners in recessions, their tax burden is positively correlated with their income providing insurance to tax payers. At the same time, the safe asset service flow rises in recessions, which provides insurance to government bond holders. This finding in our incomplete market setting with a safe asset is in sharp contrast to traditional asset pricing in which either tax payers or government bond holders can be insured, as pointed out in [Jiang et al. \(2020\)](#).

5 Volatile, Flight-to-Safety Prone Equity Markets

The presence of idiosyncratic risk and government debt as a safe asset has also implications for equity markets. We explain in this section why the diversified equity portfolio does not emerge as a safe asset and how flight to safety can generate quantitatively large additional equity return volatility.

Why Stocks Are not Safe Assets. In our model, agents can hold a diversified stock portfolio. Like government bonds, this stock portfolio is free of idiosyncratic risk and

thus allows agents to self-insure against idiosyncratic consumption fluctuations. However, unlike government bonds, stocks are poor aggregate risk hedges as they are ultimately claims to capital, which loses in value in recessions. This implies that stocks are positive- β assets in our model.

To understand why stock prices fall in times of high idiosyncratic risk, even though idiosyncratic equity risk can be diversified away, note that the marginal holder of capital in our model is always an insider who has to bear the increased idiosyncratic risk. As a consequence, when idiosyncratic risk goes up, so does the insider premium earned by the managing households, which is achieved by a reduction in the dividend that is paid to outside equity holders.³⁶ This makes stock dividends more procyclical than production cash flows, so that stocks lose value precisely when idiosyncratic risk goes up.

When evaluating the diversified stock portfolio with regard to the key characteristic of safe assets, the Good Friend Analogy, stocks fail to qualify as safe assets in the same way as government debt does. Stocks have the good friend property only partially: stocks are valuable and liquid when an agent experiences a negative idiosyncratic shock, but due to their positive β , they are not in bad aggregate times.

Flight-to-safety Volatility. While the focus of this paper is on government bonds, our model can also match the empirical mean and volatility of the excess return on the stock market in excess of government bonds. The realistic sharpe ratio is clearly a feature of recursive preferences with a high risk aversion, but the ability of our simple model to generate large return volatility in the presence of realistic levels of output variation is quite remarkable³⁷ and directly related to the existence of safe government bonds.

To gain intuition, let's abstract from the distinction between capital and outside equity³⁸ and for a moment also switch off both government spending G_t and physical capital investments I_t by putting $g = 0$ and considering the limit $\phi \rightarrow \infty$, so that

³⁶Formally, the Lagrange multiplier λ_t^i on the skin-in-the-game constraint (5) that governs the spread $\mathbb{E}_t[dr_t^{K,i}] - \mathbb{E}_t[dr_t^{E,i}]$ increases, compare Appendix A.1.

³⁷It is remarkable because we work with preferences that feature a unit elasticity of intertemporal substitution (EIS). It is well-known from the long-run risk literature that recursive preferences can also generate large return volatility, but *only* if the EIS is sufficiently larger than 1. In contrast, the mechanism we describe here works also for $\text{EIS} \leq 1$.

³⁸As we have discussed previously in this section, a state-dependent insider premium will ensure that equity values and capital values move in lockstep despite the fact that idiosyncratic equity risk can be diversified away.

$Y_t = C_t$. Then, aggregating individual households' intertemporal budget constraints yields the equation

$$q_t^K K_t + q_t^B K_t = \mathbb{E}_t \left[\int_t^\infty \frac{\int \tilde{\zeta}_s^i \eta_s^i di}{\int \tilde{\zeta}_t^i \eta_t^i di} Y_s ds \right]. \quad (24)$$

In standard Lucas-type models, government debt does not represent positive net wealth, $q_t^B = 0$, and thus equation (24) implies for such models that the value of the capital stock equals the present value of future output. In other words, in a Lucas-type economy, pricing the aggregate equity claim is equivalent to pricing the aggregate output claim.³⁹ In the presence of realistic output volatility, large volatility in capital valuations $q_t^K K_t$ is then hard to generate (and requires substantial time variation in the SDF $\int \tilde{\zeta}_s^i \eta_s^i di$). If we allow for $G_t, I_t \neq 0$, the puzzle tends to become even larger because consumption is smoother than output in the data.

Our model with $q^B \neq 0$ suggests an additional explanation for the high observed stock market volatility. When idiosyncratic risk $\tilde{\sigma}_t$ rises, there is a flight to safety that increases the value of bonds (q_t^B) and lowers the value of capital (q_t^K). Even in the absence of changes in the present value on the right-hand side of equation (24), this portfolio reallocation generates *flight-to-safety volatility* in capital valuations and thus in the stock market.

To understand how much flight-to-safety volatility matters quantitatively, we compare the excess stock return volatility in our model to the one generated by a version of the model without government debt (and primary surpluses set to zero). In that alternative version, $q_t^B = 0$ at all times and thus flight-to-safety volatility disappears.⁴⁰ We find that the average (annualized) excess return volatility in the alternative model would be 2.4% as opposed to 11.7% in our baseline model.⁴¹ We can therefore conclude that flight-to-safety volatility accounts for more than three quarters of the overall excess return volatility in our framework.

³⁹Because the equation results from aggregating individual intertemporal budget constraints, the SDF used in this pricing equation is again the weighted-average SDF as in the dynamic trading perspective to government bond valuation, not any market SDF (i.e. a SDF that prices all tradeable assets). In standard Lucas-type models there is no idiosyncratic risk so that the two coincide.

⁴⁰Except for the elimination of primary surpluses ($\check{\mu}^{B,0} = \alpha_B = 0$) and the selection of the “non-monetary” equilibrium, we keep all other parameters as in our baseline model.

⁴¹As the benchmark asset in the alternative model, we choose a zero net supply risk-free bond, the most common choice in the literature.

6 Privately Issued Safe Assets and Convenience Yields

So far, we have emphasized government debt as a safe asset. In this section, we discuss safe asset issuance by private agents. We also elaborate on the difference between service flows derived from retrading and convenience yields.⁴²

Privately Issued Safe Assets. We consider a model extension with privately issued safe assets. We discuss here merely the economic conclusions and relegate the formal details to Appendix A.6.

We assume that each agent can issue nominally risk-free bonds, just like the government. Our safe asset definition, based on the Good Friend Analogy, applies equally also to such debt instruments issued by private citizens. For any individual asset holder, government bonds and privately issued safe bonds are perfect substitutes. As a consequence, the equilibrium interest rate i_t^p that private agents have to pay on their bonds equals the government's, $i_t^p = i_t$.

In equilibrium, agents are then indifferent as to how many bonds to issue and how many privately issued bonds of other agents to hold. In Appendix A.6, we consider a simple example in which the quantity of outstanding private bonds is always proportional to the quantity of government bonds and all agents keep the relative allocation to private and government bonds in their portfolios constant, so that they must trade them in constant proportions. Just like government bonds, we can value bonds issued by some agent j (“ j -bonds”) from the dynamic trading perspective by pricing the cash flows of the portfolios of j -bonds held by all other agents $i \neq j$. The resulting equation is in complete analogy to equation (13) for government bonds. In particular, the service flow that agents derive per real unit of j -bonds outstanding is the same as for government bonds.

Overall, the model extension in Appendix A.6 highlights that privately issued bonds are equally suitable as safe assets for their holders. However, private bond issuance also comes with a short position in the bond for the issuing agent. In the same spirit as before, we can value the short position by determining the present value of net payouts that an issuer makes to all bond holders. That valuation exercise reveals that the short position generates a negative service flow for the issuing agent. This negative service flow results from the fact that the agent repays outstanding debt after negative idiosyn-

⁴²For the equations presented in this section, we revert back to the logarithmic preference specification.

cratic shocks and issues additional debt after positive idiosyncratic shocks. While the cash flows generated from this contingent bond issuance are zero on average, they are systematically correlated with marginal utility and thus tend to increase the overall riskiness of the agent’s portfolio.

Once we aggregate all long and short positions of all privately issued safe bonds, the service flows “earned” by bond holders and the service flows “paid” by bond issuers exactly cancel out.⁴³ Therefore, unlike government debt, private safe asset creation does not generate net service flows for the economy as a whole.

The previous conclusion is ultimately a manifestation of the fact that privately issued bonds do not represent net wealth as they are in zero net supply. This is in contrast to government bonds which do represent net wealth in our model. If privately issued bonds also represented net wealth, e.g. because they had a bubble component or because private agents could run Ponzi schemes, then privately issued safe assets, too, would generate a positive net service flow in the aggregate. Such a situation is impossible under our maintained assumptions about equilibrium selection, but can occur in some of the alternative equilibria discussed in Section 8.

Convenience Yields. A conclusion from the previous model extension is that $\Delta i_t := i_t^p - i_t = 0$, the yield spread between privately issued and government debt is zero. Government debt is not special. In the presence of idiosyncratic risk, a precautionary motive depresses all asset returns symmetrically. Equivalently, a service flow from re-trading can be derived from all assets that are both free of idiosyncratic risk and tradeable on liquid markets.

Such a service flow is conceptually different from a convenience yield. A convenience yield on government debt captures the special role that government bonds play in certain transactions. It can be measured by a positive yield spread $\Delta i_t > 0$ between government debt and safe corporate debt of equal maturity. In contrast, the service flow from re-trading we emphasize in this paper affects also safe corporate debt. It is therefore unrelated to the spread Δi_t .

To illustrate this difference further, we augment our model so that government debt has a convenience yield. We model the source of the convenience yield by simply putting government bond holdings in agents’ utility functions. Other mechanisms like collateral constraints require richer environments but would lead to the same conclu-

⁴³This conclusion is generally true, not just in the specific example analyzed in Appendix A.6.

sions. We present the formal model equations in Appendix A.7.

In the augmented model, we can once again price government debt according to our two valuation perspectives:

Proposition 5. *In the model with convenience yields, the value of government debt at $t = 0$ satisfies from the buy and hold perspective:*

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \lim_{T \rightarrow \infty} \left(\mathbb{E} \left[\int_0^T \bar{\zeta}_t s_t K_t dt \right] + \mathbb{E} \left[\int_0^T \bar{\zeta}_t \Delta i_t \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \right] + \mathbb{E} \left[\bar{\zeta}_T \frac{\mathcal{B}_T}{\mathcal{P}_T} \right] \right),$$

and from the dynamic trading perspective:

$$\frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E} \left[\int_0^\infty \zeta_t^{**} s_t K_t dt \right] + \mathbb{E} \left[\int_0^\infty \zeta_t^{**} \left(\Delta i_t + (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2 \right) \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \right].$$

From the latter, dynamic trading perspective, the service flows from bonds in the utility function (captured by Δi_t) and from self-insurance through retrading (captured by $(1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2$) appear symmetrically. However, the buy and hold perspective reveals an asymmetry. The convenience yield still enters the valuation explicitly as a service flow term. In contrast, the service flow from retrading is absent in this perspective. Instead, it is implicitly contained in the stochastic discount factor $\bar{\zeta}_t$ and results in a lower discount rate due to precautionary savings as well as – potentially – a bubble term.

The terms arising from the buy and hold perspective are the ones that are typically measured in empirical asset pricing. The best an empirical researcher can do when estimating a SDF based on aggregate asset price data is to identify $\bar{\zeta}_t$. When looking at yield differences between safe corporate and government bonds, the empirical researcher identifies an estimate of Δi_t . The importance of self-insurance service flows can only be determined indirectly, e.g. by finding a bubble component.⁴⁴

We interpret the empirical findings of Jiang et al. (2019) as evidence in support of such a bubble component. They conclude that an empirical estimate of the present value of surpluses and convenience yields falls short of the market value of government

⁴⁴The presence of a bubble component in the buy and hold perspective means that even at the low discount rates implied by $\bar{\zeta}_t$, cash flows $s_t K_t$ and convenience yield service flows $\Delta i_t \mathcal{B}_t / \mathcal{P}_t$ are insufficient to explain the total value of government debt. The same always remains true if we discount at the higher rates implied by ζ_t^{**} , so that the self-insurance service flow must explain the gap.

debt. As their empirical analysis is conducted from the buy and hold perspective, the gap must be explained by a bubble component.

7 Quantifying the Bubble Mining Laffer Curve

When idiosyncratic risk is large, safe asset demand may be sufficient to sustain a public debt bubble. As Brunnermeier et al. (2021a) point out, such public debt bubbles represent a fiscal resource that can be “mined” for revenue as a substitute for taxation. However, the ability to mine a bubble does not imply that a government can expand spending without limits. Bubble mining affects the sustainability of bubbles and thereby creates a “debt Laffer curve”.

In this section, we briefly revisit the Laffer curve logic and then use our calibrated model to quantify the Laffer curve for the US. The main takeaway is that the negative β property of government debt matters considerably. The (average) maximum permanent deficit is above 2% of GDP in our dynamic model but merely 0.1% if we hold idiosyncratic risk constant over time.

The Laffer curve logic follows from the following simple formula for primary deficits per unit of capital⁴⁵

$$-s_t = \check{\mu}_t^B q_t^B.$$

The first factor, $\check{\mu}_t^B$, measures revenue raised by bond issuance that is not distributed to bond holders in the form of interest payments. If it is positive, the claim of old bond holders is diluted by the issuance of new bonds, i.e., a higher $\check{\mu}_t^B$ represents a tax on existing bond holders. The second factor, q_t^B , is the tax base, the real value of existing debt (per unit of capital). Permanent deficits are possible if this tax base remains positive even for (permanently) negative primary surpluses. That this is a possibility can be seen from both perspectives to debt valuation discussed in Section 3: the value of debt remains positive despite negative surpluses if, in the buy and hold perspective (equation (13)), a positive bubble term offsets the negative surplus term, or, equivalently, in the dynamic trading perspective (equation (13)), the service flow term is sufficiently large. In this case, the tax base is positive, but it nevertheless reacts negatively to an increase in the rate of bubble mining $\check{\mu}_t^B$.⁴⁶ This negative reaction

⁴⁵This equation, in turn, follows immediately from the government budget constraint.

⁴⁶We can see analytically that higher $\check{\mu}^B$ lowers the equilibrium value of q^B in steady state, compare

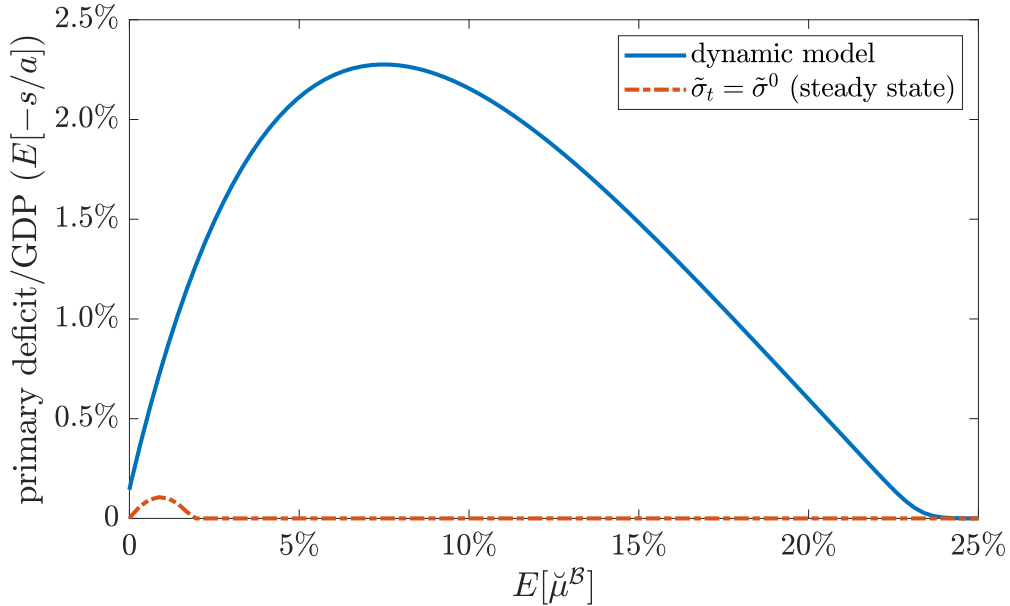


Figure 4: Debt Laffer curve for dynamic model and in steady state (constant $\tilde{\sigma}_t$) when there is a bubble on government debt. $\mathbb{E}[\check{\mu}^B]$ is varied by varying parameter $\check{\mu}^{B,0}$ while keeping all other parameters as in Table 1.

creates a Laffer curve.

The blue line in Figure 4 depicts the debt Laffer curve for our calibrated model. Specifically, the figure plots the average deficit-GDP ratio that can be sustained for different debt growth policies of the form (23) with identical α^B (identical cyclicity of debt growth and surpluses) but varying $\check{\mu}^{B,0}$, i.e. the average level of (interest-adjusted) debt growth varies across different policies on the x -axis. The implicit assumption in Figure 4 is that g remains unchanged, so that larger deficits imply smaller output taxes.⁴⁷

In Figure 4, if the bubble is mined too aggressively so that the average $\check{\mu}^B$ exceeds 7.3%, the government fails to raise additional real revenues. In particular, there is a limit to bubble mining and the government still faces a constraint on real spending. Our calibrated model suggests that the average primary deficit that can be sustained by bubble mining is bounded above by 2.3% of GDP.

the formulas in Section 2.3. Outside of the steady state, equation (7) tells us that there is a negative relationship if an upward shift in $\check{\mu}_t^B$ at all dates decreases ϑ_t . Equation (9) suggests that this is indeed the case, but additional technical considerations are required to make this a fully rigorous argument.

⁴⁷If instead the increased revenues from bubble mining are used to fund additional government expenditures (higher g), the slope of the Laffer curve is uniformly smaller.

It turns out that the negative β property is very important for the qualitative and quantitative shape of the Laffer curve depicted in Figure 4. If we abstract from counter-cyclical idiosyncratic risk and consider a constant level of $\tilde{\sigma}_t = \tilde{\sigma}^0$ instead, the resulting Laffer curve is as depicted by the red dashed line in Figure 4. This steady-state Laffer curve reveals two differences compared to the dynamic model. First, the Laffer curve is quantitatively tiny. The maximum (average) permanent deficit is merely 0.1% (and it is reached at a much lower average value of $\tilde{\mu}^B$). Second, the steady-state Laffer curve quickly decays to zero, so that the tax base is more quickly eroded as the government dilutes the claims of existing bond holders at a faster rate. Instead, in the dynamic model, agents hold on to some bonds even at very large levels of average (interest-adjusted) debt growth rates of more than 10% despite the high inflation rates that they imply. The reason is that the insurance against adverse aggregate events makes bonds attractive for agents even if they pay negative rates of return on average.

8 Equilibrium Multiplicity: Safe Asset Tautology and Loss of Safe Asset Status

If government debt has a bubble component, our selected equilibrium is not unique. Some of the alternative equilibria feature a smaller bubble or no bubble at all attached to government debt.

In this section, we clarify the relationship between bubbles and safe assets. The key takeaway is that for a bubbly safe asset – as opposed to a fundamentally safe asset – the safe asset status can be fragile and may be lost when the bubble pops. Drawing on results from Brunnermeier et al. (2021a), we also briefly summarize when bubbles can exist, how alternative model equilibria look like, and how government policy can ensure that our selected equilibrium with a public debt bubble is in fact the only possibility.

Bubbles and Safe Assets. While bubbles and safe assets are distinct concepts, there is a complementarity between bubbles and the negative β property of safe assets:

First, low- β assets can sustain bubbles more easily. A bubble on an asset is possible if, in the buy and hold perspective, the discounted terminal value does not necessarily vanish in the limit. For example, in the case of government debt, the terminal value may

not vanish if $\mathbb{E} [\bar{\xi}_T \mathcal{B}_T / \mathcal{P}_T]$ does not necessarily converge to zero as $T \rightarrow \infty$ (compare equation (10)). In the long-run, the value of government debt $\mathcal{B}_T / \mathcal{P}_T$ grows on average at the same rate as the aggregate economy, so that the discounted terminal value grows on average at the rate

$$g - r^f - \text{risk premium on gov. debt}$$

where g is the average growth rate of the economy and r^f is the average risk-free rate. The (average) risk premium of government debt scales linearly with its (average) β , at least approximately. Hence, the smaller is the β , the larger is the growth rate of the discounted terminal value and the easier it is to sustain a bubble on government debt. The same argument applies of course also to other assets than government debt.

Second, a bubble component can lower an asset's β and thereby make it safer. To understand this point, take again government debt as an example but now consider the dynamic trading perspective, equation (13). Suppose the situation is as depicted in Figure 3 with a cash flow component that has a positive β and a service flow component that has a negative β .⁴⁸ The relative contribution of the two components to the overall value of government debt determines the size and sign of the asset's β . The service flow component is proportional to the market value of government debt whereas the cash flow component does not directly depend on it. A bubble component raises the market value of the debt and thus increases the relative contribution of the service flow component, which lowers the asset's β .

The previous discussion implies that the safe asset status can be bubbly. An asset whose cash flow component has a sufficiently large β may be safe in some equilibria but not safe in others. Specifically, in an equilibrium in which the asset has a (sufficiently large) bubble component, the service flow component dominates, it has a negative total β and its required risk-adjusted rate of return is low, such that the asset can easily sustain the bubble. In a different equilibrium without a bubble component, the cash flow component dominates, the total β is positive and, as a result, the risk-adjusted rate of return so large that the asset does not appear to be able to sustain bubbles.

Bubbly safe assets give rise to the *Safe Asset Tautology*: the asset is safe in a given equilibrium because it is perceived to be safe, but there are alternative equilibria in

⁴⁸As we have explained previously, this is the relevant case to make sense of the empirical facts from the perspective of our theory.

which the asset is not safe. In these alternative equilibria, the asset has a positive β and is therefore not a good friend after negative aggregate shocks. When government debt is merely a bubbly safe asset, the safe asset status can be fragile. If markets coordinate on a different equilibrium, the bubble bursts and the government loses the safe asset status together with the fiscal space it implies.

In the remainder of this section, we briefly discuss how the government can avoid the fragility of bubbly safe assets, either by creating a fundamentally safe asset and foregoing the benefits of bubble mining or by using its ability to tax and impose regulations to select a unique bubble equilibrium. Both cases lead to the equilibrium selection we have made in previous sections.

Bubbly versus Fundamentally Safe Government Debt. Loosely speaking, rational bubbles can only exist if a suitably risk-adjusted discount rate is below the growth rate of the economy, at least on average. In our environment, a precautionary savings motive depresses discount rates and can make bubbles possible. But importantly, it depends on government policy how strong this precautionary savings motive is. If the government makes bonds more attractive by raising higher primary surpluses (equivalently, by lowering $\tilde{\mu}_t^B$), the bond wealth share ϑ_t increases, idiosyncratic risk sharing improves and the precautionary motive is dampened.

A government that makes its debt very attractive can therefore raise discount rates sufficiently above the growth rate to eliminate any space for rational bubbles.⁴⁹ Under such a policy, government debt can still be a safe asset, however. In this case, government debt is a fundamentally safe asset whose safe asset status does not require the continued belief of market participants in its safety, unlike for bubbly safe assets.

The simplest way to retain a safe asset status in the absence of a bubble is for the government to give up insurance of tax payers in recessions and make the β of the cash-flow component negative. Alternatively, there may still be a sufficient amount of remaining idiosyncratic risk for the counter-cyclical service flow component to be substantial even though the precautionary savings motive is not large enough to sustain bubbles. In this case, even a mildly pro-cyclical surplus process may be consistent with safe government debt in the absence of bubbles.

⁴⁹An immediate question is whether such a policy always exists. The answer is positive (Brunnermeier et al., 2021a, Section 6.1): a policy that generates primary surpluses that exceed a fixed (arbitrarily small) positive fraction of output in every state is always sufficient.

The Structure of Alternative Equilibria. If idiosyncratic risk is sufficiently large and in the absence of fiscal policies that ensure sufficient risk sharing, rational bubbles are possible and a number of alternative equilibria can exist.

Even if we abstract from bubbles on other assets than government debt, there are always equilibria in which the aggregate bubble is smaller than in our selected equilibrium. First, there is always also a no bubble equilibrium. This is easiest to see if the government plans to never generate positive surpluses by choosing a nonnegative $\check{\mu}_t^B$ throughout. If agents no longer believe that they can pass on the debt to someone else in the future, then it becomes worthless for them today, q^B drops to zero and the government does not collect any revenue by issuing more bonds. Second, there are many inflationary equilibria in between the stationary bubble equilibrium and the no bubble equilibrium. In all of these, the initial bubble is smaller than in the stationary bubble equilibrium and its value shrinks over time, so that it disappears asymptotically. All these equilibria correspond to alternative solutions to equation (9) than the one we have implicitly selected so far.

In addition, it is also possible to extend the model to allow for equilibria on assets other than government debt. E.g., private agents may be able to issue long-lived liabilities that circulate as bubbles or face a relaxed no-Ponzi constraint that allows them to run Ponzi schemes by indefinitely rolling over short-term liabilities. Such an extended model features an even richer set of possible alternative equilibria with regard to the distribution of the aggregate bubble on individual assets and the distribution of rents from bubble creation. Nevertheless, so long as agents do not face the prospects of idiosyncratic bubble creation opportunities in the future,⁵⁰ all these bubbly model equilibria lead to the same positive predictions for aggregates with the exception that private bubbles transfer bubble mining seigniorage away from the government to private agents. In each of these alternative bubbly equilibria, fiscal space is therefore lower than in the corresponding equilibrium with the same aggregate bubble that is fully attached to government debt. However, if the government imposes a time-0 lump sum tax whose aggregate value equals the present value of private sector bubble mining revenues, uses the proceeds to purchase private assets, and holds onto its original plans for spending g , taxes τ_t , and adjusted bond growth $\check{\mu}_t^B$,⁵¹ then the resulting equilibrium

⁵⁰Idiosyncratic bubble creation opportunities that cannot be contracted on ex ante introduce an additional source of uninsurable idiosyncratic risk and thereby affect aggregate safe asset demand.

⁵¹The government then has to trade in claims held against the private sector to satisfy its flow budget constraint.

looks precisely like the one in which the aggregate bubble is on government bonds.⁵² In this sense, the equilibria with public debt bubbles are representative for the behavior of aggregates in our model.

Selecting the Public Debt Bubble. When multiple equilibria are possible, a government with access to a richer set of policy tools than considered in this paper may still be able to select a unique equilibrium even in the presence of equilibrium bubbles. Brunnermeier et al. (2021a) analyze in a closely related model environment how fiscal policy and asset regulation can affect the set of possible equilibria and explain why such policy may be credible even under limited commitment.

The government can impose a number of specific policy measures that target bubbles on alternative assets or give an advantage to government bonds: it can eliminate private Ponzi schemes by imposing no Ponzi conditions on private agents through insolvency laws, it can tax competing bubbly assets, and it can use financial repression tools such as reserve and liquidity requirements to generate additional demand for its own liabilities. But most importantly, if a government can credibly commit to create a fundamentally safe asset off equilibrium by raising surpluses, it can both eliminate the no bubble/shrinking bubble equilibria and ensure that the bubble always remains attached to its own debt. Hence, a government with sufficient backup fiscal capacity does not need to fear a loss of the safe asset status of its debt.

9 Conclusion

In this paper we have developed a safe asset theory of government debt based on time-varying idiosyncratic insurance service flows generated by trading government bonds. Our model matches properties of US government debt qualitatively and quantitatively and can resolve the empirical puzzles emphasized by Jiang et al. (2019, 2020). The theory also features a novel explanation for the large equity return volatility based on flight to safety into government bonds.

Throughout this paper we have assumed that government bonds are traded on liq-

⁵²The wealth distribution within the private sector may be affected unless individual time 0 lump sum tax liabilities exactly equal the present value of the individual's bubble mining revenues. But these effects on the wealth distribution do not have any impact on model aggregates or the government budget in our model.

uid markets. The bubbly safe asset status rests on this assumption because the service flow that citizens derive from government debt is directly tied to their ability to trade it as they experience adverse shocks. The government through its central bank can engage as market maker of last resort so that citizens can trade the asset facing only small bid-ask spreads. This helps ensuring that government debt retains the safe asset status. Private assets do not enjoy this privilege.

References

- Aït-Sahalia, Yacine**, “Transition Densities for Interest Rate and Other Nonlinear Diffusions,” *Journal of Finance*, 1999, pp. 1361–1395.
- Aiyagari, S. Rao**, “Uninsured Idiosyncratic Risk and Aggregate Saving,” *Quarterly Journal of Economics*, 1994, 109 (3), 659–684.
- **and Ellen R. McGrattan**, “The optimum quantity of debt,” *Journal of Monetary Economics*, 1998, 42 (3), 447–469.
- Angeletos, George-Marios**, “Uninsured Idiosyncratic Investment Risk and Aggregate Saving,” *Review of Economic Dynamics*, 2007, 10 (1), 1–30.
- Bach, Laurent, Laurent E Calvet, and Paolo Sodini**, “Rich pickings? Risk, return, and skill in household wealth,” *American Economic Review*, 2020, 110 (9), 2703–47.
- Bassetto, Marco and Wei Cui**, “The fiscal theory of the price level in a world of low interest rates,” *Journal of Economic Dynamics and Control*, 2018, 89, 5–22.
- Bewley, Truman F.**, “The Optimum Quantity of Money,” in John H. Kareken and Neil Wallace, eds., *Models of Monetary Economies*, Federal Reserve Bank of Minneapolis, 1980, pp. 169–210.
- Blanchard, Olivier**, “Public debt and low interest rates,” *American Economic Review*, 2019, 109 (4), 1197–1229.
- Bloom, Nicholas, Max Floetotto, Nir Jaimovich, Itay Saporta-Eksten, and Stephen J. Terry**, “Really Uncertain Business Cycles,” *Econometrica*, May 2018, 86 (3), 1031–1065.

- Brunnermeier, Markus K. and Yuliy Sannikov**, “The I Theory of Money,” 2016. Working Paper, Princeton University.
- **and** —, “On the Optimal Inflation Rate,” *American Economic Review Papers and Proceedings*, 2016, 106 (5), 484–489.
- , **Luis Garicano, Philip Lane, Marco Pagano, Ricardo Reis, Tanos Santos, David Thesmar, Stijn Van Nieuwerburgh, and Dimitri Vayanos**, “The Sovereign-Bank Diabolic Loop and ESBies,” *American Economic Review Papers and Proceedings*, May 2016, 106 (5), 508–512.
- , **Sam Langfield, Marco Pagano, Ricardo Reis, Stijn Van Nieuwerburgh, and Dimitri Vayanos**, “ESBies: Safety in the Tranches,” *Economic Policy*, 2017, 32 (90), 175–219.
- Brunnermeier, Markus, Sebastian Merkel, and Yuliy Sannikov**, *Lecture Notes on Macro, Money and Finance: A Heterogeneous-Agent Continuous Time Approach* 2020. Princeton University.
- , —, **and** —, “The Fiscal Theory of the Price Level with a Bubble,” 2021. Working Paper, Princeton University.
- , —, **and** —, “A Safe Asset Perspective for an Integrated Policy Framework,” in Stephen J. Davis, Edward S. Robinson, and Bernard Yeung, eds., *The Asian Monetary Policy Forum: Insights for Central Banking*, 2021.
- Caballero, Ricardo J., Emmanuel Farhi, and Pierre-Olivier Gourinchas**, “The Safe Assets Shortage Conundrum,” *Journal of Economic Perspectives*, August 2017, 31 (3), 29–46.
- Constantinides, George M. and Darrell Duffie**, “Asset Pricing with Heterogeneous Consumers,” *Journal of Political Economy*, 1996, 104 (2), 219–240.
- Cooper, Russell W and John C Haltiwanger**, “On the nature of capital adjustment costs,” *The Review of Economic Studies*, 2006, 73 (3), 611–633.
- Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross**, “A Theory of the Term Structure of Interest Rates,” *Econometrica*, 1985, 53 (2), 385–408.
- Dang, Tri Vi, Gary Gorton, and Bengt Holmstrom**, “The Information Sensitivity of a Security,” in “in” 2015.

- , —, **Bengt Holmström, and Guillermo Ordoñez**, “Banks as Secret Keepers,” *American Economic Review*, April 2017, 107 (4), 1005–29.
- Diamond, Peter A.**, “National Debt in a Neoclassical Growth Model,” *American Economic Review*, 1965, 55 (5), 1126–1150.
- DiTella, Sebastian and Robert Hall**, “Risk Premium Shocks Can Create Inefficient Recessions,” 2020. Working paper, Stanford University.
- Duffie, Darrell and Larry G. Epstein**, “Stochastic differential utility,” *Econometrica: Journal of the Econometric Society*, 1992, pp. 353–394.
- Epstein, Larry G. and Stanley E. Zin**, “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework,” *Econometrica*, 1989, 57 (4), 937–969.
- Fagereng, Andreas, Luigi Guiso, Davide Malacrino, and Luigi Pistaferri**, “Heterogeneity and persistence in returns to wealth,” *Econometrica*, 2020, 88 (1), 115–170.
- Feenstra, Robert C, Robert Inklaar, and Marcel P Timmer**, “The next generation of the Penn World Table,” *American economic review*, 2015, 105 (10), 3150–82.
- Gorton, Gary and George Pennachi**, “Financial Intermediaries and Liquidity Creation,” *The Journal of Finance*, 1990, 45 (1), 49–71.
- Greenwood, Robin, Samuel Gregory Hanson, and Jeremy C. Stein**, “The Federal Reserve’s Balance Sheet as a Financial-Stability Tool,” in “Jackson Hole Symposium,” Vol. 1 Federal Reserve Bank of Kansas City Kansas City, KS 2016, pp. 335–397.
- Hassan, Tarek A, Stephan Hollander, Laurence Van Lent, and Ahmed Tahoun**, “Firm-level political risk: Measurement and effects,” *The Quarterly Journal of Economics*, 2019, 134 (4), 2135–2202.
- He, Zhiguo, Arvind Krishnamurthy, and Konstantin Milbradt**, “A model of safe asset determination,” *American Economic Review*, 2019, 109 (4), 1230–62.
- Heaton, John and Deborah J. Lucas**, “Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing,” *Journal of Political Economy*, 1996, 104 (3), 443–487.
- **and Deborah Lucas**, “Portfolio Choice and Asset Prices: The Importance of Entrepreneurial Risk,” *The Journal of Finance*, 2000, 55 (3), 1163–1198.

- and —, “Portfolio choice in the presence of background risk,” *The Economic Journal*, 2000, 110 (460), 1–26.
- Herskovic, Bernard, Bryan Kelly, Hanno Lustig, and Stijn Van Nieuwerburgh**, “The common factor in idiosyncratic volatility: Quantitative asset pricing implications,” *Journal of Financial Economics*, 2016, 119 (2), 249–283.
- Heston, Steven L.**, “A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options,” *The Review of Financial Studies*, 1993, 6 (2), 327–343.
- Jiang, Zhengyang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z. Xiaolan**, “The US Public Debt Valuation Puzzle,” December 2019. NBER Working Paper 26583.
- , —, —, —, and —, “Manufacturing Risk-Free Government Debt,” 2020.
- Kiyotaki, Nobuhiro and John Moore**, “Liquidity, Business Cycles, and Monetary Policy,” April 2008. mimeo.
- Krishnamurthy, Arvind and Annette Vissing-Jorgensen**, “The Aggregate Demand for Treasury Debt,” *Journal of Political Economy*, 2012, 120 (2), 233–267.
- Li, Ziang and Sebastian Merkel**, “Flight-to-Safety in a New Keynesian Model,” 2020. Unpublished.
- Martin, Alberto and Jaume Ventura**, “The macroeconomics of rational bubbles: a user’s guide,” *Annual Review of Economics*, 2018, 10, 505–539.
- Merkel, Sebastian**, “The Macro Implications of Narrow Banking: Financial Stability versus Growth,” 2020. Working Paper, Princeton University.
- Miao, Jianjun**, “Introduction to economic theory of bubbles,” *Journal of Mathematical Economics*, 2014, 53, 130–136.
- Reis, Ricardo**, “The constraint on public debt when $r < g$ but $g < m$,” 2021. BIS Working Paper No 939.
- Samuelson, Paul A.**, “An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money,” *Journal of Political Economy*, 1958, 66 (6), 467–482.

Tella, Sebastian Di, "Risk premia and the real effects of money," *American Economic Review*, 2020, 110 (7), 1995–2040.

Tirole, Jean, "Asset Bubbles and Overlapping Generations," *Econometrica*, 1985, 53 (5), 1071–1100.

A Appendix

A.1 Omitted Steps in the Derivation of the Model Solution

Hamiltonian of the Household Problem. We use the stochastic maximum principle to derive optimal choice conditions for the household problem. The Hamiltonian for the problem is (using the return expressions stated in the main text)

$$\begin{aligned}
 H_t^i = e^{-\rho t} \log c_t^i + \zeta_t^i \left[-c_t^i + n_t^i \left(\frac{\mathbb{E}_t[dr_t^B]}{dt} + \theta_t^{K,i} \left(\frac{\mathbb{E}_t[dr_t^{K,i}(i_t)]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} \right) \right. \right. \\
 \left. \left. + \theta_t^{E,i} \left(\frac{\mathbb{E}_t[dr_t^{E,i}]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} \right) + \theta_t^{\bar{E},i} \left(\frac{\mathbb{E}_t[dr_t^{\bar{E}}]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} \right) \right) \right] \\
 - \zeta_t^i \zeta_t^i n_t^i \left(\sigma_t^{q,B} - \left(\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i} \right) \frac{\sigma_t^\theta}{1 - \theta_t} \right) - \zeta_t^i \zeta_t^i \left(\theta_t^{K,i} + \theta_t^{E,i} \right) \tilde{\sigma}_t,
 \end{aligned}$$

where we have used $\sigma_t^{q,K} - \sigma_t^{q,B} = \frac{\sigma_t^\theta}{1 - \theta_t}$. Here ζ_t^i denotes the costate (“Lagrange multiplier”) for the net worth evolution (4), and we write the loadings of ζ_t^i with respect to the Brownian motions dZ_t and $d\tilde{Z}_t^i$ as $-\zeta_t^i \zeta_t^i$ and $-\zeta_t^i \tilde{\zeta}_t^i$, respectively.⁵³

As this is a standard portfolio choice problem, we conjecture that the value function of the problem inherits the functional form of the utility function, i.e. $V_t(n^i) = v_t + \frac{1}{\rho} \log n^i$, where v_t depends on (aggregate) investment opportunities, but not on individual net worth n^i .⁵⁴ The usual relationship between the value function and the costate, $\zeta_t^i = e^{-\rho t} V_t'(n^i)$ then implies $\zeta_t^i = e^{-\rho t} / (\rho n_t^i)$, which we can use to eliminate ζ_t^i from the Hamiltonian H_t^i .

By the stochastic maximum principle, the optimal choice $(c_t^i, l_t^i, \theta_t^{K,i}, \theta_t^{E,i}, \theta_t^{\bar{E},i})$ must maximize the Hamiltonian subject to the skin-in-the-game constraint (5).

Optimal Consumption and Investment Choice. Taking first-order conditions with respect to c_t^i and l_t^i yields the two equations

$$c_t^i = \rho n_t^i, \tag{25}$$

⁵³We use the same notation ζ_t^i for the costate in the household problem and the SDF in Section 3 because the two in fact coincide.

⁵⁴The verification argument for this conjecture is entirely standard, see e.g. Brunnermeier et al. (2021a), Appendix A.2.

$$\frac{d}{dt} \mathbb{E}_t \left[dr_t^{K,i}(\iota) \right] \Big|_{\iota=\iota_t^i} = 0.$$

The first equation is precisely the optimal consumption condition stated in the main text. The second equation implies the Tobin's q condition stated in the main text once we take the derivative in the explicit formula for $dr_t^{K,i}(\iota)$ stated in Section 2.2. We reproduce the Tobin's q condition here for the convenience of the reader:⁵⁵

$$q_t^K = 1 + \phi \iota_t. \quad (26)$$

Derivation of Equations (6), (7), and (8). Integrating the optimal consumption condition (25) across all households i yields

$$C_t = \int c_t^i di = \rho \int n_t^i di = \rho (q_t^B + q_t^K) K_t,$$

where the last equality follows from the fact that aggregate net worth consists precisely of all capital and bond wealth combined.⁵⁶

We next use $q_t^B + q_t^K = q_t^K / (1 - \vartheta_t)$ by the definition of ϑ_t to replace the right-hand side of the previous equation.

$$C_t = \frac{\rho}{1 - \vartheta_t} q_t^K K_t.$$

Substituting this into goods market clearing (2), canceling K_t , and using equation (26) to eliminate q_t^K yields the equation

$$\frac{\rho}{1 - \vartheta_t} (1 + \phi \iota_t) + \mathfrak{g} + \iota_t = a_t.$$

This is a simple linear equation for ι_t . Solving it implies equation (6) as stated in the main text. Equation (8) can then be recovered by substituting the resulting expression for ι back into equation (26). Finally, equation (7) follows by exploiting the relationship $q_t^B = \frac{\vartheta_t}{1 - \vartheta_t} q_t^K$, which is a direct consequence of the definition of ϑ_t .

⁵⁵We have already dropped the i superscript on ι because all households choose the same investment rate as argued in the main text.

⁵⁶Note that in our formulation, taxes are effectively imposed on capital holdings such that the present value of tax liabilities of households is implicitly capitalized in capital valuations. Also note that outside equity claims are in zero net supply and thus do not contribute to aggregate net worth.

Optimal Portfolio Choice. The first-order conditions for maximizing the Hamiltonian with respect to the portfolio shares $\theta_t^{K,i}$, $\theta_t^{E,i}$, and $\theta_t^{\bar{E},i}$ yields three Merton portfolio choice equations

$$\begin{aligned}\frac{\mathbb{E}_t[d r_t^{K,i}(i_t^i)]}{dt} - \frac{\mathbb{E}_t[d r_t^B]}{dt} &= -\zeta_t^i \frac{\sigma_t^\vartheta}{1 - \vartheta_t} + \tilde{\zeta}_t^i \tilde{\sigma}_t - \lambda_t^i (1 - \bar{\chi}), \\ \frac{\mathbb{E}_t[d r_t^{E,i}]}{dt} - \frac{\mathbb{E}_t[d r_t^B]}{dt} &= -\zeta_t^i \frac{\sigma_t^\vartheta}{1 - \vartheta_t} + \tilde{\zeta}_t^i \tilde{\sigma}_t - \lambda_t^i, \\ \frac{\mathbb{E}_t[d \bar{r}_t^E]}{dt} - \frac{\mathbb{E}_t[d r_t^B]}{dt} &= -\zeta_t^i \frac{\sigma_t^\vartheta}{1 - \vartheta_t}.\end{aligned}$$

Here, λ_t^i is a scaled Lagrange multiplier on the constraint (5) (skin-in-the-game constraint). Combining the last two equations and using $\frac{\mathbb{E}_t[d \bar{r}_t^E]}{dt} = \frac{\mathbb{E}_t[d r_t^{E,i}]}{dt}$ in equilibrium, we obtain a simple characterization of λ_t^i :

$$\lambda_t^i = \tilde{\zeta}_t^i \tilde{\sigma}_t.$$

As we will show below, $\tilde{\zeta}_t^i$ is always positive and so the constraint (5) must always be binding – households issue the maximum possible amount of outside equity. In particular,

$$\theta_t^{K,i} + \theta_t^{E,i} = \theta_t^{K,i} \bar{\chi}. \quad (27)$$

We now perform two substitutions in the first portfolio choice condition stated above. First, we eliminate λ_t^i on the right-hand side using the previously derived equation. Second, we plug in the expected return expressions implied by the return equations stated in Section 2.2. The condition then becomes

$$\frac{a_t - \mathfrak{g}_t - \iota_t}{q_t^K} - \frac{\mu_t^\vartheta - \check{\mu}_t^B}{1 - \vartheta_t} - \frac{(\sigma_t^{q,B} - \sigma_t^\vartheta) \sigma_t^\vartheta}{1 - \vartheta_t} = -\zeta_t^i \frac{\sigma_t^\vartheta}{1 - \vartheta_t} + \tilde{\zeta}_t^i \bar{\chi} \tilde{\sigma}_t. \quad (28)$$

Characterizing the Costate Volatility Loadings ζ_t^i and $\tilde{\zeta}_t^i$. To determine the values of ϑ_t^i and $\tilde{\zeta}_t^i$ in the previous equations, recall that, by definition, $-\zeta_t^i \zeta_t^i$ and $-\tilde{\zeta}_t^i \tilde{\zeta}_t^i$ are the loadings of $d\zeta_t^i$ with respect to dZ_t and $d\tilde{Z}_t^i$, respectively. We can use $\zeta_t^i = e^{-\rho t} / (\rho n_t^i)$ to conclude that

$$\zeta_t^i = \sigma_t^{n,i}, \quad \tilde{\zeta}_t^i = \tilde{\sigma}_t^{n,i},$$

where $\sigma_t^{n,i}$ and $\tilde{\sigma}_t^{n,i}$ are the (geometric) volatility loading of net worth n_t^i for aggregate and idiosyncratic risk, respectively. The net worth evolution (4) combined with the return expressions stated in Section 2.2 furthermore implies⁵⁷

$$\sigma_t^{n,i} = \sigma_t^{q,B} - \left(\theta_t^{K,i} + \theta_t^{E,i} + \theta_t^{\bar{E},i} \right) \frac{\sigma_t^\vartheta}{1 - \vartheta_t}, \quad \tilde{\sigma}_t^{n,i} = \left(\theta_t^{K,i} + \theta_t^{E,i} \right) \tilde{\sigma}_t.$$

We now eliminate the equity portfolio weights. For idiosyncratic volatility, we can use equation (27) to eliminate $\theta_t^{E,i}$. For aggregate volatility, we use that in equilibrium all agents face the same portfolio conditions (27) and (28) and thus optimally choose the same portfolio allocation $\theta_t^{K,i}$, $\theta_t^{E,i}$, $\theta_t^{\bar{E},i}$ (i.e., these quantities do not depend on i). Market clearing in the outside equity market then implies $\theta_t^{E,i} = -\theta_t^{\bar{E},i}$, which allows us to eliminate the sum $\theta_t^{E,i} + \theta_t^{\bar{E},i}$ in the aggregate risk loading.

By combining all equations, we obtain

$$\zeta_t^i = \sigma_t^{q,B} - \theta_t^{K,i} \frac{\sigma_t^\vartheta}{1 - \vartheta_t}, \quad \tilde{\zeta}_t^i = \theta_t^{K,i} \tilde{\chi} \tilde{\sigma}_t. \quad (29)$$

Derivation of Equation (9). We start from the portfolio choice condition (28), substitute in the costate volatility loadings as stated in equation (29), use that all households choose identical portfolios together with capital market clearing $\theta_t^{K,i} = 1 - \vartheta_t$ as well as the fact that $q_t^K = (1 - \vartheta_t)(q_t^B + q_t^K)$, and rearrange:

$$\frac{1}{1 - \vartheta_t} \frac{a_t - \mathfrak{g} - \iota_t}{q_t^B + q_t^K} - \frac{\mu_t^\vartheta - \check{\mu}_t^B}{1 - \vartheta_t} = (1 - \vartheta_t) \tilde{\chi}^2 \tilde{\sigma}_t^2.$$

By goods market clearing, the second factor in the first term on the left equals ρ . Solving the resulting equation for μ_t^ϑ yields

$$\mu_t^\vartheta = \rho + \check{\mu}_t^B - (1 - \vartheta_t)^2 \tilde{\chi}^2 \tilde{\sigma}_t^2.$$

Equation (9) follows from this equation by multiplying both sides by ϑ_t and using $\mathbb{E}_t[d\vartheta_t] = \mu_t^\vartheta \vartheta_t$.

⁵⁷Compare also the Hamiltonian stated in the beginning of this appendix. There, the same expressions enter because n_t^i is a controlled state variable.

A.2 Proof of Proposition 1

Note that, in general, because $\tilde{\zeta}_t^i = e^{-\rho t} / (\rho n_t^i)$ and $dr_t^{n,i}$ has the same stock loadings as dn_t^i/n_t^i (compare equation (4)),

$$\text{Cov}_t \left(\frac{d\tilde{\zeta}_t^i}{\tilde{\zeta}_t^i}, dr_t^{n,i} \right) = \text{Cov}_t \left(\frac{d(1/n_t^i)}{1/n_t^i}, \frac{dn_t^i}{n_t^i} \right) = - \left((\sigma_t^{n,i})^2 + (\tilde{\sigma}_t^{n,i})^2 \right)$$

Under the assumptions of the proposition, ϑ_t and thus prices q_t^K, q_t^B do not load on the aggregate shock dZ_t , $\sigma_t^\vartheta = \sigma_t^{q,B} = \sigma_t^{q,K} = 0$. In particular, $\sigma_t^{n,i} = 0$ (compare equation (29) and recall that prices of risk and net worth loadings coincide). Thus

$$\text{Cov}_t \left(\frac{d\tilde{\zeta}_t^i}{\tilde{\zeta}_t^i}, dr_t^{n,i} \right) = \text{Cov}_t \left(\frac{d(1/n_t^i)}{1/n_t^i}, \frac{dn_t^i}{n_t^i} \right) = - (\tilde{\sigma}_t^{n,i})^2 = -(1 - \vartheta)^2 \bar{\chi}^2 \tilde{\sigma}^2 < 0,$$

where the last equation follows from equation (29) and market clearing for $\theta_t^{K,i}$.

In contrast, $\sigma_t^{q,B} = 0$ implies that dr_t^B is locally deterministic (does not load on Brownian shocks), so that

$$\text{Cov}_t \left(\frac{d\tilde{\zeta}_t^i}{\tilde{\zeta}_t^i}, dr_t^B \right) = 0.$$

Comparing the two covariances reveals that the former is always strictly smaller. Thus the bond is a safe asset for agent i at all times t .

A.3 Omitted Steps in the Derivation of Equation (13)

We present here the missing steps in the derivation of equation (13) left out in the main text. There, we have used without proof equation (14), which expresses the value of the bond portfolio as the present value of trading cash flows, and equation (15), which characterizes the bond trading process. In addition, we have used that the price of idiosyncratic risk is given by

$$\tilde{\zeta}_t^i = (1 - \vartheta_t) \bar{\chi} \tilde{\sigma}_t.$$

This equation has already been derived in Appendix A.1 (compare equation (29) and the market clearing equation for $\theta_t^{K,i}$ stated in the subsequent paragraph).⁵⁸ We thus only derive equations (14) and (15) here.

Derivation of equation (14). We can write the evolution of the bond portfolio as

$$\frac{db_t^i}{b_t^i} = dr_t^B + d\Delta_t^{b,i}. \quad (30)$$

Absent trading and payouts, the bond portfolio grows at the (stochastic) bond return dr_t^B , but the actual portfolio value has to be adjusted for cash inflows $b_t^i d\Delta_t^{b,i}$ due to trading and payouts. Writing

$$dr_t^B = \mu_t^{r^B} dt + \sigma_t^{r^B} dZ_t, \quad d\Delta_t^{b,i} = \mu_t^{\Delta,i} dt + \sigma_t^{\Delta,i} dZ_t + \tilde{\sigma}_t^{\Delta,i} d\tilde{Z}_t^i$$

and using Ito's product rule, we obtain for the discounted bond wealth

$$\begin{aligned} \frac{d(\tilde{\zeta}_t^i b_t^i)}{\tilde{\zeta}_t^i b_t^i} = & \left(\underbrace{\mu_t^{r^B} - r_t^f - \zeta_t \sigma_t^{r^B}}_{=0} + \mu_t^{\Delta,i} - \zeta_t \sigma_t^{\Delta,i} - \tilde{\zeta}_t^i \tilde{\sigma}_t^{\Delta,i} \right) dt \\ & + \left(\sigma_t^{r^B} + \sigma_t^{\Delta,i} - \zeta_t \right) dZ_t + \left(\tilde{\sigma}_t^{\Delta,i} - \tilde{\zeta}_t^i \right) d\tilde{Z}_t^i. \end{aligned} \quad (31)$$

Here, the first part of the drift is zero by standard asset pricing logic because agent i is marginal in the market for government bonds. Integrating over $t \in [0, T]$, taking expectations, and rearranging yields

$$\tilde{\zeta}_0^i b_0^i = -\mathbb{E}_0 \left[\int_0^T \tilde{\zeta}_t^i b_t^i \left(\mu_t^{\Delta,i} - \zeta_t \sigma_t^{\Delta,i} - \tilde{\zeta}_t^i \tilde{\sigma}_t^{\Delta,i} \right) dt \right] + \mathbb{E}_0 \left[\tilde{\zeta}_T^i b_T^i \right].$$

Optimal behavior implies a transversality condition $\lim_{T \rightarrow \infty} \mathbb{E} \left[\tilde{\zeta}_T^i n_T^i \right] = 0$ on total wealth n_T^i of agent i as a necessary choice condition. Because total wealth consists of bond wealth and capital wealth and the latter cannot become negative, a transversality condition for bond wealth b_T^i immediately follows. Consequently, the second term converges to zero as $T \rightarrow \infty$ and we obtain equation (14) in the limit.

⁵⁸Note that we have defined $\tilde{\zeta}_t^i$ in Appendix A.1 as the costate in the household's problem whereas we use the same notation for the individual SDF in Section 3. The two are the same (up to scaling) because both measure the marginal utility of an additional unit of wealth at time t in a given state.

Derivation of equation (15). To characterize the trading process $d\Delta_t^{b,i}$, start from (30):

$$d\Delta_t^{b,i} = \frac{db_t^i}{b_t^i} - dr_t^B. \quad (32)$$

Because all agents hold the same fraction $\theta_t^i = \vartheta_t$ of their net worth in bonds, we have $b_t^i = \eta_t^i q_t^B K_t$. As η_t^i loads only on the idiosyncratic Brownian and $q_t^B K_t$ only on the aggregate Brownian, their quadratic covariation vanishes and thus Ito's product rule simply implies

$$\frac{db_t^i}{b_t^i} = \frac{d(q_t^B K_t)}{q_t^B K_t} + \frac{d\eta_t^i}{\eta_t^i}.$$

Furthermore, the return on bonds can be written as (compare equation (3))

$$dr_t^B = \frac{d(q_t^B K_t)}{q_t^B K_t} - \check{\mu}_t^B dt.$$

Substituting the previous two equations into (32) yields

$$d\Delta_t^{b,i} = \check{\mu}_t^B dt + \frac{d\eta_t^i}{\eta_t^i} = \check{\mu}_t^B dt + \sigma_t^{\eta,i} d\tilde{Z}_t^i,$$

which implies

$$\mu_t^{\Delta,i} = \check{\mu}_t^B, \quad \sigma_t^{\Delta,i} = 0, \quad \tilde{\sigma}_t^{\Delta,i} = \sigma_t^{\eta,i}.$$

The equations in formula (15) follow, if we can show that

$$\check{\mu}_t^B = -s_t / q_t^B, \quad (33)$$

$$\sigma_t^{\eta,i} = (1 - \vartheta_t) \bar{\chi} \tilde{\sigma}_t. \quad (34)$$

Equation (33) follows immediately from the government budget constraint (1) and the definition of s_t . For the proof of equation (34), note that individual net worth n_t^i and total net worth $N_t := (q_t^K + q_t^B)K_t$ have identical drifts and volatility loadings on the aggregate Brownian dZ_t , so that simply

$$\frac{d\eta_t^i}{\eta_t^i} = \frac{d(n_t^i / N_t)}{n_t^i / N_t} = \tilde{\sigma}_t^{n,i} d\tilde{Z}_t^i$$

because n_t^i loads on the idiosyncratic Brownian $d\tilde{Z}_t^i$, but N_t does not. Combining the

net worth evolution (4) with the equilibrium portfolio weights, we obtain

$$\tilde{\sigma}_t^{n,i} = (1 - \vartheta_t) \tilde{\chi} \tilde{\sigma}_t \quad (35)$$

which completes the proof of (34).

A.4 Model Solution with Stochastic Differential Utility

The model setup is identical to the one described in Section 2, except that logarithmic preferences are replaced with the utility recursion

$$V_t^i = \mathbb{E}_t \left[\int_t^\infty f(c_s^i, V_s^i) ds \right],$$

where the aggregator f is defined by

$$f(c, V) = (1 - \gamma) \rho V \left(\log(c) - \frac{1}{1 - \gamma} \log((1 - \gamma) V) \right)$$

We can solve this augmented model as we have solved the baseline model in Section 2.2 (compare also Appendix A.1). The Hamiltonian of the household problem is precisely as stated in Appendix A.1, except that the very first term $e^{-\rho t} \log c_t^i$ must be replaced with $f(c^i, V_t(n^i))$.⁵⁹

We use again a standard guess for the value function to eliminate the costate variable from the Hamiltonian. The guess here is $V_t(n^i) = v_t \frac{(n^i)^{1-\gamma}}{1-\gamma}$, where v_t is, again, a variable that does not depend on individual net worth. The relationship between the value function and the costate requires $\zeta_t^i = V_t'(n_t^i) = v_t (n_t^i)^{-\gamma}$.⁶⁰ We write μ_t^v and σ_t^v for the (geometric) drift and aggregate volatility of v_t . Note that v_t does not load on the idiosyncratic Brownian because it merely depends on aggregate conditions.

The model solution procedure follows the same steps as for the baseline model. Here, we merely highlight the differences that occur on the way.

⁵⁹On a small technical note, the resulting Hamiltonian here is a “current value Hamiltonian” whereas the one used in Appendix A.1 is a “present value Hamiltonian”. The costate must thus be discounted differently here. Otherwise, this does not affect the solution procedure.

⁶⁰It is here that the difference between “present value” and “current value” matters. For this reason, there is no time discounting term (such as $e^{-\rho t}$) in this equation, unlike in Appendix A.1.

The first difference is that the first-order condition for optimal consumption is not immediately equation (25), but instead of the more complicated form

$$v_t(n_t^i)^{-\gamma} = \partial_c f(c_t, V_t) = (1 - \gamma)\rho \frac{V_t}{c_t}.$$

However, once the value function $V_t = v_t \frac{(n_t^i)^{1-\gamma}}{1-\gamma}$ is plugged in, the condition reduces again to the familiar form of equation (25).

The second difference is in the characterization of the costate volatility loadings ζ_t^i and $\tilde{\zeta}_t^i$. Because the costate is now $\tilde{\zeta}_t^i = v_t(n_t^i)^{-\gamma}$, Ito's lemma implies

$$\zeta_t^i = \gamma \sigma_t^{n,i} - \sigma_t^v, \quad \tilde{\zeta}_t^i = \gamma \tilde{\sigma}_t^{n,i}. \quad (36)$$

The net worth volatilities $\sigma_t^{n,i}$ and $\tilde{\sigma}_t^{n,i}$ take the same form as before such that we simply need to replace the final equation (29) with the slightly more complicated form

$$\zeta_t^i = \gamma \left(\sigma_t^{q,B} - \theta_t^{K,i} \frac{\sigma_t^\vartheta}{1 - \vartheta_t} \right) - \sigma_t^v, \quad \tilde{\zeta}_t^i = \gamma \theta_t^{K,i} \tilde{\chi} \tilde{\sigma}_t.$$

The third difference is that the modified expressions for ζ_t^i and $\tilde{\zeta}_t^i$ affect the derivation and final result of equation (9). Following the same steps as in Appendix A.1, we obtain the slightly modified equation

$$\mathbb{E}_t [d\vartheta_t] = \left(\rho + \check{\mu}_t^B - \left(\sigma_t^v - (\gamma - 1) \sigma_t^{\bar{q}} \right) \sigma_t^\vartheta - \gamma (1 - \vartheta_t)^2 \tilde{\chi}^2 \tilde{\sigma}_t^2 \right) \vartheta_t dt,$$

where $\sigma_t^{\bar{q}}$ is the volatility of $\bar{q}_t := q_t^B + q_t^K$.

The fourth and final difference is that we now also have to characterize the process v_t as it affects the BSDE for ϑ_t through the term σ_t^v .⁶¹ To characterize v_t , we start from the costate equation (a necessary optimality condition by the stochastic maximum principle), which is here given by

$$\mathbb{E}_t [d\tilde{\zeta}_t^i] = - \left(\partial_V f(c_t^i, V_t^i) \tilde{\zeta}_t^i + \frac{\partial H_t^i}{\partial n_t^i} \right) dt$$

⁶¹There is no need to solve for v_t in the baseline model because there it enters the value function additively and thus only impacts total utility but not optimal choices.

$$\begin{aligned}
&= - \left((1 - \gamma)\rho \log(c_t^i/n_t^i) - \rho \log v_t^i - \rho + \mu_t^{n,i} + \frac{c_t^i}{n_t^i} - \zeta_t^i \sigma_t^{n,i} - \tilde{\zeta}_t^i \tilde{\sigma}_t^{n,i} \right) \tilde{\zeta}_t^i dt \\
&= - \left((1 - \gamma)\rho \log \rho - \rho \log v_t^i + \mu_t^{n,i} - \left(\gamma \sigma_t^{n,i} - \sigma_t^v \right) \sigma_t^{n,i} - \gamma \left(\tilde{\sigma}_t^{n,i} \right)^2 \right) \tilde{\zeta}_t^i dt,
\end{aligned} \tag{37}$$

where the last line uses $c_t^i/n_t^i = \rho$ and the price of risk formulas (36). We also know $\tilde{\zeta}_t^i = v_t(n_t^i)^{-\gamma}$ and applying Ito's lemma to this equation yields for the drift term

$$\mathbb{E}_t[d\tilde{\zeta}_t^i] = \left(\mu_t^v - \gamma \mu_t^{n,i} + \frac{\gamma(\gamma+1)}{2} \left(\left(\sigma_t^{n,i} \right)^2 + \left(\tilde{\sigma}_t^{n,i} \right)^2 \right) - \gamma \sigma_t^v \sigma_t^{n,i} \right) \tilde{\zeta}_t^i dt \tag{38}$$

Combining equations (37) and (38) and solving for μ_t^v yields

$$\begin{aligned}
\mu_t^v &= \gamma \mu_t^{n,i} - \frac{\gamma(\gamma+1)}{2} \left(\left(\sigma_t^{n,i} \right)^2 + \left(\tilde{\sigma}_t^{n,i} \right)^2 \right) + \gamma \sigma_t^v \sigma_t^{n,i} \\
&\quad - \left((1 - \gamma)\rho \log \rho - \rho \log v_t^i + \mu_t^{n,i} - \left(\gamma \sigma_t^{n,i} - \sigma_t^v \right) \sigma_t^{n,i} - \gamma \left(\tilde{\sigma}_t^{n,i} \right)^2 \right) \\
&= \rho \log v_t + (\gamma - 1) \left(\rho \log \rho + \mu_t^{n,i} - \frac{\gamma}{2} \left(\left(\sigma_t^{n,i} \right)^2 + \left(\tilde{\sigma}_t^{n,i} \right)^2 \right) + \sigma_t^v \sigma_t^{n,i} \right) \\
&= \rho \log v_t + (\gamma - 1) \left(\rho \log \rho + \mu_t^{\bar{q}} + \Phi(\iota_t) - \delta - \frac{\gamma}{2} \left(\left(\sigma_t^{\bar{q}} \right)^2 + (1 - \vartheta)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \right) + \sigma_t^v \sigma_t^{\bar{q}} \right),
\end{aligned}$$

where in the last line we use that individual net worth has the same drift and aggregate volatility as aggregate net worth $\bar{q}_t K_t$, while its idiosyncratic volatility is $\tilde{\sigma}_t^{n,i}$, as determined previously. The previous equation for μ_t^v leads to a second BSDE

$$\mathbb{E}_t[dv_t] = \mu_t^v v_t dt$$

that has to be solved numerically jointly with the BSDE for ϑ_t stated previously.

Numerical Model Solution. We solve the model numerically using a finite difference method. This is a standard approach employed in the literature to solve models of this type. Here, we only briefly outline the procedure. A more comprehensive description of the method can be found, e.g., in Brunnermeier et al. (2020), Chapter 3 (specifically Sections 3.2.6 and 3.2.7).

For our numerical solution, we impose the functional relationships $\vartheta_t = \vartheta(t, \tilde{\sigma}_t)$,

$v_t = v(t, \tilde{\sigma}_t)$ and use the known forward equation for the state variable $\tilde{\sigma}_t$ to transform the two BSDEs into partial differential equations in time t and the state $\tilde{\sigma}_t$. We choose suitable terminal guesses for the functions ϑ and v ⁶² at a finite terminal time T and solve the two PDEs backward in time using a finite difference method. We choose T sufficiently large such that an increase in T no longer changes the solutions at $t = 0$, $\vartheta(0, \cdot)$ and $v(0, \cdot)$, noticeably. These solution functions $\vartheta(0, \cdot)$ and $v(0, \cdot)$ represent our numerical approximation to the stationary (Markov) equilibrium functions $\tilde{\sigma} \mapsto \vartheta(\tilde{\sigma}), v(\tilde{\sigma})$.

A.5 Omitted Details in the Representative Agent Formulation

Derivation of Utility Representation (18). By Ito's formula,

$$\log \eta_t^i = \log \eta_0^i - \frac{1}{2} \int_0^t (\tilde{\sigma}_s^\eta)^2 ds + \int_0^t \tilde{\sigma}_s^\eta d\tilde{Z}_s^i$$

and thus

$$\begin{aligned} \int_0^\infty e^{-\rho t} \int \lambda^i \mathbb{E} [\log \eta_t^i] didt &= \int_0^\infty e^{-\rho t} \int \lambda^i \log \eta_0^i didt - \frac{1}{2} \int_0^\infty e^{-\rho t} \int \lambda^i \int_0^t (\tilde{\sigma}_s^\eta)^2 ds didt \\ &= \frac{1}{\rho} \int \lambda^i \log \eta_0^i di - \frac{1}{2} \int \lambda^i di \int_0^\infty e^{-\rho t} \int_0^t (\tilde{\sigma}_s^\eta)^2 ds dt \\ &= \frac{1}{\rho} \int \lambda^i \log \eta_0^i di - \frac{1}{2\rho} \int_0^\infty e^{-\rho t} (\tilde{\sigma}_t^\eta)^2 dt, \end{aligned}$$

where the last line uses that $\int \lambda^i di = 1$. Substituting this into equation (17) (with interchanged order of integration where necessary) implies

$$\mathcal{W}_0 = \frac{1}{\rho} \int \lambda^i \log \eta_0^i di + \mathbb{E} \left[\int_0^\infty e^{-\rho t} \left(\log C_t - \frac{1}{2\rho} (\tilde{\sigma}_t^\eta)^2 \right) dt \right]$$

With the definition $w_0 := \frac{1}{\rho} \int \lambda^i \log \eta_0^i di$, this is precisely equation (18).

Competitive Equilibrium in Representative Agent Economy. As this is a representative agent economy, we can fully characterize the allocation by determining goods and asset supplies. The problem of the representative agent only needs to be considered to

⁶²Specifically, we use the functions implied by the steady state equilibrium with $\tilde{\sigma}_t = \tilde{\sigma}^0$ forever.

determine asset prices.

The assumed growth rate process for capital K_t is the same as in the equilibrium of the incomplete markets model, so that K_t must follow precisely the same process as in that equilibrium if we start from the same initial K_0 (which we can assume w.l.o.g. as this only scales the overall size of the economy). Because $dX_t/X_t = d(q_t^B K_t)/(q_t^B K_t)$ and $X_0 = q_0^B X_0$ by the condition on initial supply, we then also have $X_t = q_t^B K_t$ for all t . Total consumption goods produced by the two “trees” in period t are

$$\begin{aligned} C_t &= ((1 - \tau_t) a_t - \iota_t) K_t - \check{\mu}_t^B X_t \\ &= (a_t - \iota_t + \tau_t a_t - \check{\mu}_t^B q_t^B) K_t \\ &= (a_t - \mathfrak{g} - \iota_t) K_t, \end{aligned}$$

where the last line follows from the government budget constraint (1) (in the incomplete markets model). The aggregate consumption goods supply is thus the same as the (endogenous) aggregate consumption process in the incomplete markets economy.

We now turn to the remaining “good”, volatility reduction. Total volatility “supply” is determined by equation (19),

$$\tilde{\sigma}_t^\eta = \frac{q_t^K K_t}{q_t^K K_t + X_t} \bar{\chi} \tilde{\sigma}_t = \frac{q_t^K K_t}{q_t^K K_t + q_t^B K_t} \bar{\chi} \tilde{\sigma}_t = (1 - \vartheta_t) \bar{\chi} \tilde{\sigma}_t.$$

This is also the same as the (endogenous) volatility of consumption shares η_t^i in the incomplete markets economy. The representative agent economy therefore generates the same allocation as the equilibrium in our incomplete markets model.

We now turn to asset prices. As this is the decision problem of a consumer with logarithmic utility, the optimal consumption rule is $C_t = \rho N_t$, exactly as for the agents in our incomplete markets economy.⁶³ This fact can be derived using the stochastic maximum principle in precisely the same way as in Appendix A.1, so that we skip the details here. Using the definition $N_t = Q_t^K K_t + P_t^X X_t$ and the supplies $X_t = q_t^B K_t$, $C_t = (q_t^B + q_t^K) K_t$ derived previously, we obtain

$$(q_t^B + q_t^K) K_t = \frac{C_t}{\rho} = Q_t^K K_t + P_t^X X_t = (Q_t^K + P_t^X q_t^B) K_t.$$

⁶³In the utility function here, there is also a second term ($\tilde{\sigma}_t^\eta$). But because it is additively separated, it does not affect the optimal consumption rule.

Therefore, if we can show $P_t^X = 1$, $Q_t^K = q_t^K$ is automatically implied. $P_t^X = 1$, in turn, follows from equation (21) and the remarks following it in the main text. Consequently, we only need to derive equation (21) to complete the equilibrium characterization.

Valuation Formula (21) for “Derivatives”. We can use standard asset pricing logic. From the perspective of the representative agent, this is an entirely standard complete markets economy with two consumption goods. The price of a single unit of an asset measured in time-zero consumption units must thus equal the sum of the present discounted value of its future marginal consumption flow dividends and the present discounted future consumption value of its marginal volatility flow dividends, both discounted with the SDF Ξ_t , the marginal rate of substitution between consumption at time t and consumption at time 0.

The consumption flow term is straightforward. One unit of derivatives at time 0 turns into X_t/X_0 units of derivatives at time t and each of them produces a consumption flow $-\check{\mu}_t^B dt$. The present discounted value of these future consumption flows is therefore

$$\mathbb{E} \left[\int_0^\infty \Xi_t \left(-\check{\mu}_t^B \frac{X_t}{X_0} \right) dt \right].$$

For the volatility flow term, note that the “marginal volatility product of derivatives” at time t is

$$\frac{\partial \tilde{\sigma}_t^\eta}{\partial X_t} = -\frac{q_t^K K_t}{(q_t^K K_t + X_t)^2} \tilde{\chi} \tilde{\sigma}_t = -\frac{\tilde{\sigma}_t^\eta}{(q_t^K + q_t^B) K_t} = -\frac{\tilde{\sigma}_t^\eta}{N_t}$$

and the marginal rate of substitution between time- t consumption and time- t volatility is

$$\frac{\partial \left(\log C_t - \frac{1}{2\rho} (\tilde{\sigma}_t^\eta)^2 \right) / \partial \tilde{\sigma}_t^\eta}{\partial \left(\log C_t - \frac{1}{2\rho} (\tilde{\sigma}_t^\eta)^2 \right) / \partial C_t} = \frac{-\tilde{\sigma}_t^\eta / \rho}{1/C_t} = -\frac{C_t}{\rho} \tilde{\sigma}_t^\eta.$$

The consumption value of the marginal volatility reduction of X_t/X_0 derivatives at

time t is therefore

$$\frac{\partial \left(\log C_t - \frac{1}{2\rho} \left(\tilde{\sigma}_t^\eta \right)^2 \right) / \partial \tilde{\sigma}_t^\eta}{\partial \left(\log C_t - \frac{1}{2\rho} \left(\tilde{\sigma}_t^\eta \right)^2 \right) / \partial C_t} \cdot \frac{\partial \tilde{\sigma}_t^\eta}{\partial X_t} \cdot \frac{X_t}{X_0} = \frac{C_t}{\rho N_t} \left(\tilde{\sigma}_t^\eta \right)^2 \frac{X_t}{X_0} = \left(\tilde{\sigma}_t^\eta \right)^2 \frac{X_t}{X_0},$$

here the last equation follows from $C_t = \rho N_t$. Consequently, the discounted value of volatility flows generates by one unit of derivatives is

$$\mathbb{E} \left[\int_0^\infty \Xi_t \left(\tilde{\sigma}_t^\eta \right)^2 \frac{X_t}{X_0} ds \right].$$

Combining the two present values and using $\tilde{\sigma}_t^\eta = (1 - \vartheta_t) \bar{\chi} \tilde{\sigma}_t$ (derived previously) yields

$$P_0^X = \mathbb{E} \left[\int_0^\infty \Xi_t \left(-\check{\mu}_t^B \frac{X_t}{X_0} \right) dt \right] + \mathbb{E} \left[\int_0^\infty \Xi_t (1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \frac{X_t}{X_0} ds \right].$$

After multiplying both sides by X_0 , we obtain equation (21).

A.6 Model Extension with Privately Issued Safe Assets

In this appendix, we present the formal details for the model extension with privately issued safe assets. We restrict attention to the baseline model from Section 2 with logarithmic preferences.

Setup and Model Solution. Each agent i issues nominally risk-free bonds (“ i -bonds”) of total real value $B_t(i) \geq 0$ and holds a real quantity $b_t^i(j) \geq 0$ of j -bonds issued by other agents $j \neq i$. The clearing conditions at all times t and for all varieties j are

$$B_t(j) = \int b_t^i(j) di.$$

We denote by i_t^p the nominal interest a household has to pay in equilibrium on its privately issued debt⁶⁴ and by $B_t^p := \int B_t(j) dj$ the aggregate quantity of privately issued

⁶⁴Theoretically, i_t^p could depend on the issuing household j . However, as all privately issued bonds are required to be nominally risk-free, it is obvious that they all have to pay the same nominal rate in equilibrium.

bonds outstanding. Because privately issued debt is nominally risk-free, its return is

$$dr_t^b = \left(i_t^p - i_t\right) dt + dr_t^B,$$

where, as before, dr_t^B is the return on government bonds (compare equation (3)). By no arbitrage, in equilibrium $i_t^p = i_t$. Thus, the yields on privately issued bonds and government bonds are identical.

We can solve household i 's problem as in the baseline model. Denote by $\theta_t^{B,i} := -B_t(i)/n_t^i \leq 0$ the negative of bond issuance as a share of net worth and by $\theta_t^{b,i}(j) := b_t^i(j)/n_t^i \geq 0$ holdings of j -bonds as a fraction of net worth. Relative to the baseline model, the household has the additional choice variables $\theta_t^{B,i}$ and $(\theta_t^{b,i}(j))_{j \in [0,1]}$ subject to the nonnegativity constraints. However, the Hamiltonian of the household's problem does not change relative to Appendix A.1: due to $dr_t^b = dr_t^B$, choices of $\theta_t^{B,i}$ and $(\theta_t^{b,i}(j))_{j \in [0,1]}$ do not affect either the expected return or the risk characteristics of the household's portfolio, such that the additional terms in the Hamiltonian cancel out.

We can draw two immediate conclusions from the previous observation. First, because the Hamiltonian remains unaffected, the model solution steps outlined in Appendix A.1 remain valid in this extended model. Consequently, all equilibria with private bond issuance must feature the same real allocation and the same prices of government bond (q_t^B) and capital (q_t^K) as in the baseline model. Second, all households are indifferent between any choice of private bond issuance and holdings of bonds issued by other agents as long as these holdings do not interfere with the optimal plans for capital holdings ($\theta_t^{K,i}$), outside equity issuance ($\theta_t^{E,i}$), and diversified equity holdings ($\theta_t^{E,i}$).

There are thus many different equilibria that all feature the same consumption allocation and valuation of government bonds, equity, and capital, but differ with regard to the quantities $B_t(j)$ of private bonds in circulation.

A Simple Example. To illustrate how privately issued bonds can serve as safe assets in precisely the same way as government bonds, we consider an example in which all agents trade private and government bonds in equal proportions.⁶⁵ Specifically, we make the following choices: (a) the aggregate real value of privately issued bonds is

⁶⁵While valuation equations for individual bond types depend on what we assume about trading of individual bonds (which is indeterminate due to indifference), none of the economic conclusions from the example crucially depend on this choice.

proportional to the value of government bonds, $B_t^p \propto q_t^B K_t$, (b) the total bonds issued by each agent j is proportional to the agent's net worth share, $B_t(j) = \eta_t^j B_t^p$, and (c) all agents hold a portfolio of j -bonds for $j \neq i$ and government bonds according to market capitalization weights.

We now discuss the debt valuation equations verbally referenced in the main text. We defer a derivation of the following equations to the end of this appendix.

For each agent i , the value of the long position $b_t^i(j)$ in j -bonds must equal the present value of future cash inflows from the portfolio of j -bonds, either due to payments made by agent j or due to trading of j -bonds. This insight leads to an equation in full analogy to equations (14) and (16) for government bonds that we have derived in the context of the dynamic trading perspective:

$$b_0^i(j) = \mathbb{E} \left[\int_0^\infty \zeta_t^i x_t b_t^i(j) dt \right] + \mathbb{E} \left[\int_0^\infty \zeta_t^i (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2 b_t^i(j) dt \right]. \quad (39)$$

Here, x_t denotes the expected net payouts made by agent j to all holders of j -bonds per real unit of j -bonds outstanding. Total expected net payouts $x_t B_t(j)$ made by agent j are the private debt counterparts of primary surpluses $s_t K_t$, which represent the net payouts made by the government to public debt holders.

Equation (39) emphasizes that the valuation of j -bonds for agent i depends on a cash flow component resulting from payouts made by agent j and a service flow component resulting from the fact that i trades j -bonds with agents other than j . When aggregating these equations for all $i \neq j$, we obtain a debt valuation equation from the dynamic trading perspective for the aggregate long position in j -bonds:⁶⁶

$$B_0(j) = \mathbb{E} \left[\int_0^\infty \zeta_t^{**} x_t B_t(j) dt \right] + \mathbb{E} \left[\int_0^\infty \zeta_t^{**} (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2 B_t(j) dt \right]. \quad (40)$$

The key takeaway is that this equation looks precisely like equation (13) for government bonds. In particular, the service flow component is identical.

Equation (40) emphasizes the similarity between government bonds and privately issued bonds for their holders. However, private bond issuance also comes with a short position in the bond for the issuer j . In the same spirit as before, we can value that short position by determining the present value of all net payouts that j makes to holders of

⁶⁶Relative to equation (39), the following equation also interchanges integrals and uses $b_t^i(j) = \eta_t^i B_t(j)$.

j -bonds,

$$-B_0(j) = \mathbb{E} \left[\int_0^\infty \zeta_t^j(-x_t) B_t(j) dt \right] + \mathbb{E} \left[\int_0^\infty \zeta_t^j \left(-(1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2 \right) B_t(j) dt \right]. \quad (41)$$

This equation illustrates that issuing bonds according to the specified issuance strategy effectively exposes the agent to negative service flows. Because $B_t(j) = \eta_t^j B_t^p$ is proportional to η_t^j , cash flows from debt issuance and repayments are systematically correlated with marginal utility in a way that increases the riskiness of j 's portfolio.

Once we integrate equations (40) and (41) over all bond types j , the integrated service flow terms on the right-hand side become identical in absolute value but have opposite sign. In other words, in the aggregate the positive service flows derived from privately issued bonds by their holders exactly cancel with the negative service flows generated for their issuers. Private safe asset creation does not generate additional net service flows for the economy.

Derivation of Equations (39), (40), and (41). In precisely the same way as in Appendix A.3, we can derive equations in analogy to equation (14) for the portfolios of j -bonds held by agents i and j :

$$b_0^i(j) = -\mathbb{E}_0 \left[\int_0^\infty \zeta_t^i b_t^i(j) \left(\mu_t^{\Delta,i}(j) - \zeta_t \sigma_t^{\Delta,i}(j) - \zeta_t \bar{\sigma}_t^{\Delta,i,i}(j) \right) dt \right], \quad (42)$$

$$-B_0^j = -\mathbb{E}_0 \left[\int_0^\infty \zeta_t^j (-B_t^j) \left(\mu_t^{\Delta,j}(j) - \zeta_t \sigma_t^{\Delta,j}(j) - \zeta_t \bar{\sigma}_t^{\Delta,j}(j) \right) dt \right]. \quad (43)$$

Here, $d\Delta^{b,i}(j)_t$ and $d\Delta^{B,j}$ are the trading processes for j -bonds of agents i and j , respectively:

$$\begin{aligned} d\Delta_t^{b,i}(j) &= \mu_t^{\Delta,i}(j) dt + \sigma_t^{\Delta,i}(j) dZ_t + \bar{\sigma}_t^{\Delta,i,i}(j) d\tilde{Z}_t^i + \bar{\sigma}_t^{\Delta,i,j}(j) d\tilde{Z}_t^j, \\ d\Delta_t^{B,j} &= \mu_t^{\Delta,j}(j) dt + \sigma_t^{\Delta,j}(j) dZ_t + \bar{\sigma}_t^{\Delta,j}(j) d\tilde{Z}_t^j. \end{aligned}$$

As in Section 3 and Appendix A.3, $b_t^i(j) d\Delta^{b,i}(j)_t$ represents the real value of new j -bonds purchased by agent i at time t (net of payouts made by agent j on existing bonds). Similarly, but with opposite sign due to the short position, $-B_t^j d\Delta^{B,j}$ represents the real value of new j -bonds (re-)purchased by agent j . In other words, $-d\Delta_t^{B,j}$ corresponds to

the payouts that the issuer j makes to bond holders.

To derive equations (39) and (41), we have to characterize the trading processes. In full analogy to Appendix A.3, these processes must satisfy

$$d\Delta_t^{b,i}(j) = \frac{db_t^i(j)}{b_t^i(j)} - dr_t^b, \quad (44)$$

$$d\Delta_t^{B,j} = \frac{dB_t^j}{B_t^j} - dr_t^b. \quad (45)$$

We first characterize the second process. By definition, $\mu_t^{\Delta,j}(j) = -x_t$ corresponds to the negative of the expected net payouts made by agent j to holders of j -bonds per real unit of bonds outstanding. To determine the volatility loadings of the trading process, we use $B_t^j = \eta_t^j B_t^p \propto \eta_t^j q_t^B K_t$, so that

$$\frac{dB_t^j}{B_t^j} = \frac{d\eta_t^j}{\eta_t^j} + \frac{d(q_t^B K_t)}{q_t^B K_t}.$$

The volatility loadings of $dr_t^b = dr_t^B$ coincide with the ones of $d(q_t^B K_t)/(q_t^B K_t)$, compare equation (3). Thus,

$$d\Delta_t^{B,j} = \text{drift terms} + \tilde{\sigma}_t^\eta d\tilde{Z}_t^j.$$

In total, we get

$$\mu_t^{\Delta,j}(j) = -x_t, \quad \sigma_t^{\Delta,j}(j) = 0, \quad \tilde{\sigma}_t^{\Delta,j}(j) = \tilde{\sigma}_t^\eta.$$

Substituting this into equation (43) and using $\tilde{\zeta}_t = \tilde{\sigma}_t^\eta = \bar{\chi}(1 - \vartheta_t)\tilde{\sigma}_t$ implies equation (41).

The previous discussion also implies (using equation (45))

$$dr_t^b = \frac{dB_t(j)}{B_t(j)} - d\Delta_t^{B,j} = x_t dt + \frac{d(q_t^B K_t)}{q_t^B K_t}$$

and substituting this into equation (44) and using $b_t^i(j) = \eta_t^i B_t^j = \eta_t^i \eta_t^j B_t^p$ implies

$$d\Delta_t^{b,i}(j) = \frac{d\eta_t^i}{\eta_t^i} + \frac{d\eta_t^j}{\eta_t^j} + \frac{dB_t^p}{B_t^p} - \left(x_t dt + \frac{d(q_t^B K_t)}{q_t^B K_t} \right)$$

$$\begin{aligned}
&= \tilde{\sigma}_t^\eta d\tilde{Z}_t^i + \tilde{\sigma}_t^\eta d\tilde{Z}_t^j + \frac{d(q_t^B K_t)}{q_t^B K_t} - x_t dt - \frac{d(q_t^B K_t)}{q_t^B K_t} \\
&= -x_t dt + \tilde{\sigma}_t^\eta d\tilde{Z}_t^i + \tilde{\sigma}_t^\eta d\tilde{Z}_t^j.
\end{aligned}$$

In other words,

$$\mu_t^{\Delta,i}(j) = -x_t, \quad \sigma_t^{\Delta,i}(j) = 0, \quad \tilde{\sigma}_t^{\Delta,i,i}(j) = \tilde{\sigma}_t^{\Delta,i,j}(j) = \tilde{\sigma}_t^\eta.$$

Substituting these equations into equation (42) implies equation (39).

It is left to derive equation (40). This equation easily follows from the previously derived equation (39) by integrating over all holders i :

$$\begin{aligned}
B_0(j) &= \int b_t^i(j) di \\
&= \int \left(\mathbb{E} \left[\int_0^\infty \zeta_t^i x_t b_t^i(j) dt \right] + \mathbb{E} \left[\int_0^\infty \zeta_t^i (1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 b_t^i(j) dt \right] \right) di \\
&= \mathbb{E} \left[\int_0^\infty \int \zeta_t^i x_t \eta_t^i B_t(j) di dt \right] + \mathbb{E} \left[\int_0^\infty \int \zeta_t^i (1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 \eta_t^i B_t(j) di dt \right] \\
&= \mathbb{E} \left[\int_0^\infty \int \zeta_t^i \eta_t^i di \cdot x_t B_t(j) dt \right] + \mathbb{E} \left[\int_0^\infty \int \zeta_t^i \eta_t^i di \cdot (1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 B_t(j) dt \right] \\
&= \mathbb{E} \left[\int_0^\infty \zeta_t^{**} x_t B_t(j) dt \right] + \mathbb{E} \left[\int_0^\infty \zeta_t^{**} (1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 B_t(j) dt \right].
\end{aligned}$$

A.7 Model Extension with Convenience Yields

In this appendix, we present the model extension with bonds in the utility function to generate a convenience yield and derive the two debt valuation equations stated in Section 6

Setup and Equilibrium Characterization. To keep equations as simple as possible, we only consider the case of logarithmic consumption preferences and introduce separable logarithmic bond utility as in Di Tella (2020). Each agent i maximizes

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} \left((1 - v) \log c_t^i + v \log b_t^i \right) dt \right],$$

where

$$b_t^i = (1 - \theta_t^{K,i} - \theta_t^{E,i} - \theta_t^{\bar{E},i}) n_t^i$$

are real government bond holdings of the agent as in Section 3. v measures the utility share derived from bond holdings. For $v = 0$, the model collapses to the baseline model. As in the main text, but unlike in Appendix A.6, we assume here that the gross holdings or privately issued nominal debt are zero, so that all bonds are government bonds. So long as privately issued bonds do not provide utility, this assumption is without loss of generality.

However, as in Appendix A.6, we use the notation i_t^p to denote the (shadow) nominal short rate on such privately issued bonds. As these bonds do not enter utility, the spread $\Delta i_t := i_t^p - i_t$ can be positive in this augmented model. It captures the convenience yield on government bonds.

The augmented model has almost the same equilibrium solution as our baseline model. ι , q^B , and q^K are given by the equations

$$\iota_t = \frac{(1 - \vartheta_t)(a_t - \mathfrak{g}) - (1 - v)\rho}{1 - \vartheta_t + \phi(1 - v)\rho}, \quad (46)$$

$$q_t^B = \vartheta_t \frac{1 + \phi(a_t - \mathfrak{g})}{1 - \vartheta_t + \phi(1 - v)\rho}, \quad (47)$$

$$q_t^K = (1 - \vartheta_t) \frac{1 + \phi(a_t - \mathfrak{g})}{1 - \vartheta_t + \phi(1 - v)\rho} \quad (48)$$

as a function of the bond wealth share ϑ_t . The latter is determined by the dynamic equation

$$\mathbb{E}_t [d\vartheta_t] = \left(\rho + \check{\mu}_t^B - \Delta i_t - (1 - \vartheta_t)^2 \bar{\chi}^2 \bar{\sigma}_t^2 \right) \vartheta_t dt, \quad (49)$$

where $\Delta i_t = v\rho/\vartheta_t$ is the equilibrium convenience yield on government bonds. This equation differs from equation (9) only by the presence of the convenience yield term Δi_t , which raises the equilibrium level of ϑ_t .

We present a proof of equations (46)–(48) and (49) at the end of this appendix.

Debt Valuation Equations (Proposition 5). We next sketch the derivations of the two debt valuation equations stated in the main text. The derivation steps are in complete analogy to the ones presented in Section 3 for the baseline model.

The valuation from the buy and hold perspective starts again from the government

flow budget constraint (1) and follows precisely the same steps as in Section 3 up to the derivation of equation (11) stated in the main text and restated here for convenience:

$$\bar{\zeta}_0^i \frac{\mathcal{B}_0}{\mathcal{P}_0} = \mathbb{E} \left[\int_0^T \bar{\zeta}_t^i s_t K_t dt \right] - \mathbb{E} \left[\int_0^T \mathcal{B}_t \left(d \left(\bar{\zeta}_t^i / \mathcal{P}_t \right) + i_t \bar{\zeta}_t^i / \mathcal{P}_t dt \right) \right] + \mathbb{E} \left[\bar{\zeta}_T^i \frac{\mathcal{B}_T}{\mathcal{P}_T} \right].$$

From here on, the derivation departs slightly. Because the nominal SDF $\bar{\zeta}_t^i / \mathcal{P}_t$ in this model does *not* price nominal government debt but nominal private debt, it decays on average at rate $i_t^p = i_t + \Delta i_t$, and the second term does not vanish. Instead, we obtain

$$-\mathbb{E} \left[\int_0^T \mathcal{B}_t \left(d \left(\bar{\zeta}_t^i / \mathcal{P}_t \right) + i_t \bar{\zeta}_t^i / \mathcal{P}_t dt \right) \right] = \mathbb{E} \left[\int_0^T \bar{\zeta}_t^i \Delta i_t \frac{\mathcal{B}_t}{\mathcal{P}_t} dt \right],$$

which is the present value of convenience yield service flows derived from government debt between $t = 0$ and $t = T$. From here on, the derivation is again analogous to the one in Section 3. Once we replace $\bar{\zeta}_t^i$ with $\bar{\zeta}_t$ and take the limit $T \rightarrow \infty$, we arrive at the equation stated in Section 6.

The valuation from the dynamic trading perspective proceeds precisely as in Section 3. The only difference is that the derivation no longer results in the intermediate equation (14) but in the slightly modified version

$$b_0^i = -\mathbb{E} \left[\int_0^\infty \bar{\zeta}_t^i b_t^i \left(-\Delta i_t + \mu_t^{\Delta i} - \zeta_t \sigma_t^{\Delta i} - \bar{\zeta}_t^i \tilde{\sigma}_t^{\Delta i} \right) dt \right],$$

where the term $-\Delta i_t$ is new. After replacing equation (14) with this variant and otherwise following the steps outlined in Section 3, we obtain the valuation equation from the dynamic trading perspective stated in Section 6.

To understand where the additional term $-\Delta i_t$ comes from, note that also the derivation steps for equation (14) in Appendix A.3 remain unchanged except for one detail: in that appendix, we have used in equation (31) that

$$\mu_t^{r^B} - r_t^f - \zeta_t \sigma_t^{r^B} = 0$$

by standard asset pricing logic. That argument is valid if the SDF $\bar{\zeta}_t^i$ prices the government bond, so that the expected return $\mu_t^{r^B}$ equals the risk-adjusted required return $r_t^f + \zeta_t \sigma_t^{r^B}$. Due to the presence of utility services from government bonds, this is not

true anymore in the augmented model. The expected return on a privately issued bond $\mu_t^{r^B} + \Delta i_t$ still equals the required return, but the expected return on the government bond is lower by Δi_t . Consequently, we must use the modified relationship

$$\mu_t^{r^B} - r_t^f - \zeta_t \sigma_t^{r^B} = -\Delta i_t$$

in equation (31). This explains the additional term $-\Delta i_t$ above.

Model Solution Details. The model solution follows the same steps as in Appendix A.1. The difference here is that the term $\log c_t^i$ in the Hamiltonian must be replaced with

$$(1 - v) \log c_t^i + v \log \left(1 - \theta_t^{K,i} - \theta_t^{E,i} - \theta_t^{\bar{E},i} \right) + v \log n_t^i.$$

We only discuss how this affects the solution without repeating all steps from Appendix A.1 explicitly.

With the same conjecture for the value function (and thus for ζ_t^i) as in Appendix A.1, the first-order condition for optimal consumption becomes

$$c_t^i = (1 - v) \rho n_t^i$$

while the first-order condition for the optimal investment choice remains unaffected. Following the aggregation steps in Appendix A.1, we obtain again equations (6), (7), and (8) for ι_t , q_t^B , and q_t^K from the maintext with the difference that ρ in these equations, which represents the consumption-wealth ratio, must be replaced with $(1 - v) \rho$. With this replacement, these equations take the form equations (46), (47), and (48).

The first-order conditions for the portfolio shares $\theta_t^{K,i}$, $\theta_t^{E,i}$, and $\theta_t^{\bar{E},i}$ are the same as in Appendix A.1 except that there is an additional term⁶⁷

$$\frac{\rho v}{1 - \theta_t^{K,i} - \theta_t^{E,i} - \theta_t^{\bar{E},i}} = \rho v \frac{n_t^i}{b_t^i} = \Delta i_t$$

on the right-hand side of all three conditions that is due to the marginal utility of bond

⁶⁷The last equality follows from the fact that a hypothetical zero net supply nominal bond not entering the utility function but with otherwise identical risk profile would only have this term in the first-order condition for its excess return.

holdings:

$$\begin{aligned}\frac{\mathbb{E}_t[dr_t^{K,i}(i_t^i)]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} &= -\zeta_t^i \frac{\sigma_t^\vartheta}{1 - \vartheta_t} + \tilde{\zeta}_t^i \tilde{\sigma}_t - \lambda_t^i (1 - \bar{\chi}) + \Delta i_t, \\ \frac{\mathbb{E}_t[dr_t^{E,i}]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} &= -\zeta_t^i \frac{\sigma_t^\vartheta}{1 - \vartheta_t} + \tilde{\zeta}_t^i \tilde{\sigma}_t - \lambda_t^i + \Delta i_t, \\ \frac{\mathbb{E}_t[dr_t^E]}{dt} - \frac{\mathbb{E}_t[dr_t^B]}{dt} &= -\zeta_t^i \frac{\sigma_t^\vartheta}{1 - \vartheta_t} + \Delta i_t.\end{aligned}$$

From here, we can follow the same steps as in Appendix A.1, which yield again equation (27), but a modified version of equation (28):

$$\frac{a_t - \mathfrak{g} - \iota_t}{q_t^K} - \frac{\mu_t^\vartheta - \check{\mu}_t^B}{1 - \vartheta_t} - \frac{(\sigma_t^{q,B} - \sigma_t^\vartheta) \sigma_t^\vartheta}{1 - \vartheta_t} = -\zeta_t^i \frac{\sigma_t^\vartheta}{1 - \vartheta_t} + \tilde{\zeta}_t^i \bar{\chi} \tilde{\sigma}_t + \Delta i_t.$$

Replacing equation (28) with the previous one but following otherwise the steps in Appendix A.1 yields for μ_t^ϑ

$$\mu_t^\vartheta = (1 - v)\rho + \check{\mu}_t^B - (1 - \vartheta_t)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 - (1 - \vartheta_t)\Delta i_t.$$

To bring this into the form (49), note that $\Delta i_t = \rho v \frac{n_t^i}{b_t^i} = \frac{\rho v}{\vartheta_t}$ in equilibrium and hence

$$(1 - \vartheta_t)\Delta i_t + v\rho = \frac{\rho v}{\vartheta_t} - \rho v + v\rho = \frac{v\rho}{\vartheta_t} = \Delta i_t.$$

The previous equation therefore simplifies to

$$\mu_t^\vartheta = \rho + \check{\mu}_t^B - (1 - \vartheta)^2 \bar{\chi}^2 \tilde{\sigma}_t^2 - \Delta i_t.$$

Multiplying both sides by ϑ_t yields equation (49).

A.8 Calibration Details

A.8.1 Data Sources and Definitions

The data series for the CIV factor (Herskovic et al., 2016) have been retrieved from Bernard Herskovic's website (<https://bernardherskovic.com/data/>). That series (column "CIV") represents an annualized return variance measure of the common idiosyn-

cratic volatility in stock returns.

All other data used in this paper have been retrieved from the FRED database maintained by the Federal Reserve Bank of St. Louis (<https://fred.stlouisfed.org/>). We briefly describe next how we map model quantities into FRED data series.

For the macro aggregates Y , C , I , and G , we use quarterly data from 1970Q1 to 2019Q4. Output is defined as $Y = C + I + G$ (in particular, exclusive of net exports) while we define the three series C , I , and G as follows:

- In line with the business cycle literature, we exclude consumption of durable goods from our consumption measure. To compute C , we start from real personal consumption expenditures (FRED code PCECC96) and subtract real expenditures for durable goods. We identify the latter by multiplying total real consumption expenditures by the ratio of nominal expenditures for durable goods (PCDG) and nominal total consumption expenditures (PCEC).
- We define investment I as the sum of two components: (1) real gross private domestic investment (GPDIC1) net of the change in private inventories (CBIC1) and (2) real consumption expenditures for durable goods (measured as described previously). We include durables in investment as we have removed them from consumption but they nevertheless represent an important part of overall private expenditures.⁶⁸
- We government spending G as real government consumption expenditures and gross investment (GCEC1).

The ratios of primary surpluses and government debt to GDP, S/Y and $q^B K/Y$, respectively, are measured from nominal data. We use again quarterly data series from 1970Q1 to 2019Q4. We define the nominal primary surplus as current receipts (FGRECPT) minus current expenditures (FGEXPND) but add back current interest expenditures (A091RC1Q027SBEA) of the federal government. We define nominal debt as the market value of marketable treasury debt (MVMTD027MNFRBDAL). We compute the ratios S/Y and $q^B K/Y$ by dividing both nominal primary surpluses and nominal

⁶⁸Excluding durables altogether from our measures of economic activity does not substantially change our computed data moments: it lowers the volatility of output somewhat but otherwise only marginally affects results.

debt by nominal GDP (GDP).⁶⁹

Data on the capital stock to compute the capital-output ratio is based on the Penn World Tables (Feenstra et al., 2015) and only available annually. We again choose the time period from 1970 to 2019. The capital-output ratio $q^K K/Y$ is defined as capital stock at constant national prices (RKNANPUSA666NRUG) divided by real GDP at constant national prices (RGDPNAUSA666NRUG), both for the US.

For returns on bonds and equity, we use monthly data from February 1970 to December 2019.⁷⁰ We first construct monthly log returns from these data sources as follows:⁷¹

- We measure the return on government debt using data on the market yield on treasury securities at 5-year constant maturity (DGS5). We chose the 5-year maturity as this approximately reflects the average duration of federal debt. We convert the yield data into (holding period) returns using the well-known formula

$$r_{t+1}^T = Ty_t^T - (T - 1)y_{t+1}^{T-1}$$

that relates the log holding period return r_{t+1}^T over the period from t to $t + 1$ of a bond with time to maturity of T at date t to the log yield y_t^T of a T -period bond at t and the log y_{t+1}^{T-1} of a $T - 1$ -period bond at $t + 1$. To operationalize this formula, we approximate the unknown 59-month yield y_{t+1}^{T-1} with the observed 60-month yield y_{t+1}^T . This procedure generates a series \hat{r}_t^B of monthly log returns for government bonds.

- As a proxy for the total equity market, we take the Wilshire 5000 index. We compute monthly log returns by dividing successive end-of-month values of the total market index (Wilshire 5000 Total Market Index, FRED series WILL5000IND), which includes dividend reinvestments, and then taking natural logarithms. As market returns are based on leveraged equity returns, this procedure yields a series $\hat{r}_t^{E, \text{leverage}}$ of leveraged monthly log returns for equity.

⁶⁹Unlike for our time series of macro aggregates, we do not correct the GDP measure for components not in the model. Doing so would have only a minor impact on the resulting numbers. Not doing so is also consistent with how we compute the capital-output ratio below.

⁷⁰February 1970 is the first month at which some of the required series are available on FRED.

⁷¹To be precise, the following definitions are for nominal returns while the returns in the model are real. However, for the purpose of computing return differentials, as we do, this distinction is irrelevant.

Based on these data series, we construct the sample estimates for $\mathbb{E} [d\bar{r}^E - dr^B]$ and $\sigma (d\bar{r}^E - dr^B)$ reported in Table 2 as follows. We first define for leveraged returns:

$$\begin{aligned}\mathbb{E} [d\bar{r}^{E,\text{leverage}}] &= 12 \cdot \text{sample mean} (\hat{r}^{E,\text{leverage}}) + \frac{12}{2} \cdot \text{sample var} (\hat{r}^{E,\text{leverage}}), \\ \mathbb{E} [dr^B] &= 12 \cdot \text{sample mean} (\hat{r}^B) + \frac{12}{2} \cdot \text{sample var} (\hat{r}^B), \\ \sigma^2 (d\bar{r}^{E,\text{leverage}} - dr^B) &= 12 \cdot \text{sample var} (\hat{r}^{E,\text{leverage}} - \hat{r}^B).\end{aligned}$$

However, the model counterpart $d\bar{r}^E$ of the market equity return is closer to a delevered equity return. The theoretical relationship between the delevered equity return $d\bar{r}^E$ and the leveraged return $d\bar{r}^{E,\text{leverage}}$ is

$$d\bar{r}^E = dr^B + \frac{1}{\ell} (d\bar{r}^{E,\text{leverage}} - dr^B),$$

where $\ell \geq 1$ is financial leverage as measured by the ratio of total assets to equity. We therefore define:

$$\begin{aligned}\mathbb{E} [d\bar{r}^E - dr^B] &= \frac{1}{\ell} \left(\mathbb{E} [d\bar{r}^{E,\text{leverage}}] - \mathbb{E} [dr^B] \right), \\ \sigma (d\bar{r}^E - dr^B) &= \frac{1}{\ell} \sigma (d\bar{r}^{E,\text{leverage}} - dr^B).\end{aligned}$$

We use $\ell = 1.5$ to compute delevered equity returns.

A.8.2 Calibration of the Exogenous $\tilde{\sigma}_t$ Process

We estimate the coefficients $\tilde{\sigma}^0$, ψ , and σ of the idiosyncratic risk process (22) such that it matches the observed CIV series. Here, we first describe the details of the estimation procedure and then explain why CIV is a suitable data counterpart for idiosyncratic risk $\tilde{\sigma}_t^2$ in the model.

Parameters Estimation. We use a maximum likelihood estimation (MLE) to determine $\tilde{\sigma}^0$, ψ , and σ based on a monthly CIV sample from January 1945 to December 2019. MLE is straightforward here because the conditional density of the CIR process $\tilde{\sigma}_t^2$ has a known closed-form expression (e.g. Ait-Sahalia (1999), equation (20)).

While not directly targeted by MLE, the estimated process generates first and sec-

ond ergodic moments of $\tilde{\sigma}_t$, 0.5078 and 0.1701, respectively, that closely match their empirical counterparts (based on square roots of the CIV sample), 0.4950 and 0.1817, respectively.

CIV as a Model-consistent Measure of $\tilde{\sigma}_t^2$. We briefly outline why CIV indeed measures $\tilde{\sigma}_t^2$. [Herskovic et al. \(2016\)](#) construct CIV as the cross-sectional mean of the idiosyncratic return variance of individual stocks in their sample. The idiosyncratic return variance of an individual stock, in turn, is defined as the variance of the residual of a factor regression on the market factor.

In our model, this procedure broadly amounts to a (population) regression of the type

$$dr_t^{E,i} - r_t^f dt = \alpha_t^i + \beta_t^i \left(d\bar{r}_t^E - r_t^f dt \right) + \varepsilon_t^i$$

for stocks issued by all agents i . Comparing the return expressions for $dr_t^{E,i}$ and $d\bar{r}_t^E$ stated in Section 2.2, it is clear that this regression yields $\alpha_t^i = 0$, $\beta_t^i = 1$ and $\varepsilon_t^i = \tilde{\sigma}_t d\tilde{Z}_t^i$. The variance of each individual residual ε_t^i therefore exactly equals $\tilde{\sigma}_t^2$, and so does the cross-sectional mean over all residual variances. In other words, if the real-world data was generated by the model, measured CIV at time t would exactly correspond to $\tilde{\sigma}_t^2$.

A.8.3 Calibration of Remaining Model Parameters

The calibration choices for χ and δ are explained in the main text. The remaining nine parameters, γ , ρ , a^0 , \mathfrak{g} , $\check{\mu}^{B,0}$, α^a , α^B , ϕ , ι^0 , are chosen to match twelve moments as described in the main text. We briefly explain here (heuristically) how these moments identify the model parameters.

First, given the estimated $\tilde{\sigma}_t$ process, the capital productivity process

$$a_t = a(\tilde{\sigma}_t) = a^0 - \alpha^a (\tilde{\sigma}_t - \tilde{\sigma}^0)$$

is exogenous and fully determined by the two parameters a^0 and α^a . While output $Y_t = a_t K_t$ still contains an endogenous term K_t , the capital stock is slow-moving such that most of the variation in HP-filtered output is due to variation in a_t . Therefore, the parameter α^a is effectively determined by the target moment $\sigma(Y)$.

Second, because \mathfrak{g} is constant, the variability of output left for private uses, $Y - G$, is also determined by the parameter α^a . By the aggregate resource constraint $Y - G =$

$C + I$, so that the choice of α^a also constrains the variation of the sum of consumption and investment. The parameter ϕ effectively controls how much of that variation is absorbed by the individual components of that sum. While in principle the full details of the model matter for the dynamics of investment opportunities, ϕ controls to which extent changes in investment opportunities change actual physical investment as opposed to simply driving up or down capital valuations. For $\phi \rightarrow 0$, investment reacts a lot while for $\phi \rightarrow \infty$, investment is fixed and only prices react. Therefore, the two relative volatilities $\sigma(C)/\sigma(Y)$ and $\sigma(I)/\sigma(Y)$ effectively determine ϕ .⁷²

Third, the ratio of primary surpluses to output is given by

$$S_t/Y_t = -\check{\mu}_t^B \frac{q_t^B}{a_t} = -\left(\check{\mu}^{B,0} + \alpha^B(\check{\sigma}_t - \check{\sigma}^0)\right) \frac{q_t^B}{a_t}.$$

While the dynamics of this variable depend on the endogenous price q_t^B , the parameter α^B is nevertheless able to control the overall volatility of S_t/Y_t .⁷³ The parameter α^B is therefore determined by the moment $\sigma(S/Y)$.

Fourth, the six average ratio targets in the calibration effectively determine the five parameters ρ , a^0 , \mathfrak{g} , $\check{\mu}^{B,0}$, and ι^0 . To see this, we explain how, in the stochastic steady state of the model, the five parameters map directly into functions of target ratios and how this mapping can be inverted to obtain the parameters. While we do not target the stochastic steady state but the ergodic mean when matching moments, the two are quantitatively very close.

The identity $C + I + G = Y$ and the level targets for C/Y and G/Y imply $I/Y = 1 - C/Y - G/Y$. We can thus write for capital productivity a^0 in the stochastic steady state

$$a^0 = \frac{Y}{K} = \frac{I/K}{I/Y} = \frac{I/K}{1 - C/Y - G/Y}.$$

This determines a^0 as a function of targets. Due to $G = \mathfrak{g}K$, we obtain immediately also

$$\mathfrak{g} = G/Y \cdot a^0.$$

⁷²This does not imply that we are always able to pick ϕ in a way that matches both relative volatilities precisely. It just means that if model dynamics are such that they can be matched at all, then this works only for one value of ϕ .

⁷³This is not a rigorous theoretical statement but an empirical one based on observed numerical model solutions.

Because G/Y is a target and a^0 has already been determined, this equation determines g .

Next, ρ represents the ratio of consumption to total wealth in the model, that is

$$\rho = \frac{C}{(q^B + q^K)K} = \frac{C/Y}{q^B K/Y + q^K K/Y}$$

and the right-hand expression is a function of targeted ratios. Hence, the targets also determine ρ .

By the government budget constraint, the policy variable $\check{\mu}^B$ in the stochastic steady state must satisfy

$$\check{\mu}^{B,0} = -\frac{s}{q^B} = -\frac{S/Y}{q^B K/Y}$$

and, again, the right-hand expression is a function of targeted ratios.

Finally, the capital price in the stochastic steady state can be related to the capital-output ratio by the equation $q^{K,0} = q^K K/Y \cdot a^0$. Because a^0 is a function of targeted ratios, so is $q^{K,0}$. It is easy to show that the investment rate is $I/K = \iota^0 + \frac{q^{K,0}-1}{\phi}$. This expression only depends on $q^{K,0}$ and the parameters ι^0 and ϕ . For any given parameter ϕ , ι^0 is therefore determined by targets through the equation

$$\iota^0 = I/K - \frac{q^{K,0} - 1}{\phi}.$$

We remark that the six average ratios do not only identify the five parameters ρ , a^0 , g , $\check{\mu}^{B,0}$, and ι^0 (in the stochastic steady state) but also the average value ϑ^0 of the endogenous variable ϑ_t , namely

$$\vartheta^0 = \frac{q^B}{q^B + q^K} = \frac{q^B K/Y}{q^B K/Y + q^K K/Y}.$$

This generates an implicit target that must be somehow matched by varying parameters other than ρ , a^0 , g , $\check{\mu}^{B,0}$, and ι^0 in order to match all six average ratios.

Fifth, because ρ , $\bar{\chi}$, and the dynamics of $\check{\mu}^B$ and $\tilde{\sigma}_t$ are already determined by external calibration choices or the targeted average ratios, the counterpart of equation (9) in Appendix A.4 implies that this implicit target ϑ^0 for the average value of ϑ_t must be matched by a sufficient size of the risk premium terms in that equation. The only “free”

variables in these terms are σ_t^v and γ and the former is effectively also determined by γ (once ρ , $\bar{\chi}$, and the dynamics of $\tilde{\sigma}_t$ are fixed). In fact, the risk premium terms are strictly increasing in γ given the remaining parameter choices. Therefore, the implicit target ϑ^0 is only achieved for a specific value of γ . At the same time, γ affects also the average equity premium $\mathbb{E}[d\bar{r}^E - dr^B]$ and the equity sharpe ratio $\mathbb{E}[d\bar{r}^E - dr^B]/\sigma(d\bar{r}^E - dr^B)$. The parameter γ is thus certainly identified by the set of target moments, but it is generally not possible to match all of them.