Redistributive Fiscal Policy and Marginal Propensities to Consume

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Abstract

Fiscal stimulus during the Great Recession consisted mainly of transfers, rather than government purchases. This paper analyzes the role of marginal propensities to consume (MPCs) in shaping the effect of such policies. I build a tractable continuous-time New Keynesian model with heterogeneous overlapping generations (OLG) which allows for arbitrary MPC heterogeneity. I provide a complete analytical characterization of output multipliers for arbitrary policy paths of fiscal transfers. When consumers with a low MPC receive a transfer, they save most of it, which allows them to consume more in future periods. As a result, I show that the role of MPCs is mainly to determine the timing of the fiscal stimulus: high MPCs front-load the stimulus, low MPCs back-load it. The relation between the timing of the stimulus and the cumulative effect on output (measured by the present value) is, however, ambiguous. Indeed, I show that transfers to low-MPC consumers may generate a higher cumulative effect on output. From a normative perspective, however, there is no ambiguity: with larger differences in MPCs, optimal policy can obtain macro stabilization with a smaller welfare loss because transfers create less cross-sectional consumption dispersion. Finally, I undertake quantitative exercises with a standard incomplete markets model. The numerical results are consistent with my analytical OLG model.

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1 Introduction

In the aftermath of the Great Recession, economists and policymakers have taken a renewed interest in the role of fiscal policy as a tool for macroeconomic stabilization. While typically this role is played by monetary rather than fiscal policy, the former can be constrained when the policy rate is effectively at zero (commonly known as the zero lower bound, or ZLB). In light of recent experiences with a binding ZLB and with interest rates still today at historically low levels, fiscal policy as a tool for macro stabilization has become an area of intense economic research.

Fiscal policy can stimulate aggregate demand during a recession either through direct government purchases or through redistributions between households with different marginal propensities to consume (MPCs). While most of the empirical and theoretical literature on fiscal stimulus focuses on government purchases, most of the fiscal expansion during the Great Recession consisted of transfers. Between 2007 and 2009, government expenditure in the US increased by 4.4% of GDP, of which 3.4% consisted of transfers (Oh and Reis (2012)). Thus, analyzing the effects of redistributive policies is a matter of great practical importance.

It is well known that MPCs are a key determinant of the effect of such policies. In this paper, I will show that their role is more nuanced than has previously been recognized. Suppose we want to compare the effect of giving a transfer to a household with a high MPC with the effect of giving the same transfer to another household with a lower MPC. The starting point of my analysis is a simple, yet crucial, observation: because a lower MPC implies that a larger fraction of the transfer goes into savings, in the next period the lower-MPC household will have more financial assets. With more financial assets, this household will be wealthier and will consume more. Therefore, lower-MPC agents will be able to sustain a higher level of consumption over a longer period of time. This means that the concepts of MPC and persistence of the redistribution are inevitably intertwined.

This interaction between MPCs and persistence is not captured by the standard IS-LM model, in which MPCs have no dynamic effects. However, since in the data we observe that output can remain below potential for a considerable period of time (for example, in the aftermath of the Great Recession, CBO estimates indicate that output in the US remained below potential from 2008 to 2016), it is important to understand the dynamic effects of fiscal transfers. In particular, we would like to understand whether in dynamic models it is still true that transfer multipliers are proportional to the difference in MPCs between the households that receive the transfer and the households that pay for the transfer (as is the case in the IS-LM).

This paper sheds light on this issue by analyzing fiscal transfers in the context of a New Keynesian model with heterogeneous agents. My model incorporates arbitrary MPC heterogeneity, thus allowing us to study the interactions between MPCs, the accumulation of financial assets and the persistence of the fiscal stimulus. The modeling device used to obtain such heterogeneity while preserving tractability is the incorporation of two types of overlapping generations, each with a different planning horizon.

The main contribution of this paper is to clarify the role of MPCs in shaping the effects of redistributive fiscal policies. Since MPCs and the accumulation of financial assets are two sides of the same coin, my model will show that the crucial role of MPCs is to determine the timing of the stimulus to aggregate demand. That is, high MPCs generate a front-loaded stimulus, while lower MPCs tend to back-load the stimulus.

In an initial example, I will show that if the real interest rate is exogenously fixed and the government makes a transfer from a Ricardian to a non-Ricardian agent, the MPC of the latter is completely irrelevant to determining the cumulative effect of the redistribution on aggregate output (where the cumulative effect is measured by the present value). This can be interpreted as a neutrality result: in this case, MPCs matter in determining the timing of the stimulus, but they do not matter in determining the cumulative effect on output. This neutrality result provides a useful benchmark, since we can then analyze how deviating from it allows MPCs to affect the size of the transfer multipliers.

MPCs *will* affect the cumulative effect on output when there are features in the model that interact with the timing of the stimulus. That is, if the model has features that favor a more front-loaded stimulus, then transfers to high-MPC households will tend to have higher multipliers, while the opposite will happen if there are features that favor back-loading.

One such feature that interacts with the timing of the stimulus is the endogenous response of monetary policy. It is well known that in New Keynesian models, the response of monetary policy is a key determinant of fiscal multipliers, and this will be true in my model as well. What will be especially important here, though, is how the monetary response depends on how front- or back-loaded the stimulus is.

I will show that the relation between MPCs and transfer multipliers is ambiguous. Depending on how the monetary policy response interacts with the timing of the stimulus, it is in fact possible for transfers to higher-MPC agents to have lower multipliers – a result that contradicts the intuition from the Keynesian cross.

A case in which this can occur is a liquidity trap in an economy with sticky prices. In a liquidity trap, higher aggregate demand lowers the real interest rate by increasing inflation (i.e., there is an inverted Taylor principle). As shown by Farhi and Werning (2016), the more back-loaded the stimulus to aggregate demand, the stronger the effect on output because of a forward guidance effect. Therefore, since a transfer to a low-MPC agent generates a back-loaded stimulus, it can have a very large multiplier.

While in the baseline model I will consider government policies that directly transfer resources between agents, the same results are also useful for characterizing other government policies that have redistributive effects. In an extension, I consider the case of households that have a heterogeneous interest rate exposure. In this case, the total effect of a monetary shock can be decomposed into a direct effect and a redistribution, and the latter can be characterized by the transfer multipliers derived in the baseline model.

After characterizing transfer multipliers from a positive perspective, I will turn to a normative analysis. Another contribution of this paper is an analysis of the relation between MPCs and welfare when the government can optimally choose the timing of transfers. In this case, there is no ambiguity: with larger differences in MPCs, the planner can obtain macro stabilization with a smaller welfare loss because transfers create less cross-sectional consumption dispersion.

A methodological contribution of this paper is to expand our toolkit to deal with heterogeneity in New Keynesian models. While previous papers have introduced OLG into a New Keynesian framework (e.g., Galí (2016)), this is to the best of my knowledge the first one to have multiple OLG dynasties with heterogeneous horizons.

By introducing OLG dynasties we can preserve analytical tractability while obtaining a consumption behavior that closely resembles that of credit-constrained households. A high death rate increases households' MPCs and limits their capacity to borrow from future labor income; both of these features are analogous to the behavior of households with low liquid assets in models with uninsurable idiosyncratic risk and borrowing constraints.

Of course, just as there are similarities, there are also differences between binding credit constraints and the death event; for example, agents can self-insure against the former by accumulating assets, but cannot insure against the latter. Therefore, I complement my theoretical analysis with quantitative exercises in a standard incomplete-markets Bewley-Huggett-Aiyagari model with nominal rigidities (commonly known as HANK, using the terminology introduced by Kaplan, Moll, and Violante (2018)). In these numerical exercises, I find a role for MPCs that is similar to their role in the analytical OLG model.

In Section 2 of the paper, I set up the model and in Section 3 I provide an analytical characterization of transfer mutlipliers. Section 4 considers some extensions of the baseline model. In Section 5 I do a welfare analysis, clarifying the relation between MPCs and welfare under an optimal redistributive fiscal policy. Finally, Section 6 presents a quantitative analysis with an incomplete markets model à la Bewley-Huggett-Aiyagari with nominal rigidities.

Related literature. This paper builds on a long literature that studies fiscal multipliers. Early contributions to this literature, such as Aiyagari, Christiano, and Eichenbaum (1992) and Baxter and King (1993), analyze fiscal multipliers in neoclassical models with a representative agent. Woodford (2011) incorporates nominal rigidities into the analysis and emphasizes the monetary policy response to the fiscal stimulus as a key determinant of the size of fiscal multipliers. He finds that multipliers can be large in liquidity traps because there is an inverted Taylor principle. The monetary policy response to the stimulus will be crucial in my model as well, and I will also find large fiscal multipliers when the ZLB binds. However, my focus will be on the interaction between monetary policy and MPC heterogeneity, which cannot be analyzed within Woodford (2011)'s representative agent framework.

Galí, Lopez-Salido, and Valles (2007) study fiscal multipliers in a model that departs from the representative agent assumption by allowing for two types of agents, one of them Ricardian and the other one Hand-to-Mouth. This Two-Agent New Keynesian (TANK) framework is a tractable way to allow for MPC heterogeneity while preserving tractability. However, it only allows for a very particular type of heterogeneity, so it is not clear how to extend its conclusions to more general cases. This is one of the main issues I will address in this paper.

Building on Galí, Lopez-Salido, and Valles (2007), Farhi and Werning (2016) also use a TANK model, focusing on fiscal multipliers when the economy is in a liquidity trap. My approach will be similar to theirs in using a continuous-time New Keynesian model and characterizing the multipliers for arbitrary policy paths (rather than analyzing a particular autorregressive shock).

Two other recent papers that analyze fiscal policy within the TANK model are Giambattista and Pennings (2017) and Bilbiie, Monacelli, and Perotti (2013). The former compares the effects of transfers with government purchases and the latter focuses on the role of government debt. Meanwhile, Mehrotra (2018) analyzes fiscal multipliers in a spender-saver model with a debt-elastic interest rate spread. My main departure from this literature will be to allow for a more general form of MPC heterogeneity.

My model will be similar to the TANK in that I will have two OLG dynasties that have heterogeneous MPCs. In fact, my model will turn out to be a generalization of the TANK: if we take the death rate of one dynasty to be zero and the other one approaches infinity, we obtain the TANK as a special case.

This paper is also related to the very active literature on New Keynesian models with heterogeneous agents (HANK). These models incorporate uninsurable idiosyncratic risk and borrowing constraints à la Bewley-Huggett-Aiyagari into a New Keynesian framework, thus allowing for richer heterogeneity than TANK models do. Within this literature, Hagedorn, Manovskii, and Mitman (2019) focus particularly on government expenditure multipliers, while McKay and Reis (2016) and Oh and Reis (2012) focus on redistributive fiscal policies. Auclert and Rognlie (2018) analyze the implications of changes in the wealth distribution for aggregate demand, and show that partial equilibrium sufficient statistics can be converted into general equilibrium effects using numerical multipliers. Given the complexity of these HANK models, these papers take a mostly numerical approach. I will provide instead an analytical characterization, which will help clarify the role of MPC heterogeneity.

Auclert, Rognlie, and Straub (2018) study government purchases multipliers in a fairly general HANK model. They emphasize the importance not just of MPCs from current income, but MPCs from future income as well. They denote this as intertemporal MPCs (iMPCs) and show that the mapping of partial equilibrium effects into general equilibrium effects generally depends on these iMPCs. My paper instead will focus on transfers rather than government purchases. In the theoretical section of my paper, I will use a more parsimonious model than theirs, which will allow me to provide a full characterization of the general equilibrium effects of transfers. This analytical characterization will uncover several new results, in particular the ambiguity in the relation between MPCs and transfer multipliers, and the implications of MPC heterogeneity for optimal policy.

My model is also closely related to Werning (2015). In his paper, he shows that under certain assumptions, models with Bewley-Huggett-Aiyagari households admit an as-if representative agent with a time-varying discount factor. My model will turn out to be a particular instance of Werning (2015)'s framework, admitting the same type of as-if representation.

Another recent contribution to this literature is Auclert (2017), who analyzes the redistributive effects of monetary policy in a model with heterogeneous agents. While my main focus will be on fiscal policy, I will show in a simple extension that the same transfer multipliers described in the baseline model can be used to characterize the redistributive effects of other government policies. In particular, I will illustrate this point with the redistribution channel of monetary policy studied by Auclert (2017).

Finally, this paper is related to the empirical literature that uses exogenous variation in household income to estimate MPCs. This includes, for example, Parker et al. (2013), Souleles, Parker, and Johnson (2006), Misra and Surico (2014), Jappelli and Pistaferri (2009) and Holm, Natvik, and Fagereng (2017). While this literature tends to focus on the immediate reaction of consumption to changes in income, I will emphasize the dynamic response. That is, I will analyze both the dynamics after an unanticipated transfer and the effects of transfers that are fully anticipated by households.

2 A Model of Heterogeneous MPCs

In this section I set up the baseline model that I will use to analyze the effects of redistributive fiscal policies. The model departs from the standard representative agent New Keynesian framework by incorporating two dynasties of overlapping generations that are heterogeneous in their death probabilities. This heterogeneity in planning horizons creates MPC heterogeneity, which will allow us to analyze the interaction between MPCs and transfer multipliers while preserving analytical tractability.

Environment. Time is continuous, $t \geq 0$, and there is no aggregate uncertainty. There are two types of agents that make consumption and labor choices. Each type of agent is a dynasty of overlapping generations as in Blanchard (1985)'s perpetual youth model, with types differing in their death probabilities, λ_i. Without loss of generality assume that $\lambda_1 < \lambda_2$. There is a mass χ and $1 - \chi$ of households of type 1 and 2 respectively.

Preferences of household h of type $i \in \{1, 2\}$ are given by

$$U_{i,t}^{h}=\int_{t}^{\infty}e^{-(\rho+\lambda_{i})(s-t)}\left[\ln C_{i,s}^{h}-\frac{\left(N_{i,s}^{h}\right)^{1+\phi}}{1+\phi}\right]ds,$$

where $N_{i,t}^{h}$ are hours worked, and the discount factor is given by the sum of the strictly defined discount ρ (which is common across types) and the idiosyncratic death probability λ_i.[1] Agents consume a CES aggregator of varieties indexed by $j\in[0,1]$, defined as

$$C_{i,t}^{h}\equiv\left(\int_{0}^{1}C_{i,t}^{h}\left(j\right)^{\frac{\varepsilon-1}{\varepsilon}}dj\right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

One could also introduce government purchases into the utility function, but for simplicity I will assume all throughout the paper that government purchases are completely wasteful.

The only factor of production is labor. The firm that produces a typical product $j\in[0,1]$ uses a linear technology

$$Y_{t}\left(j\right)=AN_{t}\left(j\right)$$

and chooses its price subject to Calvo frictions. I will generally use the normalization $A=1$.[2]

Households' problem. Assume that insurance companies operate in perfectly competitive annuity markets, and these firms can tell apart the type of agents and price accordingly. Therefore, households will buy annuities that reflect their idiosyncratic death probabilities.

The budget constraint (in real terms) of an individual household h of type $i\in\{1,2\}$ is

$$\dot{B}_{i,t}^{h}=\left(r_{t}+\lambda_{i}\right)B_{i,t}^{h}+\frac{W_{t}}{P_{t}}N_{i,t}^{h}-\frac{1}{P_{t}}\int_{0}^{1}P_{t}\left(j\right)C_{i,t}^{h}\left(j\right)dj-\left(T_{i,t}+T_{t}^{g}+T_{i,t}^{rebate,h}\right),\qquad(1)$$

where $B_{i,t}^{h}$ are risk-free bonds that pay a real interest rate $r_t\equiv i_t-\pi_t$ plus the return from the annuities λ_i, and W_t is the nominal wage. As usual, I am defining the price index as $P_{t}=\left(\int_{0}^{1}P_{t}\left(j\right)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}}$.

The last term of equation (1), $T_{i,t}+T_{t}^{g}+T_{i,t}^{rebate,h}$, requires some more explanation. This term consists of lump-sum taxes, which I separate (without loss of generality) into three parts: $T_{i,t}$ is a purely redistributive component which is levied from one type of agent and immediately transferred to the other type of agent, and will be the policy instrument on which I will focus most of the analysis throughout the paper; T_t^g is a tax that is used to pay for government purchases, and I will assume this tax is the same for all households so that

[1] While we could also generate MPC heterogeneity by allowing agents to differ both in their parameter ρ and their parameter λ, here I will assume that all the heterogeneity comes from the death probability. This captures the view that MPC heterogeneity mainly stems from constraints and not preferences, and this view is consistent with the recent TANK and HANK literature.

[2] I only depart from the simplifying assumption that $A=1$ in Section 5 when I do welfare analysis. In that case I will use a time-varying TFP as a way to generate a time-varying natural allocation.

all the redistributive effect of fiscal policy loads onto $T_{i,t}$; and $T^{rebate,h}_{i,t}$ is used to rebate firms' profits (which are fully taxed) back to households.[3] I assume that $T^{rebate,h}_{i,t}$ is proportional to hours worked by household h, so the total real wage received by households is equal to the marginal productivity of labor, which I denote as $\hat{A}_t \equiv \frac{A}{\int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} dj}$.[4]

Since the utility function is logarithmic in consumption and separable between consumption and labor, we find that for any given path of labor $\left\{N^h_{i,t}\right\}$, consumption of household h is

$$C^h_{i,t} = (\rho + \lambda_i)\left[B^h_{i,t} + \mathcal{H}^{h,(r+\lambda_i)}_{i,t} - \left(\mathcal{T}^{r+\lambda_i}_{i,t} + \mathcal{T}^{g,r+\lambda_i}_{t}\right)\right],$$

where for any variable X, I use calligraphic font to denote the PDV of the corresponding variable, with the superscript indicating the interest rate used to do the discounting,[5] so that $\mathcal{X}^{r+\lambda_i}_{i,t} \equiv \int_t^\infty e^{-\int_t^s (r_l+\lambda_i)dl} X_{i,s} ds$. Agents consume at a rate $\rho + \lambda_i$ out of their wealth, which is composed of financial wealth ($B_{i,t}$) and human wealth ($\mathcal{H}^{h,(r+\lambda_i)}_{i,t}$), net of taxes ($\mathcal{T}^{r+\lambda_i}_{i,t} + \mathcal{T}^{g,r+\lambda_i}_{t}$). Note then that the MPC out of current income is $\rho + \lambda_i$.

The choice of optimal labor supply can be written as an intratemporal problem that does not depend on λ_i (conditional on consumption):

$$\frac{W_t}{P_t} = C^h_{i,t}\left(N^h_{i,t}\right)^\phi. \tag{2}$$

Aggregation. We can aggregate the budget constraints in equation (1) for all the households of the same type alive at time t to obtain an aggregate law of motion for bond holdings of type i agents:

$$\dot{B}_{i,t} = r_t B_{i,t} + \hat{A}_t N_{i,t} - C_{i,t} - \left(T_{i,t} + T^g_t\right), \tag{3}$$

where variables without an h superscript (e.g. $B_{i,t}$, $N_{i,t}$) denote the average for all households of type i.

Since households' consumption is linear in wealth, we can express the aggregate consumption of type $i \in \{1,2\}$ agents as

$$C_{i,t} = (\rho + \lambda_i)\left[B_{i,t} + \mathcal{H}^{r+\lambda_i}_{i,t} - \left(\mathcal{T}^{r+\lambda_i}_{i,t} + \mathcal{T}^{g,r+\lambda_i}_{t}\right)\right].$$

[3]If we did not assume that corporate profits are rebated back to households, changes in aggregate demand would have redistributive effects because the change in profits would be asymmetrically shared across types of agents. By rebating profits back to households, we ensure that $T_{i,t}$ is the only source of redistribution, which simplifies the analysis. The redistribution channel through firm profits has already been studied before, for example, in Bilbiie (2008).

[4]Note however that from the perspective of the individual household the rebates are lump-sum, so the labor supply decision is made according to the wage $\frac{W_t}{P_t}$ instead of $\hat{A}_t$.

[5]In the case of human wealth, define $H^h_{i,t} \equiv \hat{A}_t N^h_{i,t}$ to be the labor income of household h at time t, so that the PDV is

$$\mathcal{H}^{h,(r+\lambda_i)}_{i,t} = \int_t^\infty e^{-\int_t^s (r_l+\lambda_i)dl} \hat{A}_s N^h_{i,s} ds$$

We can differentiate this consumption equation to obtain an Euler equation that is distorted by bond holdings:

$$\dot{C}_{i,t} = (r_t - \rho) C_{i,t} - \Lambda_i B_{i,t}, \tag{4}$$

where I define

$$\Lambda_i \equiv \lambda_i (\rho + \lambda_i).$$

The second term on the right-hand side of equation (4) is the crucial departure of this model from the RANK model. This term is telling us that short-sighted perpetual youth agents will run down their financial assets faster than would be implied by the standard Euler equation of a Ricardian agent.

Firms' problem. Monopolistically competitive firms set their prices subject to Calvo frictions. They choose their reset price $\hat{P}_t(j)$ to maximize their discounted profits:

$$\max_{\hat{P}_t(j)} \int_t^{\infty} e^{-\zeta(s-t)-\int_t^s i_x dx} \left(\hat{P}_t(j) Y_{s|t} - \left(1+\tau^L\right) W_t \frac{Y_{s|t}}{A} \right) ds, \tag{5}$$

where $Y_{s|t} \equiv \left(\frac{\hat{P}_t(j)}{P_s}\right)^{-\varepsilon} Y_s$ and ζ is the flow of firms that can reset their price per unit time. I assume that the government sets a labor tax $\tau^L < 0$ that offsets the inefficiency produced by market power under monopolistic competition. This assumption is inconsequential for the positive characterization of fiscal multipliers, but will be important for the normative analysis in Section 5.

Government. The government controls fiscal and monetary policy. I assume that at $t = 0$ the government makes an unexpected announcement of its policies, and from there onwards there is perfect foresight.

The budget constraint of the government is

$$\dot{B}_t^g = r_t B_t^g + T_t^g - G_t + [\chi T_{1,t} + (1-\chi) T_{2,t}], \tag{6}$$

where B_t^g are the government's financial assets (with $B_0^g = 0$). The last term is the redistributive component of taxation which satisfies

$$\chi T_{1,t} + (1-\chi) T_{2,t} = 0$$

since the revenue from one type of agent is immediately transferred to the other type. I will assume that the government has a balanced budget, so it sets $T_t^g = G_t,\ B_t^g = 0\ \forall t$.

The government will set the policy nominal rate according to a rule

$$i_t = i\left(\frac{Y_t}{Y^*}, \pi_t, t\right)$$

that depends on the deviation of output from steady state and on inflation. I am allowing the monetary policy rule to be time-dependent so that we can allow for the case in which there is a "regime change", for example from a liquidity trap to a neoclassical regime.

In order to guarantee determinacy of the equilibrium, I will assume (as in Farhi and Werning (2016)) that there exists some time $\widetilde{T}$ such that for $t \geq \widetilde{T}$ the real interest rate will be the natural rate which implements the neoclassical equilibrium. In this way, the equilibrium allocation at time $\widetilde{T}$ acts as the "anchor" of the economy, allowing us to work backwards from that uniquely determined allocation[6]. I will sometimes assume that $\widetilde{T}$ is infinite, in which case this is only a modeling device to obtain uniqueness of the equilibrium, while in other instances I will allow for a finite $\widetilde{T}$ to analyze an economy that exits a liquidity trap in finite time.

Discussion We can now discuss the role of overlapping generations in this model. The MPC out of current income is $\rho+\lambda_i$, so the heterogeneity in the death probability immediately translates into MPC heterogeneity. Since households discount their future labor income at rate $r_t + \lambda_i$, the death rate also limits households' borrowing capacity. These two features establish a close connection between the perpetual youth model and the standard incomplete markets models. In the latter, households that are faced with uninsurable idiosyncratic risk and are close to their borrowing constraint also display high MPCs and have limited capacity to borrow from their future labor income.

The benefit of using an OLG framework instead of an incomplete markets model is that consumption aggregates linearly across households of the same type. Therefore, we can characterize the dynamics of macro variables as if we had only two agents. In particular, the wealth distribution across different types will be summarized by a single variable, $b_{1,t}$, while in an incomplete markets model the wealth distribution is an infinite-dimensional object.

As indicated above, I will assume without loss of generality that $\lambda_2 > \lambda_1$. Intuitively, I want to think of type 1 agents as those who are further away from their borrowing constraints, and the redistributive fiscal policy will transfer resources from these agents to those who are more borrowing constrained and therefore have higher MPCs.

Equilibrium. Now that we have described the setup of the model, we can define an equilibrium in this economy.

[6] Alternatively, we could assume that after time $\tilde{T}$ the Taylor rule will respond with sufficient strength to increases in inflation and the output gap so as to obtain local determinacy.

Definition 1. *Given taxes, government bonds and purchases* $\{T_{i,t}, T_t^g, B_t^g, G_t\}_{t\geq 0, i\in\{1,2\}}$, *a monetary policy rule, and initial bond holdings* $\{B_{i,0}^h\}_{i\in\{1,2\}}$, *an equilibrium is a path for consumption, bonds, labor and prices* $\{C_{i,t}^h, B_{i,t}^h, N_{i,t}^h, \pi_t\}_{t\geq 0, i\in\{1,2\}}$ *such that*

(i) households maximize utility, where household optimality requires satisfying the budget constraint (3), the Euler equation (4), and the labor supply equation (2),

(ii) firms chose their reset price to maximize profits according to (5),

(iii) and the goods and bonds markets clear:

$$\chi C_{1,t} + (1-\chi) C_{2,t} + G_t = Y_t = \hat{A}_t N_t,$$
$$\chi B_{1,t} + (1-\chi) B_{2,t} + B_t^g = 0.$$

Steady state. I will log-linearize the equations around a steady state with no inflation, no redistributive taxes and no government purchases – i.e., $T_i^* = T^{g*} = G^* = \pi^* = 0$, where starred variables are used to describe the steady state.

Looking at equation (4) we can immediately see that the only possible steady state has an interest rate $i^* = r^* = \rho$ and agents hold no debt (i.e., $B_i^* = 0$ for $i = 1, 2$),[7] so despite having overlapping generations with heterogeneous death probabilities, the steady state is the same as if we had an infinitely-lived representative agent with discount factor ρ. The intuition is that, since there is no idiosyncratic risk (other than the death event), there are no shocks that households can insure against by holding financial assets. From equation (4) we get that the λ_i's do not distort the Euler equation when agents do not have savings. Therefore, we obtain a symmetric steady state[8] such that $C_i^* = N_i^* = \left(\frac{W}{P}\right)^* = 1$ for $i = 1, 2$.

Log-linearization. I use lower case letters to denote log-deviations from the steady state, except for bonds, government purchases, and taxes, which will be expressed as a fraction of steady state output:[9]

$$b_{i,t} \equiv \frac{B_{i,t}}{Y^*}, \quad g_t \equiv \frac{G_t}{Y^*}, \quad \tau_{i,t} \equiv \frac{T_{i,t}}{Y^*}.$$

I will assume that the economy always converges back to the same steady state; this means that fiscal policies must be of a temporary nature. Since the main case of interest is

[7] To get $\dot{C}_i = 0$ in (4) it must be that either $r^* - \rho = B_i^* = 0$, or $C_i^* = \frac{\Lambda_i}{r^*-\rho} B_i^*$ but since B_1^* and B_2^* must have opposite signs to obtain market clearing in the bonds market, this latter case is not compatible with a steady state in which both agents have positive consumption.

[8] This result relies on the assumption that net aggregate assets in this economy are zero. If there were a source of outside liquidity (e.g., Lucas trees, or government debt), then the steady state would no longer be symmetric. With positive net aggregate assets, some agents would need to hold non-zero assets. From equation (4) we can see that the value of λ_i would then affect consumption decisions in steady state, and thus the symmetry would be broken.

[9] In the particular case of taxes, instead of a lower case t I will use the letter τ to avoid confusion with the time variable.

the use of fiscal policy for macroeconomic stabilization during liquidity traps, this restriction seems unimportant. It would be relevant, however, if we were focusing on the effects of fiscal policy on the long-run distribution of wealth.

The full system of log-linear equations is left for the Appendix. Combining some of those conditions, we can write the system in a more concise form as a 5-equation New Keynesian model:

$$\dot{c}_t = i_t - \pi_t - \rho + \chi \left(\Lambda_2 - \Lambda_1\right) b_{1,t}, \tag{7}$$

$$\dot{c}_{1,t} = i_t - \pi_t - \rho - \Lambda_1 b_{1,t}, \tag{8}$$

$$\dot{b}_{1,t} = \rho b_{1,t} + \frac{1+\phi}{\phi}\left(c_t - c_{1,t}\right) - \tau_{1,t}, \tag{9}$$

$$\dot{\pi}_t = \rho\pi_t - \mu\left(c_t + \frac{\phi}{1+\phi} g_t\right), \tag{10}$$

$$i_t = \rho + \kappa_{y,t}\left(c_t + g_t\right) + \kappa_{\pi,t}\pi_t. \tag{11}$$

Equations (7) and (8) are the Euler equations for aggregate consumption and consumption of agent 1, respectively. Equation (9) is the law of motion of bonds held by type 1 agents. Note that the term $(c_t - c_{1,t})$ that appears in equation (9) reflects the difference between wage income (which depends on aggregate product) and individual consumption. Equation (10) is the NKPC and equation (11) is the Taylor rule.

Since we are using a log-linear approximation to the equilibrium conditions, redistributive fiscal policy (captured by the path for $\{\tau_{1,t}\}_{t=0}^{\infty}$) and government purchases ($\{g_t\}_{t=0}^{\infty}$) have additive effects, so we can analyze each of them separately.[10] The main focus of this paper is redistributive policies, so in Section 3 I will provide a detailed analytical characterization of the output effect of such policies. The analysis of government purchases multipliers will be considered as an extension in Section 4, both because it is not my main focus and because the heterogeneous OLG structure of my model does not play any relevant role in shaping the effects of government purchases financed with symmetric taxes.

3 Transfer Multipliers

I adopt as the main object of interest the cumulative effect of the redistribution on aggregate output. In particular, I calculate this cumulative effect as the present discounted value (PDV) of the change in aggregate output, using the steady state interest rate to do the discounting - i.e. $\int_0^{\infty} e^{-\rho t} c_t dt$. So rather than analyzing the effect on output at some arbitrary point in time, I will use the PDV as a natural summary statistic to capture the dynamic effects of transfers. Analyzing the effect on the PDV of output will also be crucial

[10]That is, we can first analyze the effect of transfers while assuming $g_t = 0\ \forall t$, and then focus on government purchases while setting $\tau_{i,t} = 0\ \forall i, t$.

to understand the relation between MPCs and welfare once we turn to a normative analysis in Section 5.

Given that we are using a log-linear approximation to the equilibrium, we can express the effect of transfers as present-value multipliers:[11]

$$\int_0^\infty e^{-\rho t} c_t dt = \int_0^\infty m_t^T (\lambda_1, \lambda_2) \times \left(e^{-\rho t} \tau_{1,t}\right) dt.$$

I will refer to $\left\{m_t^T (\lambda_1, \lambda_2)\right\}_{t=0}^\infty$ as the transfer multipliers, which are a function of both time t and the death probabilities λ_1, λ_2.[12] Note that on the right-hand side we have $e^{-\rho t}\tau_{1,t}$ (instead of $\tau_{1,t}$) so that the multipliers apply to transfers of the same PDV.

To start getting some intuition about what determines these multipliers, let us first consider the simplest possible example: there is a one-time redistribution at time $t = 0$, with the interest rate permanently fixed at $r_t = \rho$.

3.1 A neutrality result: one-time transfers in an infinitely lasting liquidity trap with fixed prices

Let us assume that we have an exogenous interest rate $i_t = \rho$ and prices are perfectly rigid (so $\pi_t = 0 \ \forall t$). Suppose that the government does a one-time redistribution at time $t = 0$.[13] Let us also take $\widetilde{T} \to \infty$. Then, we can think of this example as an economy with fixed prices that is stuck forever in a liquidity trap.

Let us first consider the case in which $\lambda_1 = 0$ so that agent 1 is Ricardian. In this case, the multiplier is

$$m_0^T (0, \lambda_2) = \frac{\phi}{1+\phi}, \qquad \forall \lambda_2 > 0.$$

The curious thing is that this multiplier does not depend on λ_2, which means that the difference in MPCs is irrelevant to determine the effect of the transfer on the PDV of output. Having a Ricardian agent plays a crucial role to obtain this result. Since $\lambda_1 = 0$, agent 1's consumption path is completely pinned down by the path of interest rates and the steady state level of consumption, regardless of fiscal policy.[14] Thus, agent 1's consumption does not depend on fiscal policy, implying that for the intertemporal budget constraint to be satisfied,

[11] Since I will always be talking about present-value multipliers, I will henceforth just refer to them as multipliers.

[12] When the government sets a tax $\tau_{1,t}$ it is actually levying only $\chi\tau_{1,t}$ in taxes since only agents of type 1 are paying the tax. Thus, the multipliers m_t^T describe the effect on the PDV of output per dollar charged to type 1 agents. If we wanted to have a multiplier per dollar of government revenue, we would have to multiply m_t^T by $\frac{1}{\chi}$.

[13] A one-time redistribution can be equivalently interpreted as a change in initial bond holdings, or as a path for $\{\tau_{1,t}\}_{t=0}^\infty$ that has a Dirac measure at time $t = 0$.

[14] That is, with the Euler equation we can work backwards from the steady state to obtain the whole consumption path, which will only depend on real interest rates.

the PDV of the life-time income (after taxes) of the agent cannot depend on fiscal policy either. This means that the PDV of aggregate output has to adjust so that the income of the Ricardian agent increases exactly to offset the taxes levied by the government.

To understand this better, suppose for a moment that instead of having a competitive labor market we had that each agent supplies inelastically 1 unit of labor and there is a symmetric labor rationing rule (that is, all households work the same hours, and total hours are demand determined). Since labor is the only input of production and all agents work the same number of hours, the human wealth of agent 1 is equal to the PDV of aggregate consumption.[15] Therefore, the total wealth of agent 1 at time $t = 0$ is given by $B_{1,0} + \mathcal{C}_0^r$, where the first term are the initial bond holdings of the Ricardian agent and the second term is the PDV of aggregate consumption. If we take any two initial values of bond holdings, say $\breve{B}_{1,0} \neq \hat{B}_{1,0}$, it must be that $\breve{B}_{1,0} + \breve{\mathcal{C}}_0^r = \hat{B}_{1,0} + \hat{\mathcal{C}}_0^r$ since the life-time income of agent 1 must be constant to be able to afford the same path of consumption. Therefore, reordering terms we get that the change in the PDV of consumption must be equal to the change in initial bond holdings:

$$\breve{\mathcal{C}}_0^r - \hat{\mathcal{C}}_0^r = -\left(\breve{B}_{1,0} - \hat{B}_{1,0}\right).$$

In this case, we would get a transfer multiplier equal to 1,[16] so why do we get a multiplier $\frac{\phi}{1+\phi} < 1$ when we have a competitive labor market instead of a symmetric labor rationing rule? The reason for this is that wealth effects generate an asymmetric response in labor supply between the two agents. Firstly, note that when $\phi \to \infty$, wealth effects on labor supply are weak and therefore both agents' labor supply responds (almost) symmetrically to the fiscal stimulus. Therefore, the transfer multiplier converges to 1 as in the case with symmetric rationing in the labor market. When instead ϕ is small, there are strong wealth effects on labor supply, so the asymmetry in consumption introduced by the redistribution also leads to a large asymmetry in labor supply. Agent 1 works more and therefore receives (as labor income) a larger fraction of aggregate output than in the symmetric case, then to keep his lifetime income fixed it must be that aggregate output is increasing less than in the symmetric case. This implies that the multiplier must be lower than 1.

While λ_2 may have no effect on the PDV of output when agent 1 is Ricardian, it does have a more subtle effect on the equilibrium path for output. The role of λ_2 here is to determine the timing of the stimulus. When λ_2 is low, type 2 agents do not increase their consumption much at $t = 0$, but this means that they will have more financial assets in future periods. This allows them to maintain their consumption level above steady state for a long period of time. If instead λ_2 is high, there is a large immediate increase in consumption, but it fades away fast.

This is illustrated in Figure 1. The plot displays typical paths for $\{c_t\}_{t=0}^{\infty}$ for two different

[15] This is also true for agent 2 of course, but I am focusing on agent 1.

[16] Note that a change in initial bond holdings is equivalent to a transfer at time $t = 0$.

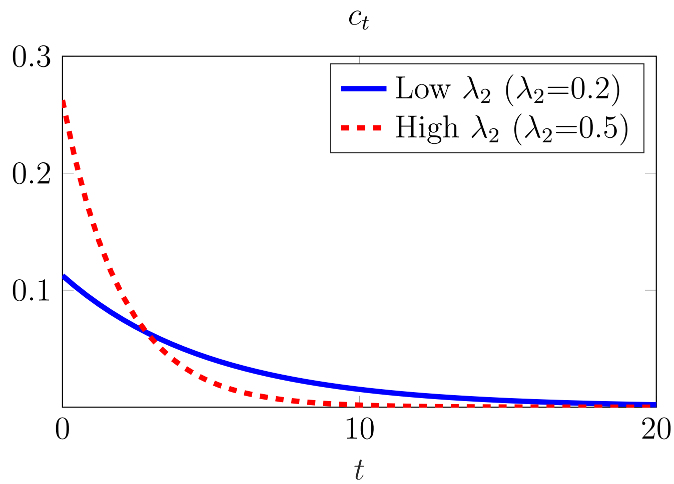

Figure 1 – Typical path for consumption c_t after a unit transfer from agent 1 to agent 2 at $t = 0$ in an economy with rigid prices, an infinitely lasting liquidity trap, fixed interest rates and a Ricardian agent 1.

values of λ_2. We can see that with a low λ_2 the increase in aggregate consumption is initially smaller, but is more persistent that with a high λ_2.

This can be interpreted as a neutrality result: in this case, MPCs matter in determining the timing of the stimulus, but they do not matter in determining the cumulative effect on output. This provides a useful benchmark, since we can now analyze how deviating from it allows MPCs to affect the size of the transfer multipliers.

Note that this initial example displays a discontinuity. When agent 1 is Ricardian, if we had $\lambda_2 = \lambda_1 = 0$ redistributive fiscal policy would have no effect on output (i.e. the transfer multiplier would be zero), but for any $\lambda_2 > \lambda_1 = 0$ we get the same transfer multiplier $\frac{\phi}{1+\phi}$. This discontinuity is, however, a knife-edge result for two reasons: one is that we are taking agent 1 to be exactly Ricardian, and the other has to do with the way we take the limit $\widetilde{T} \to \infty$. The analysis of this latter consideration is postponed till Section 3.4.2, where I will study an economy that exits the liquidity trap in finite time.

Let us now consider $\lambda_1 > 0$ so that neither of the agents are Ricardian. In this case, we can show (see Appendix) that the transfer multiplier is given by

$$m_0^T(\lambda_1, \lambda_2) = \theta(\lambda_1, \lambda_2) \frac{\phi}{1+\phi},$$

where

$$\theta(\lambda_1, \lambda_2) \equiv 1 - \frac{\Lambda_1}{L}$$

and L is defined as a weighted average $L \equiv (1-\chi)\Lambda_1 + \chi\Lambda_2$. Figure 2 plots $\theta(\lambda_1, \lambda_2)$ as a function of λ_2 for fixed values of λ_1. If $\lambda_1 > 0$, we have that $\theta < 1$ and θ is increasing in λ_2. The intuition is the following. In the case $\lambda_1 = 0$, when the government takes resources

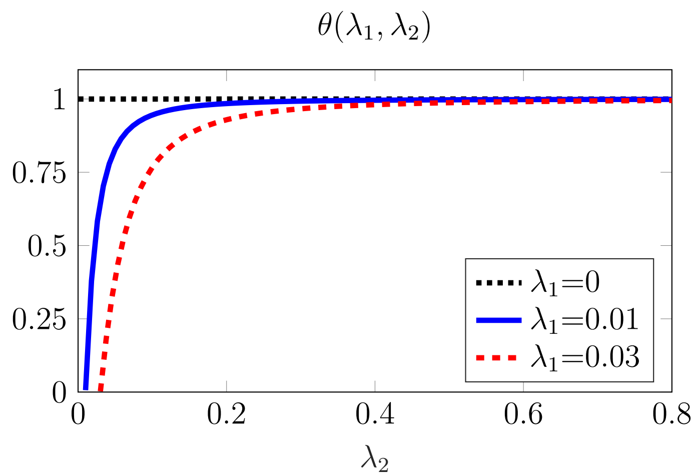

Figure 2 – Transfer multiplier corresponding to the example with a fixed interest rate, rigid prices, and an infinitely lasting liquidity trap (function $\theta(\lambda_1, \lambda_2)$)

away from the Ricardian agent at time $t = 0$, he "recovers" these resources through the general equilibrium stimulus to the economy which increases his future labor income. Being Ricardian he is indifferent about the timing in which he obtains this extra income. However, when $\lambda_1 > 0$ agent 1 is shortsighted, so he will under-react today to future increases in output. This means that the more backloaded the stimulus to the economy, the less the agent will consume today, leading to lower aggregate demand. We can think of the ratio $\frac{\Lambda_1}{L}$ that appears in the definition of $\theta(\lambda_1, \lambda_2)$ as measuring how much agent 1 under-reacts to the increase in future output from the redistributive fiscal policy.

Note that $\theta(\lambda_1, \lambda_2)$ satisfies

$$\begin{cases} \lim_{\lambda_2 \to \infty} \theta(\lambda_1, \lambda_2) = 1 \\ \lim_{\lambda_2 \to \lambda_1^+} \theta(\lambda_1, \lambda_2) = 0 \end{cases}, \quad \forall \lambda_1 > 0.$$

The first of these limits tells us that when agent 2 spends all the transfer immediately (i.e. agent 2 is approximately Hand-to-Mouth, or HtM) there is no under-reaction on the part of agent 1, so we get the same multiplier as when $\lambda_1 = 0$. On the other hand, the second limit tells us that when $\lambda_2 \to \lambda_1^+$, agent 2's demand is so backloaded that agent 1's under-reaction to future income leads him to decrease his demand in such a way that output remains unchanged. This means that we no longer find the same discontinuity as we had when agent 1 was Ricardian.

Yet, although knife-edge, there is also a sense in which the case $\lambda_1 = 0$ is telling us something relevant. We can see in Figure 2 that the function $\theta(\lambda_1, \lambda_2)$ starts at zero (when $\lambda_2 \to \lambda_1 > 0$), increases very fast and then becomes almost flat. So as long as we are considering cases in which one of the agents is close to Ricardian ($\lambda_1 \cong 0$) and the other one is far from being Ricardian relative to agent 1, we find that the transfer multiplier is quite

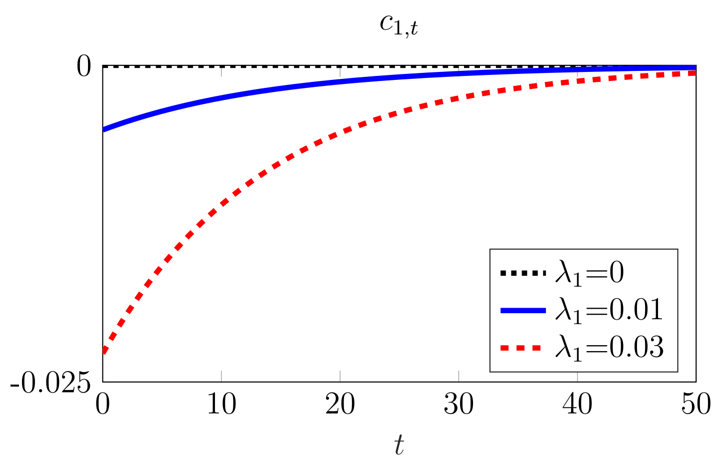

Figure 3 – Typical path for consumption $c_{1,t}$ after a unit transfer from agent 1 to agent 2 at $t = 0$ in an economy with rigid prices, an infinitely lasting liquidity trap and fixed interest rates.

insensitive to the MPC of agent 2.[17]

It is also important to understand how the path of consumption of type 1 agents is affected by the transfer. Figure 3 displays the path for $\{c_{1,t}\}_{t=0}^{\infty}$ (after a one-time transfer of unit size at time $t = 0$) for different values of λ_1 and a fixed value of λ_2. When $\lambda_1 = 0$, the consumption of agent 1 is unaffected by the transfer. However, as λ_1 increases, there is a stronger under-reaction of agent 1 to future labor income, and therefore consumption of agent 1 declines more as a result of the transfer.

3.2 Back to the general case

Now that we understand the baseline scenario with fixed interest rates and prices we can address the general case. One useful property of this model is that the equilibrium path of bond holdings does not depend on monetary policy.

Lemma 1. *For given initial bond holdings $b_{1,0}$ and a path of transfers $\{\tau_{1,t}\}_{t=0}^{\infty}$, the equilibrium path for $\{b_{1,t}\}_{t=0}^{\infty}$ solves the second order linear differential equation*

$$\dot{b}_{1,t} = \rho b_{1,t} - \frac{1+\phi}{\phi} L \int_t^{\infty} b_{1,s} ds - \tau_{1,t},$$

where $L \equiv (1-\chi)\Lambda_1 + \chi\Lambda_2$. Therefore, the equilibrium path for bonds $\{b_{1,t}\}_{t=0}^{\infty}$ does not

[17] We can see this more formally by computing the derivative

$$\frac{\partial\theta}{\partial\Lambda_2} = \frac{\chi\Lambda_1}{(\chi\Lambda_2 + (1-\chi)\Lambda_1)^2}$$

and noting that it will be close to zero if λ_2 is large relative to λ_1.

depend on monetary policy.

If we replace the bond holdings obtained in Lemma 1 into equation (7), we are left with a smaller system (that we can solve by hand) that characterizes the equilibrium paths for $\{c_t, \pi_t, i_t\}$:

$$\dot{c}_t = r_t - \varrho_t, \tag{12}$$

$$\dot{\pi}_t = \rho\pi_t - \mu c_t, \tag{13}$$

$$i_t = \rho + \kappa_{y,t}c_t + \kappa_{\pi,t}\pi_t, \tag{14}$$

where I am defining

$$\varrho_t \equiv \rho - \chi\left(\Lambda_2 - \Lambda_1\right) b_{1,t}.$$

From Lemma 1 we know that $\{\varrho_t\}_{t=0}^{\infty}$ does not depend on monetary policy. This means that this economy with heterogeneous perpetual youth dynasties admits an as-if representative agent (RA) as in Werning (2015), where $\{\varrho_t\}_{t=0}^{\infty}$ is the discount factor of the RA. The nice thing about this model is that from Lemma 1 we can obtain an analytical characterization of the as-if discount factor, while in Bewley-Huggett-Aiyagari economies it is typically impossible to obtain such a characterization even when they admit an as-if RA. Since the elasticity of output with respect to the interest rate will not depend on the death probabilities, this model will be subject to the "forward guidance puzzle" in the same way as the standard RANK model.

While we could directly solve (12)-(14) for particular specifications of monetary policy, I will take a more indirect route which I believe is more helpful to interpret the results. Instead of directly solving for the transfer multipliers, what I will do is express them as a function of the transfer multipliers that we would obtain in a model in which agent 2 is Hand-to-Mouth instead of a perpetual youth dynasty.

In the Appendix, I show that when we have a Hand-to-Mouth agent 2, the equilibrium is characterized by the following system of equations

$$\dot{c}_t = r_t - \left(\rho - \frac{\phi}{1+\phi}\dot{\tau}_{1,t}\right), \tag{15}$$

$$\dot{\pi}_t = \rho\pi_t - \mu c_t, \tag{16}$$

$$i_t = \rho + \kappa_{y,t}c_t + \kappa_{\pi,t}\pi_t. \tag{17}$$

That is, the NKPC and Taylor rule are unchanged with respect to equations (13) and (14), and the demand side of the economy is now characterized by equation (15) instead of (12).[18]

[18]Equation (15) can in fact be obtained as the limit of equation (12) when we take the limit $\lambda_2 \to \infty$. See Appendix for a proof.

Note that this system does not depend on λ_1, λ_2. By construction, it correspond to an economy in which agent 2 if HtM, so of course it cannot depend on λ_2, but a priori it could depend on λ_1. However, when agent 2 is HtM agent 1 cannot have any bond holdings (from market clearing, if agent 2 has zero bonds, then so does agent 1), then from equation (4) we can see that λ_1 does not affect the Euler equation of agent 1.

This means that the system (15)-(17) is the same as if we had a standard TANK model. Given that we are using a log-linear approximation to the equilibrium of the TANK model, we can again characterize the effect of transfers on the PDV of output through multipliers:

$$\int_0^\infty e^{-\rho t} c_t dt = \int_0^\infty m_t^{TANK} \times \left(e^{-\rho t} \tau_{1,t}\right) dt.$$

I will refer to $\left\{m_t^{TANK}\right\}_{t=0}^\infty$ as the TANK-transfer multipliers to distinguish them from the transfer multipliers $\left\{m_t^T\right\}_{t=0}^\infty$ that we defined before.

So what determines the path for $\left\{m_t^{TANK}\right\}_{t=0}^\infty$? In this version of the model, the TANK-transfer multipliers are completely determined by monetary policy. In a more general setup, there could be other features of the model that also affect the time path of TANK-transfer multipliers, but since for simplicity we are using a bare-bones NK model, the TANK-transfer multipliers will be fully determined by the specification of monetary policy.[19]

The behavior of the TANK-transfer multipliers has been analyzed in previous papers, for example in Farhi and Werning (2016). Therefore, by expressing the multipliers in my OLG model as a function of the TANK-transfer multipliers we can establish a connection with an object that has already been studied in the literature.

Proposition 1 describes the relation between the transfer multipliers in the OLG model and the TANK-transfer multipliers.

Definition 2. *Denote as $\left\{b_{1,t|\bar{t}}\right\}_{t=0}^\infty$ the equilibrium path of bonds (characterized in Lemma 1) when there is only a transfer at time $t = \bar{t}$ of magnitude $\tau_{1,\bar{t}} = e^{\rho \bar{t}}$.*

Proposition 1. *The transfer multipliers in the OLG model are proportional to a weighted average of the TANK-transfer multipliers:*

$$m_{\bar{t}}^T\left(\lambda_1, \lambda_2\right) = \theta\left(\lambda_1, \lambda_2\right) \int_0^\infty \omega_{t|\bar{t}}\left(\lambda_1, \lambda_2\right) m_t^{TANK} dt,$$

where the weights satisfy $\int_0^\infty \omega_{t|\bar{t}}\left(\lambda_1, \lambda_2\right) dt = 1$ and are defined as

$$\omega_{t|\bar{t}}\left(\lambda_1, \lambda_2\right) = \frac{\partial\left(e^{-\rho t} b_{1,t|\bar{t}}\right)}{\partial t}.$$

[19] For example, in Section 4 I will analyze a version of the model with capital and investment, and in that case we will find that the TANK-transfer multipliers depend on the investment technology.

Proposition 1 gives us a useful decomposition of the transfer multipliers in the OLG model into three elements: the TANK-transfer multipliers, the weights[20] $\left\{\omega_{t|\bar{t}}\right\}_{t=0}^{\infty}$, and the coefficient of proportionality θ. The TANK-transfer multipliers act as a "sufficient statistic" for monetary policy and they do not depend on λ_1, λ_2. Meanwhile, the weights $\left\{\omega_{t|\bar{t}}\right\}_{t=0}^{\infty}$ and the coefficient θ are determined by λ_1, λ_2 but they do not depend on monetary policy.

As discussed before, there is some subtlety to the role of MPCs here. Giving a transfer to a low MPC agent means that the stimulus is small at the time of the transfer, but is more persistent over time. The weights $\left\{\omega_{t|\bar{t}}\right\}_{t=0}^{\infty}$ inform us about the timing in which the stimulus takes place. To obtain the transfer multiplier, we have to compute a weighted average of TANK-transfer multipliers that puts more weight on the periods in which the stimulus is stronger.

In Proposition 1 the weights $\left\{\omega_{t|\bar{t}}\right\}_{t=0}^{\infty}$ are defined as the derivative of the present value of financial assets held by agent 1. Intuitively, this says that the stimulus is stronger in the periods in which agent 2 is running down his financial assets at a faster pace.

Lastly, the coefficient θ is the same as we had in the initial example, and we know that it is always lower than 1. Therefore, we can think of θ as applying a "penalty" (relative to the weighted average of TANK-transfer multipliers) for backloaded stimulus due to the shortsightedness of perpetual youth agents. It is worth noting that the same penalty is applied regardless of the timing of transfers. A priori, one could have expected to obtain a different result: we could have conjectured (incorrectly) that redistributions that take place further in the future should carry a heavier penalty since agents are short-sighted. The reason why this conjecture is incorrect is the following: while it is true that perpetual youth agents discount more heavily increases in output that take place further in the future, they also discount the taxes that will be levied in the future to pay for those transfers. Since I am assuming that the government has a balanced budget so that taxation and redistribution take place at the same time, then both effects cancel out, and the same penalty is applied regardless of the timing of the transfer.

Figure 4 plots typical paths for $\left\{b_{1,t|\bar{t}}\right\}_{t=0}^{\infty}$ (top row) and weights $\left\{\omega_{t|\bar{t}}\right\}_{t=0}^{\infty}$ (bottom row) when there is a redistribution at some time $\bar{t}$. On the left-hand column we take $\bar{t} = 0$ while on the right-hand column we take $\bar{t} \gg 0$. In each case, the plot is done for two different values of λ_2.

When there is a redistribution at time $\bar{t} = 0$, agent 1's bond holdings are initially negative and converge monotonically to zero as agent 2 increases his consumption and agent 1 decreases his. In this case, the weights $\omega_{t|\bar{t}}$ peak at time $t = 0$ and also decay monotonically. Importantly, when λ_2 is higher the distribution of the weights $\omega_{t|\bar{t}}$ is more concentrated around $t = 0$; intuitively, since a more short-sighted agent 2 will run down his financial assets at a faster rate, we have to give more weight to the TANK-transfer multipliers that are closer to time $t = 0$.

[20] To simplify the notation, I will generally omit the dependence of θ and $\omega_{t|\bar{t}}$ on λ_1, λ_2.

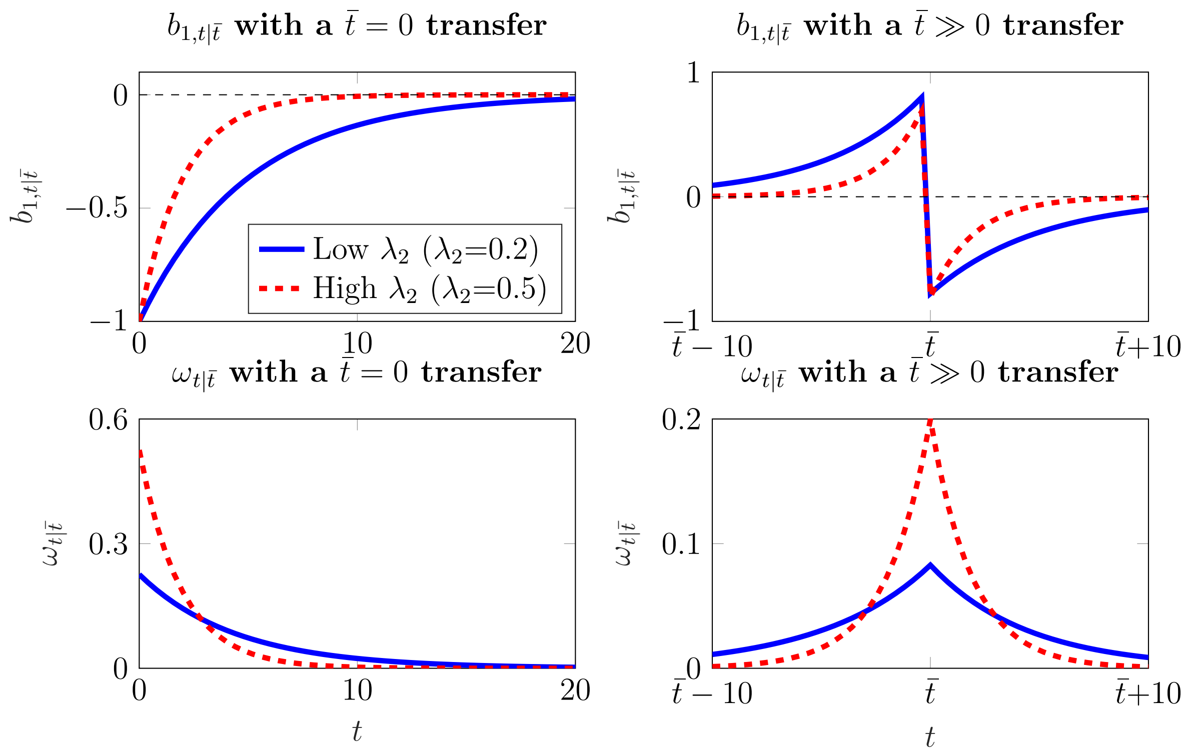

Figure 4 – Typical paths for bond holdings, $b_{1,t|\bar{t}}$, and weights, $\omega_{t|\bar{t}}$, when there is a one time redistribution from agent 1 to agent 2 at time $\bar{t}$. On the left-hand column we set $\bar{t} = 0$ and on the right-hand column we set $\bar{t} \gg 0$.

When there is a transfer at time $\bar{t} \gg 0$, since agents know from $t = 0$ that there will be a transfer in the future and they have access to financial markets, they can adjust their consumption even before the transfer actually takes place. In particular, agent 2 will increase consumption and agent 1 will decrease consumption in anticipation of the future transfer. This means that before $\bar{t}$ agent 2 will accumulate debt while agent 1 builds up a positive stock of financial assets. At the time of the transfer agent 1 jumps from having a positive financial position to a negative one, and from there onwards we converge monotonically back to the steady state with no debts. In this case, the weights $\omega_{t|\bar{t}}$ will be computing an average of TANK-transfer multipliers that is both forward- and backward-looking relative to the time $\bar{t}$ of the transfer. Just as in the left-hand side graphs, we find that when λ_2 is higher the distribution of weights is more concentrated around the time of the transfer.

With these results in mind, we can now address the question of how the transfer multipliers depend on the difference in MPCs between the two agents. From a policy perspective, it is particularly interesting to characterize the sign of the derivative $\frac{\partial m_t^T(\lambda_1,\lambda_2)}{\partial \lambda_2}$ as this will tell us how the effect on the PDV of output will depend on the MPC of the agents that we are targeting to receive the government transfers. Proposition 2 characterizes this derivative under the assumption that the TANK-transfer multipliers are exponentially increasing or decreasing. Although this simplifying assumption does not apply to every possible monetary policy, it does help build intuition and will apply in some of the cases of interest analyzed in Section 3.4.

Proposition 2. *Suppose that the TANK-transfer multipliers are of the form* $m_t^{TANK} = m_0^{TANK} e^{\xi t}$*, with* $m_0^{TANK} > 0,\ \xi \in \mathbb{R}$*. Then, the transfer multipliers in the OLG model are such that*

- *for transfers done at time* $\bar{t} = 0$*, the derivative* $\frac{\partial m_0^T(\lambda_1,\lambda_2)}{\partial \lambda_2}$ *is positive (negative) if* ξ *is below (above) a threshold* $\Upsilon(\lambda_1, \lambda_2)$*, where this threshold satisfies* $\Upsilon \geq 0$*,* $\frac{\partial \Upsilon}{\partial \lambda_2} < 0$*, and* $\lim_{\lambda_2 \to \infty} \Upsilon(\lambda_1, \lambda_2) = 0\ \forall \lambda_1 > 0$.

- *for transfers done at some time* $\bar{t} \gg 0$ *we have*

$$\frac{\partial m_{\bar{t}}^T(\lambda_1, \lambda_2)}{\partial \lambda_2} \begin{cases} > 0 \text{ if } \xi \in (\gamma, \bar{\gamma}) \\ = 0 \text{ if } \xi \in \{\gamma, \bar{\gamma}\} \\ < 0 \text{ if } \xi \notin [\gamma, \bar{\gamma}] \end{cases},$$

where we define

$$\gamma \equiv \frac{\rho - \sqrt{\rho^2 + 4\frac{1+\phi}{\phi}\Lambda_1}}{2}, \quad \bar{\gamma} \equiv \frac{\rho + \sqrt{\rho^2 + 4\frac{1+\phi}{\phi}\Lambda_1}}{2}.$$

This is an important result, so let us discuss in detail what Proposition 2 is saying. Consider first the case of a redistribution done at time $t = 0$. Take any fixed value $\lambda_1 = \bar{\lambda}_1$, and let us analyze how $m_0^T(\bar{\lambda}_1, \lambda_2)$ depends on λ_2. Using Proposition 1, we can think of the increase in λ_2 as having two effects on the transfer multiplier $m_0^T(\bar{\lambda}_1, \lambda_2)$:

(a) There is an increases in the coefficient $\theta(\lambda_1, \lambda_2)$, which intuitively means that, since the transfer is spent faster, there is a lower "penalty" due to the under-reaction of short-sighted agents to future income. This effect always works towards increasing $m_0^T(\bar{\lambda}_1, \lambda_2)$ regardless of the value of ξ.

(b) As we saw in Figure 4, the weights $\omega_{t|\bar{t}=0}$ become more concentrated around $t = 0$, which means that when the transfer is spent faster we have to put more weight on the TANK-transfer multipliers that are closer to $t = 0$. The direction of this effect depends on the sign of ξ: when $\xi < 0$, it works towards increasing $m_0^T(\bar{\lambda}_1, \lambda_2)$ because the weighted average puts more weight on higher values of the TANK-transfer multipliers, and the opposite happens when $\xi > 0$.

Therefore, when $\xi < 0$ we find that both effects (a) and (b) work in the same direction, so $m_0^T(\bar{\lambda}_1, \lambda_2)$ will be monotonically increasing in λ_2. I will refer to this as case (*i*), and it is depicted in panel (*i*) of Figure 5.

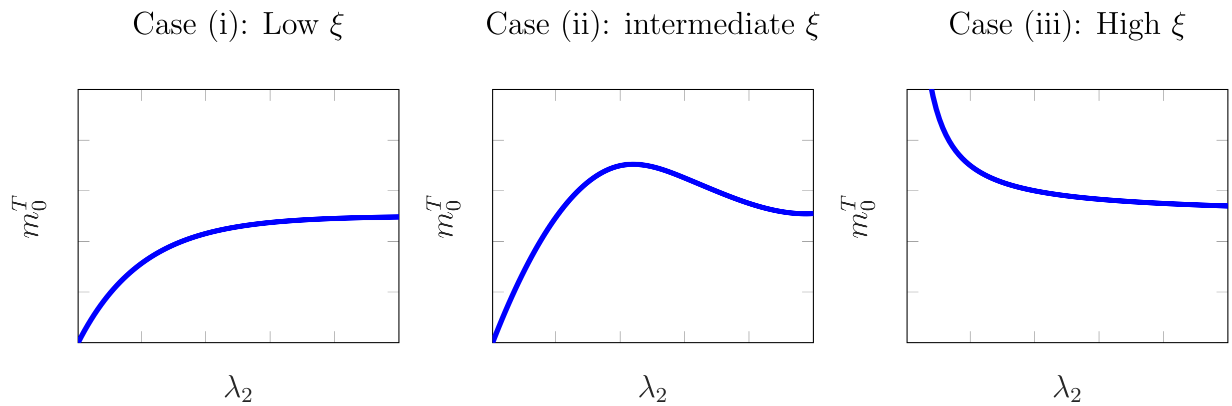

Figure 5 – The three panels depict the three possible shapes of the multiplier m_0^T as a function of λ_2, for a fixed value of $\lambda_1 > 0$. In Case (i) the TANK-transfer multipliers are decreasing ($\xi < 0$); in Case (ii) the TANK-transfer multipliers are increasing but the growth rate is not too large (intermediate ξ); while in Case (iii) the TANK-transfer multipliers are increasing at a high rate (high ξ).

When $\xi > 0$ however, we find that effects (a) and (b) work in opposite directions, so the total effect is ambiguous. From the first part of Proposition 2, we can see that there are two possible cases. If ξ is sufficiently large, effect (b) will always dominate, so $m_0^T\left(\bar{\lambda}_1, \lambda_2\right)$ is monotonically decreasing with respect to λ_2; the parameter region in which this case occurs is $\xi > \Upsilon\left(\bar{\lambda}_1, \bar{\lambda}_1\right)$.[21] This case is depicted in panel (iii) of Figure 5. The other possibility to consider is $\xi \in \left(0, \Upsilon\left(\bar{\lambda}_1, \bar{\lambda}_1\right)\right)$; in this parameter region we find that effect (a) dominates for low values of λ_2 (so $m_0^T\left(\bar{\lambda}_1, \lambda_2\right)$ is increasing) while effect (b) dominates for high values of λ_2 (so $m_0^T\left(\bar{\lambda}_1, \lambda_2\right)$ is decreasing).[22] This last case is depicted in panel (ii) of Figure 5.

Therefore, the "conventional wisdom" that the effect of a redistribution on output is stronger when the difference in MPCs is higher only holds when the TANK-transfer multipliers decrease over time. Meanwhile, when the TANK-transfer multipliers are increasing, the relation between MPCs and output can in fact be reversed. The reason for this reversal is that giving a transfer to a low MPC agent will generate a long-lived increase in consumption, and in situations in which back-loaded stimuli have high TANK-transfer multipliers, this will result in high transfer multipliers in the OLG model. As we will see in Section 3.4.1 an empirically relevant case in which this happens is when there is a liquidity trap and prices are sticky.

Let us now consider the second part of Proposition 2. If the redistribution is done at some

[21] Since the threshold $\Upsilon\left(\bar{\lambda}_1, \lambda_2\right)$ is decreasing in λ_2, the condition $\xi > \Upsilon\left(\bar{\lambda}_1, \bar{\lambda}_1\right)$ immediately implies that $\xi > \Upsilon\left(\bar{\lambda}_1, \lambda_2\right)\ \forall \lambda_2$.

[22] Using the fact that $\Upsilon\left(\bar{\lambda}_1, \lambda_2\right)$ is decreasing in λ_2 and the limit $\lim_{\lambda_2 \to \infty} \Upsilon\left(\bar{\lambda}_1, \lambda_2\right) = 0$ we know there must exist some $\tilde{\lambda}_2 \in \left(\bar{\lambda}_1, \infty\right)$ such that $m_0^T\left(\bar{\lambda}_1, \lambda_2\right)$ is increasing if $\lambda_2 < \tilde{\lambda}_2$ and decreasing if $\lambda_2 > \tilde{\lambda}_2$.

This shape of $m_0^T\left(\bar{\lambda}_1, \lambda_2\right)$ as a function of λ_2 is very related to the behavior of θ, as depicted in Figure 2. When λ_2 is close to $\bar{\lambda}_1$ the function θ is very steep with respect to λ_2 (so effect (a) is strong), but for higher values of λ_2 it becomes very flat (so effect (a) weakens).

time $\bar{t} \gg 0$, increasing λ_2 has the same two effects as before: there is an increase in θ, and the weights $\omega_{t|\bar{t}}$ become more concentrated around the time of the transfer $t = \bar{t}$. However, the direction of this latter effect is not straightforward. The reason is that, when the transfer takes place at $\bar{t} \gg 0$, the weighted average is both forward- and backward-looking (with respect to $\bar{t}$) since agent 2 will start increasing his expenditure even before receiving the transfer. Therefore, when the weights $\omega_{t|\bar{t}}$ get more concentrated around $\bar{t}$, we are reducing the weight from both high and low values of the TANK-transfer multipliers, which makes the direction of this effect ambiguous. The second part of Proposition 2 tells us that the transfer multiplier will decrease with λ_2 if ξ is outside of the interval $(\gamma, \bar{\gamma})$. The intuition for having such an interval is the following: if the TANK-transfer multipliers are fast increasing or decreasing (i.e. ξ large in absolute value), an increase in λ_2, by concentrating the weights $\omega_{t|\bar{t}}$ around time $\bar{t}$, is putting less weight on the highest TANK-transfer multipliers (which could either be to the right of $\bar{t}$ if $\xi > 0$ or to the left of $\bar{t}$ if $\xi < 0$), which decreases the transfer multiplier.

In fact, in the case $\lambda_1 = 0$, we have that $(\gamma, \bar{\gamma}) = (0, \rho)$. The parameter ρ is a small positive number since it is equal to the steady state interest rate (so it would be about 0.02 if time is measured in years). Therefore, in the case with a Ricardian agent we would get that the transfer multiplier $m^T_{\bar{t}}$ is decreasing in λ_2 for any value of ξ except in the narrow interval $(0, \rho)$.

3.3 Discussion of assumptions

In this section I will discuss the role of some of the simplifying assumptions made in the baseline model. I will first discuss the assumption that redistributive fiscal policies are always budget-balanced. Then, I will consider the case of more general household preferences (i.e., preferences with an elasticity of intertemporal substitution different than one), and finally I will consider the possibility of having more than two types of households.

3.3.1 Deficit-financed transfers

In the baseline model I assumed that redistributive fiscal policies are always budget-balanced. We know that in practice, however, these policies are generally deficit-financed rather than a direct transfer between households. Therefore, it is important to understand the relation between deficit-financed and budget-balanced transfers.

Proposition 3 establishes that any arbitrary path of taxes $\{T_{1,t}, T_{2,t}\}_{t=0}^{\infty}$ can always be decomposed into a budget-balanced transfer from type 1 to type 2 agents and a redistribution across generations of type 1 agents. Since I am studying the equilibrium under a log-linear approximation, the effect of these two components is additive. The output effect of the former component is characterized by the transfer multipliers that we studied above. Meanwhile, if agent 1 is Ricardian, the latter component (i.e., the redistribution across generations of type 1) has no effect on output. Therefore, when $\lambda_1 = 0$ assuming budget-balanced transfers is without loss of generality. If $\lambda_1 > 0$, though, the balanced budget assumption is not irrelevant. However, the output effect of changing the timing of taxes in OLG models has been extensively studied in the literature (e.g., Blanchard (1985)), so in this paper I will focus the analysis on the redistribution across types.

Proposition 3. *Suppose the government sets an arbitrary path of taxes $\{T_{1,t}, T_{2,t}\}_{t=0}^{\infty}$. We can do a decomposition of $T_{1,t}$ as $T_{1,t} = \left(T_{1,t} - \hat{T}_{1,t}\right) + \hat{T}_{1,t}$ where $\hat{T}_{1,t}$ satisfies:*

(i) $\chi \hat{T}_{1,t} + (1-\chi) T_{2,t} = 0$, *and*

(ii) $\int_0^{\infty} e^{-\int_0^t r_s ds} \left(T_{1,t} - \hat{T}_{1,t}\right) dt = 0.$

Therefore, $\left\{\hat{T}_{1,t}, T_{2,t}\right\}_{t=0}^{\infty}$ is a budget-balanced transfer and $\left\{T_{1,t} - \hat{T}_{1,t}\right\}_{t=0}^{\infty}$ is a transfer across generations of type 1 agents.

Proof. Define $\hat{T}_{1,t} = -\frac{1-\chi}{\chi} T_{2,t}$. By construction it satisfies $\chi \hat{T}_{1,t} + (1-\chi) T_{2,t} = 0$. For part (ii) of the proposition, we have that

$$\int_0^{\infty} e^{-\int_0^s r_s ds} \left(T_{1,t} - \hat{T}_{1,t}\right) dt = \frac{1}{\chi} \int_0^{\infty} e^{-\int_0^s r_s ds} \left(\chi T_{1,t} + (1-\chi) T_{2,t}\right) dt$$

From the governments' budget constraint (6) it is immediate that the right-hand side of this expression must be equal to zero. □

3.3.2 Isoelastic utility

Assuming logarithmic preferences has the benefit that the death rate λ_i also has a clear interpretation as an MPC (recall that the MPC out of current income of type i agents is $\rho + \lambda_i$). If we use more general preferences, we lose this straightforward interpretation, but otherwise the assumption of logarithmic preferences is inconsequential.

Suppose that we had isoelastic preferences $u(C) = \frac{C^{1-\sigma}}{1-\sigma}$. In this case, we can obtain an Euler equation for type i agents that is analogous to equation (4):[23]

$$\dot{C}_{i,t} = \frac{1}{\sigma} (r_t - \rho) C_{i,t} - \frac{\lambda_i}{\Delta_{i,t}} B_{i,t}, \tag{18}$$

$$\dot{\Delta}_{i,t} = -1 - \frac{1}{\sigma} \left[(1-\sigma)(r_t + \lambda_i) - (\rho + \lambda_i)\right] \Delta_{i,t}. \tag{19}$$

The steady state is not affected by σ, since both types of agents hold zero bonds and the interest rate is equal to ρ. Therefore, we can log-linearize around the same steady state

[23]This result is derived in Section III of Blanchard (1985).

we had in the baseline model (with the only caveat that now we have one more variable, $\Delta_i^* = (\rho + \lambda_i)^{-1}$). Since $B_i^* = 0$, when we do a first order approximation of equation (18) we can replace $\Delta_{i,t}$ for Δ_i^*, and therefore equation (19) is irrelevant to characterize consumption dynamics (up to a first order). The log-linear approximation of (18) is

$$\dot{c}_{i,t} = \frac{1}{\sigma}(r_t - \rho) - \Lambda_i b_{i,t}.$$

We can see from this equation that the parameter σ only affects the elasticity of consumption with respect to the interest rate, but has no interaction with λ_i. This implies that all the results that we obtain in the case $\sigma = 1$ have a straightforward generalization to the case $\sigma \neq 1$.

3.3.3 $I \geq 2$ types of agents

Up to now we have assumed that there are only two types of agents. This simplifying assumption is inconsequential from a conceptual perspective, but has the benefit that it allows us to characterize the equilibrium in closed form.

Suppose now that we have $I \geq 2$ types of OLG dynasties, each with a death probability $\lambda_i \geq 0$ and each with mass χ_i (with $\sum_{i=1}^{I} \chi_i = 1$). The Euler equation and the budget constraint for each type of agent are

$$\dot{c}_{i,t} = i_t - \pi_t - \rho - \Lambda_i b_{i,t}, \qquad for\ i = 1, ..., I$$

$$\dot{b}_{i,t} = \rho b_{i,t} + \frac{1+\phi}{\phi}(c_t - c_{i,t}) - \tau_{i,t}, \ . \qquad for\ i = 1, ..., I$$

Just as we did before, we can combine these expressions to obtain a path for bonds $\{b_{1,t}, ..., b_{I,t}\}_{t=0}^{\infty}$ that does not depend on monetary policy. We can then write the equilibrium as in (12)-(14), where the as-if discount factor is now a function of $\{b_{1,t}, ..., b_{I,t}\}_{t=0}^{\infty}$, and use equations (15)-(17) to establish a relation between the multipliers in the OLG model and the multipliers in the TANK model, as we did in Proposition 1. The only problem with having I agents instead of 2 is that the equilibrium path for bonds does not have a closed form solution, which makes it more challenging to provide an analytical characterization of the equilibrium.

3.4 Transfer multipliers with particular monetary policy rules

Using the general characterization of the transfer multipliers that we derived in section 3.2, it is quite straightforward to analyze the behavior of these multipliers under specific assumptions on monetary policy. For each particular monetary policy rule, what we have to do is to first compute the TANK-transfer multipliers, and then calculate a weighted average

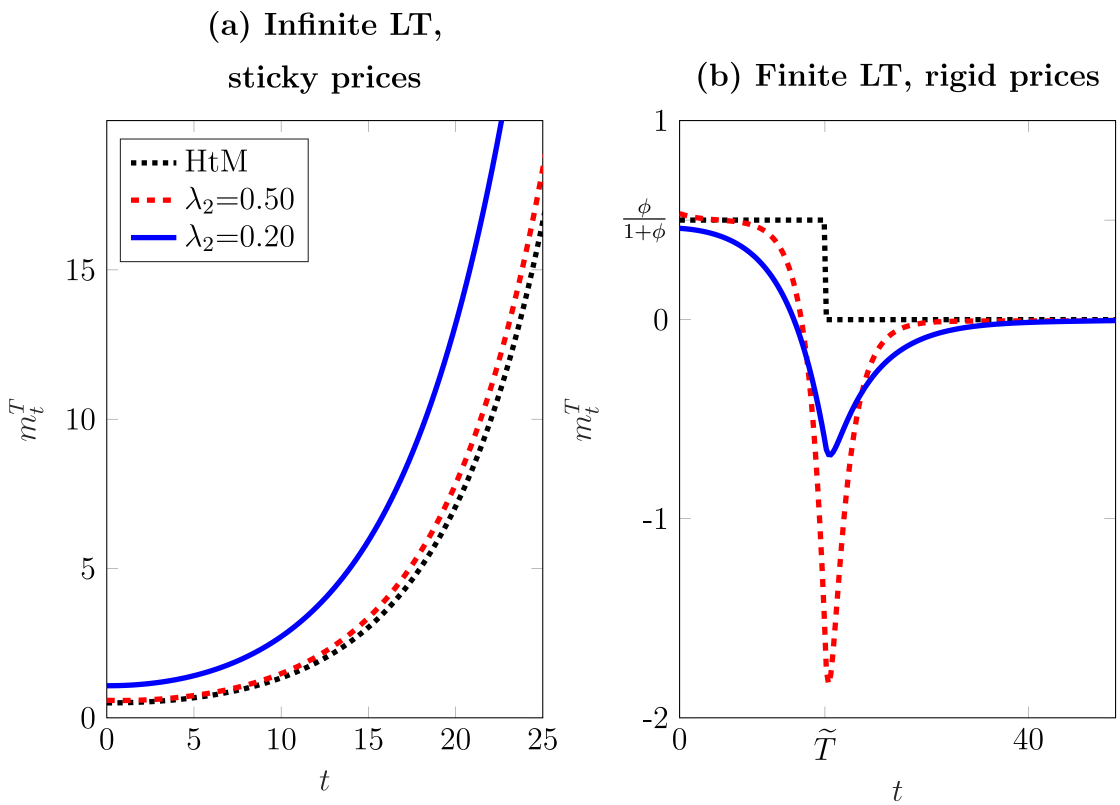

Figure 6 – The two graphs plot the paths for m_t^{TANK} and m_t^T as a function of t, for the two monetary policy rules analyzed in Section 3.4. In each case, the paths for m_t^T are plotted for two values of λ_2 and assume $\lambda_1 = 0$. Note: LT stands for liquidity trap.

to obtain the transfer multipliers in the OLG model. In each case, I will provide an analytical expression for the TANK-transfer multipliers and for the transfer multiplier for time $t = 0$. For the latter, I will write it in the form

$$m_0^T = \Gamma \times \theta \frac{\phi}{1+\phi}$$

so that by characterizing the coefficient Γ we can easily compare the multipliers across the different assumptions on monetary policy. The analytical expression for the transfer multipliers for an arbitrary $\bar{t}$ is usually somewhat complicated and therefore not too helpful to gain intuition, so I will not provide these expressions. The monetary policies that I will consider are: an infinitely lasting liquidity trap with sticky prices, and an economy that exits the liquidity trap in finite time with rigid prices. I leave for the Appendix the case of a standard Taylor rule. For each of these policies I plot in panels (a) and (b), respectively, of Figure 6 the TANK-transfer multipliers together with the transfer multipliers for different values of λ_2 (the plots assume that $\lambda_1 = 0$).

3.4.1 Infinitely lasting liquidity trap, with sticky prices

Suppose that the nominal rate is fixed at $i_t = \rho$ and prices are sticky. This case is meant to represent a liquidity trap in which the Central Bank is not able to adjust the nominal rate in response to shocks. This is analogous to the case analyzed in Farhi and Werning (2016).

With a HtM agent 2, the system of equations is

$$\dot{c}_t = -\pi_t + \frac{\phi}{1+\phi}\dot{\tau}_{1,t},$$
$$\dot{\pi}_t = \rho\pi_t - \mu c_t.$$

Denote the eigenvalues of the system as

$$\nu \equiv \frac{\rho - \sqrt{\rho^2 + 4\mu}}{2}, \quad \bar{\nu} \equiv \frac{\rho + \sqrt{\rho^2 + 4\mu}}{2}.$$

where $\nu < 0 < \bar{\nu}$.

We can show that the TANK-transfer multipliers are

$$m_t^{TANK} = \frac{|\nu|\, e^{\bar{\nu}t} + \bar{\nu}e^{-|\nu|t}}{|\nu| + \bar{\nu}}\frac{\phi}{1+\phi}.$$

These multipliers are increasing in t and take a value $\frac{\phi}{1+\phi}$ in $t = 0$. The reason for this behavior is a standard forward guidance effect. Suppose that we make a transfer to the HtM agent at some time $\bar{t}$. With a fixed nominal rate, the increase in aggregate demand leads to a *drop* in real interest rates by increasing inflation (i.e., we have an "inverted Taylor principle"). Since a decrease in interest rates at time $\bar{t}$ leads to an increase in demand from agent 1 for every period $t \in [0, \bar{t}]$, we get that the larger $\bar{t}$, the larger the TANK-transfer multiplier.

Computing the weighted average of the TANK-transfer multipliers to obtain the transfer multiplier in the OLG economy for $t = 0$, we find that

$$m_0^T(\lambda_1, \lambda_2) = \Gamma(\lambda_1, \lambda_2) \times \theta(\lambda_1, \lambda_2)\frac{\phi}{1+\phi},$$

where

$$\Gamma(\lambda_1, \lambda_2) \equiv \left(1 - \frac{\mu}{\frac{1+\phi}{\phi}L}\right)^{-1}.$$

In this case, the coefficient Γ is higher than one.[24] As explained above, when the economy is in a liquidity trap and prices are sticky, we have an inverted Taylor principle, so the increase in inflation due to the fiscal stimulus amplifies the direct effect by reducing the real interest

[24] We need to assume that $\mu < \frac{1+\phi}{\phi}L$, otherwise the weighted average does not converge.

rate.

To analyze how the transfer multipliers depend on λ_2, let us assume for simplicity that $\lambda_1 = 0$, but the generalization to $\lambda_1 \geq 0$ would be straightforward. For the multiplier m_0^T we *cannot* immediately apply Proposition 2 because the TANK-transfer multipliers are a weighted average of two exponentials, while the proposition allowed for only one exponential. However, by looking at the expression for Γ above, it is immediate that it is decreasing in λ_2. Since a more impatient agent 2 will tend to front-load consumption, the amplification effect due to the inverted Taylor principle is weaker, so we find that larger differences in MPCs lead to weaker effects on output.

For transfers that take place at a time $\bar{t} \gg 0$, we *can* apply Proposition 2 because, although the TANK-transfer multipliers are a weighted average of two exponentials, one of the two exponentials is decreasing in t, so for large $\bar{t}$ we can assume it is zero. Since the growth rate of the exponential that does not converge to zero is $\bar{\nu} > \rho$, from the second part of Proposition 2 we have that the transfer multipliers are decreasing in λ_2.[25]

Panel (a) of Figure 6 plots the TANK-transfer multipliers and the transfer multipliers for various values of λ_2. We verify graphically that when λ_2 increases, the whole function $\left\{m_t^T\right\}_{t=0}^{\infty}$ shifts downwards.

While I have focused the analysis on the impact of transfers on the present value of output, in this case it is even possible for a higher λ_2 to be associated with a smaller value of aggregate consumption at the time of the transfer. If we make a transfer from agent 1 to agent 2 at $t = 0$, the partial equilibrium effect on c_0 is increasing in λ_2, but the forward guidance effect generated by the stimulus can be sufficiently strong to overpower the partial equilibrium effect. The Appendix provides an example of parameter values in which we get this inverse relation between the difference in MPCs and c_0.

3.4.2 Finite liquidity trap, with rigid prices

In the case that we just analyzed, the nominal interest rate was fixed forever, but it would be more realistic to assume that the economy will exit the liquidity trap in finite time. Thus, let us assume that $\widetilde{T}$ is finite - that is, at a finite time $\widetilde{T}$ the economy will jump from a fixed nominal interest rate regime to a neoclassical regime. To isolate the effects of a finite liquidity trap from the effects of the "inverted Taylor principle" that we considered in Section 3.4.1, I will assume here that prices are completely rigid.

For $t \geq \widetilde{T}$ we have $c_t = 0$, since monetary policy will respond in such a way that aggregate output is fully stabilized. For $t \in \left[0, \widetilde{T}\right]$ let us assume that the interest rate is fixed at ρ.

[25]Recall from Proposition 2 that when $\lambda_1 = 0$, we find that the multipliers m_t^T are decreasing in λ_2 whenever the growth rate ξ of the TANK-transfer multiplier is such that $\xi \notin [0, \rho]$.

If agent 2 is HtM, integrating equation (15) we find that

$$\int_0^\infty e^{-\rho t} c_t dt = \left[\int_0^{\widetilde{T}} e^{-\rho t} \tau_{1,t} dt - \frac{1 - e^{-\rho \widetilde{T}}}{\rho} \tau_{1,\widetilde{T}} \right] \frac{\phi}{1+\phi}.$$

The transfers done before $\widetilde{T}$ have the same TANK-transfer multiplier ($\frac{\phi}{1+\phi}$) as if the liquidity trap were infinitely-lasting: since the HtM agent 2 spends the transfers immediately, it is irrelevant how long after the transfer the economy will exit the liquidity trap. Meanwhile, the transfers done after $\widetilde{T}$ have a zero TANK-transfer multiplier because the monetary authority will adjust the interest rate to cancel them out.

Transfers at $t = \widetilde{T}$ have a very particular effect. Even though transfers are a flow over time, so that the money transferred at any single instant has zero measure, the transfers that take place at the instant $t = \widetilde{T}$ have a non-zero effect on the PDV of output. The reason why we get this result is that, because the interest rate path for $t \in \left[0, \bar{T}\right]$ is fixed, the path of agent 1's consumption for $t \in \left[0, \widetilde{T}\right]$ is fully determined by his consumption at time $t = \widetilde{T}$ (that is, $c_{1,\widetilde{T}}$ pins down agent 1's consumption path).[26] In this case, we get that the transfers $\tau_{1,\widetilde{T}}$ have a *negative* effect on the PDV of output because a transfer from agent 1 to agent 2 will lead to an increase in the interest rate, which depresses agent 1's consumption all throughout the liquidity trap, which in turn decreases agent 2's consumption by reducing his labor income.

If we compute the transfer multiplier for $t = 0$ we find[27]

$$m_0^T (\Lambda_1, \Lambda_2) = \Gamma\left(\lambda_1, \lambda_2, \widetilde{T}\right) \times \theta(\lambda_1, \lambda_2) \frac{\phi}{1+\phi},$$

where

$$\Gamma\left(\lambda_1, \lambda_2, \widetilde{T}\right) \equiv 1 - \frac{\bar{v} e^{-|v|\widetilde{T}} - |v| e^{-\bar{v}\widetilde{T}}}{\rho}. \tag{20}$$

In this case Γ is lower than 1. Intuitively, Γ tells us what fraction of the fiscal redistribution is going to be spent during the liquidity trap. It is only the expenditure that takes place before $\widetilde{T}$ that stimulates output, since the expenditure that is left for after the economy exits the liquidity trap is "canceled out" by the increase in interest rates.

[26] While this reasoning may seem to imply that the government could obtain any fiscal stimulus it wants with zero measure transfers, that could require arbitrarily low interest rates at time $t = \widetilde{T}$. For simplicity I am assuming that from $t = \widetilde{T}$ onwards the monetary authority can set any interest rate it needs to keep output at potential, but in a more realistic setup we would have to allow for an occasionally binding ZLB, which would limit the stimulus that can be obtained with these zero measure transfers.

[27] Contrary to the TANK-transfer multipliers, the transfer multipliers are such that any zero-measure transfer has a null effect on output. The reason for this dichotomy is that the OLG agents run down their financial wealth smoothly over time for any finite λ, while the HtM agents immediately spend all of the transfers they receive.

If we look at how Γ changes with $\widetilde{T}$ (for fixed λ_1, λ_2), we find that

$$\lim_{\widetilde{T} \to 0} \Gamma = 0, \quad \lim_{\widetilde{T} \to \infty} \Gamma = 1, \quad \frac{\partial \Gamma}{\partial \widetilde{T}} > 0 \; \forall \lambda_1, \lambda_2.$$

If $\widetilde{T}$ is small, then most of the expenditure will take place after $\widetilde{T}$, so the transfer multiplier will be close to zero. The larger $\widetilde{T}$ is, the larger the fraction of expenditure that will take place during the liquidity trap, so Γ is increasing in $\widetilde{T}$.

Meanwhile, if we look at how Γ depends on λ_2, we find that

$$\frac{\partial \Gamma}{\partial \lambda_2} > 0,$$

which was also to be expected, since a more impatient agent 2 will spend the transfer faster, and therefore more of that expenditure will occur during the liquidity trap.

The reader might recall that, when we analyzed the initial example, it was mentioned that one of the reasons we found a discontinuity at $\lambda_1 = \lambda_2 = 0$ had to do with the way we were taking the limit $\widetilde{T} \to \infty$. Expression (20) clarifies exactly when we get such a discontinuity: if $\lambda_1 = 0$ and we think of $\widetilde{T}$ as being a function of λ_2, when we take the limit $\lambda_2 \to 0$, we need $\lim_{\lambda_2 \to 0} |\upsilon| \widetilde{T} = +\infty$ for Γ to converge to 1. That is, as we take λ_2 closer and closer to zero, we need $\widetilde{T}$ to grow to infinity sufficiently fast so that all of the transfer is spent during the liquidity trap. If, on the contrary, we have $\lim_{\lambda_2 \to 0} |\upsilon| \widetilde{T} = 0$, then the transfer multiplier converges to zero as $\lambda_2 \to 0$. That is, when $\widetilde{T}$ is either fixed or grows slowly as $\lambda_2 \to 0$, part of agent 2's expenditure is left for after the end of the liquidity trap, which eliminates the discontinuity.

Lastly, let us discuss the behavior of the transfer multipliers for transfers done at times other than $t = 0$. Panel (b) of Figure 6 plots the transfer multipliers for various values of λ_2 (assuming $\lambda_1 = 0$). We can see that transfers done early on have multipliers close to $\frac{\phi}{1+\phi}$ because they tend to be fully spent during the liquidity trap, while transfers done at times closer to $\widetilde{T}$ have lower multipliers. Transfers done shortly before $t = \widetilde{T}$ can have negative multipliers because when a large part of the transfer is not spent by time $\widetilde{T}$, monetary policy will respond by increasing the interest rate after the end of the liquidity trap, which will lead to a drop in $c_{1,\widetilde{T}}$, which in turn depresses the whole path of consumption of agent 1 during the liquidity trap.

4 Extensions

In this section I will consider some extensions of the analysis we did in Section 3. I will first characterize government purchases multipliers, and establish a relation with transfer multipliers. We will see that, when there are no transfers, the model behaves in the same way as the standard representative agent New Keynesian model. Therefore, I will analyze government purchases for completeness, but the heterogeneous OLG structure of my model does not play any relevant role in shaping the effects of government purchases financed with symmetric taxes.

I will then consider a simple example where monetary policy has a redistribution channel. The purpose of this extension is to show that the transfer multipliers analyzed in Section 3 are not just useful to characterize fiscal transfers, but are also useful to study other government policies that have redistributive effects.

Finally, I will analyze a version of the model with capital and investment. I will show that in this case the TANK-transfer multipliers are exponentially increasing over time. Therefore, we find that, just as in a liquidity trap with sticky prices, transfer multipliers decrease with the difference in MPCs between agents.

4.1 Government purchases multipliers

So far we have focused on the effect of redistributive fiscal policy while assuming that $g_t = 0\ \forall t$. Now consider the opposite case: assume that $\tau_{1,t} = 0\ \forall t$ and let us analyze the effect of government purchases. For now, let us maintain the assumption that the government always has a balanced budget. I will focus on the effect of g_t on the PDV of output (which now includes both consumption and government purchases), which, given the linearity of the model can be expressed as

$$\int_0^\infty e^{-\rho t} y_t dt = \int_0^\infty m_t^G \times \left(e^{-\rho t} g_t\right) dt,$$

where I will refer to $\left\{m_t^G\right\}_{t=0}^\infty$ as the Government purchases multipliers (G multipliers for short).

Since the government charges the same taxes to all agents to pay for its purchases, there will not be any debts in equilibrium. Recalling that in the perpetual youth model the Euler equation is only distorted by bond holdings, we find that when $\tau_{1,t} = 0\ \forall t$ the parameters λ_1, λ_2 do not affect the equilibrium. Therefore, the G multipliers will be the same as in a standard RANK model.

Just as we did when we characterized the transfer multipliers, rather than directly solving for the G multipliers it is helpful to find a relation between them and the TANK-transfer multipliers. Without redistributive transfers, we can write the system (7)-(11) more concisely in matrix form as

$$\begin{bmatrix} \dot{y}_t \\ \dot{\pi}_t \end{bmatrix} = \begin{bmatrix} \kappa_{y,t} & (\kappa_{\pi,t} - 1) \\ -\mu & \rho \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \dot{g}_t \\ \mu \frac{1}{1+\phi} g_t \end{bmatrix}. \tag{21}$$

Note that the homogeneous part of the system (21) is exactly the same as in the model with a HtM agent 2, described by equations (15)-(17). However, with government purchases we have a non-homogeneous term in the NKPC that did not appear in the case of transfers to a HtM agent 2. This tells us that government purchases not only shift the aggregate demand curve, but they also shift the NKPC.

For any given level of output y_t, when there are more government purchases there must be less private consumption (due to the accounting identity $y_t = c_t + g_t$), so this means consumers are poorer and therefore supply more labor (because leisure is a normal good), which in turn depresses wages and prices. Depending on how monetary policy responds to inflation, this shift in the NKPC can lead to either an increase or a decrease in real interest rates, which would decrease or increase the G multipliers respectively.

To see this more clearly, let us make the simplifying assumption that monetary policy is

$$i_t = \rho + \kappa_\pi \pi_t.$$

Proposition 4 establishes the relation between the G multipliers and the TANK-transfer multipliers with this monetary policy rule. What we find is that when the nominal interest rate responds more than one to one to inflation, then the shift in the NKPC due to the negative wealth effect on workers, which tends to generate less inflation, has a positive impact on output. On the contrary, when the nominal interest rate responds less than one to one with inflation (so there is an inverted Taylor principle, as would be the case in a liquidity trap for example), the shift in the NKPC dampens the effect of government purchases as compared to transfers.

Proposition 4. *If monetary policy follows the Taylor rule $i_t = \rho + \kappa_\pi \pi_t$, then the G multipliers are*

$$m_t^G = \chi \frac{1+\phi}{\phi} \left(\frac{1}{\chi} m_t^{TANK} \right) + \Psi(t), \tag{22}$$

where the sign of $\Psi(t)$ is given by

$$sign\{\Psi(t)\} = sign\{\kappa_\pi - 1\}.$$

Note that, in order to compare the effects of transfers with the effect of government purchases, it does not quite make sense to directly compare m_t^G with m_t^{TANK} because the former is a multiplier *per dollar of government expenditure* while the latter is a multiplier *per dollar charged to type 1 agents*. To make them comparable, we can express both multipliers on a *per dollar of government expenditure* basis, which requires multiplying m_t^{TANK} by $\frac{1}{\chi}$. That is why equation (22) has $\frac{1}{\chi} m_t^{TANK}$ on the right-hand side.

The G multipliers and the TANK-transfer multipliers differ for two reasons related to wealth effects in labor supply. In the first term of (22), we have to multiply the TANK-transfer multiplier by $\frac{1+\phi}{\phi}$ because government purchases are financed with symmetric taxes, so there is a symmetric response in labor supply, while in the case of transfers the labor supply response of both agents is asymmetric (the agent that receives the transfer works less than the agent that pays the tax).

Meanwhile, the second term of (22), $\Psi(t)$, appears because of the shift in the NKPC that I explained above. If $\kappa_\pi = 1$, the NKPC is irrelevant to determine the path of output (i.e. inflation does not appear in the first equation of (21)), so the term $\Psi(t)$ is zero. When $\kappa_\pi > (<)1$, since higher government purchases are associated with lower inflation -for a given level of y_t-, they will also be associated through the Taylor rule with lower (higher) interest rates and therefore higher (lower) output, so we get a positive (negative) value for $\Psi(t)$.

Even when these two effects are shut down (which we can obtain by taking $\kappa_\pi = 1$, and either $\phi \to \infty$ or assuming symmetric labor rationing), we find that $m_t^G = \chi\left(\frac{1}{\chi}m_t^{TANK}\right)$, so when compared on a *per dollar of government expenditure* basis, the G-multipliers are smaller than the TANK-transfer multipliers. The reason for this is that there is a Keynesian multiplier effect on the transfer that does not apply to government purchases. That is, when we transfer \$1 from agent 1 to a HtM agent, there is a "first-round" in which output increases by \$1, which in turn increases the labor income of the HtM agent, leading to a "second-round" of output increase, which again increases the labor income of the HtM agent, and so on *ad infinitum*. However, this multiplier effect is not present with a budget-balanced increase in government purchases.

Relation between transfer multipliers and government purchases financed up-front. The last point I want to make about the comparison between transfers and purchases is that there is a close connection between transfers at $t = 0$ in the OLG model and government purchases that are financed by an up-front tax. Assume that prices are rigid and there is symmetric labor rationing, so that transfers and purchases do not differ due to their different effects on the labor market. Let us consider two economies: one is the economy with two OLG dynasties that we have been discussing so far, and another one is an economy with only type 1 agents and the government (i.e. this second economy is a special case of the baseline model with $\chi = 1$). If in this second economy (i.e. the one with only type 1 agents and the government) we tax all agents by \$1 at $t = 0$ and then the government spends it according to

$$e^{-\rho t}g_t = \frac{\partial\left(e^{-\rho t}b_{1,t|0}\right)}{\partial t}$$

we can show that the effect on the PDV of output is equal to $m_0^T(\lambda_1, \lambda_2)$. In some sense, this result is almost trivial, since we are replacing transfers to a perpetual youth agent 2 with taxes charged by a government whose purchases replicate the consumption behavior of that agent 2. Yet, there is still a useful intuition here: transferring resources to an agent who is not HtM creates a time lag between the tax and the stimulus to output, very much akin to a government that taxes up-front and then spends those resources over time. If $\lambda_1 = 0$, the timing of taxes would be irrelevant, but if $\lambda_1 > 0$ agents are non-Ricardian and therefore taxing up-front tends to decrease output.

This analogy between transfers and government purchases will prove useful for the quantitative exercises in Section 6, as I will use it to get some clean comparisons between the behavior of perpetual youth and Bewley-Huggett-Aiyagari (BHA) agents.

4.2 Redistributive effect of government policies

While the analysis in Section 3 was done under the assumption that the government directly implements a transfer between types of agents, in many cases we can apply the same results to other government policies that have a redistributive effect.

An example of this is a monetary policy shock that has a redistribution channel due to agents' heterogeneous exposure to interest rates, as in Auclert (2017). In this case, we can decompose the total effect of the monetary policy shock into two parts: an interest rate shock and a redistribution. The latter can be characterized using the transfer multipliers described in Section 3.

I will show this in a particularly simple example. Suppose that in the steady state of our OLG economy, type 1 agents have a long position in a bond with long duration and they have a short position in a bond with short duration, such that the value of their net position is zero. For example, we could assume that the long duration bond pays dividends $\delta_t^L = \left(\alpha^L + \rho\right) e^{-\alpha^L t}$ and the short duration bond pays dividends $\delta_t^S = \left(\alpha^S + \rho\right) e^{-\alpha^S t}$, with $\alpha^S > \alpha^L > 0$. When the interest rate is $r_t = \rho \; \forall t$ the value of a unit of each of these bonds is $V_0^j = 1, \; j \in \{L, S\}$, so if type 1 agents own and owe one unit of each bond respectively, they will have a zero net financial position. Suppose now that there is a contractionary monetary shock at $t = 0$:

$$i_t = \rho + \kappa_y c_t + \kappa_\pi \pi_t + \varepsilon_t, \text{ where } \varepsilon_t = \varepsilon_0 e^{-\eta t}, \; \varepsilon_0 > 0.$$

This shock will affect the value of the long and short bonds asymmetrically. Denoting the values of the financial assets after the shock as $\breve{V}_0^L, \breve{V}_0^S$, type 1 agents will now have a non-zero initial financial position $\breve{B}_{1,0} = \breve{V}_0^L - \breve{V}_0^S$. The value of $\breve{B}_{1,0}$ is endogenous since it depends on the policy rate, which in turn depends on the endogenous variables c_t, π_t.

Under a log linear approximation, the equilibrium of this economy can be obtained as the sum of two solutions:

1. the equilibrium when we have the monetary policy shock ε_0, but there are zero initial bond holdings, $b_{1,0} = 0$; plus

2. the equilibrium when there is no monetary policy shock (i.e. $\varepsilon_0 = 0$) but with initial bond holdings $b_{1,0} = \frac{\breve{B}_{1,0}}{Y^*}$.

That is, we can decompose the equilibrium as the sum of a monetary policy shock plus a redistribution. The first of these effects will be the same as in a standard RANK model.[28] Meanwhile, the second effect is a redistribution just as the ones we have been analyzing so far and therefore its effect is described by the transfer multiplier m_0^T.

Thus, this example shows that the transfer multipliers obtained in Section 3 have a wider applicability that the direct transfer policies considered before.

4.3 Capital and investment

In Proposition 2 we saw that a necessary condition for the relation between MPCs and the multiplier m_0^T to become negative is for the TANK-transfer multipliers to increase over time. Then in Section 3.4.1 we found an example in which this occurs. However, the reader may wonder if this is a rare occurrence or if there are other (empirically relevant) circumstances in which the TANK-transfer multipliers are also increasing over time. I will briefly discuss here another case in which we obtain increasing TANK-transfer multipliers, by introducing capital and investment. The Appendix provides more details about this version of the model.

To simplify things, I will assume that the production technology is linear in capital

$$Y_t = AK_t$$

and consider only transfers done at $t = 0$. Given that there is no labor income, perpetual youth agents are no longer "credit constrained", in the sense that they discount the future dividends that accrue to the capital they own at a rate r instead of a rate $r+\lambda$.[29] This means that in this version of the model the perpetual youth structure plays no role, and we could actually assume that agents are Ricardian with heterogeneous discount factors, denoted as $\rho_i \equiv \rho + \lambda_i$. Therefore, the steady state is not symmetric anymore but instead will have agent 1 owning all capital.

The law of motion for capital is

$$\frac{\dot{K}_t}{K_t} = \varphi\left(\iota_t\right),$$

where ι_t is investment per unit of capital, and $\varphi\left(.\right)$ is increasing and strictly concave to represent adjustment costs, and satisfies $\varphi'\left(0\right) = 1$. For simplicity, I will assume that there is no depreciation. Households can freely trade capital at a price of Q_t per unit.

When there is a recession, suppose that capital is under-utilized, so actual production will be $Y_t = A\eta_t K_t$, where η_t is the utilization rate. Let us also assume that the price of the consumption good is fully rigid (and normalized to 1) and that the interest rate is

[28] Since there are no bonds at $t = 0$, and we know already that monetary policy does not affect the path of bonds, then bonds will be zero in every time period.

[29] This is because they can sell their capital, but cannot sell the future labor income of their descendants.

constant at $r_t = \rho_1$ $\forall t$ (so that TANK-transfer multipliers will only be time dependent due to investment, not due to monetary policy). The price of capital however is assumed to be perfectly flexible. A simple New Keynesian micro-foundation for this version of the model can be found in Caballero and Simsek (2019).

Assume that $A = \rho_1$ so that we have a well-defined steady state with no investment. In steady state, the value of capital is $Q^* = 1$, but the stock of capital is not pinned down by the equilibrium equations, so assume without loss that there is some arbitrary value K^* around which we will log-linearize the equilibrium.

Since investing in capital is risk-free, households must be indifferent between investing in capital and in bonds, so both must yield the same return. Therefore, we get the following arbitrage condition:

$$r_t = \frac{C_t}{Q_t K_t} + \varphi\left(Q_t\right) + \frac{\dot{Q}_t}{Q_t}. \tag{23}$$

The first term of (23) can be interpreted as the dividend payments of capital, the second term is the growth rate of the stock of capital, and the last term is the appreciation of the price of the asset. Adding up the three terms we get the total return to holding capital, which must be the same as the interest rate of bonds.

With this setup, we can show that the TANK-transfer multipliers are[30]

$$\frac{1}{\chi} m_t^{TANK} = 1 + \frac{e^{(1+\rho_1)t} - 1}{1 + \rho_1},$$

so we find that they are increasing over time. As shown in the arbitrage condition above, when the government transfers resources to a HtM agent, the increase in consumption also increases the dividends to capital, thus raising the price of capital and therefore investment. Given that we are using an AK model, any increase in investment has a permanent effect on capital, thus increasing steady state consumption. Now, why is it that more backloaded transfers have a stronger effect on consumption? The reason is similar to a forward guidance effect: a transfer that takes place at some time $\bar{t}$ increases the price of capital not only at time $\bar{t}$ but also for every period $t \in [0, \bar{t}]$, which in turn increases investment for all of those periods. Therefore, a more backloaded transfer increases investment for a longer time interval, thus having a stronger permanent effect on the stock of capital.

Finally, we can express the $t = 0$ transfer multiplier as a weighted average of the TANK-transfer multipliers:

$$m_0^T\left(\lambda_1, \lambda_2\right) = \int_0^\infty \omega_{t|0} m_t^{TANK} dt,$$

where the weights $\omega_{t|0}$ satisfy $\int_0^\infty \omega_{t|0} dt = 1$ and are defined as

$$\omega_{t|0} \equiv \rho_2 e^{-\rho_2 t}.$$

[30] In this case, the multipliers describe the effect on the PDV of consumption, not output.

Note that in this case we do not have a "penalty" coefficient θ because without labor income both agents behave as Ricardian, so they are indifferent about the timing of their income as long as the PDV is the same. Since the multipliers $\left\{m_t^{TANK}\right\}_{t=0}^{\infty}$ are increasing over time and the weights $\left\{\omega_{t|0}\right\}_{t=0}^{\infty}$ become more concentrated around $t = 0$ when ρ_2 is higher, we can immediately conclude that a higher MPC of agent 2 leads to a lower present-value transfer multiplier, just as we had in the case of a liquidity trap with sticky prices in Section 3.4.1.

5 Welfare

In Section 2 we did a positive analysis, considering the effects of transfers on output, but we did not say anything about which policies were desirable from a welfare perspective. In this Section, I want to analyze the effects of redistributive fiscal policy on welfare, with a particular focus on the meaning of MPCs for optimal policy.

I will continue to use the same model as in Section 2. For simplicity, let us assume that there are no government purchases and focus on redistributive policy. Since the steady state of the model is efficient, I will take a linear-quadratic approach: I will derive a second-order approximation to welfare and analyze optimal policy using the log-linear approximation to the equilibrium equations that we have been using so far.

I will assume that the social welfare function (SWF) is utilitarian, with equal weights for all agents (this means giving equal weight both to agents of different types and to different generations of the same type). If we denote by V_i^t the lifetime utility of a household of type i born at time t, and by V_i^{-0} the lifetime utility of a household of type i that was already alive at time $t = 0$, the SWF is

$$SWF = \chi \left[V_1^{-0} + \lambda \int_0^{\infty} e^{-\rho t} V_1^t dt\right] + \\ + (1-\chi)\left[V_2^{-0} + \lambda \int_0^{\infty} e^{-\rho t} V_2^t dt\right].$$

That is, we have to sum the lifetime utility of the households who were already alive at time $t = 0$ and the utility of the households that are born at every instant t. An equivalent, but more convenient, way to express this same SWF is

$$SWF = \int_0^{\infty} e^{-\rho t}\left[\chi u_{1,t} + (1-\chi) u_{2,t}\right] dt,$$

where

$$u_{i,t} = \int_0^1 u\left(C_{i,t}^h, N_{i,t}^h\right) dh,$$

where $h \in [0,1]$ indexes the households of type i alive at time t. That is, we can obtain the SWF by first summing period utilities over the cross-section of households alive at every instant t, and then summing this over time.

Taking a second order approximation to the social welfare function around the steady state we obtain the loss function[31]

$$\mathcal{L}_0\left(\{x_t, x_{1,t}, \pi_t, b_{1,t}\}_{t=0}^{\infty}; \lambda_1, \lambda_2\right) = \int_0^{\infty} e^{-\rho t} \left\{ \begin{array}{c} \frac{\varepsilon}{\zeta(\rho+\zeta)}\pi_t^2 + (1+\phi)\, x_t^2 + \\ +\frac{\chi}{1-\chi}\frac{1+\phi}{\phi}\left[(x_t - x_{1,t})^2 + L b_{1,t}^2\right] \end{array} \right\} dt, \tag{24}$$

where $x_t, x_{1,t}$ are the aggregate output gap and the gap in consumption of agent 1, defined as

$$x_t \equiv c_t - \check{c}_t,$$
$$x_{1,t} \equiv c_{1,t} - \check{c}_{1,t}.$$

I define the natural allocation $\{\check{c}_t, \check{c}_{1,t}\}_{t=0}^{\infty}$ and the natural interest rate $\{\check{r}_t\}_{t=0}^{\infty}$ as the equilibrium that we obtain when there are no redistributive taxes and monetary policy implements the same interest rate as in a neoclassical regime with no inflation. In this section, I will allow the TFP A_t to be time dependent in order to create time variation in the natural allocation and interest rate.[32]

The social planner cares both about macro stabilization and about distribution. The terms on the first line of the loss function (the ones corresponding to π^2 and x^2) are familiar from the New Keynesian literature,[33] and reflect the planner's desire to have low inflation and low output gap. The terms on the second line appear because redistributive fiscal policy creates cross-sectional heterogeneity in consumption and labor. This dispersion can be decomposed into a "between" and a "within" components. The "between" component reflects the dispersion of consumption between the type 1 and type 2 agents and is captured by the term $(x_t - x_{1,t})^2$, while the "within" component captures heterogeneous consumption across generations of the same type. This "within" heterogeneity is captured by the square of bond holdings, $b_{1,t}^2$, because what distinguishes old generations from newborns is that the

[31] As far as I am aware, the only other paper that uses a second order approximation to the welfare function with perpetual youth agents is Nisticò (2016).

[32] As discussed before, when there are no redistributive taxes (and there are no initial debts) the model behaves in the same way as the standard RANK model, so $\{\check{c}_t, \check{c}_{i,t}, \check{r}_t\}$ will be such that $\check{c}_{i,t} = \check{c}_t = a_t$,and $\check{r}_t = \dot{a}_t + \rho$.

Note that there is a subtle difference between what I am defining here to be the natural equilibrium and what we usually call natural equilibrium in the RANK model. Here, the natural equilibrium is not just the allocation that we would get with flexible prices (i.e. in a neoclassical regime), it is the allocation that we would get with both flexible prices *and* no transfers nor initial debts.

[33] See, for example, Galí (2015).

former hold bonds while the latter are born with no financial wealth.[34]

We now need to write the equilibrium equations (7)-(11) in terms of gaps. In the Appendix, I show that the system that we obtain is:

$$\dot{x}_t = i_t - \pi_t - \check{r}_t + \chi\left(\Lambda_2 - \Lambda_1\right) b_{1,t}, \tag{25}$$

$$\dot{x}_{1,t} = i_t - \pi_t - \check{r}_t - \Lambda_1 b_{1,t}, \tag{26}$$

$$\dot{b}_{1,t} = \rho b_{1,t} + \frac{1+\phi}{\phi}\left(x_t - x_{1,t}\right) - \tau_{1,t}, \tag{27}$$

$$\dot{\pi}_t = \rho \pi_t - \mu x_t, \tag{28}$$

$$i_t = \rho + \kappa_{y,t} x_t + \kappa_{\pi,t} \pi_t. \tag{29}$$

If there were no constraints on monetary policy, the planner would be able to implement the natural allocation and therefore have a zero welfare loss.[35] However, if we have a binding lower bound on interest rates for example, the natural allocation can no longer be implemented and there will be a trade off between macro stabilization and asymmetry in the distribution. By looking at loss function (24), we can already anticipate some results about optimal policy. In general, doing some redistributive fiscal policy will be optimal, since in the equilibrium without government intervention the terms in the second line are zero (symmetric allocation) while the terms in the first line are non zero, so on the margin there is zero cost of using redistribution to obtain macro stabilization. However, it will not be optimal to use fiscal policy to get full macro stabilization either, since we would then have zeros in the first line and positive values in the second line, so we could reduce the asymmetries in the distribution with zero marginal cost on the macro stabilization side.

Let us consider the optimal policy problem of a planner that has full commitment to chose redistributive policy, but takes the monetary policy rule as given[36]. The optimization

[34] A possible concern is that when we "penalize" heterogeneity across generations, we are taking the OLG structure of the model too literally instead of thinking of it just as a modeling device. While this is a valid concern, perhaps an alternative interpretation of the term $b_{1,t}^2$ is that it penalizes the size of the redistribution, since larger transfers will be associated with larger bond holdings. In a more general model, one could incorporate features like distortive taxation that make larger redistributions costlier.

[35] Even if we introduced a cost-push shock in the NKPC as is common in the literature, there would still be no role for fiscal policy when monetary policy is unconstrained. This result is immediate from noting that the NKPC establishes a relation only between π_t and x_t, so if this is the only restriction faced by the planner, it would always be optimal to have an allocation with zero consumption dispersion.

[36] Since in most countries the institutional setup is such that fiscal and monetary policy are decided by different institutions (i.e. executive and legislative branches of government decide on fiscal policy, while an independent Central Bank decides on monetary policy), we can interpret the problem being analyzed here as that of a government that has no influence on the Central Bank's policy.

problem is

$$\mathcal{L}_0^*(\lambda_1,\lambda_2) \equiv \min_{\{x_t,x_{1,t},\pi_t,b_{1,t},i_t,\tau_{1,t}\}_{t=0}^{\infty}} \mathcal{L}_0\left(\{x_t,x_{1,t},\pi_t,b_{1,t}\}_{t=0}^{\infty};\lambda\right) \quad (30)$$
$$s.t.\ Eqs\ (25)-(29)$$
$$\{\check{r}_t\}_{t=0}^{\infty}\ given$$

where $\mathcal{L}_0^*(\lambda_1,\lambda_2)$ is the minimized loss from the perspective of time $t=0$, parametrizing the problem by (λ_1,λ_2). Note that the law of motion for bonds (27) is actually a redundant constraint, as we can think of the bonds $b_{1,t}$ as the control variable, and once we solve for $b_{1,t}$ we can find the transfers $\tau_{1,t}$ as a "residual".

Before actually solving problem (30) under particular assumptions on the natural interest rate and monetary policy, we can draw some general conclusions as to *what* MPCs mean for optimal policy and welfare. In particular, in Proposition 5 I will analyze how the minimized loss $\mathcal{L}_0^*(\lambda_1,\lambda_2)$ depends on the parameter λ_2 - i.e. how it depends on the MPC of the agent that we are targeting with the redistributive policy. Proposition 5 shows that, if $\lambda_2''>\lambda_2'$, we can attain a smaller loss $\mathcal{L}_0^*(\lambda_1,\lambda_2'')<\mathcal{L}_0^*(\lambda_1,\lambda_2')$.

Proposition 5. *Consider the optimal policy problem (5), and take any two arbitrary values $\lambda_2''>\lambda_2'$. Then, the optimized loss is such that $\mathcal{L}_0^*(\lambda_1,\lambda_2'')\leq\mathcal{L}_0^*(\lambda_1,\lambda_2')$, with a strict inequality whenever the optimal path of $\{b_{1,t}\}$ is not zero when $\lambda_2=\lambda_2'$.*

The formal proof of Proposition 5 is left for the Appendix, but I will explain here the intuition behind it. Suppose that, when $\lambda_2=\lambda_2'$, it is optimal for the planner to implement some aggregate output gap and inflation $\{x_t',\pi_t'\}_{t=0}^{\infty}$. If instead we had a higher λ_2, say $\lambda_2''>\lambda_2'$, the planner could still implement the same macro allocation $\{x_t',\pi_t'\}_{t=0}^{\infty}$ by choosing a more backloaded timing of transfers.[37] We can show that this policy would generate a smaller consumption dispersion, thus obtaining a smaller welfare loss (i.e., the terms of the loss function (24) that correspond to x^2 and π^2 would be the same as with λ_2', but the terms corresponding to $(x-x_1)^2$ and b_1^2 would be smaller).

But why do we get a smaller consumption dispersion with a higher λ_2? For the "between" dispersion, the function $\theta(\lambda_1,\lambda_2)$ plays a crucial role. The reader may recall that in Proposition 1 we found that the same θ affects the transfer multipliers regardless of the time in which the transfer takes place. This means that θ imposes a constraint on optimal policy that the planner cannot "work around" by changing the timing of transfers. In particular, a θ closer to zero means that a transfer from agent 1 to agent 2 will lead to a larger decline in the consumption of agent 1, thus generating a larger "between" consumption dispersion (this mechanism was illustrated by Figure 3).

[37]Note that I am not claiming that implementing the same macro allocation $\{x_t',\pi_t'\}_{t=0}^{\infty}$ is optimal, I am just claiming that this policy is within the planner's choice set.

Meanwhile, the "within" consumption dispersion directly penalizes bond holdings $\{b_{1,t}\}_{t=0}^{\infty}$ because they generate inter-generational consumption dispersion. A larger λ_2 means that smaller bond holdings are needed to induce type 2 agents to increase their consumption, so macro stabilization can be obtained with less "within" consumption dispersion.

We discussed extensively in Section 2 that while "conventional wisdom" tells us that distributing towards high MPC agents leads to a larger effect on output, this is not necessarily true and there are cases of interest in which this relation between MPCs and PDV of output can actually be reversed. However, when it comes to welfare, we find an unambiguous answer: if we have $\lambda_2'' > \lambda_2'$, we can attain a smaller loss $\mathcal{L}_0^*(\lambda_1, \lambda_2'') < \mathcal{L}_0^*(\lambda_1, \lambda_2')$. The reason why we get these different results is that in Section 2 we were analyzing the effects of transfers with a *given* timing, so the interaction between how fast the agents consumed out of the transfer and the time path of the TANK-transfer multipliers created an ambiguous relation between MPCs and output. Meanwhile, the proof of Proposition 5 is based on comparing policies that implement the *same aggregate output gap*, but the timing of transfers will be different for different values of λ_2.

Let us now make some specific assumptions about the natural interest rate and monetary policy so that we can explicitly solve the optimal policy problem in (30). For tractability, I will consider a particularly simple economy with fixed nominal rate $i = \rho$, rigid prices, and with a natural rate $\check{r}_s = \rho + (\check{r}_0 - \rho) e^{-\delta s}$, with $\check{r}_0 < \rho$, $\delta > 0$. So in this case the natural rate is initially below ρ, and as time goes by it converges exponentially to ρ. If monetary policy were unconstrained, there would be no welfare loss, but here I am assuming that there is a lower bound ρ that constraints the monetary authority from setting the interest rate equal to the natural rate.

I will first solve the problem under the ad-hoc assumption that the planner does not care about the within component of heterogeneity. The OLG structure of the model is a modeling device used to generate MPC heterogeneity, so it might be reasonable to assume that a planner would only be concerned about the between component of heterogeneity. The reason why I initially make this ad-hoc assumption is that it will allow us to obtain a simple rule for the optimal redistributive fiscal policy which has a straightforward intuitive interpretation.

If there is no government intervention, there will be a recession, which is deeper at the beginning and then output converges back to its full employment level. Denote this output gap when there are no transfers as $\left\{x_t^{NT}\right\}_{t=0}^{\infty}$ (where NT stands for no transfers; this path is displayed in Figure 7). The government can implement transfers from type 1 agents to type 2 agents in order to dampen the recession, but this will lead to an allocation in which consumption is heterogeneous across households. Given this trade-off between macro stabilization and dispersion, the government will choose an interior optimum that provides some macro stabilization but does not bring output all the way to its natural level.

In particular, with the optimal redistribution, we find that (where I am denoting the

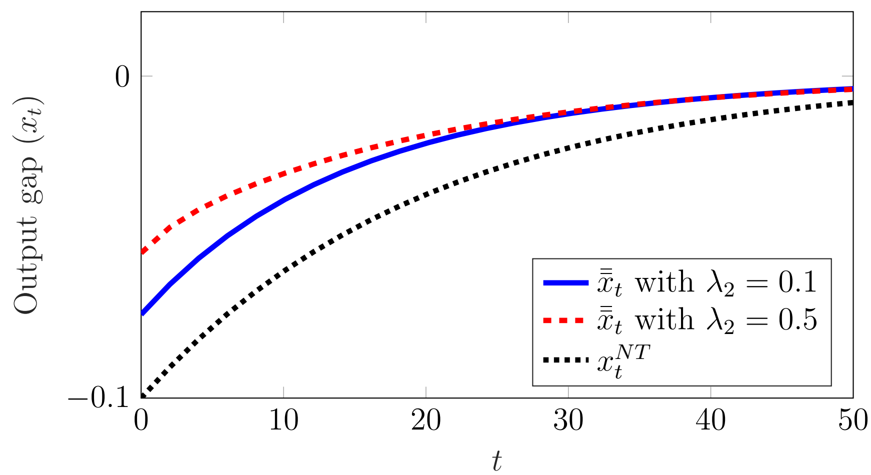

Figure 7 – The graph shows the path for $\{x_t^{NT}\}_{t=0}^{\infty}$ and $\{\bar{\bar{x}}_t\}_{t=0}^{\infty}$ (i.e., the output gap without transfers and the optimal path for the output gap when the planner cares both about between and within heterogeneity) under the assumption that prices are rigid, the interest rate is fixed at ρ and the natural interest rate is $\check{r}_t = \rho + (\check{r}_0 - \rho)\, e^{-\delta t}$. The blue and red (dashed) lines repeat the same exercise for two different values of λ_2.

optimal output gap as $\{\bar{x}_t\}_{t=0}^{\infty}$)

$$\bar{x}_t = \alpha x_t^{NT}, \tag{31}$$

where

$$\alpha \equiv \left(1 + \phi \frac{1-\chi}{\chi} \left[\theta\left(\lambda_1, \lambda_2\right)\right]^2\right)^{-1}.$$

Since $\alpha \in (0,1)$, this implies that there will be some macro stabilization, but it will not be optimal to fully stabilize output. As explained above, the value of θ plays a crucial role here because it determines how strongly the consumption of agent 1 declines when there is a transfer (which introduces a wedge between the consumptions of types 1 and 2). Since θ is increasing in λ_2, we find that when λ_2 is higher, it becomes "cheaper" for the planner to obtain macro stabilization because agent 1's consumption does not decline as much when there is a transfer. For this reason, when λ_2 is higher the planner optimally chooses more macro stabilization.

Let us now reintroduce the "within" component of heterogeneity into the planner's objective function. This makes it even costlier for the planner to do transfers. I will denote the optimal path for the output gap as $\{\bar{\bar{x}}_t\}_{t=0}^{\infty}$. In this case there is no simple expression that characterizes the optimal redistributive policy, so Figure 7 displays a numerical solution, together with the path for the output gap without transfers (i.e., the figure displays the output gap without transfers $\{x_t^{NT}\}_{t=0}^{\infty}$ and the optimal path when the planner cares both about between and within heterogeneity $\{\bar{\bar{x}}_t\}_{t=0}^{\infty}$). I repeat the same exercise for two values of λ_2. We see that with a low λ_2 the optimal path for output is closer to the one with no government intervention, while in the case with a larger λ_2 the planner uses redistributive fiscal policy to bring output closer to its natural level.[38]

6 A Bewley-Huggett-Aiyagari Incomplete Markets Model

In this last Section of the paper, I will analyze the effect of transfers in an incomplete markets model with nominal rigidities. This model features precautionary savings, and as a result we can no longer aggregate individual consumption functions. For this reason, it will be necessary to complement analytical results with numerical computations. The objectives for using a BHA model are twofold: I want to quantitatively analyze fiscal multipliers in a plausibly calibrated model, and I also want to show that many of the features that we found in the OLG model are also present in an incomplete markets setup.

The economy is populated by a continuum (mass 1) of ex-ante identical households that are exposed to uninsurable idiosyncratic labor risk. Their utility function is

$$U_0^h = E\left\{\sum_{t=0}^{\infty}\beta^t\left[\ln\left(C_t^h\right)+\frac{\left(N_t^h\right)^{1+\phi}}{1+\phi}\right]\right\}.$$

Assume that there is a symmetric labor rationing rule, so we do not need to specify the disutility of labor. Assume also (for now) that prices are perfectly rigid and the gross interest rate is fixed at its steady state level R^*. There is no aggregate uncertainty, so we can interpret the model as if there is a zero measure shock at $t = 0$ and perfect foresight after that.

Households' idiosyncratic productivity, z_t^h, follows an autorregressive process in logs:

$$\ln\left(z_{t+1}^h\right) = \psi\ln\left(z_t^h\right) + \varepsilon_t^h,$$

where ε is normally distributed with variance ς^2.[39]

The aggregate production function is Cobb-Douglas:

$$Y_t = K^{\delta}N_t^{1-\delta},$$

where K is a fixed level of capital and will be the only source of liquidity. Factors of production are paid their marginal productivity. Normalize steady state output to $Y^* = 1$.

The government levies a proportional tax on firms' output, τ_t^Y. Therefore, the total income that accrues to labor is $\left(1-\tau_t^Y\right)(1-\delta)Y_t$, and the total income that accrues to capital is $\left(1-\tau_t^Y\right)\delta Y_t$.

[38] In the Appendix, Figure 15 compares the paths $\bar{x}_t$,$\bar{\bar{x}}_t$. Both $\bar{x}_t$ and $\bar{\bar{x}}_t$ display similar patterns: with a higher λ_2 it is optimal to use redistributive policy to bring output closer to its natural level.

[39] The mean of the shock ε is chosen so that $E[z] = 1$.

Households are not allowed to borrow. Assume also that households are only allowed to trade bonds that have the same dividend stream as capital. Therefore, although we will write the household's optimization problem as if households own bonds B_t^h, they are actually owning a fraction of the capital stock. This assumption is only relevant for the effects of an unexpected shock at time $t = 0$, because an unexpected shock changes the price of capital (due to the change in the future stream of dividends) while it would not affect the price of bonds. We therefore need to specify who owns the stock of capital at $t = 0$.

The assumptions that I have made in the setup of this economy will give us two useful results that will help compare the numerical simulations with the analytical OLG model: one result is that a budget balanced increase in G_t has a unit multiplier for any given path of real interest rates[40]; and the second result is that there exists an as-if representative agent as in Werning (2015).

The utility maximization problem of an individual household h is

$$\max_{\left\{C_t^h, B_t^h\right\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t \ln\left(C_t^h\right)\right]$$
$$s.t.\ C_t^h + \frac{B_{t+1}^h}{R_t} = z_t^h\left(1-\tau_t^Y\right)(1-\delta)\,Y_t + B_t^h$$
$$B_t^h \geq 0,\ B_0^h\ given$$

Assume that in steady state there are no taxes, no government expenditure, and no government bonds.

6.1 Calibration

A period is interpreted to be a quarter. The discount factor is chosen to match a steady state annual interest rate of 2%. The AR(1) labor income process is taken from Flodén and Lindé (2001) and Guerrieri and Lorenzoni (2017): we set $\psi = 0.966$ and $\varsigma^2 = 0.017$.[41] The capital share is set at $\delta = 0.029$ so that the value of the stock of capital in steady state is 1.44 times annual GDP. Table 1 summarizes the calibrated parameter values and their corresponding targets.

With this calibration, we have that 15.3% of agents are borrowing constrained, and the average quarterly MPC[42] in the economy is 19.2%.

[40] This is a result shown by Woodford (2011) in a representative agent model, and Auclert, Rognlie, and Straub (2018) extend the same result to an incomplete markets model.

[41] The continuous process is discretized into a 7-state discrete process following the Rowenhorst method.

[42] This MPC corresponds to the increase in consumption during the quarter immediately after a household receives a marginal increase in its cash-in-hand.

Parameter	Meaning	Value	Target/Source
β	Discount factor	0.989	2% annual interest rate
ψ	Persistence of labor shock	0.966	Flodén and Lindé (2001)
ς^2	Variance of labor shock	0.017	Flodén and Lindé (2001)
δ	Capital Share	0.029	Assets equal to 144% of annual GDP

Table 1 – Calibrated Parameters

6.2 Partial equilibrium response

To understand how consumption responds to changes in output, let us first do a partial equilibrium exercise. Suppose that there is a one-time exogenous increase in aggregate output at some time $\bar{t} \geq 0$, and in all other periods output stays at 1. That is, suppose that

$$Y_t = \begin{cases} 1+\epsilon & if\ t = \bar{t} \\ 1 & if\ t \neq \bar{t} \end{cases}$$

and let us look at the response of consumption for various values of $\bar{t}$. This is a partial equilibrium exercise in that I do not require consumption to equal output (so it can be interpreted as the consumption response of a small open economy).

Figure 8 displays the increase in aggregate consumption demand, taking $\epsilon = 0.01$ (i.e. output increases by 1%) and repeating the exercise for $\bar{t} = 0, 8, 16, 32, 40$. The figure shows that consumption has a peak at the time of the transfer, but the increase in consumption also propagates forwards and backwards (with respect to $\bar{t}$), just as we had in the OLG model. Agents who are borrowing constrained cannot borrow out of their future labor income, so they cannot adjust their consumption until the shock actually takes place. However, the rest of the agents, who have a positive amount of financial wealth, can start increasing their consumption even before the shock, generating the backwards propagation that we see in the plot.

This exercise is closely connected to Auclert, Rognlie, and Straub (2018). In their paper, they define intertemporal-MPCs (iMPCs) as the consumption response to increases in income at different points in time. Figure 8 can be interpreted as displaying the iMPCs in my BHA model. Auclert, Rognlie, and Straub (2018) find that the tent-shaped partial equilibrium response that we observe in Figure 8 is a common feature across various heterogeneous-agents models.

6.3 Government expenditure financed with an up-front tax

We had discussed before that there is a close connection between a transfer at time $t = 0$ and a path of government expenditure that is financed with an up-front tax at $t = 0$. Intuitively, when the agent who receives the transfer spends it over time (i.e. the agent is

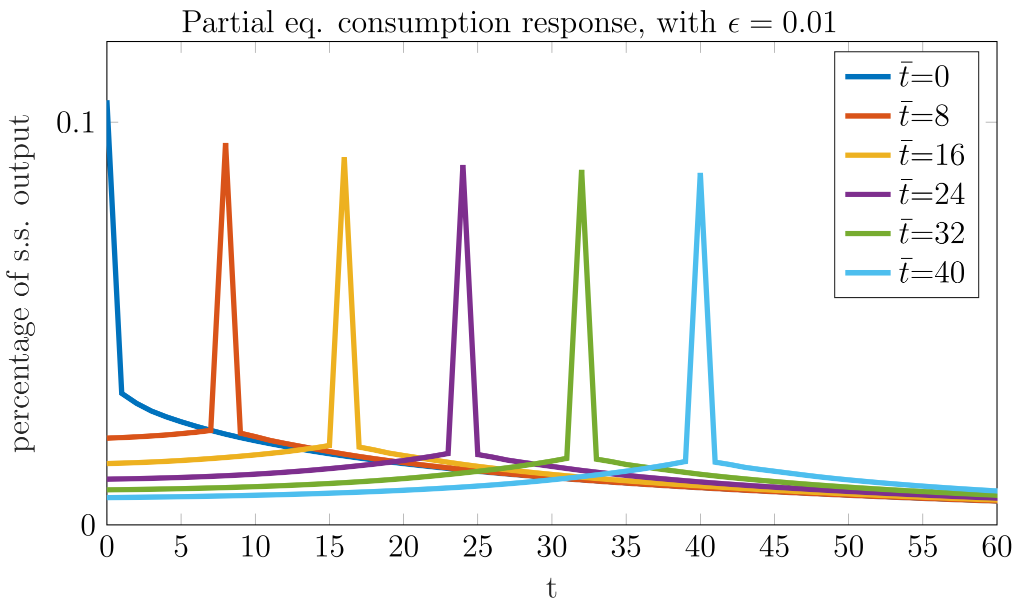

Figure 8 – Path for consumption demand (partial equilibrium) when income increases by a factor of $1 + \epsilon$ at time $\bar{t}$.

not HtM), there is a time-lag between the time of the transfer and the time in which the expenditure takes place. This same time-lag occurs when the government levies taxes at $t = 0$ to finance future expenditure.

We can exploit this relation to do an exercise in the incomplete markets model that closely resembles the transfer in the OLG model. Assume that the government levies a tax at $t = 0$ that generates a revenue $\varkappa_0$ (as described above, the tax is proportional to output), and spends it according to[43]

$$G_t = \left[R\left(1 - \lambda_g\right)\right]^t G_0.$$

That is, the parameter λ_g controls the rate at which the government spends the resources it obtained with the $t = 0$ tax.

Define the government purchases multiplier as

$$\text{Gov. Purchases Multiplier} = \frac{\sum_{t=0}^{\infty} \frac{1}{R^t}\left(Y_t - 1\right)}{\varkappa_0}.$$

Figure 9 plots the multiplier as a function of λ_g, assuming $\varkappa_0 = 0.01$ (i.e. the government raises a revenue at $t = 0$ of 1% of steady state quarterly output). The behavior of this function is very similar to the function $\theta\left(\lambda_1, \lambda_2\right)$ in the OLG model, displayed in Figure 2. The intuition behind the behavior of these two functions (i.e. Figures 2 and 9) is the same:

[43]This is of course not a realistic way to describe a fiscal stimulus. In practice, fiscal stimulus programs during recessions are mostly deficit-financed, while here I am making the exact opposite assumption: the government taxes up-front and spends later. The point of this exercise is to help understand the mechanisms at play in the BHA model and how they relate to the OLG model analyzed before.

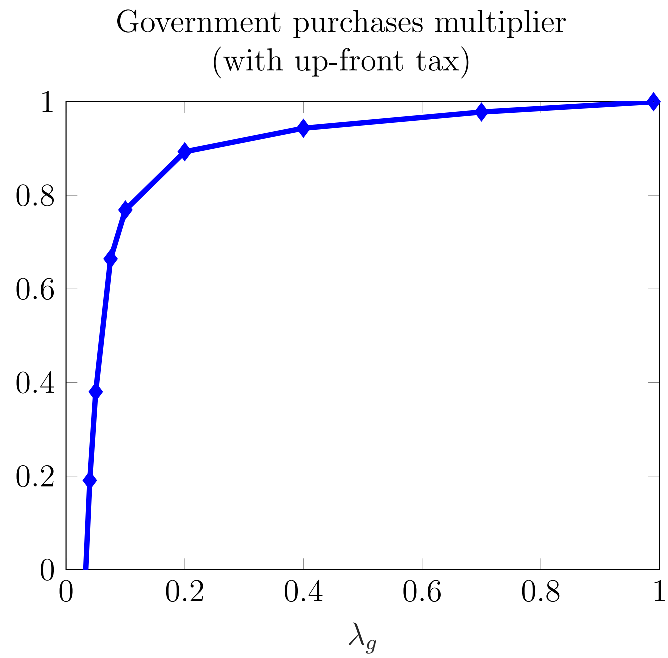

Figure 9 – Multiplier associated with a one-time tax at $t = 0$, which is spent by the government at rate λ_g.

because of the limited pledgeability of future labor income, households can only partially bring to $t = 0$ the increase in income generated by a future fiscal stimulus; therefore, the more backloaded the fiscal stimulus, the lower the multiplier. Note also that, similarly to the θ function, the multiplier in Figure 9 increases very steeply for low values of λ_g, and then becomes very flat. In the flat region, the λ_g determines the timing of the stimulus, but has little effect on the cumulative effect on output. As λ_g goes to 1, we obtain a multiplier equal to 1, which is consistent with previous results in the literature and with the OLG model we used before.

6.4 Transfer multiplier

Let us now consider a redistribution at $t = 0$. For the exercise to be as comparable as possible with the OLG model, I want to transfer resources from low to high MPC households. In this model, the MPC is closely related to financial assets, so I will assume that we have a redistribution from the top of the asset distribution to the bottom of the asset distribution. For this exercise, government expenditure is set to zero.

I assume, arbitrarily, that the redistributive tax is paid by the top 10% of the asset distribution, and takes the following form

$$T_0^h = \max\left\{\tau^{redist}\left(B_0^h - B^{p90}\right)\ ,\ 0\right\},$$

where B^{p90} is the 90th percentile of the asset distribution. The revenue from this tax is immediately transferred to poorer agents. I will consider transfers to different percentiles of the asset distribution in order to create variation in the MPC of the agents who receive the transfer. In particular, the transfers take the following form

$$S_0^h = \begin{cases} \max\left\{M_0 - \left(B_0^h - B_0^p\right)\ ;\ 0\right\} & if\ B_0^h \geq B_0^p \\ 0 & if\ B_0^h < B_0^p \end{cases},$$

where B_0^p is the $p-th$ percentile of the asset distribution. That is, we give a fixed amount M_0 as a subsidy (starting at the $p-th$ percentile), and this subsidy is faced out as assets increase. I will repeat the exercise for various values of p.[44]

Then, the net transfer received by each household h is $Net\ Transfer_0^h = S_0^h - T_0^h$. In order to describe the change in aggregate demand generated by this redistribution, we can compute:

$$\Delta(p) = \frac{Cov\left\{MPC_0^h\ ; Net\ Transfer_0^h\right\}}{\int_h T_0^h dF_h^*},$$

where dF_h^* is the steady state joint distribution of productivity and assets, and MPC_0^h is the (quarterly) MPC of the household evaluated at its initial asset holdings. That is, $\Delta(p)$ gives us the change in MPC (at time $t = 0$) per dollar transferred[45].

Define the transfer multiplier as

$$\text{Transfer Multiplier} = \frac{\sum_{t=0}^{\infty} \frac{1}{R^t}(Y_t - 1)}{\int T_0^h dF_h^*}.$$

In Figure 10, I do this exercise with $\tau^{redist} = 0.01$, which raises a revenue of approximately 1% of quarterly steady state output (think of this as a normalization, although the model is not exactly linear). The left-hand side panel shows the transfer multiplier when we give the transfer to various different percentiles of the asset distribution. I am putting $\Delta(p)$ on the horizontal axis so that the graph has a similar interpretation as Figure 9 (the reader should bear in mind however that $\Delta(p)$ only reflects the increase in MPC at $t = 0$, while in Figure 9 the parameter λ_g is a constant rate). Higher values of $\Delta(p)$ have higher multipliers because transfers to high MPC agents lead to a more front-loaded stimulus. As predicted by the analytical model, multipliers can be well above 1[46]. In particular, when the transfer is given to agents at the very bottom of the distribution (i.e. $p = 0$), we find that for each dollar transferred there is an increase in aggregate consumption of 1.6 dollars.

[44] The value of M_0 is chosen so that the redistribution program is budget balanced.

[45] Note however that this is only the MPC at time $t = 0$, but contrary to the OLG model now the MPC changes over time.

[46] While in the analytical model we found that (with a fixed interest rate) $m_t^T \in [0, 1]$, recall that these transfer multipliers have to be multiplied by $\frac{1}{\chi}$ to express them on a per-dollar-of-government-revenue basis.

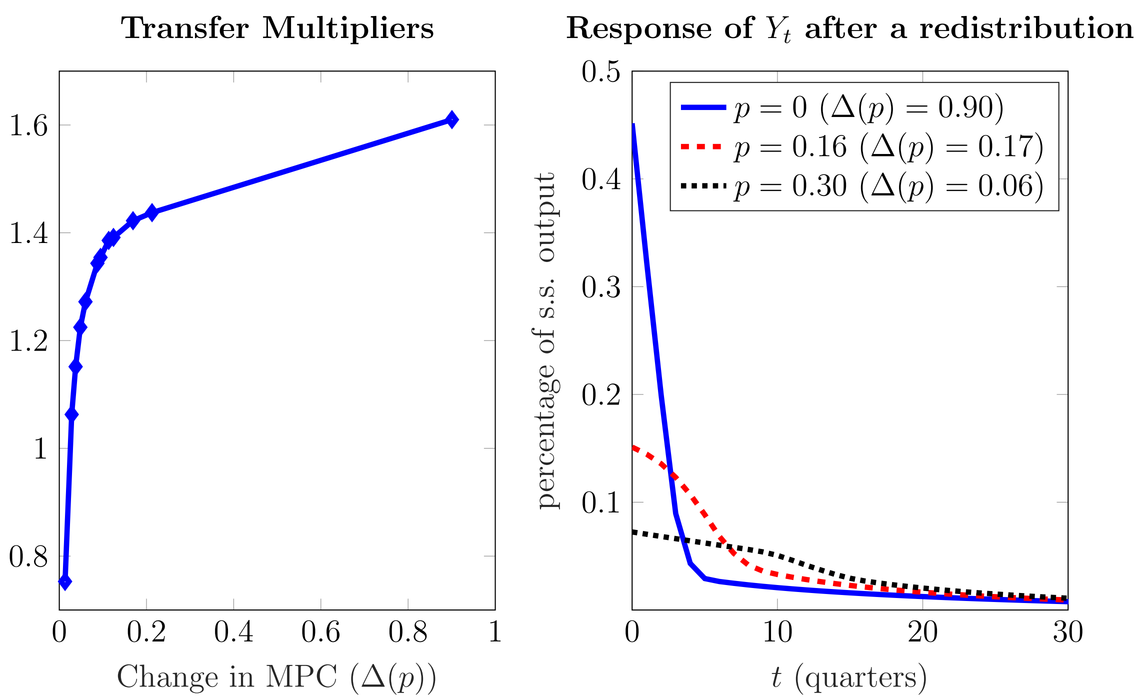

Figure 10 – Left: transfer multipliers corresponding to a $t = 0$ wealth redistribution. Right: path for Y_t after the redistribution.

The right-hand side panel of Figure 10 displays the path of output after the redistribution, for different percentiles p. Just as we found in the OLG model, transfers to agents with a higher MPC lead to a larger stimulus initially, but this stimulus fades away faster. When the transfer (of 1% of output) is given to the poorest agents ($p = 0$), output increases by 0.45% in the first period, but the stimulus only lasts a few quarters. When the transfer is given to agents at the 30th percentile however, the initial effect is much smaller (0.07% of output), but it is much more persistent (output is still 0.05% above steady state 3 years after the initial shock).

6.5 Endogenous interest rate

Suppose now that the interest rate is no longer fixed, but instead follows the following rule

$$R_t = R^* \left(\frac{Y_t}{Y^*} \right)^{\kappa_{y,t}}.$$

I continue to assume that prices are rigid.

The government chooses a path of purchases $\{G_t\}_{t \geq 0}$, and maintains a balanced budget in every period (recall that the government finances its expenditure with a tax proportional on output).

I want to analyze the effect of a transfer at $t = 0$, just as we did in the previous section.

Since this model satisfies the conditions described in Werning (2015) for the existence of an as-if representative agent, and G_t has a unit multiplier for any given path of interest rates[47], the demand side of the economy can be described by an aggregate Euler equation

$$Y_{t+1} = \beta_t \left(\left\{ B_0^h \right\}_h \right) R_t C_t + G_{t+1}, \tag{32}$$

where $\beta_t \left(\left\{ B_0^h \right\}_h \right)$ is the discount factor of the as-if representative agent, which depends on the initial distribution of assets, but does not depend on monetary policy nor the path for $\{G_t\}_{t \geq 0}$.

While we can solve this system numerically for specific parameters of the Taylor rule, it might be somewhat hard to see the relation with the OLG model. In order to establish a clearer connection between the analytical results and the numerical analysis, I want to show that the transfer multiplier in the incomplete markets model can be written as proportional to a weighted average of the government purchases multipliers that we would obtain in a standard RANK model.[48]

Linearizing the Euler equation with respect to aggregate variables, and expressing the Taylor rule in log deviations, we find that the equilibrium is described by two equations

$$\begin{aligned} y_{t+1} - y_t &= r_t - \varrho_t \left(\left\{ B_0^h \right\}_h \right) + g_{t+1} - g_t \\ r_t &= r^* + \kappa_{y,t} y_t \end{aligned}$$

where lower case letters denote deviations from steady state, and we define $r_t \equiv \ln(R_t)$ and $\varrho_t \equiv -\ln(\beta_t)$.

Given that this system is linear in government purchases, we can express their effect on output through a path of multipliers $\left\{ m_t^G \right\}$:

$$\sum_{t=0}^{\infty} \frac{1}{(R^*)^t} y_t = \sum_{t=0}^{\infty} m_t^G \times \frac{1}{(R^*)^t} g_t.$$

Note that these multipliers are exactly the same as if we had a representative agent.

As above, denote by $\int_h T_0^h dF_h^*$ the total revenue raised by the government to redistribute at $t = 0$. In the Appendix I show that for any redistribution, the transfer multiplier can be written as

$$\text{Transfer Multiplier} = \frac{\sum_{t=0}^{\infty} \frac{1}{(R^*)^t} (Y_t - 1)}{\int T_0^h dF_h^*} = \theta \sum_{t=0}^{\infty} \omega_t m_t^G,$$

[47]The intuition for this latter result is that if output increases in the same amount as taxes (recall I am assuming balanced-budget), then all agents end up having the same net income as if we had $G = 0$, so their demand for consumption must be the same.

[48]While we could also use HtM multipliers as we did in the OLG model, it might seem somewhat odd to introduce a HtM agent in a Bewley-Huggett-Aiyagari model. To avoid this, I will just express the transfer multiplier as a weighted average of government purchases multipliers.

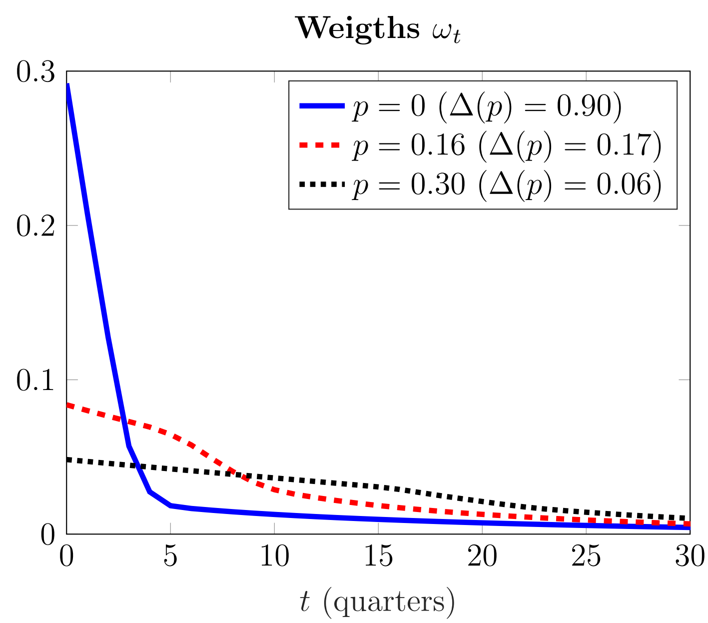

Figure 11 – Weights ω_t used to express the transfer multiplier as a weighted average of government purchases multipliers. The different lines correspond to redistributions that target different percentiles of the wealth distribution.

where

$$\theta \equiv \frac{1}{\int_h T_0^h dF_h^*} \sum_{t=0}^{\infty} \sum_{s=t}^{\infty} \frac{1}{(R^*)^t} (\varrho_s - \varrho^*), \text{ and } \omega_t \equiv \frac{\frac{1}{(R^*)^t} \sum_{s=t}^{\infty} (\varrho_s - \varrho^*)}{\sum_{t=0}^{\infty} \sum_{s=t}^{\infty} \frac{1}{(R^*)^t} (\varrho_s - \varrho^*)}.$$

Just as in the OLG model, the transfer multiplier is proportional to a weighted average of government purchases multipliers. The multipliers $\{m_t^G\}$ are acting as a sufficient statistic for monetary policy and they do not depend on the initial distribution $\{B_0^h\}_h$. Meanwhile, the coefficient θ and the weights $\{\omega_t\}$ depend on the initial distribution but they do not depend on monetary policy.

Note that in the special case $\kappa_y = 0$ (i.e. when the interest rate is fixed at $R_t = R^* \ \forall t$), the multipliers are $m_t^G = 1 \ \forall t$. Therefore, the transfer multiplier in this case is just θ. This means that the left-hand panel of Figure 10 can be interpreted as displaying the value of θ for each corresponding redistribution. Meanwhile, Figure 11 displays the weights ω_t corresponding to the same three redistributions considered in the right-hand panel of Figure 10. We can see that a transfer to agents with lower MPCs puts more weight on later periods, just as we had in the OLG model.

Suppose we want to analyze the transfer multipliers when $\kappa_y = 0.13$.[49] As explained above, we need to compute the weighted average of government expenditure multipliers.

[49]This implies a moderate reaction of the monetary authority to the output gap, and is commonly used in the New Keynesian literature (e.g., Galí (2015)).

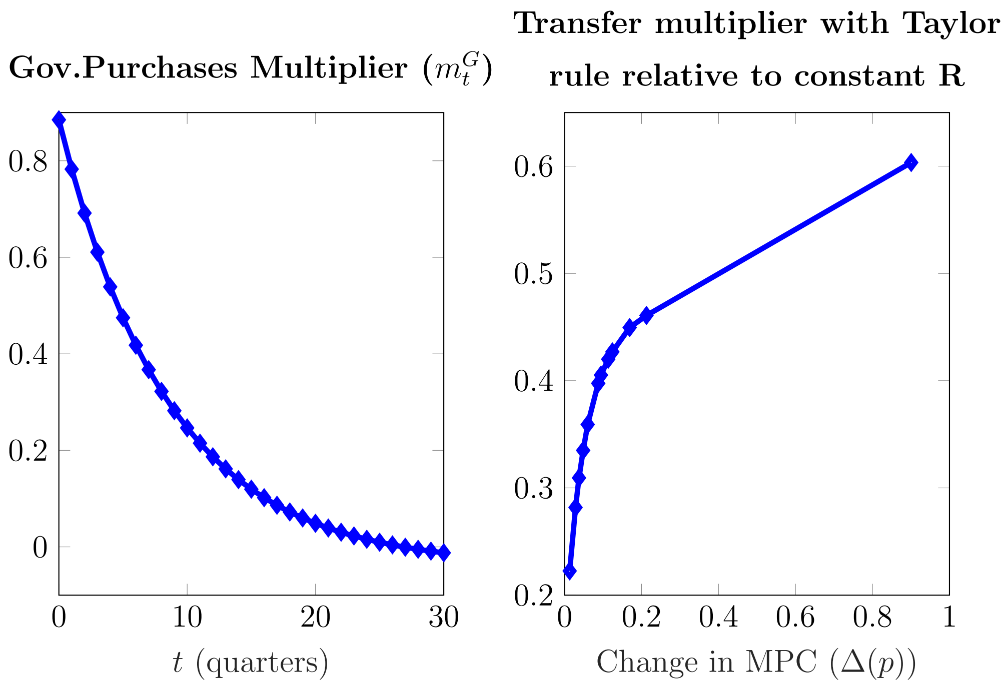

Figure 12 – Left: government expenditure multipliers m_t^G. Right: transfer multipliers with a $t = 0$ redistribution, expressed as a ratio with respect to the case in which interest rates are fixed.

The left-hand panel of Figure 12 displays these multipliers $\{m_t^G\}_{t\geq 0}$. The multipliers are decreasing over time because of a forward guidance effect.[50] Since the m_t^G multipliers are decreasing, a transfer that generates a more backloaded stimulus will result in lower transfer multipliers. In the right-hand panel of Figure 12 I show the transfer multipliers corresponding to the same redistributions analyzed in Figure 10, expressed relative to the multipliers obtained in that same Figure (that is, relative to the case in which the interest rate is fixed, so the θ cancels out, and we are left only with the weighted average $\sum_{t=0}^{\infty} \omega_t m_t^G$). The transfer multiplier corresponding to a redistribution that is concentrated on the poorest households ($p = 0$, which generates an increase in MPC of 0.9 for every dollar transferred) decreases by 40% with the Taylor rule with respect to the case with a constant R. Meanwhile, the transfer multiplier corresponding to a redistribution that generates an increase of the MPC of 0.2 would fall by 55% because it generates a more backloaded stimulus and therefore a stronger forward guidance effect.

[50] That is, a stimulus to aggregate demand at time $\bar{t}$ leads to an increase in the real interest rate which depresses private consumption in all periods $t \in [0; \bar{t}]$.

6.6 Sticky prices

Let us now consider a more general incomplete markets model with sticky prices. This version is a natural extension of the rigid prices model, incorporating Calvo pricing to generate a Phillips curve.

Households now choose their labor supply, in addition to consumption and savings. Their labor income is the product of their work effort (N_t^h) and their idiosyncratic productivity (z_t^h). The consumption that enters the utility function is an aggregator of varieties, $C_{i,t}^h \equiv \left(\int_0^1 C_{i,t}^h(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}$, where the different varieties are produced by monopolistically competitive firms. Just as in Section 2, I will assume that profits are taxed and rebated to households, so that the total compensation of all factors of production (i.e., labor and capital) is their respective marginal product.

I will assume that monetary policy is such that the nominal interest rate is fixed for the first $\tilde{T}$ periods, and after that (i.e., for all $t > \tilde{T}$) monetary policy follows a Taylor rule such that the nominal interest rate increases with inflation: $1 + i = R^* (1 + \pi_t)^{\kappa_\pi}$.

Calibration. I set $\phi = 2$ to match a Frisch elasticity of 0.5. The probability that a firm gets to change its price in each quarter is 0.15 (so that the average duration of a price is about 6 quarters). The elasticity of substitution across varieties is set at $\varepsilon = 6$, so that the markup is 1.2. Finally, I set $\kappa_\pi = 1.5$, and I arbitrarily set the duration of the liquidity trap to be $\tilde{T} = 16$ quarters.

I compute the present value transfer multipliers corresponding to the same redistributions analyzed in Figure 10. These multipliers are displayed in Figure 13. We can observe that the relation between MPCs and transfer multipliers is non-monotonic. This means that, just as in the OLG model, when the economy is in a liquidity trap it is possible for transfers to low-MPC households to have a stronger effect on output. The mechanism is the same as in the OLG model: transfers to low-MPC households generate a back-loaded stimulus, which has a powerful effect on output due to a forward guidance effect. Note that even with a liquidity trap of an moderate duration (4 years) this forward guidance effect is strong enough to generate large multipliers of around 4.

7 Conclusion

This paper contributes to our understanding of the role of MPC heterogeneity in shaping how output responds to redistributive fiscal policies. I analyzed the effect of fiscal transfers in a New Keynesian overlapping generations model that allows for arbitrary MPC heterogeneity. I found an expression that decomposes the transfer multipliers into the effect of MPCs and the

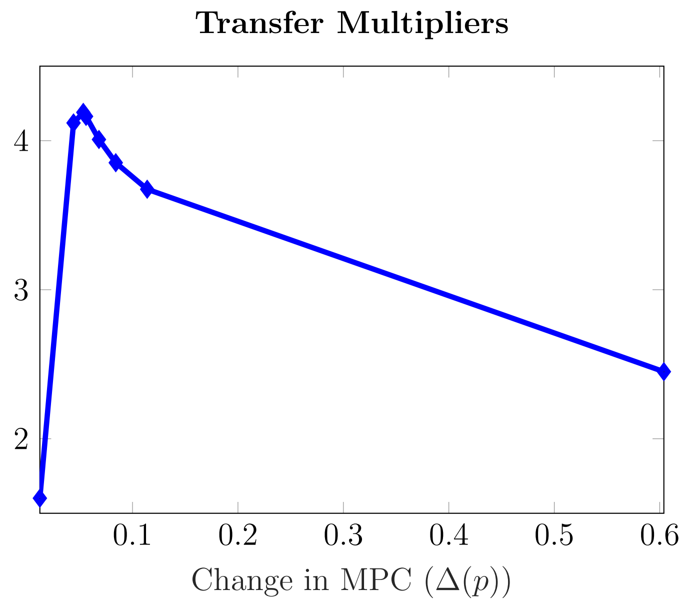

Figure 13 – Transfer multipliers corresponding to a $t = 0$ wealth redistribution in the incomplete markets model with sticky prices and a liquidity trap that lasts for $\tilde{T} = 16$ quarters.

monetary policy response. In particular, I showed that transfer multipliers are proportional to a weighted average (over time) of the multipliers in a standard TANK model.

This decomposition makes it clear that, in a dynamic setting, MPCs determine the persistence of the change in the wealth distribution induced by transfers. Therefore, transfers to high MPC agents tend to generate a front-loaded stimulus while transfers to low MPC agents tend to back-load the stimulus. This analysis also showed that it is possible for transfers to agents with low MPCs to have higher transfer multipliers.

I then turned from a positive to a normative analysis, and in this case found that there is no ambiguity in the relation between MPCs and welfare: larger differences in MPCs always allow the planner to obtain macro stabilization with better risk sharing. This dichotomy between the results obtained in the positive analysis and in the normative one stem from the fact that in the positive analysis we took the timing of transfers as given, while in the normative analysis the timing of transfers was chosen optimally by the social planner.

Finally, I undertook some numerical exercises with an incomplete-markets model à-la Bewley-Huggett-Aiyagari. In these exercises, the relation between transfer multipliers and the timing of consumption displayed similar patterns as in the analytical model. This should provide some reassurance that what we learnt from the OLG model is also useful to improve our understanding of transfers in an incomplete-markets framework.

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A Proofs for Section 2

A.1 Log-linearization of the equilibrium conditions

The nonlinear equations are

$$\begin{aligned}
\dot{C}_{i,t} &= (r_t - \rho) C_{i,t} - (\rho + \lambda_i) \lambda_i B_{i,t} \\
\dot{B}_{i,t} &= r_t B_{i,t} + \hat{A}_t N_{i,t} - C_{i,t} - (T_{i,t} + G_t) \\
\frac{W_t}{P_t} &= C^h_{i,t} \left(N^h_{i,t}\right)^\phi \\
C_t &= \chi C_{1,t} + (1-\chi) C_{2,t} \\
N_t &= \chi N_{1,t} + (1-\chi) N_{2,t} \\
C_t + G_t &= \hat{A}_t N_t
\end{aligned}$$

A log-linear approximation of the first two equations immediately gives us:

$$\begin{aligned}
\dot{c}_{i,t} &= r_t - \rho - \lambda_2 (\rho + \lambda_2) b_{i,t} \\
\dot{b}_{i,t} &= \rho b_{i,t} - (c_{i,t} - a_t) + n_{i,t} - \tau_{i,t} - g_t
\end{aligned}$$

From the third equation we get

$$\varpi_t = c^h_{i,t} + \phi n^h_{i,t}$$

where we are defining $\varpi_t \equiv w_t - p_t$. Since in steady state all households consume and work the same, we can integrate this equation across the households of the same type:

$$\begin{aligned}
\varpi_t &= \int_{h \in i} c^h_{i,t} dh + \phi \int_{h \in i} n^h_{i,t} dh \\
&= c_{i,t} + \phi n_{i,t}
\end{aligned}$$

and if we sum over the different types of agents we get

$$\begin{aligned}
\varpi_t &= \chi [c_{1,t} + \phi n_{1,t}] + (1-\chi) [c_{2,t} + \phi n_{2,t}] \\
&= c_t + \phi n_t
\end{aligned}$$

The log-linear approximation to $C_t + G_t = \hat{A}_t N_t$ is

$$c_t + g_t = a_t + n_t$$

because, as is well-known, to a first order we have that $\hat{A}_t = A_t$. Finally, the usual solution to the firms' profit maximization problem with Calvo pricing gives us that

$$\dot{\pi}_t = \rho\pi_t - \zeta(\rho+\zeta)(\varpi_t - a_t)$$

so replacing the previous equations we get

$$\dot{\pi}_t = \rho\pi_t - \zeta(\rho+\zeta)\left[(1+\phi)(c_t - a_t) + \phi g_t\right]$$

Therefore, the log-linear system is (where $\mu \equiv \zeta(\rho+\zeta)(1+\phi)$):

$$\begin{aligned}
\dot{c}_{i,t} &= r_t - \rho - \Lambda_i b_{i,t}, \;\; i = 1,2 && (33)\\
\dot{b}_{i,t} &= \rho b_{i,t} - (c_{i,t} - a_t) + n_{i,t} - \tau_{i,t} - g_t, \;\; i = 1,2 && (34)\\
\varpi_t &= c_{i,t} + \phi n_{i,t}, \;\; i = 1,2 && (35)\\
\varpi_t &= c_t + \phi n_t && (36)\\
c_t &= \chi c_{1,t} + (1-\chi)c_{2,t} && (37)\\
n_t &= \chi n_{1,t} + (1-\chi)n_{2,t} && (38)\\
c_t + g_t &= a_t + n_t && (39)\\
0 &= \chi b_{1,t} + (1-\chi)b_{2,t} && (40)\\
\dot{\pi}_t &= \rho\pi_t - \mu\left(c_t - a_t + \frac{\phi}{1+\phi}g_t\right) && (41)\\
i_t &= \rho + \kappa_{y,t}(c_t + g_t) + \kappa_{\pi,t}\pi_t && (42)
\end{aligned}$$

where I am defining $\varpi_t \equiv w_t - p_t$. Equation (33) is the Euler equation of agent $i \in \{1,2\}$, equation (34) is the budget constraint, (35) and (36) are the individual and aggregate labor supply curves respectively, (37) and (38) aggregate consumption and labor across types, (39) and (40) are the market clearing conditions for the goods market and bond market respectively, (41) is the NKPC, and (42) is a linear approximation to the monetary policy rule.

To reduce the number of equations, I want to substitute out $n_{i,t}$ and n_t from the system. In the law of motion for bonds, replace

$$\begin{aligned}
n_{i,t} &= \frac{\varpi_t - c_{i,t}}{\phi}\\
&= \frac{\{c_t + \phi n_t\} - c_{i,t}}{\phi}\\
&= \frac{(1+\phi)c_t + \phi g_t - \phi a_t - c_{i,t}}{\phi}
\end{aligned}$$

to obtain

$$\dot{b}_{i,t} = \rho b_{i,t} + \frac{1+\phi}{\phi}(c_t - c_{i,t}) - \tau_{i,t}, \;\; i = 1,2$$

Using that $c_t = \chi c_{1,t} + (1-\chi)c_{2,t}$ and $0 = \chi b_{1,t} + (1-\chi)b_{2,t}$, we can obtain an Euler equation for aggregate consumption:

$$\begin{aligned}\dot{c}_t &= \chi \dot{c}_{1,t} + (1-\chi)\,\dot{c}_{2,t} \\ &= \chi\left[r_t - \rho - \lambda_1(\rho+\lambda_1)\,b_{1,t}\right] + (1-\chi)\left[r_t - \rho - \lambda_2(\rho+\lambda_2)\,b_{2,t}\right] \\ &= r_t - \rho + \chi\left[\lambda_2(\rho+\lambda_2) - \lambda_1(\rho+\lambda_1)\right] b_{1,t}\end{aligned}$$

Therefore, we obtain a reduced system:

$$\begin{aligned}\dot{c}_t &= r_t - \rho + \chi(\Lambda_2 - \Lambda_1)\,b_{1,t} \\ \dot{c}_{1,t} &= r_t - \rho - \Lambda_1 b_{1,t} \\ \dot{b}_{i,t} &= \rho b_{i,t} + \frac{1+\phi}{\phi}(c_t - c_{i,t}) - \tau_{i,t} \\ \dot{\pi}_t &= \rho\pi_t - \mu\left(c_t - a_t + \frac{\phi}{1+\phi} g_t\right) \\ i_t &= \rho + \kappa_{y,t}(c_t + g_t) + \kappa_{\pi,t}\pi_t\end{aligned}$$

B Proofs for Section 3

B.1 A neutrality result

Obtaining the multiplier $m_0^T(\lambda_1, \lambda_2)$ is very straightforward. Using the aggregate Euler equation and Lemma 1 we have

$$\begin{aligned}c_t &= -\chi(\Lambda_2 - \Lambda_1)\int_t^\infty b_{1,s}\,ds \\ &= \chi(\Lambda_2 - \Lambda_1)\,b_{1,0}\frac{e^{\upsilon t}}{\upsilon}\end{aligned}$$

Integrating

$$\begin{aligned}\int_0^\infty e^{-\rho t} c_t\,dt &= \int_0^\infty e^{-\rho t}\chi(\Lambda_2 - \Lambda_1)\,b_{1,0}\frac{1}{\upsilon}e^{\upsilon t}\,dt \\ &= \chi(\Lambda_2 - \Lambda_1)\,b_{1,0}\frac{1}{\upsilon}\frac{1}{\rho - \upsilon} \\ &= \frac{\chi(\Lambda_2 - \Lambda_1)}{(1-\chi)\Lambda_1 + \chi\Lambda_2}\frac{\phi}{1+\phi}(-b_{1,0})\end{aligned}$$

B.2 Proof of Lemma 1

Taking the difference between equations (7) and (8), and integrating we get

$$\int_t^\infty (\dot{c}_s - \dot{c}_{1,s})\, ds = [\chi\lambda_2(\rho+\lambda_2) + (1-\chi)\lambda_1(\rho+\lambda_1)] \int_t^\infty b_{1,s} ds$$

$$c_t - c_{1,t} = -[\chi\lambda_2(\rho+\lambda_2) + (1-\chi)\lambda_1(\rho+\lambda_1)] \int_t^\infty b_{1,s} ds$$

Replacing in (9)

$$\dot{b}_{1,t} = \rho b_{1,t} - \frac{1+\phi}{\phi} L \int_t^\infty b_{1,s} ds - \tau_{1,t}$$

where $L \equiv (1-\chi)\lambda_1(\rho+\lambda_1) + \chi\lambda_2(\rho+\lambda_2)$. This is the second order differential equation in Lemma 1. Define $\beta_{1,t} \equiv \int_t^\infty b_{1,s} ds$, so that $\dot{\beta}_{1,t} = -b_{1,t}$. Then we get a system of equations

$$\dot{\beta}_{1,t} = -b_{1,t}$$
$$\dot{b}_{1,t} = \rho b_{1,t} - \frac{1+\phi}{\phi} L\beta_{1,t} - \tau_{1,t}$$

In matrix form

$$\begin{bmatrix} \dot{\beta}_{1,t} \\ \dot{b}_{1,t} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -\frac{1+\phi}{\phi}L & \rho \end{bmatrix} \begin{bmatrix} \beta_{1,t} \\ b_{1,t} \end{bmatrix} - \tau_{1,t} E_2$$

Using vector notation

$$\dot{Z}_t - MZ_t = -\tau_{1,t} E_2$$

where $Z_t \equiv \begin{bmatrix} \beta_{1,t} \\ b_{1,t} \end{bmatrix}$ and $M \equiv \begin{bmatrix} 0 & -1 \\ -\frac{1+\phi}{\phi}L & \rho \end{bmatrix}$. The terminal conditions for the differential equations are $b_{1,0}$ given and $\lim_{t\to\infty} Z_t = 0$ since we are assuming that the economy will eventually converge back to the steady state. The general solution to a non-homogeneous linear system of differential equations can be written as a particular solution to the non-homogeneous system and the general solution to the homogeneous system.

Let us first find a particular solution to the non-homogeneous system. We have that

$$\dot{Z}_s - MZ_s = -\tau_{1,s} E_2$$

$$\int_t^{\bar{t}} \left(e^{-Ms} Z_s\right)' ds = -\int_t^{\bar{t}} e^{-Ms} E_2 \tau_{1,s} ds$$

Assuming that $Z_{\bar{t}} = 0$ and taking $\bar{t} \to \infty$, we have

$$Z_t^{particular} = \int_t^\infty e^{-M(s-t)} E_2 \tau_{1,s} ds$$

To compute this we need to find the eigenvalues and eigenvectors of M. The eigenvalues are

$$\upsilon \equiv \frac{\rho - \sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}}{2}, \quad \bar{\upsilon} \equiv \frac{\rho + \sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}}{2}$$

Note that $\upsilon < 0 < \bar{\upsilon}$. Meanwhile, the matrix of eigenvectors is

$$\begin{bmatrix} 1 & 1 \\ -\upsilon & -\bar{\upsilon} \end{bmatrix}$$

where the first column corresponds to υ and the second column corresponds to $\bar{\upsilon}$.

The inverse is

$$\begin{bmatrix} 1 & 1 \\ -\upsilon & -\bar{\upsilon} \end{bmatrix}^{-1} = \frac{1}{-\bar{\upsilon} + \upsilon} \begin{bmatrix} -\bar{\upsilon} & -1 \\ \upsilon & 1 \end{bmatrix}$$

Then, we have that

$$\begin{aligned} Z_t^{Particular} &= \int_t^\infty e^{-M(s-t)} E_2 \tau_{1,s} ds \\ &= \frac{1}{-(\bar{\upsilon} - \upsilon)} \begin{bmatrix} 1 & 1 \\ -\upsilon & -\bar{\upsilon} \end{bmatrix} \begin{bmatrix} \int_t^\infty e^{-\upsilon(s-t)} \tau_{1,s} ds & 0 \\ 0 & \int_t^\infty e^{-\bar{\upsilon}(s-t)} \tau_{1,s} ds \end{bmatrix} \begin{bmatrix} -\bar{\upsilon} & -1 \\ \upsilon & 1 \end{bmatrix} E_2 \\ &= \frac{1}{-(\bar{\upsilon} - \upsilon)} \begin{bmatrix} 1 & 1 \\ -\upsilon & -\bar{\upsilon} \end{bmatrix} \begin{bmatrix} -\int_t^\infty e^{-\upsilon(s-t)} \tau_{1,s} ds \\ \int_t^\infty e^{-\bar{\upsilon}(s-t)} \tau_{1,s} ds \end{bmatrix} \\ &= \frac{1}{-(\bar{\upsilon} - \upsilon)} \begin{bmatrix} \int_t^\infty \left(e^{-\bar{\upsilon}(s-t)} - e^{-\upsilon(s-t)} \right) \tau_{1,s} ds \\ \upsilon \int_t^\infty e^{-\upsilon(s-t)} \tau_{1,s} ds - \bar{\upsilon} \int_t^\infty e^{-\bar{\upsilon}(s-t)} \tau_{1,s} ds \end{bmatrix} \end{aligned}$$

The general solution of the homogeneous system is

$$Z_t^{general\ Homog.} = \alpha_\upsilon \begin{bmatrix} -\frac{1}{\upsilon} \\ 1 \end{bmatrix} e^{\upsilon s} + \alpha_{\bar{\upsilon}} \begin{bmatrix} -\frac{1}{\bar{\upsilon}} \\ 1 \end{bmatrix} e^{\bar{\upsilon} s}, \quad \alpha_\upsilon, \alpha_{\bar{\upsilon}} \in \mathbb{R}$$

so the general solution to the non-homogeneous system is

$$Z_t^{general\ solution} = \alpha_\upsilon \begin{bmatrix} -\frac{1}{\upsilon} \\ 1 \end{bmatrix} e^{\upsilon t} + \alpha_{\bar{\upsilon}} \begin{bmatrix} -\frac{1}{\bar{\upsilon}} \\ 1 \end{bmatrix} e^{\bar{\upsilon} t} - \frac{1}{\bar{\upsilon} - \upsilon} \begin{bmatrix} 1 & 1 \\ -\upsilon & -\bar{\upsilon} \end{bmatrix} \begin{bmatrix} -\int_t^\infty e^{-\upsilon(s-t)} \tau_{1,s} ds \\ \int_t^\infty e^{-\bar{\upsilon}(s-t)} \tau_{1,s} ds \end{bmatrix}$$

To satisfy $\lim_{t\to\infty} Z_t = 0$, it must be that $\alpha_{\bar{\upsilon}} = 0$. Evaluating at $t = 0$, and imposing the initial condition $b_{1,0}$ we find

$$\begin{aligned} b_{1,0} &= \alpha_\upsilon + \frac{1}{\bar{\upsilon} - \upsilon} \left[-\upsilon \int_0^\infty e^{-\upsilon(s-t)} \tau_{1,s} ds + \bar{\upsilon} \int_0^\infty e^{-\bar{\upsilon}(s-t)} \tau_{1,s} ds \right] \\ \Rightarrow \alpha_\upsilon &= b_{1,0} + \frac{1}{\bar{\upsilon} - \upsilon} \int_0^\infty \left(\upsilon e^{-\upsilon(s-t)} - \bar{\upsilon} e^{-\bar{\upsilon}(s-t)} \right) \tau_{1,s} ds \end{aligned}$$

so finally we obtain the solution

$$Z_t = \left(b_{1,0} + \frac{1}{(\bar{\upsilon}-\upsilon)}\int_0^\infty \left(\upsilon e^{-\upsilon s} - \bar{\upsilon} e^{-\bar{\upsilon} s}\right)\tau_{1,s}ds\right)\left[\begin{array}{c} -\frac{1}{\upsilon} \\ 1 \end{array}\right] e^{\upsilon t} +$$

$$-\frac{1}{\bar{\upsilon}-\upsilon}\left[\begin{array}{c} \int_t^\infty \left(e^{-\bar{\upsilon}(s-t)} - e^{-\upsilon(s-t)}\right)\tau_{1,s}ds \\ \int_t^\infty \left(\upsilon e^{-\upsilon(s-t)} - \bar{\upsilon} e^{-\bar{\upsilon}(s-t)}\right)\tau_{1,s}ds \end{array}\right]$$

$$= \left[\begin{array}{c} -\frac{1}{\upsilon} b_{1,0} e^{\upsilon t} - \frac{1}{(\bar{\upsilon}-\upsilon)}\left(\int_0^t \left(e^{-\upsilon(s-t)} - \frac{\bar{\upsilon}}{\upsilon} e^{-\bar{\upsilon} s} e^{\upsilon t}\right)\tau_{1,s}ds + \int_t^\infty e^{-\bar{\upsilon}(s-t)}\left(1 - \frac{\bar{\upsilon}}{\upsilon} e^{(\upsilon-\bar{\upsilon})t}\right)\tau_{1,s}ds\right) \\ b_{1,0} e^{\upsilon t} + \frac{1}{(\bar{\upsilon}-\upsilon)}\left(\int_0^t \left(\upsilon e^{-\upsilon(s-t)} - \bar{\upsilon} e^{-\bar{\upsilon} s} e^{\upsilon t}\right)\tau_{1,s}ds + \int_t^\infty \bar{\upsilon} e^{-\bar{\upsilon}(s-t)}\left(1 - e^{(\upsilon-\bar{\upsilon})t}\right)\tau_{1,s}ds\right) \end{array}\right]$$

From Lemma 1 we can derive the following corollary.

Corollary 1. *If $\bar{t} = 0$, we have*

$$b_{1,t|0} = -e^{\upsilon t}$$

If there is a transfer only at some time $t = \bar{t} \gg 0$, we approximately have

$$b_{1,t|\bar{t}} = \frac{\bar{\upsilon} e^{-\bar{\upsilon}(\bar{t}-t)} I_{(t<\bar{t})} + \upsilon e^{-\upsilon(\bar{t}-t)} I_{(t\geq\bar{t})}}{\bar{\upsilon}-\upsilon} e^{\rho\bar{t}}$$

where $I_{(.)}$ is the indicator function.

The second part of Corollary 1 provides an expression for the equilibrium path of bonds when the transfers are done at a time $t = \bar{t} \gg 0$. I interpret $\bar{t} \gg 0$ as being sufficiently large so that one can assume that time has started in the infinite past, so any "border effects" from having time starting at $t = 0$ are negligible.

Let us provide a proof for this Corollary. The first part is immediate from our previous derivations. For the second part, setting $b_{1,0} = 0,\ \tau_{1,t} = 0\ \forall t \neq \bar{t}$, we have that

$$\left[\begin{array}{c} \beta_{1,t} \\ b_{1,t} \end{array}\right] = \left[\begin{array}{c} -\frac{1}{(\bar{\upsilon}-\upsilon)} e^{-\bar{\upsilon}(\bar{t}-t)}\left(1 - \frac{\bar{\upsilon}}{\upsilon} e^{(\upsilon-\bar{\upsilon})t}\right) \\ \frac{1}{(\bar{\upsilon}-\upsilon)}\bar{\upsilon} e^{-\bar{\upsilon}(\bar{t}-t)}\left(1 - e^{(\upsilon-\bar{\upsilon})t}\right) \end{array}\right]\tau_{1,\bar{t}} I_{(t<\bar{t})} + \left[\begin{array}{c} -\frac{1}{(\bar{\upsilon}-\upsilon)}\left(e^{-\upsilon(\bar{t}-t)} - \frac{\bar{\upsilon}}{\upsilon} e^{-\bar{\upsilon}\bar{t}} e^{\upsilon t}\right) \\ \frac{1}{(\bar{\upsilon}-\upsilon)}\left(\upsilon e^{-\upsilon(\bar{t}-t)} - \bar{\upsilon} e^{-\bar{\upsilon}\bar{t}} e^{\upsilon t}\right) \end{array}\right]\tau_{1,\bar{t}} I_{(t\geq\bar{t})}$$

$$= \frac{1}{\bar{\upsilon}-\upsilon}\left\{\left[\begin{array}{c} -\left(e^{-\bar{\upsilon}(\bar{t}-t)} - \frac{\bar{\upsilon}}{\upsilon} e^{-\bar{\upsilon}\bar{t}} e^{\upsilon t}\right) \\ \bar{\upsilon}\left(e^{-\bar{\upsilon}(\bar{t}-t)} - e^{-\bar{\upsilon}\bar{t}} e^{\upsilon t}\right) \end{array}\right] I_{(t<\bar{t})} + \left[\begin{array}{c} -e^{-\upsilon(\bar{t}-t)} + \frac{\bar{\upsilon}}{\upsilon} e^{-\bar{\upsilon}\bar{t}} e^{\upsilon t} \\ \upsilon e^{-\upsilon(\bar{t}-t)} - \bar{\upsilon} e^{-\bar{\upsilon}\bar{t}} e^{\upsilon t} \end{array}\right] I_{(t\geq\bar{t})}\right\}\tau_{1,\bar{t}}$$

where $I_{(.)}$ is the indicator function.

If $\bar{t} \gg 0$, we can approximate $e^{-\bar{\upsilon}\bar{t}} e^{\upsilon t} \cong 0$, so we get

$$\left[\begin{array}{c} \beta_{1,t} \\ b_{1,t} \end{array}\right] = \frac{1}{\bar{\upsilon}-\upsilon}\left\{\left[\begin{array}{c} -e^{-\bar{\upsilon}(\bar{t}-t)} \\ \bar{\upsilon} e^{-\bar{\upsilon}(\bar{t}-t)} \end{array}\right] I_{(t<\bar{t})} + \left[\begin{array}{c} -e^{-\upsilon(\bar{t}-t)} \\ \upsilon e^{-\upsilon(\bar{t}-t)} \end{array}\right] I_{(t\geq\bar{t})}\right\}\tau_{1,\bar{t}}$$

B.3 HtM log-linear system

When agent 2 is HtM instead of a perpetual youth dynasty, the only equation that changes is the one that describes the consumption of agent 2. Now his consumption is given by

$$C_{2,t} = \hat{A}_t N_{2,t} - T_{2,t}$$

A log-linear approximation to this equation is

$$c_{2,t} = n_{2,t} - \tau_{2,t}$$

Just as we had before, we can substitute out $n_{i,t}$ using that

$$n_{i,t} = \frac{(1+\phi)\, c_t - c_{i,t}}{\phi}$$

so we obtain

$$c_{2,t} = c_t - \frac{\phi}{1+\phi}\tau_{2,t}$$

Differentiating this equation, and replacing in the equation for aggregate consumption we have

$$\begin{aligned}
\dot{c}_t &= \chi \dot{c}_{1,t} + (1-\chi)\, \dot{c}_{2,t} \\
&= \chi\,(r_t - \rho) + (1-\chi)\left(\dot{c}_t - \frac{\phi}{1+\phi}\dot{\tau}_{2,t}\right) \\
&= r_t - \rho + \frac{\phi}{1+\phi}\dot{\tau}_{1,t}
\end{aligned}$$

Therefore, the log-linear system is

$$\begin{aligned}
\dot{c}_t &= i_t - \pi_t - \rho + \frac{\phi}{1+\phi}\dot{\tau}_{1,t} \\
\dot{c}_{1,t} &= i_t - \pi_t - \rho \\
\dot{\pi}_t &= \rho\pi_t - \mu c_t \\
i_t &= \rho + \kappa_{y,t} c_t + \kappa_{\pi,t}\pi_t
\end{aligned}$$

I will now prove that we can obtain equation (15) as the limit of equation (12) when we take the limit $\lambda_2 \to \infty$. What we have to prove is that

$$\lim_{\lambda_2 \to \infty} \chi\,(\Lambda_2 - \Lambda_1)\, b_{1,t} = \frac{\phi}{1+\phi}\dot{\tau}_{1,t}$$

We know from Lemma 1 that for any path of transfers $\{\tau_{1,t}\}_{t=0}^{\infty}$ the equilibrium path for bonds is

$$b_{1,t} = \frac{1}{\bar{\upsilon} - \upsilon}\left[\int_0^t \left(\upsilon e^{-\upsilon(s-t)} - \bar{\upsilon} e^{-\bar{\upsilon}s} e^{\upsilon t}\right) \tau_{1,s} ds + \int_t^{\infty} \bar{\upsilon} e^{-\bar{\upsilon}(s-t)} \left(1 - e^{(\upsilon - \bar{\upsilon})t}\right) \tau_{1,s} ds\right]$$

Integrating by parts, we can write this as

$$\chi\,(\Lambda_2 - \Lambda_1)\, b_{1,t} = \begin{array}{l} -\int_0^t \frac{\chi(\Lambda_2 - \Lambda_1)}{\bar{\upsilon} - \upsilon} \left(e^{\upsilon t} e^{-\bar{\upsilon} s} - e^{\upsilon(t-s)}\right) \dot{\tau}_{1,s} ds + \\ + \int_t^{\infty} \frac{\chi(\Lambda_2 - \Lambda_1)}{\bar{\upsilon} - \upsilon} \left(e^{-\bar{\upsilon}(s-t)} - e^{-\bar{\upsilon}s} e^{\upsilon t}\right) \dot{\tau}_{1,s} ds \end{array}$$

Note that the first and last term converge to zero uniformly as $\lambda_2 \to \infty$, so that

$$\lim_{\lambda_2\to\infty} \chi\left(\Lambda_2 - \Lambda_1\right) b_{1,t} = \lim_{\lambda_2\to\infty} \int_0^t \frac{\chi\left(\Lambda_2 - \Lambda_1\right)}{\bar{\upsilon} - \upsilon} e^{\upsilon(t-s)} \dot{\tau}_{1,s} ds + \int_t^\infty \frac{\chi\left(\Lambda_2 - \Lambda_1\right)}{\bar{\upsilon} - \upsilon} e^{-\bar{\upsilon}(s-t)} \dot{\tau}_{1,s} ds$$

Integrating again by parts, we obtain

$$\begin{aligned} \lim_{\lambda_2\to\infty} \chi\left(\Lambda_2 - \Lambda_1\right) b_{1,t} &= \lim_{\lambda_2\to\infty} \frac{\chi\left(\Lambda_2 - \Lambda_1\right)}{\bar{\upsilon} - \upsilon} \left[\frac{1}{-\upsilon} \dot{\tau}_{1,t} + \frac{1}{\upsilon} e^{\upsilon t} \dot{\tau}_{1,0} + \int_0^t \frac{1}{\upsilon} e^{\upsilon(t-s)} \ddot{\tau}_{1,s} ds\right] + \\ &+ \frac{\chi\left(\Lambda_2 - \Lambda_1\right)}{\bar{\upsilon} - \upsilon} \left[\frac{1}{\bar{\upsilon}} \dot{\tau}_{1,t} + \int_t^\infty \frac{1}{\bar{\upsilon}} e^{-\bar{\upsilon}(s-t)} \ddot{\tau}_{1,s} ds\right] \end{aligned}$$

The terms with the second order derivatives converge to zero (there is point-wise convergence of bounded functions), so that

$$\begin{aligned} \lim_{\lambda_2\to\infty} \chi\left(\Lambda_2 - \Lambda_1\right) b_{1,t} &= \lim_{\lambda_2\to\infty} \frac{\chi\left(\Lambda_2 - \Lambda_1\right)}{\bar{\upsilon} - \upsilon} \left(\frac{1}{\bar{\upsilon}} - \frac{1}{\upsilon}\right) \dot{\tau}_{1,t} \\ &= \lim_{\lambda_2\to\infty} -\frac{\chi\left(\Lambda_2 - \Lambda_1\right)}{\bar{\upsilon}\upsilon} \dot{\tau}_{1,t} \\ &= \lim_{\lambda_2\to\infty} \frac{\chi\left(\Lambda_2 - \Lambda_1\right)}{\frac{1+\phi}{\phi} L} \dot{\tau}_{1,t} = \frac{\phi}{1+\phi} \dot{\tau}_{1,t} \end{aligned}$$

which is what we wanted to prove. Therefore, the equilibrium of the model with perpetual youth agents converges to the TANK model when we take $\lambda_2 \to \infty$.

B.4 Proof of Proposition 1

Comparing the system of equations (12)-(14) corresponding to the perpetual youth model and the system (15)-(17) corresponding to the case in which agent 2 is HtM, it is easy to see that for any path of transfers in the former, we can construct a path of transfers in the latter that will implement the same equilibrium $\{c_t, \pi_t, i_t\}_{t=0}^\infty$. We will exploit this equivalence to write the transfer multipliers as a function of the TANK-transfer multipliers.

Suppose that at some time $\bar{t}$ there is a transfer of size $\tau_{1,\bar{t}} = e^{\rho \bar{t}}$. This generates a path of $b_{1,t}, \beta_{1,t}$ as described by Lemma 1. By definition, the change in output will be

$$\int_0^\infty e^{-\rho t} c_t dt = m_{\bar{t}}^T \left(\lambda_1, \lambda_2\right)$$

If we construct transfers that generate the same equilibrium in the model with a HtM agent 2 by setting

$$\dot{\tau}_{1,t}^{HtM} = \frac{1+\phi}{\phi} \chi\left(\Lambda_2 - \Lambda_1\right) b_{1,t}$$

it will also be true that

$$\int_0^\infty e^{-\rho t} c_t dt = \int_0^\infty m_t^{TANK} e^{-\rho t} \tau_{1,t}^{HtM} dt$$

so we have that

$$m_{\bar{t}}^{T}\left(\lambda_1,\lambda_2\right)=\int_0^{\infty} m_t^{TANK}e^{-\rho t}\tau_{1,t}^{HtM}dt$$

Integrating by parts we have that

$$m_{\bar{t}}^{T}\left(\lambda_1,\lambda_2\right)=\int_0^{t}\left(e^{-\rho s}m_s^{TANK}\right)ds\tau_{1,t}^{HtM}|_{t=0}^{\infty}-\int_0^{\infty}\int_0^{t}\left(e^{-\rho s}m_s^{TANK}\right)ds\dot{\tau}_{1,t}^{HtM}dt$$

The first term is zero as long as $\lim_{t\to\infty}\int_0^t\left(e^{-\rho s}m_s^{TANK}\right)ds\tau_{1,t}^{HtM}=0$. By construction, $\lim_{t\to\infty}\tau_{1,t}^{HtM}=0$, this condition is immediately satisfied unless $\lim_{t\to\infty}\int_0^t\left(e^{-\rho s}m_s^{TANK}\right)ds=\infty$. In this latter case, using L'Hopital's rule, and the fact that (by Lemma 1) $\tau_{1,t}^{HtM}$ is proportional to $e^{\upsilon t}$ for $t\geq\bar{t}$, we find that

$$\lim_{t\to\infty}\frac{\int_0^t\left(e^{-\rho s}m_s^{TANK}\right)ds}{\frac{1}{\tau_{1,t}^{HtM}}}=\lim_{t\to\infty}\frac{e^{-\rho t}m_t^{TANK}}{-\frac{1}{\left(\tau_{1,t}^{HtM}\right)^2}\dot{\tau}_{1,t}^{HtM}}\propto\lim_{t\to\infty}-e^{(\upsilon-\rho)t}m_t^{TANK}$$

From now on, I will assume that the TANK-transfer multipliers grow sufficiently slowly so that $e^{(\upsilon-\rho)t}m_t^{TANK}$ converges to zero. In general this may require some assumptions on the parameters of the model.

Under this assumption, we get

$$\begin{aligned}m_{\bar{t}}^{T}\left(\lambda_1,\lambda_2\right)&=-\int_0^{\infty}\int_0^{t}\left(e^{-\rho s}m_s^{TANK}\right)ds\dot{\tau}_{1,t}^{HtM}dt\\&=-\frac{1+\phi}{\phi}\chi\left(\Lambda_2-\Lambda_1\right)\int_0^{\infty}\int_0^{t}\left(e^{-\rho s}m_s^{TANK}\right)b_{1,t}^{PY}dsdt\end{aligned}$$

Changing the order of integration

$$\begin{aligned}m_{\bar{t}}^{T}\left(\lambda_1,\lambda_2\right)&=\frac{\chi\left(\Lambda_2-\Lambda_1\right)}{L}\int_0^{\infty}e^{-\rho s}\frac{1+\phi}{\phi}L\left(-\beta_{1,s}\right)m_s^{TANK}ds\\&=\left(1+\frac{\Lambda_1}{\chi\left(\Lambda_2-\Lambda_1\right)}\right)^{-1}\int_0^{\infty}e^{-\rho s}\frac{1+\phi}{\phi}L\left(-\beta_{1,s}\right)m_s^{TANK}ds\end{aligned}$$

Define $\omega_s\equiv e^{-\rho s}\frac{1+\phi}{\phi}L\left(-\beta_{1,s}\right)$. Let us now prove that these weights integrate to 1.

Recall from before that if we take the difference between (7) and (8) and integrate, we get

$$c_t-c_{1,t}=-L\beta_{1,t}$$

so from this equation we get

$$\int_0^{\infty}e^{-\rho t}\left(c_t-c_{1,t}\right)dt=-L\int_0^{\infty}e^{-\rho t}\beta_{1,t}dt$$

From equation (9), with the condition $b_{1,0} = 0$ we have

$$\int_0^\infty e^{-\rho t}\tau_{i,t}dt = \frac{1+\phi}{\phi}\int_0^\infty e^{-\rho t}\left(c_t - c_{i,t}\right)dt$$

so combining these last two equations and using that by construction $\int_0^\infty e^{-\rho t}\tau_{i,t}^{PY}dt = 1$, we obtain

$$\begin{aligned}\frac{\phi}{1+\phi} &= -L\int_0^\infty e^{-\rho t}\beta_{1,t}dt\\ \Rightarrow 1 &= \int_0^\infty \frac{1+\phi}{\phi}Le^{-\rho t}\left(-\beta_{1,t}\right)dt\end{aligned}$$

which is what we wanted to prove.

The last thing I want to show is that the weights can also be written as $\omega_s = \left.\frac{\partial\left(e^{-\rho x}b_{1,x|t}\right)}{\partial x}\right|_{x=s}$. Using the expression above, we have

$$\omega_s = \frac{1+\phi}{\phi}Le^{-\rho t}\left(-\beta_{1,t}\right) = \frac{1+\phi}{\phi}\int_0^\infty e^{-\rho t}\left(c_t - c_{1,t}\right)dt$$

From equation (9) we get

$$\left(e^{-\rho t}b_{1,t}\right)' = \frac{1+\phi}{\phi}e^{-\rho t}\left(c_t - c_{1,t}\right) - e^{-\rho t}\tau_{1,t}$$

If there is a transfer only at time $\bar{t}$, then we can drop the term $e^{-\rho t}\tau_{1,t}$ since it is zero everywhere except at $\bar{t}$ so it will not affect the derivative.

B.5 Proof of Proposition 2

Suppose that the TANK-transfer multipliers are of the form $m_t^{TANK} = m_0^{TANK}e^{\xi t}$.
If there is a transfer only at $t = 0$, we know from Corollary 1 that

$$\beta_{1,t} = -\frac{1}{\upsilon}b_{1,0}e^{\upsilon t}$$

so setting $b_{1,0} = -1$ we have that the weights used to compute the weighted average are

$$\omega_t = \frac{1+\phi}{\phi}L\left(-\frac{1}{\upsilon}\right)e^{(\upsilon-\rho)t}$$

Therefore, the transfer multiplier at time $t = 0$ is

$$\begin{aligned}
m_0^T(\lambda_1, \lambda_2) &= \left(1 + \frac{1}{\chi}\frac{\Lambda_1}{\Lambda_2 - \Lambda_1}\right)^{-1} \int_0^\infty \frac{1+\phi}{\phi} L \left(-\frac{1}{\upsilon}\right) e^{(\upsilon - \rho)t} m_0^{TANK} e^{\xi t} dt \\
&= \left(1 - \frac{\Lambda_1}{L}\right) \frac{1+\phi}{\phi} L m_0^{TANK} \left(-\frac{1}{\upsilon}\right) \frac{1}{\rho - \upsilon - \xi} \\
&= m_0^{TANK} \frac{L - \Lambda_1}{L + \frac{\phi}{1+\phi} \frac{\rho - \sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}}{2} \xi}
\end{aligned}$$

Differentiating wrt Λ_2

$$\frac{\partial m_0^T(\lambda_1, \lambda_2)}{\partial \Lambda_2} = m_0^{TANK} \frac{\partial L}{\partial \Lambda_2} \left\{ \frac{\frac{\phi}{1+\phi}\frac{\rho - \sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}}{2}\xi + \Lambda_1 + \frac{\xi(L - \Lambda_1)}{\sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}}}{\left(L + \frac{\phi}{1+\phi}\frac{\rho - \sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}}{2}\xi\right)^2} \right\}$$

The sign of this derivative is

$$\begin{aligned}
sign\left\{\frac{\partial m_0^T(\lambda_1, \lambda_2)}{\partial \Lambda_2}\right\} &= sign\left\{ \left(\frac{\phi}{1+\phi} \frac{\rho - \sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}}{2} + \frac{L - \Lambda_1}{\sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}} \right) \xi + \Lambda_1 + \right\} \\
&= sign\left\{ -\xi + \frac{\Lambda_1}{\frac{\frac{\phi}{1+\phi}\frac{1}{2}\left(\sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L} - \rho\right)^2 + 2\Lambda_1}{2\sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}}} \right\}
\end{aligned}$$

Define

$$\Upsilon(\lambda_1, \lambda_2) \equiv \Lambda_1 \frac{2\sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}}{\frac{\phi}{1+\phi}\frac{1}{2}\left(\sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L} - \rho\right)^2 + 2\Lambda_1}$$

It is trivial that $\Upsilon \geq 0$, and that $\lim_{\lambda_2 \to \infty} \Upsilon\left(\bar{\lambda}_1, \lambda_2\right) = 0$. To see how it depends on λ_2, take the derivative

$$\begin{aligned}
\frac{\partial \Upsilon}{\partial \Lambda_2} &= \frac{\partial \Upsilon}{\partial L}\frac{\partial L}{\partial \Lambda_2} \\
&= 2\Lambda_1 \frac{\partial\left(\sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}\right)}{\partial L} \frac{\left[\begin{array}{c} \frac{\phi}{1+\phi}\frac{1}{2}\left(\sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L} - \rho\right)^2 + 2\Lambda_1 + \\ -\sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}\frac{\phi}{1+\phi}\left(\sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L} - \rho\right) \end{array}\right]}{\left(\frac{\phi}{1+\phi}\frac{1}{2}\left(\sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L} - \rho\right)^2 + 2\Lambda_1\right)^2} \frac{\partial L}{\partial \Lambda_2} \\
&\propto 2(\Lambda_1 - L) < 0
\end{aligned}$$

which proves the first part of the Proposition.

If there is a transfer only at $\bar{t} \gg 0$, we know from Corollary 1 that

$$\beta_{1,t} = \frac{1}{\bar{\upsilon} - \upsilon} \left\{ -e^{-\bar{\upsilon}(\bar{t}-t)} I_{(t<\bar{t})} + -e^{-\upsilon(\bar{t}-t)} I_{(t\geq\bar{t})} \right\} e^{\rho\bar{t}}$$

so the weights used to compute the weighted average are

$$\omega_t = \frac{1+\phi}{\phi} L e^{-\rho(t-\bar{t})} \frac{1}{\bar{\upsilon} - \upsilon} \left\{ e^{-\bar{\upsilon}(\bar{t}-t)} I_{(t<\bar{t})} + e^{-\upsilon(\bar{t}-t)} I_{(t\geq\bar{t})} \right\}$$

and therefore

$$\begin{aligned} m_{\bar{t}}^{T}(\lambda_1, \lambda_2) &= \left(1 - \frac{\Lambda_1}{L}\right) \int_0^\infty \frac{1+\phi}{\phi} L e^{-\rho(t-\bar{t})} \frac{1}{\bar{\upsilon} - \upsilon} \left\{ e^{-\bar{\upsilon}(\bar{t}-t)} I_{(t<\bar{t})} + e^{-\upsilon(\bar{t}-t)} I_{(t\geq\bar{t})} \right\} m_0^{TANK} e^{\xi t} dt \\ &= (L - \Lambda_1) \frac{1+\phi}{\phi} \frac{1}{\sqrt{\rho^2 + 4\frac{1+\phi}{\phi} L}} m_0^{TANK} \left\{ \frac{e^{\upsilon\bar{t}} - e^{\xi\bar{t}}}{\upsilon - \xi} + \frac{e^{\xi\bar{t}}}{\bar{\upsilon} - \xi} \right\} \end{aligned}$$

Since we are assuming $\bar{t} \gg 0$, we have that $e^{\upsilon\bar{t}} \cong 0$, so that

$$\begin{aligned} m_t^{T}(\lambda_1, \lambda_2) &= (L - \Lambda_1) \frac{1+\phi}{\phi} \frac{1}{\sqrt{\rho^2 + 4\frac{1+\phi}{\phi} L}} m_0^{TANK} e^{\xi\bar{t}} \left\{ \frac{-\bar{\upsilon} + \upsilon}{(\upsilon - \xi)(\bar{\upsilon} - \xi)} \right\} \\ &= e^{\xi\bar{t}} \frac{L - \Lambda_1}{L - \frac{\phi}{1+\phi}\xi(\xi - \rho)} \end{aligned}$$

Differentiating wrt λ_2

$$\frac{\partial m_t^{T}(\lambda_1, \lambda_2)}{\partial \lambda_2} \propto \frac{-\frac{\phi}{1+\phi}\xi(\xi - \rho) + \Lambda_1}{\left(L - \frac{\phi}{1+\phi}\xi(\xi - \rho)\right)^2}$$

so that

$$sign\left\{ \frac{\partial m_t^{T}(\lambda_1, \lambda_2)}{\partial \lambda_2} \right\} = sign\left\{ \frac{1+\phi}{\phi}\Lambda_1 - \xi(\xi - \rho) \right\}$$

From this expression we immediately get the second part of the Proposition.

B.6 Taylor rule

In the main text, I considered the case of a liquidity trap (of both finite and infinite duration). Let us here consider the case in which monetary policy follows a standard Taylor rule of the form:

$$i_t = \rho + \kappa_y c_t + \pi_t, \text{ with } \kappa_y > \rho.$$

The only reason why I am assuming that the Taylor rule responds one-to-one with inflation is to simplify the calculations: in this case computing the TANK-transfer multipliers is very straightforward because inflation drops out of the differential equation that characterizes the path for $\{c_t\}$. However, we would obtain the same qualitative results if we allowed for a more general Taylor rule.

We can show (see calculations below) that the TANK-transfer multipliers are

$$m_t^{TANK} = \frac{\kappa_y e^{(\rho-\kappa_y)t} - \rho}{\kappa_y - \rho} \frac{\phi}{1+\phi}. \tag{43}$$

Note that these multipliers take a value $\frac{\phi}{1+\phi}$ at $t = 0$ and are monotonically decreasing. The reason why they decrease is a typical forward guidance effect. Suppose that we make a transfer to the HtM agent at some time $\bar{t}$. The larger $\bar{t}$, the more backloaded the increase in interest rates in response to the increase in output. Since an increase in interest rates at time $\bar{t}$ leads to a drop in demand from agent 1 for every period $t \in [0, \bar{t}]$, we get that the larger $\bar{t}$, the lower the TANK-transfer multiplier. In fact, note that the TANK-transfer multipliers become negative for $t > 1$, which is telling us that the forward guidance effect overpowers the direct effect of the fiscal stimulus.

If we compute the weighted average of the TANK-transfer multipliers to obtain the transfer multiplier for $t = 0$, we find that

$$m_0^T(\lambda_1, \lambda_2) = \Gamma(\lambda_1, \lambda_2, \kappa_y)\, \theta(\lambda_1, \lambda_2) \frac{\phi}{1+\phi}, \tag{44}$$

where

$$\Gamma(\lambda_1, \lambda_2, \kappa_y) \equiv \left(1 + \frac{\phi}{1+\phi} \frac{\bar{v}}{L} \kappa_y\right)^{-1}.$$

It is straightforward that Γ is lower than 1, so compared with the initial example we now get a lower transfer multiplier. This is not too surprising, since the monetary authority is dampening the fiscal stimulus by raising interest rates.

In order to analyze how the transfer multipliers depend on the difference between the MPCs of the two agents, we can immediately apply Proposition 2. For simplicity, I will discuss this under the assumption that $\lambda_1 = 0$, but we could also consider the general case $\lambda_1 \geq 0$. Since the TANK-transfer multipliers in (43) are decreasing, the first part of Proposition 2 tells us that $m_0^T(0, \lambda_2)$ is increasing with respect to λ_2.[51] Thus, this case is like panel (i) of Figure 5. However, for transfers done at time $\bar{t} \gg 0$, from the second part of Proposition 2 we know that $m_{\bar{t}}^T(0, \lambda_2)$ is *decreasing* in λ_2.

In Figure 14 we have a plot of the TANK-transfer multipliers and the transfer multipliers for various values of λ_2 when the monetary policy follows a Taylor rule. As we can see, $\frac{\partial m_t^T(0,\lambda_2)}{\partial \lambda_2}$ depends on t, being positive for t near zero and then the sign gets inverted as t gets larger.

[51] Although Proposition 2 assumed that the TANK-transfer multipliers were exponential, it is trivial to see that we can also apply it to the case in which $\lambda_1 = 0$ and the TANK-transfer multipliers are an affine transformation of an exponential.

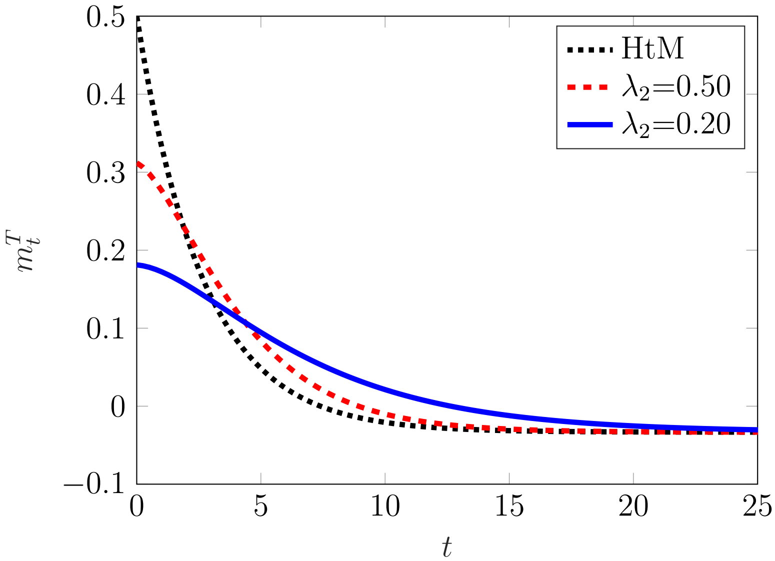

Figure 14 – The graph plots the paths for m_t^{TANK} and m_t^T as a function of t when monetary policy follows a Taylor rule. The paths for m_t^T are plotted for two values of λ_2 and assume $\lambda_1 = 0$.

Calculations Replacing the Taylor rule into the equations for the case with a HtM agent 2 we have:

$$\begin{aligned}
\dot{c}_t &= \kappa_y c_t + \frac{\phi}{1+\phi}\dot{\tau}_{1,t} \\
\int_t^\infty \left(e^{-\kappa_y s} c_s\right)' ds &= \frac{\phi}{1+\phi}\int_t^\infty e^{-\kappa_y s}\dot{\tau}_{1,s} ds \\
-e^{-\kappa_y t} c_t &= \frac{\phi}{1+\phi}\left[e^{-\kappa_y s}\tau_{1,s}|_t^\infty + \kappa_y \int_t^\infty e^{-\kappa_y s}\tau_{1,s} ds\right] \\
c_t &= \frac{\phi}{1+\phi}\left[\tau_{1,t} - \kappa_y \int_t^\infty e^{-\kappa_y (s-t)}\tau_{1,s} ds\right]
\end{aligned}$$

so its discounted value is

$$\begin{aligned}
\int_0^\infty e^{-\rho t} c_t dt &= \int_0^\infty e^{-\rho t}\frac{\phi}{1+\phi}\left[\tau_{1,t} - \kappa_y \int_t^\infty e^{-\kappa_y (s-t)}\tau_{1,s} ds\right] dt \\
&= \int_0^\infty e^{-\rho t}\left(\frac{\phi}{1+\phi}\frac{\kappa_y e^{(\rho-\kappa_y)t} - \rho}{\kappa_y - \rho}\right)\tau_{1,t} dt
\end{aligned}$$

So the TANK-transfer multipliers are

$$m_t^{TANK} = \frac{\phi}{1+\phi}\frac{\kappa_y e^{(\rho-\kappa_y)t} - \rho}{\kappa_y - \rho}$$

To obtain the multiplier m_0^T we have to compute the weighted average:

$$
\begin{aligned}
m_0^T(\lambda_1,\lambda_2) &= (L-\Lambda_1)\int_0^\infty \frac{1+\phi}{\phi}e^{-\rho s}\left(-\beta_{1,s|t}\right)m_s^{TANK}ds \\
&= (L-\Lambda_1)\int_0^\infty \frac{1+\phi}{\phi}e^{-\rho s}\left(-\frac{1}{\upsilon}\right)e^{\upsilon s}\frac{\phi}{1+\phi}\frac{\kappa_y e^{(\rho-\kappa_y)s}-\rho}{\kappa_y-\rho}ds \\
&= \left(1+\frac{\phi}{1+\phi}\frac{\bar{\upsilon}}{L}\kappa_y\right)^{-1}\theta\frac{\phi}{1+\phi}
\end{aligned}
$$

B.7 Infinitely lasting Liquidity Trap, with sticky prices

In this case, the equations with a HtM agent 2 and $i_t=\rho$ are

$$
\begin{aligned}
\dot{c}_t &= -\pi_t + \frac{\phi}{1+\phi}\dot{\tau}_{1,t} \\
\dot{\pi}_t &= \rho\pi_t - \mu c_t
\end{aligned}
$$

In matrix form

$$
\begin{bmatrix}\dot{c}_t\\ \dot{\pi}_t\end{bmatrix} = \begin{bmatrix}0 & -1\\ -\mu & \rho\end{bmatrix}\begin{bmatrix}c_t\\ \pi_t\end{bmatrix} + \begin{bmatrix}\frac{\phi}{1+\phi}\dot{\tau}_{1,t}\\ 0\end{bmatrix}
$$

Using vector notation, we have

$$
\dot{Z}_t = MZ_t + D_t
$$

where $Z_t \equiv \begin{bmatrix}c_t\\ \pi_t\end{bmatrix}$, $M \equiv \begin{bmatrix}0 & -1\\ -\mu & \rho\end{bmatrix}$, and $D_t \equiv \begin{bmatrix}\frac{\phi}{1+\phi}\dot{\tau}_{1,t}\\ 0\end{bmatrix}$. Solving the differential equation:

$$
\begin{aligned}
\dot{Z}_t &= MZ_t + D_t \\
\int_{t_1}^{t_2}\left(e^{-Ms}Z_s\right)'ds &= \int_{t_1}^{t_2} e^{-Ms}D_s ds \\
e^{-Mt_2}Z_{t_2} - e^{-Mt_1}Z_{t_1} &= \int_{t_1}^{t_2} e^{-Ms}D_s ds
\end{aligned}
$$

Assuming that $Z_{t_2}=0$ and taking $t_2\to\infty$

$$
Z_t = -\int_t^\infty e^{-M(s-t)}D_s ds
$$

I need to find the eigenvalues:

$$
\begin{aligned}
\det\left(\begin{bmatrix}-k & -1\\ -\mu & \rho-k\end{bmatrix}\right) &= 0 \\
k^2-\rho k-\mu &= 0
\end{aligned}
$$

Denote the eigenvalues as

$$\nu \equiv \frac{\rho - \sqrt{\rho^2 + 4\mu}}{2}, \quad \bar{\nu} \equiv \frac{\rho + \sqrt{\rho^2 + 4\mu}}{2}$$

so the matrix of eigenvectors is

$$\begin{bmatrix} 1 & 1 \\ -\nu & -\bar{\nu} \end{bmatrix}$$

where the first column is the eigenvector corresponding to ν and the second column is the eigenvector corresponding to $\bar{\nu}$.

Its inverse is

$$-\frac{1}{\bar{\nu}-\nu}\begin{bmatrix} -\bar{\nu} & -1 \\ \nu & 1 \end{bmatrix}$$

Therefore, we get that consumption is

$$\begin{aligned} c_t &= -\int_t^{\infty} E_1' \begin{bmatrix} 1 & 1 \\ -\nu & -\bar{\nu} \end{bmatrix} \begin{bmatrix} e^{-\nu(s-t)} & 0 \\ 0 & e^{-\bar{\nu}(s-t)} \end{bmatrix} \left(-\frac{1}{\bar{\nu}-\nu}\right) \begin{bmatrix} -\bar{\nu} & -1 \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \frac{\phi}{1+\phi}\dot{\tau}_{1,s} \\ 0 \end{bmatrix} ds \\ &= \frac{\phi}{1+\phi}\left[\tau_{1,t} + \frac{1}{\bar{\nu}-\nu}\int_t^{\infty} \mu\left(-e^{-\bar{\nu}(s-t)} + e^{-\nu(s-t)}\right)\tau_{1,s} ds\right] \end{aligned}$$

were I am assuming that $\lim_{t\to\infty}\tau_{1,t}e^{-\nu t} = 0$.

So the PDV is

$$\begin{aligned} \int_0^{\infty} e^{-\rho t} c_t dt &= \frac{\phi}{1+\phi}\left[\int_0^{\infty} e^{-\rho t}\tau_{1,t} dt + \frac{\mu}{\bar{\nu}-\nu}\int_0^{\infty}\int_t^{\infty} e^{-\rho t}\left(-e^{-\bar{\nu}(s-t)} + e^{-\nu(s-t)}\right)\tau_{1,s} ds dt\right] \\ &= \frac{\phi}{1+\phi}\left[\int_0^{\infty} e^{-\rho t}\tau_{1,t} dt + \frac{\mu}{\bar{\nu}-\nu}\int_0^{\infty}\left(-\frac{e^{-\bar{\nu}s} - e^{-\rho s}}{\rho-\bar{\nu}} + \frac{e^{-\nu s} - e^{-\rho s}}{\rho-\nu}\right)\tau_{1,s} ds\right] \\ &= \int_0^{\infty} e^{-\rho t}\frac{\phi}{1+\phi}\left\{\frac{|\nu|\, e^{\bar{\nu}t} + \bar{\nu}e^{-|\nu|t}}{\bar{\nu}+|\nu|}\right\}\tau_{1,t} dt \end{aligned}$$

To obtain the multiplier m_0^T we have to compute the weighted average:

$$\begin{aligned} m_t^T(\lambda_1,\lambda_2) &= \theta(\lambda_1,\lambda_2)\int_0^{\infty}\frac{1+\phi}{\phi}Le^{-\rho s}\left(-\frac{1}{\upsilon}\right)e^{\upsilon s}\left(\frac{|\nu|\, e^{\bar{\nu}s} + \bar{\nu}e^{-|\nu|s}}{|\nu|+\bar{\nu}}\frac{\phi}{1+\phi}\right) ds \\ &= \theta(\lambda_1,\lambda_2)\frac{1+\phi}{\phi}L\left(-\frac{1}{\upsilon}\right)\frac{(-\upsilon+\rho) + (|\nu|-\bar{\nu})}{(-\upsilon+\rho-\bar{\nu})(-\upsilon+\rho+|\nu|)}\frac{\phi}{1+\phi} \\ &= \theta(\lambda_1,\lambda_2)\left(1 - \frac{\mu}{\frac{1+\phi}{\phi}L}\right)^{-1}\frac{\phi}{1+\phi} \end{aligned}$$

To conclude this section, I want to provide an example to prove that a higher λ_2 can not only be associated with lower PDV of output, but also with a lower PDV at the time of the transfer. To find the effect on c_0, we don't need to solve the system of differential equations all over again, we can just use the equivalence between the OLG model and the TANK model when $\frac{\phi}{1+\phi}\dot{\tau}_{1,t} = \chi(\Lambda_2 - \Lambda_1)b_{1,t}$. Using this, we get

$$
\begin{aligned}
c_t &= \frac{\chi}{\bar{\nu}-\nu}\left(\Lambda_2-\Lambda_1\right)\int_0^{\infty}\left[\bar{\nu}e^{-\nu s}-\nu e^{-\bar{\nu}s}\right]e^{\upsilon s}ds \\
&= \frac{\chi}{\bar{\nu}-\nu}\left(\Lambda_2-\Lambda_1\right)\left[\frac{\bar{\nu}}{|\upsilon|-|\upsilon|}-\frac{\nu}{\bar{\upsilon}+|\upsilon|}\right]
\end{aligned}
$$

If we plot this as a function of λ_2, using for example the following parameter values: $\chi = 0.5,\ \rho = 0.02,\ \lambda_1 = 0,\ \mu = 0.1$, we find that this function is decreasing over some range of λ_2.

B.8 Finite liquidity trap, with rigid prices

With a HtM agent 2, we have

$$
\begin{aligned}
\int_t^{\widetilde{T}}\dot{c}_s ds &= \frac{\phi}{1+\phi}\int_t^{\widetilde{T}}\dot{\tau}_{1,s}ds \\
c_t &= \frac{\phi}{1+\phi}\left[\tau_{1,t}-\tau_{1,\widetilde{T}}\right]
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\int_0^{\infty}e^{-\rho t}c_t dt &= \int_0^{\widetilde{T}}e^{-\rho t}c_t dt \\
&= \left[\int_0^{\widetilde{T}}e^{-\rho t}\tau_{1,t}dt-\frac{1-e^{-\rho\widetilde{T}}}{\rho}\tau_{1,\widetilde{T}}\right]\frac{\phi}{1+\phi}
\end{aligned}
$$

When agent 2 is perpetual youth, from equation (7) and the equilibrium path for $\beta_{1,t}$ that we have from Corollary 1, we get

$$
\begin{aligned}
\int_t^{\widetilde{T}}\dot{c}_s ds &= \chi\left(\Lambda_2-\Lambda_1\right)\int_t^{\widetilde{T}}b_{1,s}ds \\
c_t &= -\chi\left(\Lambda_2-\Lambda_1\right)\left[\beta_{1,t}-\beta_{1,\widetilde{T}}\right] \\
c_t &= \chi\left(\Lambda_2-\Lambda_1\right)\left(-\frac{1}{\upsilon}\right)e^{\upsilon t}\left(1-e^{\upsilon\left(\widetilde{T}-t\right)}\right)
\end{aligned}
$$

so the PDV is

$$\begin{aligned}\int_0^{\widetilde{T}} e^{-\rho t} c_t dt &= \chi\left(\Lambda_2-\Lambda_1\right)\left(-\frac{1}{\upsilon}\right)\left(\int_0^{\widetilde{T}} e^{(\upsilon-\rho)t}dt - e^{\upsilon\widetilde{T}}\int_0^{\widetilde{T}} e^{-\rho t}dt\right) \\ &= \chi\left(\Lambda_2-\Lambda_1\right)\left(-\frac{1}{\upsilon}\right)\left(\frac{1-e^{(\upsilon-\rho)\widetilde{T}}}{\rho-\upsilon} - e^{\upsilon\widetilde{T}}\frac{1-e^{-\rho\widetilde{T}}}{\rho}\right) \\ &= \theta\left(\Lambda_1,\Lambda_2\right)\left(1-\frac{\bar{\upsilon}e^{-|\upsilon|\widetilde{T}}-|\upsilon|\,e^{-\bar{\upsilon}\widetilde{T}}}{\rho}\right)\frac{\phi}{1+\phi}\end{aligned}$$

Define $\Gamma \equiv 1-\frac{\bar{\upsilon}e^{-|\upsilon|\widetilde{T}}-|\upsilon|e^{-\bar{\upsilon}\widetilde{T}}}{\rho}$. The behavior of Γ as a function of $\widetilde{T}$ is trivial. To see that it is increasing wrt λ_2, we can take the derivative:

$$\begin{aligned}\frac{\partial\left[\bar{\upsilon}e^{-|\upsilon|\widetilde{T}}-|\upsilon|\,e^{-\bar{\upsilon}\widetilde{T}}\right]}{\partial\Lambda_2} &= \left[\frac{\partial\bar{\upsilon}}{\partial L}e^{-|\upsilon|\widetilde{T}} - \bar{\upsilon}e^{-|\upsilon|\widetilde{T}}\widetilde{T}\frac{\partial|\upsilon|}{\partial L} - \frac{\partial|\upsilon|}{\partial L}e^{-\bar{\upsilon}\widetilde{T}} + |\upsilon|\,e^{-\bar{\upsilon}\widetilde{T}}\widetilde{T}\frac{\partial\bar{\upsilon}}{\partial L}\right]\frac{\partial L}{\partial\Lambda_2} \\ &= \left[\left(1-\bar{\upsilon}\widetilde{T}\right)\left(e^{-|\upsilon|\widetilde{T}}-e^{-\bar{\upsilon}\widetilde{T}}\right)-\rho\widetilde{T}e^{-\bar{\upsilon}\widetilde{T}}\right]\frac{\partial\bar{\upsilon}}{\partial L}\frac{\partial L}{\partial\Lambda_2}\end{aligned}$$

I want to show that this expression is negative. If $1-\bar{\upsilon}\widetilde{T}<0$ then it is trivial that all the expression is negative. So let us analyze the non-trivial case $1-\bar{\upsilon}\widetilde{T}>0$. Note that

$$\left(1-\bar{\upsilon}\widetilde{T}\right)\left(e^{-|\upsilon|\widetilde{T}}-e^{-\bar{\upsilon}\widetilde{T}}\right)-\rho\widetilde{T}e^{-\bar{\upsilon}\widetilde{T}}<0 \iff e^{\rho\widetilde{T}}-1<\frac{\rho\widetilde{T}}{1-\bar{\upsilon}\widetilde{T}}$$

Since $\frac{\rho\widetilde{T}}{1-\rho\widetilde{T}}<\frac{\rho\widetilde{T}}{1-\bar{\upsilon}\widetilde{T}}$, a sufficient condition for the above condition to be satisfied is

$$e^{\rho\widetilde{T}}-1<\frac{\rho\widetilde{T}}{1-\rho\widetilde{T}}$$

Define $z\equiv\rho\widetilde{T}$, so we need to show that

$$1-\bar{\upsilon}\widetilde{T}<1-\rho\widetilde{T}$$

$$e^z-1<\frac{z}{1-z}$$

for $z\in(0,1)$ (note that $1-\bar{\upsilon}\widetilde{T}>0\Rightarrow 1-z>0$). We can easily check this condition by plotting the two functions.

C Proofs for Section 4

C.1 Government Purchases Multipliers

The system with government purchases is

$$\begin{bmatrix} \dot{y}_t \\ \dot{\pi}_t \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\pi - 1 \\ -\mu & \rho \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} + \dot{g}_t E_1' + \mu \frac{1}{1+\phi} g_t E_2' \tag{45}$$

Given the linearity of the model, we can find the solution to two non-homogeneous systems, and the add them up. Let us start with the case in which there is only a non-homogeneous term in the first equation (i.e. $\dot{g}_t$). The system will coincide with the case in which we have a HtM agent 2 described by (15)-(16) if we set $\tau_{1,t} = \frac{1+\phi}{\phi} g_t$, so we get that

$$\int_0^\infty e^{-\rho t} y_t dt = \frac{1+\phi}{\phi} \int_0^\infty e^{-\rho t} m_t^{TANK} g_t dt$$

Let us now consider the case in which there is only a non-homogeneous term in the second equation (i.e. $\mu \frac{1}{1+\phi} g_t$).

Using vector notation, we have

$$\dot{Z}_t = M Z_t + D_t$$

where $Z_t \equiv \begin{bmatrix} y_t \\ \pi_t \end{bmatrix}$, $M \equiv \begin{bmatrix} 0 & \kappa_\pi - 1 \\ -\mu & \rho \end{bmatrix}$ and $D_t \equiv \begin{bmatrix} 0 \\ \mu \frac{1}{1+\phi} g_t \end{bmatrix}$. Solving the differential equation:

$$\begin{aligned} \dot{Z}_t &= M Z_t + D_t \\ \int_{t_1}^{t_2} \left(e^{-Ms} Z_s\right)' ds &= \int_{t_1}^{t_2} e^{-Ms} D_s ds \\ e^{-Mt_2} Z_{t_2} - e^{-Mt_1} Z_{t_1} &= \int_{t_1}^{t_2} e^{-Ms} D_s ds \end{aligned}$$

Assuming that $Z_{t_2} = 0$ and taking $t_2 \to \infty$

$$Z_t = -\int_t^\infty e^{-M(s-t)} D_s ds$$

I need to find the eigenvalues:

$$\begin{aligned} \det\left(\begin{bmatrix} -k & (\kappa_\pi - 1) \\ -\mu & \rho - k \end{bmatrix}\right) &= 0 \\ k^2 - \rho k + \mu\left(\kappa_\pi - 1\right) &= 0 \end{aligned}$$

Denote the eigenvalues as

$$\eta \equiv \frac{\rho - \sqrt{\rho^2 - 4\mu\left(\kappa_\pi - 1\right)}}{2}, \quad \bar{\eta} \equiv \frac{\rho + \sqrt{\rho^2 - 4\mu\left(\kappa_\pi - 1\right)}}{2}$$

and the matrix of eigenvectors is

$$\begin{bmatrix} 1 & 1 \\ \frac{\eta}{\kappa_\pi - 1} & \frac{\bar{\eta}}{\kappa_\pi - 1} \end{bmatrix}$$

and its inverse is

$$\frac{\kappa_\pi - 1}{\bar{\eta} - \eta}\begin{bmatrix} \frac{\bar{\eta}}{\kappa_\pi - 1} & -1 \\ -\frac{\eta}{\kappa_\pi - 1} & 1 \end{bmatrix}$$

so we find that

$$\begin{aligned} c_t &= -\int_t^\infty E_1' \begin{bmatrix} 1 & 1 \\ \frac{\eta}{\kappa_\pi - 1} & \frac{\bar{\eta}}{\kappa_\pi - 1} \end{bmatrix} \begin{bmatrix} e^{-\eta(s-t)} & 0 \\ 0 & e^{-\bar{\eta}(s-t)} \end{bmatrix} \frac{\kappa_\pi - 1}{\bar{\eta} - \eta} \begin{bmatrix} \frac{\bar{\eta}}{\kappa_\pi - 1} & -1 \\ -\frac{\eta}{\kappa_\pi - 1} & 1 \end{bmatrix} E_2 \left(\mu \frac{1}{1+\phi} g_s\right) ds \\ &= \int_t^\infty \frac{\kappa_\pi - 1}{\bar{\eta} - \eta}\left[e^{-\eta(s-t)} - e^{-\bar{\eta}(s-t)}\right]\left(\mu \frac{1}{1+\phi} g_s\right) ds \end{aligned}$$

and therefore

$$\begin{aligned} \int_0^\infty e^{-\rho t} c_t dt &= \int_0^\infty \int_t^\infty e^{-\rho t} \frac{\kappa_\pi - 1}{\bar{\eta} - \eta}\left[e^{-\eta(s-t)} - e^{-\bar{\eta}(s-t)}\right]\left(\mu \frac{1}{1+\phi} g_s\right) ds dt \\ &= \int_0^\infty \frac{\kappa_\pi - 1}{\bar{\eta} - \eta}\left[e^{-\eta s}\int_0^s e^{(\eta-\rho)t} dt - e^{-\bar{\eta} s}\int_0^s e^{(\bar{\eta}-\rho)t} dt\right]\left(\mu \frac{1}{1+\phi} g_s\right) ds \\ &= \int_0^\infty e^{-\rho s}\frac{1}{1+\phi}\left(1 - \frac{\bar{\eta} e^{\eta s} - \eta e^{\bar{\eta} s}}{\bar{\eta} - \eta}\right) g_s ds \end{aligned}$$

Combining with our previous results, we find that the G multipliers are

$$m_t^G = \frac{1+\phi}{\phi} m_t^{TANK} + \Psi\left(t\right)$$

where we define

$$\Psi\left(t\right) \equiv \frac{1}{1+\phi}\left(1 - \frac{\bar{\eta} e^{\eta t} - \eta e^{\bar{\eta} t}}{\bar{\eta} - \eta}\right)$$

Note that this function satisfies

$$\Psi\left(0\right) = 0$$

and

$$\Psi'\left(t\right) = -\frac{\bar{\eta}\eta}{1+\phi}\frac{e^{\eta t} - e^{\bar{\eta} t}}{\bar{\eta} - \eta}$$

If $\kappa_\pi - 1 < 0$, we have $\eta < 0 < \bar{\eta}$, and therefore $\Psi'\left(t\right) < 0$. Meanwhile, if $\kappa_\pi - 1 > 0$, we have that $0 < \eta < \bar{\eta}$, and therefore $\Psi'\left(t\right) > 0$. This implies that

$$sign\left\{\Psi\left(t\right)\right\} = sign\left\{\kappa_\pi - 1\right\}$$

Equivalence between transfer multipliers and government purchases financed up-front In order to show the equivalence between transfers and government purchases $g_t = -(\upsilon - \rho) e^{\upsilon t}$, we need to show that both generate the same system of differential equations.

Consider first a transfer from type 1 to type 2 agents. Since we are assuming $\kappa_\pi = 1$, consumption is characterized by the equation

$$\begin{aligned} \dot{c}_t &= r_t - \rho + \chi (\Lambda_2 - \Lambda_1) b_{1,t} \\ &= \kappa_{y,t} c_t + \chi (\Lambda_2 - \Lambda_1) \left(-e^{\upsilon t}\right) \end{aligned}$$

where

$$\upsilon = \frac{\rho - \sqrt{\rho^2 + 4L}}{2}$$

In the case in which we only have type 1 agents and the government, if the government raises a \$1 tax at $t = 0$ and then spends according to $g_t = (|\upsilon| + \rho) e^{\upsilon t}$, we find that the equation that characterizes output is

$$\begin{aligned} \dot{y}_t &= \dot{c}_t + \dot{g}_t \\ &= r_t - \rho - \Lambda_1 b_{1,t} + \dot{g}_t \\ &= \kappa_{y,t} y_t - \Lambda_1 b_{1,t} + \upsilon (|\upsilon| + \rho) e^{\upsilon t} \end{aligned}$$

It must be that $b_{1,t} = -b_t^g$ to obtain market clearing in the bonds market, and we can obtain b_t^g from

$$\begin{aligned} \dot{b}_t^g &= \rho b_t^g - g_t \\ \int_t^\infty \left(e^{-\rho s} b_s^g\right)' ds &= \int_t^\infty e^{-\rho s} (\upsilon - \rho) e^{\upsilon s} ds \\ b_t^g &= e^{\upsilon t} \end{aligned}$$

Replacing above, we find

$$\begin{aligned} \dot{c}_t &= r_t - \rho + \chi (\Lambda_2 - \Lambda_1) b_{1,t} \\ &= \kappa_{y,t} c_t + \chi (\Lambda_2 - \Lambda_1) \left(-e^{\upsilon t}\right) \end{aligned}$$

so we obtain the same equation as before.

C.2 Redistributive effect of government policies

The proof that we can decompose a monetary shock into a redistribution channel and a purely monetary effect is almost trivial. The system of equations that characterizes the equilibrium is

$$\begin{aligned}
\dot{c}_t &= \kappa_{y,t} c_t + (\kappa_{\pi,t} - 1)\,\pi_t + \varepsilon_t + \chi\,(\Lambda_2 - \Lambda_1)\, b_{1,0} e^{\upsilon t} \\
\dot{\pi}_t &= \rho \pi_t - \mu c_t \\
\dot{\varepsilon}_t &= -\eta \varepsilon_t
\end{aligned}$$

and $b_{1,0}$ is itself a function of the endogeneous real interest rate path. Suppose that we solve this system and find that this endogenous equilibrium value is $\breve{b}_{1,0}$. We can then solve this system with two different sets of initial conditions: $(b_{1,0}, \varepsilon_0) = \left(\breve{b}_{1,0}, 0\right)$ and $(b_{1,0}, \varepsilon_0) = (0, \varepsilon_0)$, the former corresponding to the purely redistributive effect, and the latter corresponding to the purely monetary effect of the shock. The sum of these two solutions satisfies $(b_{1,0}, \varepsilon_0) = \left(\breve{b}_{1,0}, \varepsilon_0\right)$, so it is the actual solution to the system above.

C.3 Capital and Investment

Note that the optimal choice of investment is a purely static decision, that solves

$$\max_{\iota_t} Q_t^K \phi(\iota_t) - \iota_t$$

so the optimal investment satisfies

$$\phi'\left(\iota\left(Q_t^K\right)\right) = \frac{1}{Q_t^K}$$

Let us first obtain the TANK-transfer multipliers. The arbitrage condition between capital and bonds (using that $r = \rho_1$) is

$$\rho_1 = \frac{C_t}{Q_t K_t} + \phi(Q_t) + \frac{\dot{Q}_t}{Q_t}$$

Consumption of agent 1 is given by

$$C_{1,t} = \rho_1 \left(Q_t K_{1,t} - \mathcal{T}_{1,t}^r\right)$$

while consumption of the HtM agent (who in this case has no labor income) is simply

$$C_{2,t} = -T_{2,t}$$

This means that aggregate consumption is (where I am using that $\chi T_{1,t} + (1-\chi)\,T_{2,t} = 0$)

$$\begin{aligned}
C_t &= \chi C_{1,t} + (1-\chi)\, C_{2,t} \\
&= \rho_1 Q_t K_t - \chi\left(\rho_1 \mathcal{T}_{1,t}^r - T_{1,t}\right)
\end{aligned}$$

Replacing in the arbitrage condition we obtain

$$\frac{\dot{Q}_t}{Q_t} = \chi \frac{\rho_1 \mathcal{T}^r_{1,t} - T_{1,t}}{Q_t K_t} - \phi\left(Q_t\right)$$

Log-linearizing and solving the differential equation we find

$$\begin{aligned} \dot{q}_t &= -\phi'\left(0\right) q_t + \chi \frac{\rho_1 \mathcal{T}^r_{1,t} - T_{1,t}}{K^*} \\ \Rightarrow q_t &= -\chi \int_t^\infty e^{\phi'(0)(s-t)} \frac{\rho_1 \mathcal{T}^r_{1,s} - T_{1,s}}{K^*} ds \end{aligned}$$

Note that by definition

$$\dot{\mathcal{T}}^r_{1,s} = -T_{1,s} + \rho_1 \mathcal{T}^r_{1,s}$$

so replacing above and integrating by parts (I assume as usual that the transfers fade out over time, in this case we need $\lim_{s\to\infty} e^{\phi'(0)s} \mathcal{T}^r_{1,s}$)

$$\begin{aligned} q_t &= -\chi \int_t^\infty e^{\phi'(0)(s-t)} \frac{\dot{\mathcal{T}}^r_{1,s}}{K^*} ds \\ q_t &= -\frac{\chi}{K^*}\left[e^{\phi'(0)(s-t)} \mathcal{T}^r_{1,s}\Big|_t^\infty - \phi'\left(0\right) \int_t^\infty e^{\phi'(0)(s-t)} \mathcal{T}^r_{1,s} ds \right] \\ q_t &= \frac{\chi}{K^*}\left[\mathcal{T}^r_{1,t} + \phi'\left(0\right) \int_t^\infty e^{\phi'(0)(s-t)} \mathcal{T}^r_{1,s} ds \right] \end{aligned}$$

Note that since the dividends paid out to capital owners (net of investment costs) are equal to aggregate consumption, it must be that the value of the stock of capital at $t = 0$ is equal to the PDV of consumption at $t = 0$:

$$\begin{aligned} Q_0 K^* &= \int_0^\infty e^{-\rho_1 t} C_t dt \\ \frac{Q^* K^*}{C^*} \frac{Q_0 - Q^*}{Q^*} &= \int_0^\infty e^{-\rho_1 t} \left(\frac{C_t - C^*}{C^*} \right) dt \\ q_0 &= \rho_1 \int_0^\infty e^{-\rho_1 t} c_t dt \end{aligned}$$

Combining this with our previous result, we find

$$\begin{aligned} q_0 = \rho_1 \int_0^\infty e^{-\rho_1 t} c_t dt &= \frac{\chi}{K^*}\left[\mathcal{T}^r_{1,0} + \phi'\left(0\right) \int_0^\infty e^{\phi'(0)s} \left(\int_s^\infty e^{-\rho_1(x-s)} T_{1,x} dx \right) ds \right] \\ \int_0^\infty e^{-\rho_1 t} c_t dt &= \chi \int_0^\infty e^{-\rho_1 x} \left[1 + \frac{\phi'\left(0\right)}{\phi'\left(0\right) + \rho_1} \left(e^{(\phi'(0)+\rho_1)x} - 1 \right) \right] \left(\frac{T_{1,x}}{\rho_1 K^*} \right) dx \end{aligned}$$

Therefore, we find that the TANK-transfer multipliers are

$$m_t^{TANK} = \chi \left[1 + \frac{\phi'(0)}{\phi'(0) + \rho_1}\left(e^{(\phi'(0)+\rho_1)t} - 1\right)\right]$$

Now we need to show that we can express the transfer multipliers as a weighted average of the TANK-transfer multipliers. The consumption of each agent is

$$C_{i,t} = \rho_i W_{i,t}$$

where $W_{i,t}$ is their respective wealth. Therefore, total consumption is given by

$$C_t = \chi\rho_1 W_{1,t} + (1-\chi)\rho_2 W_{2,t}$$

Differentiating this expression we get

$$\begin{aligned}
\dot{C}_t &= \chi\rho_1 \dot{W}_{1,t} + (1-\chi)\rho_2 \dot{W}_{2,t} \\
\dot{C}_t &= \chi\rho_1 (r_t - \rho_1) W_{1,t} + (1-\chi)\rho_2 (r_t - \rho_2) W_{2,t} \\
\frac{\dot{C}_t}{C_t} &= r_t - \left[\rho_1 \frac{\chi\rho_1\zeta_{1,t}}{\bar{\rho}_t} + \rho_2 \frac{(1-\chi)\rho_2\zeta_{2,t}}{\bar{\rho}_t}\right]
\end{aligned}$$

where

$$\begin{aligned}
\zeta_{i,t} &\equiv \frac{W_{i,t}}{Q_t K_t} \\
\bar{\rho}_t &\equiv \chi\rho_1\zeta_{1,t} + (1-\chi)\rho_2\zeta_{2,t}
\end{aligned}$$

so ζ_i is the wealth of agent 1 relativ to aggregate wealth and $\bar{\rho}$ is a wealth-weighted average discount factor.

Using that $r_t = \rho_1$ and $\chi\zeta_{1,t} + (1-\chi)\zeta_{2,t} = 1$:

$$\frac{\dot{C}_t}{C_t} = \frac{(\rho_1 - \rho_2)\rho_2}{(\rho_1 - \rho_2)\chi\zeta_{1,t} + \rho_2}\left(1 - \chi\zeta_{1,t}\right)$$

Log-linearizing we find

$$\dot{c}_t = \frac{(\rho_2 - \rho_1)\rho_2}{\rho_1} z_{1,t}$$

where $z_{1,t} = \ln\left(\frac{\zeta_{1,t}}{\zeta_1^*}\right)$. From the definition of $\zeta_{1,t}$, we have

$$\begin{aligned}
\frac{\dot{\zeta}_{1,t}}{\zeta_{1,t}} &= \frac{\dot{W}_{1,t}}{W_{1,t}} - \left(\frac{\dot{Q}_t}{Q_t} + \frac{\dot{K}_t}{K_t}\right) \\
&= -\left(\frac{\dot{Q}_t}{Q_t} + \phi\left(Q_t\right)\right) \\
&= -\left(\rho_1 - \frac{C_t}{Q_t K_t}\right) \\
&= -\left(\rho_1 - \bar{\rho}_t\right) \\
&= \left(\rho_2 - \rho_1\right)\left(1 - \chi\zeta_{1,t}\right)
\end{aligned}$$

Log-linearizing we find

$$\begin{aligned}
\dot{z}_{1,t} &= \left(\rho_1 - \rho_2\right) z_{1,t} \\
z_{1,t} &= z_{1,0} e^{(\rho_1 - \rho_2)t}
\end{aligned}$$

and therefore

$$\dot{c}_t = \frac{\left(\rho_2 - \rho_1\right)\rho_2}{\rho_1} z_{1,0} e^{(\rho_1 - \rho_2)t}$$

In the case with an HtM type 2 agent, we have that (where I am using $r_t = \rho_1$ to obtain $\dot{C}_{1,t} = 0$ and $\chi T_{1,t} + (1-\chi) T_{2,t} = 0$):

$$\begin{aligned}
\dot{C}_t &= \chi \dot{C}_{1,t} + (1-\chi)\dot{C}_{2,t} \\
\frac{\dot{C}_t}{C_t} &= \chi \frac{\dot{T}_{1,t}^{HtM}}{C_t}
\end{aligned}$$

so log-linearizing we obtain

$$\dot{c}_t = \chi \dot{\tau}_{1,t}^{HtM}$$

If we choose the path $\{\tau_{1,t}\}_{t=0}^{\infty}$ to replicate the path for $\dot{c}_t$ with perpetual youth agents, then the effect on output will be the same. We have to set

$$\begin{aligned}
\chi \dot{\tau}_{1,t}^{HtM} &= \frac{\left(\rho_2 - \rho_1\right)\rho_2}{\rho_1} z_{1,0} e^{(\rho_1 - \rho_2)t} \\
\Rightarrow \tau_{1,t}^{HtM} &= \frac{1}{\chi}\frac{\rho_2}{\rho_1} e^{(\rho_1 - \rho_2)t}\left(-z_{1,0}\right)
\end{aligned}$$

Note that

$$\begin{aligned} z_{1,0} &= \frac{W_{1,0}}{Q_0 K^*} \\ &\cong \frac{W_{1,0} - W_1^*}{W_1^*} - \frac{Q_0 - Q^*}{Q^*} \\ &= \frac{\left[\frac{1}{\chi} Q_0 K^* - T_{1,0}\right] - \frac{1}{\chi} Q^* K^*}{\frac{1}{\chi} Q^* K^*} - \frac{Q_0 - Q^*}{Q^*} \\ &= -\frac{T_{1,0}}{\frac{1}{\chi} Q^* K^*} \\ &= -\frac{T_{1,0}}{\frac{1}{\chi} Y^*} \frac{Y^*}{Q^* K^*} \\ &= -\chi \rho_1 \tau_{1,0} \end{aligned}$$

so replacing above we find

$$\tau_{1,t}^{HtM} = \rho_2 \tau_{1,0} e^{(\rho_1 - \rho_2)t}$$

Assuming $\tau_{1,0} = 1$, we find that the effect on output is given by

$$\int_0^\infty m_t^{TANK} \times \left(e^{-\rho_1 t} \tau_{1,t}^{HtM}\right) dt = \int_0^\infty m_t^{TANK} \times \rho_2 e^{-\rho_2 t} dt$$

which is what we wanted to prove.

D Proofs for Section 5

D.1 Social Welfare Function

The initial definition of the SWF is

$$\begin{aligned} SWF = \chi \left[V_1^{-0} + \lambda \int_0^\infty e^{-\rho t} V_1^t dt\right] + \\ + (1-\chi) \left[V_2^{-0} + \lambda \int_0^\infty e^{-\rho t} V_2^t dt\right] \end{aligned}$$

Replacing the V's for the explicit expression for the utility function we get (where $C_{i,s}^t, N_{i,s}^t$ are the consumption and labor at time s of agents born at time t, and $C_{i,s}^{-0}, N_{i,s}^{-0}$ correspond to agents already alive at time $t = 0$):

$$\begin{aligned} V_i^{-0} + \lambda \int_0^\infty e^{-\rho t} V_i^t dt &= \int_0^\infty e^{-(\rho+\lambda_i)s} u\left(C_{i,s}^{-0}, N_{i,s}^{-0}\right) ds + \lambda \int_0^\infty e^{-\rho t} \int_t^\infty e^{-(\rho+\lambda_i)(s-t)} u\left(C_{i,s}^t, N_{i,s}^t\right) ds dt \\ &= \int_0^\infty e^{-\rho s} \left[e^{-\lambda_i s} u\left(C_{i,s}^{-0}, N_{i,s}^{-0}\right) + \int_0^s \lambda e^{-\lambda_i (s-t)} u\left(C_{i,s}^t, N_{i,s}^t\right) dt\right] ds \end{aligned}$$

Note that the expression between brackets is the cross-sectional sum of utilities of all type i agents alive at time s: there are $e^{-\lambda_i s}$ agents from the initial generation and there is a density $\lambda e^{-\lambda_i(s-t)}$ of agents born at time t who are still alive at time s.

D.2 Derivation of linear model in gaps

If inflation is zero, we obtain from the NKPC that

$$\check{c}_t = a_t$$

so aggregate product only depends on productivity, and therefore the NKPC in gaps is

$$\begin{aligned}\dot{\pi}_t &= \rho\pi_t - \mu\left(c_t - \check{c}_t\right)\\ &= \rho\pi_t - \mu x_t\end{aligned}$$

Define the "reference" or "natural" interest rate as the one that obtains zero inflation when there are no debts, so that

$$\dot{\check{c}}_t = \check{r}_t - \rho$$

Then, the law of motion for x_t is

$$\begin{aligned}\dot{x}_t &= \dot{c}_t - \dot{\check{c}}_t\\ &= \left(i_t - \pi_t - \check{r}_t\right) + \chi\left(\Lambda_2 - \Lambda_1\right) b_{1,t}\end{aligned}$$

Since both agents consume the same when there are no debts, we have that $\check{c}_t = \check{c}_{1,t} = a_t$, so we can therefore write agent 1's Euler equation and the law of motion for bonds, respectively, as

$$\begin{aligned}\dot{x}_{1,t} &= i_t - \pi_t - \check{r}_t - \Lambda_1 b_{1,t}\\ \dot{b}_{1,t} &= \rho b_{1,t} + \frac{1+\phi}{\phi}\left(x_t - x_{1,t}\right) - \tau_{1,t}\end{aligned}$$

D.3 Derivation of loss function

The per period utility function of each agent is

$$u\left(C_{i,t}^h, N_{i,t}^h\right) = \ln\left(C_{i,t}^h\right) - \frac{\left(N_{i,t}^h\right)^{1+\phi}}{1+\phi}$$

A second order approximation to this function around (C^*, N^*) gives us

$$\begin{aligned} u\left(C_t, N_t\right) - U^* &\cong u_c^* C^* \left(\frac{C_t - C^*}{C^*}\right) + u_n^* N^* \left(\frac{N_t - N^*}{N^*}\right) + \\ &+ \frac{1}{2} u_{cc}^* C^{*2} \left(\frac{C_t - C^*}{C^*}\right)^2 + \frac{1}{2} u_{nn}^* N^{*2} \left(\frac{N_t - N^*}{N^*}\right)^2 \\ &= u_c^* C^* \times c_t + u_n^* N^* \times \left(n_t + \frac{1+\phi}{2} n_t^2\right) \end{aligned}$$

where I am using that

$$\begin{aligned} \frac{C_t - C^*}{C^*} &\cong \frac{C^*\left(1 + c_t + \frac{1}{2} c_t^2\right) - C^*}{C^*} \\ &= c_t + \frac{1}{2} c_t^2 \end{aligned}$$

Using the efficiency of the steady state, we get that $u_n = -u_c \frac{C^*}{N^*}$. Therefore, integrating across all the generations of type i agents alive at time t we find

$$\frac{u_{i,t} - u^*}{u_c^* C^*} = \int_0^1 c_{i,t}\left(h\right) dh - \int_0^1 n_{i,t}\left(h\right) dh - \frac{1+\phi}{2} \int_0^1 \left(n_{i,t}\left(h\right)\right)^2 dh$$

Using that for any variable defined as an integral $Z_t = \int Z_t\left(h\right) dh$, we have up to a second-order[52]

$$z_t = E_h z_t\left(h\right) + \frac{1}{2} Var_h z_t\left(h\right)$$

so we get that

$$\begin{aligned} \frac{u_{i,t} - u^*}{u_c^* C^*} &= E_h c_{i,t}\left(h\right) - E_h n_{i,t}\left(h\right) - \frac{1+\phi}{2} E_h \left[\left(n_{i,t}\left(h\right)\right)^2\right] \\ &= c_{i,t} - \frac{1}{2} Var_h c_{i,t}\left(h\right) - \left(n_{i,t} - \frac{1}{2} Var_h n_{i,t}\left(h\right)\right) - \frac{1+\phi}{2}\left(Var_h\left(n_{i,t}\left(h\right)\right) + E_h\left[n_{i,t}\left(h\right)\right]^2\right) \\ &= c_{i,t} - \frac{1}{2} Var_h c_{i,t}\left(h\right) - n_{i,t} - \frac{\phi}{2} Var_h\left(n_{i,t}\left(h\right)\right) - \frac{1+\phi}{2} n_{i,t}^2 \end{aligned}$$

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$$\begin{aligned} \frac{Z_t - Z}{Z} &= \int \frac{Z_t\left(j\right) - Z}{Z} dj \\ z_t + \frac{1}{2} z_t^2 &= \int z_t\left(j\right) + \frac{1}{2}\left(z_t\left(j\right)\right)^2 dj \\ z_t + \frac{1}{2} z_t^2 &= E_j z_t\left(j\right) + \frac{1}{2} E_j\left[\left(z_t\left(j\right)\right)^2\right] \\ z_t + \frac{1}{2} z_t^2 &= E_j z_t\left(j\right) + \frac{1}{2}\left\{Var_j z_t\left(j\right) + \left(E_j z_t\left(j\right)\right)^2\right\} \\ z_t &= E_j z_t\left(j\right) + \frac{1}{2} Var_j z_t\left(j\right) + \frac{1}{2}\left\{\left(E_j z_t\left(j\right)\right)^2 - z_t^2\right\} \end{aligned}$$

and the last term is zero up to a second order.

where in the last step I used that $E_h[n_{i,t}(h)] = n_{i,t}$ up to a first order (since it is squared, we can use this first order approximation).

For every agent, we have that up to a first order

$$\varpi_t = c_{i,t}(h) + \phi n_{i,t}(h)$$

We can use this to express the variance of labor in terms of the variance of consumption

$$\begin{aligned} Var_h n_{i,t}(h) &= Var_h\left(\frac{\varpi_t - c_{i,t}(h)}{\phi}\right) \\ &= \frac{1}{\phi^2} Var_h(c_{i,t}(h)) \end{aligned}$$

so replacing above

$$\frac{u_{i,t} - u^*}{u_c^* C^*} = c_{i,t} - \frac{1+\phi}{2\phi} Var_h c_{i,t}(h) - n_{i,t} - \frac{1+\phi}{2} n_{i,t}^2$$

Now we can sum across both types of agents and all generations alive at time t to get

$$\begin{aligned} \frac{u_t - u^*}{u_c^* C^*} &= \chi \frac{u_{1,t} - u^*}{u_c^* C^*} + (1-\chi)\frac{u_{2,t} - u^*}{u_c^* C^*} \\ &= \chi\left[c_{1,t} - \frac{1+\phi}{2\phi} Var_h c_{1,t}(h) - n_{1,t} - \frac{1+\phi}{2} n_{1,t}^2\right] + \\ &+ (1-\chi)\left[c_{2,t} - \frac{1+\phi}{2\phi} Var_h c_{2,t}(h) - n_{2,t} - \frac{1+\phi}{2} n_{2,t}^2\right] \\ &= [\chi c_{1,t} + (1-\chi) c_{2,t}] - [\chi n_{1,t} + (1-\chi) n_{2,t}] + \\ &- \frac{1+\phi}{2\phi}[\chi Var_h c_{1,t}(h) + (1-\chi) Var_h c_{2,t}(h)] + \\ &- \frac{1+\phi}{2}\left[\chi (n_{1,t})^2 + (1-\chi)(n_{2,t})^2\right] \end{aligned}$$

By definition, we have that

$$\begin{aligned} C_t &= \chi C_{1,t} + (1-\chi) C_{2,t} \\ N_t &= \chi N_{1,t} + (1-\chi) N_{2,t} \end{aligned}$$

so doing a second order approximation we get

$$\begin{aligned} c_t + \frac{1}{2}(c_t)^2 &\cong \chi\left[c_{1,t} + \frac{1}{2}(c_{1,t})^2\right] + (1-\chi)\left[c_{2,t} + \frac{1}{2}(c_{2,t})^2\right] \\ c_t + \frac{1}{2}(c_t)^2 &\cong [\chi c_{1,t} + (1-\chi) c_{2,t}] + \frac{1}{2}\left[\chi (c_{1,t})^2 + (1-\chi)(c_{2,t})^2\right] \end{aligned}$$

and analogously

$$n_t + \frac{1}{2}(n_t)^2 = [\chi n_{1,t} + (1-\chi) n_{2,t}] + \frac{1}{2}\left[\chi (n_{1,t})^2 + (1-\chi)(n_{2,t})^2\right]$$

From the aggregate production function we have

$$n_t = c_t + d_t - a_t$$

where

$$\begin{aligned} d_t &= \ln D_t \\ &= \ln\left(\int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon}\right) \end{aligned}$$

Replacing these expressions in the SWF we get

$$\begin{aligned} \frac{u_t - u^*}{u_c^* C^*} &= \left[c_t + \frac{1}{2}c_t^2 - \frac{1}{2}\left(\chi (c_{1,t})^2 + (1-\chi)(c_{2,t})^2\right)\right] - \left[n_t + \frac{1}{2}(n_t)^2 - \frac{1}{2}\left(\chi (n_{1,t})^2 + (1-\chi)(n_{2,t})^2\right)\right] \\ &- \frac{1+\phi}{2\phi}\left[\chi Var_h c_{1,t}(h) + (1-\chi) Var_h c_{2,t}(h)\right] - \frac{1+\phi}{2}\left[\chi (n_{1,t})^2 + (1-\chi)(n_{2,t})^2\right] \\ -2\frac{u_t - u^*}{u_c^* C^*} &= \left(\chi (c_{1,t})^2 + (1-\chi)(c_{2,t})^2\right) + 2d_t - 2c_t a_t \\ &+ \frac{1+\phi}{\phi}\left[\chi Var_h c_{1,t}(h) + (1-\chi) Var_h c_{2,t}(h)\right] + \phi\left[\chi (n_{1,t})^2 + (1-\chi)(n_{2,t})^2\right] + t.i.p. \\ &= (1+\phi) c_t^2 + \left(\chi (c_{1,t})^2 + (1-\chi)(c_{2,t})^2 - c_t^2\right) + \phi\left[\chi (n_{1,t})^2 + (1-\chi)(n_{2,t})^2 - c_t^2\right] - 2c_t a_t \\ &+ \varepsilon Var_j\{\ln P_t(j)\} + \frac{1+\phi}{\phi}\left[\chi Var_h c_{1,t}(h) + (1-\chi) Var_h c_{2,t}(h)\right] + t.i.p. \end{aligned}$$

Using that $n_t = c_t - a_t$ up to a first order, and that $\check{c}_t = a_t$

$$\begin{aligned} -2\frac{u_t - u^*}{u_c^* C^*} &= (1+\phi) c_t^2 + \left(\chi (c_{1,t})^2 + (1-\chi)(c_{2,t})^2 - c_t^2\right) + \\ &+ \phi\left[\chi (n_{1,t})^2 + (1-\chi)(n_{2,t})^2 - n_t^2\right] - (1+\phi) 2c_t \check{c}_t \\ &+ \varepsilon Var_j\{\ln P_t(j)\} + \frac{1+\phi}{\phi}\left[\chi Var_h c_{1,t}(h) + (1-\chi) Var_h c_{2,t}(h)\right] + t.i.p. \end{aligned}$$

Using that

$$\begin{aligned} c_t^2 &= (\chi c_{1,t} + (1-\chi) c_{2,t})^2 \\ &= \chi^2 (c_{1,t})^2 + 2\chi(1-\chi) c_{1,t} c_{2,t} + (1-\chi)^2 (c_{2,t})^2 \end{aligned}$$

we have that

$$
\begin{aligned}
\chi \left(c_{1,t}\right)^2 + (1-\chi)\left(c_{2,t}\right)^2 - c_t^2 &= \chi (1-\chi) \left(c_{1,t} - c_{2,t}\right)^2 \\
&= \chi (1-\chi) \left(c_{1,t} - \frac{c_t - \chi c_{1,t}}{(1-\chi)} \right)^2 \\
&= \frac{\chi}{1-\chi} \left(c_t - c_{1,t}\right)^2 \\
&= \frac{\chi}{1-\chi} \left(x_t - x_{1,t}\right)^2
\end{aligned}
$$

In an analogous way we get that

$$
\chi \left(n_{1,t}\right)^2 + (1-\chi)\left(n_{2,t}\right)^2 - n_t^2 = \frac{\chi}{1-\chi} \left(n_t - n_{1,t}\right)^2
$$

From the labor supply function, we have that

$$
\begin{aligned}
n_t - n_{1,t} &= \frac{\varpi_t - c_t}{\phi} - \frac{\varpi_t - c_{1,t}}{\phi} \\
&= \frac{c_{1,t} - c_t}{\phi}
\end{aligned}
$$

so that

$$
\chi \left(n_{1,t}\right)^2 + (1-\chi)\left(n_{2,t}\right)^2 - n_t^2 = \frac{\chi}{1-\chi} \frac{1}{\phi^2} \left(x_t - x_{1,t}\right)^2
$$

Replacing in the loss function:

$$
\begin{aligned}
-2\frac{u_t - u^*}{u_c^* C^*} = {}& (1+\phi)\, x_t^2 + \frac{1+\phi}{\phi} \frac{\chi}{1-\chi} \left(x_t - x_{1,t}\right)^2 \\
& + \varepsilon Var_j \left\{ \ln P_t(j) \right\} + \frac{1+\phi}{\phi} \left[\chi Var_h c_{1,t}(h) + (1-\chi) Var_h c_{2,t}(h) \right] + t.i.p.
\end{aligned}
$$

The last part of the derivation of the loss function consists on expressiong $Var_j \{\ln P_t(j)\}$ and $Var_h (c_{i,t}(h))$ as functions of π_t^2 and $b_{i,t}^2$ respectively.

Suppose that at every instant a flow of ζ firms can reset their prices. Therefore, in a short interval of length δ, we have $\zeta\delta$ firms resetting their price, while $1-\zeta\delta$ keep the same price. We can compute the variance of the prices using the law of total variance

$$
\Delta_t \equiv Var_i \{\ln P_t(i)\} = E\{Var_i \{\ln P_{t-\delta}(i)\}\, ; 0\} + Var\{E[\ln P_{t-\delta}(i)] \; ; \; E[\ln P_t^*(i)]\}
$$

where the zero in the first term appears because all the firms that reset their price at time

t set the same price[53].

$$\Delta_t = (1-\zeta\delta)\,\Delta_{t-\delta} + \zeta\delta\,[1-\zeta\delta]\,\{E[\ln P_{t-\delta}(i)] - E[\ln P_t^*(i)]\}^2$$

Subtracting the steady state price on the RHS, we get

$$E[\ln P_{t-\delta}(i)] - E[\ln P_t^*(i)] \cong E_i\left[\breve{P}_{t-\delta} - \breve{P}_t^*\right]$$

From the definition of the price index we had

$$P_t^{1-\upsilon} = [1-\zeta\delta]\,P_{t-\delta}^{1-\upsilon} + \zeta\delta P_t^{*1-\upsilon}$$

Log-linearizing we get

$$\begin{aligned} \breve{P}_t &= [1-\zeta\delta]\,\breve{P}_{t-\delta} + \zeta\delta\breve{P}_t^* \\ \breve{P}_t - \breve{P}_{t-\delta} &= \zeta\delta\left(\breve{P}_t^* - \breve{P}_{t-\delta}\right) \end{aligned}$$

Therefore, we have

$$\{E[\ln P_{t-\delta}(i)] - E[\ln P_t^*(i)]\}^2 = \left\{\frac{\breve{P}_t - \breve{P}_{t-\delta}}{\zeta\delta}\right\}^2$$

Replacing above and taking the limit $\delta \to 0$

$$\begin{aligned} \dot{\Delta}_t = \lim_{\delta\to 0}\frac{\Delta_t - \Delta_{t-\delta}}{\delta} &= \lim_{\delta\to 0}\left\{-\zeta\Delta_{t-\delta} + \zeta\,[1-\zeta\delta]\left\{\frac{\breve{P}_t - \breve{P}_{t-\delta}}{\zeta\delta}\right\}^2\right\} \\ &= \lim_{\delta\to 0}\left\{-\zeta\Delta_t + \frac{1}{\zeta}\left\{\frac{\left(\frac{P_t-P}{P}\right) - \left(\frac{P_{t-\delta}-P}{P}\right)}{\delta}\right\}^2\right\} \\ &= \lim_{\delta\to 0}\left\{-\zeta\Delta_{t-\delta} + \frac{1}{\zeta}\left\{\frac{1}{P}\frac{P_t - P_{t-\delta}}{\delta}\right\}^2\right\} \\ &= -\zeta\Delta_t + \frac{1}{\zeta}\pi_t^2 \end{aligned}$$

[53]Recall that when a random variable can take only two values, say Z_1, Z_2 with probabilities $p, (1-p)$ respectively, the variance of the variable is

$$\begin{aligned} Var(Z) &= E\left[Z^2\right] - E[Z]^2 \\ &= \left(pZ_1^2 + (1-p)\,Z_2^2\right) - (pZ_1 + (1-p)\,Z_2)^2 \\ &= pZ_1^2 + (1-p)\,Z_2^2 - \left(p^2 Z_1^2 + (1-p)^2 Z_2^2 + 2p\,(1-p)\,Z_1 Z_2\right) \\ &= p\,(1-p)\,(Z_1 - Z_2) \end{aligned}$$

Solving the differential equation we get

$$\int_0^t \left(e^{\zeta s}\Delta_s\right)' ds = \int_0^t e^{\zeta s}\frac{1}{\zeta}\pi_s^2 ds$$
$$\Delta_t = e^{-\zeta t}\Delta_0 + \frac{1}{\zeta}\int_0^t e^{\zeta(s-t)}\pi_s^2 ds$$

The discounted value then is (up to a constant)

$$\begin{aligned}\int_0^\infty e^{-\rho t}\Delta_t dt &= \int_0^\infty \int_0^t e^{-\rho t}\frac{1}{\zeta}e^{\zeta(s-t)}\pi_s^2 ds dt \\ &= \frac{1}{\zeta}\int_0^\infty \int_s^\infty e^{-(\rho+\zeta)t}dt e^{\zeta s}\pi_s^2 ds \\ &= \frac{1}{\zeta\left(\rho+\zeta\right)}\int_0^\infty e^{-\rho s}\pi_s^2 ds\end{aligned}$$

Now we have to do something similar for $Var_h\left(c_{i,t}\left(h\right)\right)$. Using the law of total variance:

$$\Delta_{i,t}^c \equiv Var_h\left(c_{i,t}\left(h\right)\right) = E\left\{Var_h\left\{c_{i,t}\left(h\right)|born\ before\ t\right\};0\right\}+Var\left\{E_h\left[c_{i,t}\left(h\right)|born\ before\ t\right]\ ;\ E\left[\breve{c}_{i,t}\right]\right\}$$

where $\breve{c}_t$ is the consumption of the agents that are born at time t, and the zero in the first term appears because all the agents that are born at time t have the same consumption. The law of motion for $\Delta_{i,t}^c$ is

$$\Delta_{i,t}^c = \left(1-\lambda\delta\right)\Delta_{i,t-\delta}^c + \lambda\delta\left[1-\lambda\delta\right]\left\{E_h\left[c_{i,t}\left(h\right)|born\ before\ t\right] - \breve{c}_{i,t}\right\}^2$$

where I am using that $Var_h\left\{c_{i,t}\left(h\right)|born\ before\ t\right\} = Var_h\left\{c_{i,t-\delta}\left(h\right)\right\}$ because all agents who were already alive change their consumption at the same rate $\frac{\dot{C}_{i,t}^h}{C_{i,t}^h} = r_t - \rho$, so in logs the variance remains the same.

Up to a first order we have

$$c_{i,t} = \lambda\delta\breve{c}_{i,t} + \left(1-\lambda\delta\right)E_h\left[c_{i,t}|born\ before\ t\right]$$
$$c_{i,t} - \breve{c}_{i,t} = \left(1-\lambda\delta\right)\left(E_h\left[c_{i,t}|born\ before\ t\right] - \breve{c}_{i,t}\right)$$

Therefore, we have

$$\begin{aligned}
\dot{\Delta}_{i,t}^{c} = \lim_{\delta\to 0}\frac{\Delta_{i,t}^{c}-\Delta_{i,t-\delta}^{c}}{\delta} &= \lim_{\delta\to 0} -\lambda\Delta_{i,t-\delta}^{c} + \lambda\left[1-\lambda\delta\right]\left\{E_h\left[c_{i,t}\left(h\right)\left|born\ before\ t\right.\right]-\breve{c}_{i,t}\right\}^2 \\
&= \lim_{\delta\to 0} -\lambda\Delta_{t-\delta}^{c} + \lambda\left[1-\lambda\delta\right]\left\{\frac{c_{i,t}-\breve{c}_{i,t}}{1-\lambda\delta}\right\}^2 \\
&= \lim_{\delta\to 0} -\lambda\Delta_{t-\delta}^{c} + \lambda\frac{1}{1-\lambda\delta}\left\{\frac{C_{i,t}-C^*}{C^*}-\frac{\breve{C}_{i,t}-C^*}{C^*}\right\}^2 \\
&= \lim_{\delta\to 0} -\lambda\Delta_{t-\delta}^{c} + \lambda\frac{1}{1-\lambda\delta}\left\{\frac{C_{i,t}-\breve{C}_{i,t}}{C^*}\right\}^2
\end{aligned}$$

Since generations born before t and the newborns have the same labor income and only differ in their bond holdings (and using that consumption is linear in wealth), we have that

$$C_{i,t}-\breve{C}_{i,t} = \left(\rho+\lambda\right)B_{i,t}$$

Replacing above

$$\dot{\Delta}_{i,t}^{c} = -\lambda\Delta_{t-\delta}^{c} + \lambda\left(\rho+\lambda\right)^2 b_{i,t}^2$$

Solving the differential equation

$$\int_0^t \left(e^{\lambda s}\Delta_{i,s}^{c}\right)' ds = \lambda\left(\rho+\lambda\right)^2\int_0^t e^{\lambda s}\left(b_{i,t}\right)^2 ds$$

$$\Delta_{i,t}^{c} = e^{-\lambda t}\Delta_{i,0}^{c} + \lambda\left(\rho+\lambda\right)^2\int_0^t e^{\lambda(s-t)}\left(b_{i,t}\right)^2 ds$$

The discounted value then is (up to a constant)

$$\begin{aligned}
\int_0^\infty e^{-\rho t}\Delta_{i,t}^{c}dt &= \lambda\left(\rho+\lambda\right)^2\int_0^\infty\int_0^t e^{-\rho t}e^{\lambda(s-t)}\left(b_{i,t}\right)^2 dsdt \\
&= \lambda\left(\rho+\lambda\right)^2\int_0^\infty\int_s^\infty e^{-\rho t}e^{\lambda(s-t)}\left(b_{i,t}\right)^2 dtds \\
&= \lambda\left(\rho+\lambda\right)\int_0^\infty e^{-\rho s}\left(b_{i,t}\right)^2 ds
\end{aligned}$$

Recall that

$$b_{2,t} = -\frac{\chi}{\left(1-\chi\right)}b_{1,t}$$

So finally we get the loss function

$$L=\int_0^\infty e^{-\rho t}\left\{\begin{array}{c}(1+\phi)\,x_t^2+\frac{1+\phi}{\phi}\frac{\chi}{1-\chi}\left(x_t-x_{1,t}\right)^2+\\+\varepsilon\frac{1}{\zeta(\rho+\zeta)}\pi_t^2+\frac{1+\phi}{\phi}\left[\chi\lambda_1\left(\rho+\lambda_1\right)b_{1,t}^2+(1-\chi)\left(\lambda_2\left(\rho+\lambda_2\right)b_{2,t}^2\right)\right]\end{array}\right\}dt$$

$$=\int_0^\infty e^{-\rho t}\left\{\begin{array}{c}\frac{\varepsilon}{\zeta(\rho+\zeta)}\pi_t^2+(1+\phi)\,x_t^2+\\+\frac{1+\phi}{\phi}\frac{\chi}{1-\chi}\left[\left(x_t-x_{1,t}\right)^2+Lb_{1,t}^2\right]\end{array}\right\}dt$$

D.4 Proof of Proposition 5

Let $\left\{b_{1,t}'\right\}_{t=0}^\infty$ and $\left\{b_{1,t}''\right\}_{t=0}^\infty$denote the optimal path for bond holdings when $\lambda_2=\lambda_2'$ and when $\lambda_2=\lambda_2''$, respectively. Suppose that, when $\lambda_2=\lambda_2''$, we were to choose the path of bonds $\breve{b}_{1,t}=\frac{\Lambda_2'-\Lambda_1}{\Lambda_2''-\Lambda_1}b_{1,t}'$. This path $\breve{b}_{1,t}$ is not necessarily optimal (i.e. it could be that $\breve{b}_{1,t}\neq b_{1,t}''$), but it *is* one of the many feasible choices for the planner, so if we can show that this choice attains a lower loss than $\mathcal{L}_0^*\left(\lambda_1,\lambda_2'\right)$, then we would immediately have that $\mathcal{L}_0^*\left(\lambda_1,\lambda_2''\right)<\mathcal{L}_0^*\left(\lambda_1,\lambda_2'\right)$ (the only case in which the inequality would not be strict is if $b_{1,t}''=0$ since in that case we would have $\breve{b}_{1,t}=b_{1,t}''$ and this would induce a symmetric allocation regardless of the value of λ_2).

Note that by choosing $\breve{b}_{1,t}$ when $\lambda_2=\lambda_2''$, from equations (25), (28), (29) we can see that we would implement the optimal macro allocation $\{x_t',\pi_t'\}_{t=0}^\infty$ corresponding to $\lambda_2=\lambda_2'$, so the welfare loss coming from the macro variables will be the same in both cases. Then, we have to compare the welfare loss coming from the heterogeneity of the allocation.

For the within heterogeneity term, we have

$$L''\breve{b}_{1,t}^2=\left[\chi\Lambda_2''+(1-\chi)\,\Lambda_1\right]\breve{b}_{1,t}^2=\left[\chi\Lambda_2''+(1-\chi)\,\Lambda_1\right]\left(\frac{\Lambda_2'-\Lambda_1}{\Lambda_2''-\Lambda_1}b_{1,t}'\right)^2$$
$$<\left[\chi\Lambda_2''+(1-\chi)\,\Lambda_1\right]\frac{\Lambda_2'-\Lambda_1}{\Lambda_2''-\Lambda_1}\left(b_{1,t}'\right)^2<\left[\chi\Lambda_2'+(1-\chi)\,\Lambda_1\right]\left(b_{1,t}'\right)^2=L'\left(b_{1,t}'\right)^2$$

where the last inequality can be easily verified by noting that

$$\left[\chi\Lambda_2''+(1-\chi)\,\Lambda_1\right]\frac{\Lambda_2'-\Lambda_1}{\Lambda_2''-\Lambda_1}<\left[\chi\Lambda_2'+(1-\chi)\,\Lambda_1\right]\iff f\left(\Lambda_2''\right)<f\left(\Lambda_2'\right)$$

where I define $f\left(z\right)\equiv\frac{\chi z+(1-\chi)\Lambda_1}{z-\Lambda_1}$, and $f\left(z\right)$ is clearly a decreasing function.

For the between heterogeneity term, from equations (25) and (26) we find $\left(x_t-x_{1,t}\right)^2=L^2\left(\int_t^\infty b_{1,s}ds\right)^2$, and we have that

$$\left(L''\right)^2\left(\int_t^\infty\breve{b}_{1,s}ds\right)^2=\left[\chi\Lambda_2''+(1-\chi)\,\Lambda_1\right]^2\left(\frac{\Lambda_2'-\Lambda_1}{\Lambda_2''-\Lambda_1}\right)^2\left(\int_t^\infty b_{1,t}'ds\right)^2$$
$$<\left[\chi\Lambda_2'+(1-\chi)\,\Lambda_1\right]^2\left(\int_t^\infty b_{1,t}'ds\right)^2$$

where the inequality is the same as I proved before.

The last thing I want to prove in this section is that, to obtain the path $\left\{\breve{b}_{1,t}\right\}_{t=0}^{\infty}$, we need a path of transfers $\{\breve{\tau}_{1,t}\}_{t=0}^{\infty}$ that is more backloaded than $\left\{\tau'_{1,t}\right\}_{t=0}^{\infty}$. The intuition is that a higher λ_2 reduces the time lag between the transfer and the expenditure, so to obtain the same fiscal stimulus with a higher λ_2 we need to backload the transfers.

From the budget constraint we have

$$\begin{aligned}\breve{\tau}_{1,t} &= \rho\breve{b}_{1,t} + \frac{1+\phi}{\phi}\left(x'_t - \breve{x}_{1,t}\right) - \dot{\breve{b}}_{1,t} \\ &= \rho\frac{\Lambda'_2 - \Lambda_1}{\Lambda''_2 - \Lambda_1} b'_{1,t} + \frac{1+\phi}{\phi}\left(x'_t - \breve{x}_{1,t}\right) - \frac{\Lambda'_2 - \Lambda_1}{\Lambda''_2 - \Lambda_1}\dot{b}'_{1,t}\end{aligned}$$

Using that

$$\begin{aligned}\dot{\breve{x}}_{1,t} &= -\left(\breve{r}_0 - \rho\right)\frac{e^{-\gamma t}}{\gamma} - \Lambda_1 \breve{b}_{1,t} \\ &= \dot{x}'_{1,t} + \Lambda_1\left(\frac{\Lambda''_2 - \Lambda'_2}{\Lambda''_2 - \Lambda_1}\right) b'_{1,t}\end{aligned}$$

and integrating on both sides we get

$$\breve{x}_{1,t} = x'_{1,t} - \Lambda_1\left(\frac{\Lambda''_2 - \Lambda'_2}{\Lambda''_2 - \Lambda_1}\right)\int_t^{\infty} b'_{1,s} ds$$

Replacing above we obtain

$$\begin{aligned}\breve{\tau}_{1,t} &= \rho\frac{\Lambda'_2 - \Lambda_1}{\Lambda''_2 - \Lambda_1} b'_{1,t} + \frac{1+\phi}{\phi}\left(x'_t - x'_{1,t} + \Lambda_1\left(\frac{\Lambda''_2 - \Lambda'_2}{\Lambda''_2 - \Lambda_1}\right)\int_t^{\infty} b'_{1,s} ds\right) - \frac{\Lambda'_2 - \Lambda_1}{\Lambda''_2 - \Lambda_1}\dot{b}'_{1,t} \\ &= \frac{\Lambda'_2 - \Lambda_1}{\Lambda''_2 - \Lambda_1}\left\{\tau'_{1,t} + \frac{1+\phi}{\phi}\left(\Lambda''_2 - \Lambda'_2\right)\chi\left(-\int_t^{\infty} b'_{1,s} ds\right)\right\} \\ &= \frac{\Lambda'_2 - \Lambda_1}{\Lambda''_2 - \Lambda_1}\tau'_{1,t} + \left(1 - \frac{\Lambda'_2 - \Lambda_1}{\Lambda''_2 - \Lambda_1}\right)\left(\Lambda'_2 - \Lambda_1\right)\frac{1+\phi}{\phi}\chi\left(-\int_t^{\infty} b'_{1,s} ds\right)\end{aligned}$$

Since we have written $\breve{\tau}_{1,t}$ as a weighted average of $\tau'_{1,t}$ and $\int_t^{\infty} b'_{1,s} ds$, we need to prove that the latter is more backloaded than the former. For simplicity, I will show this under the simplifying assumption that $\tau'_{1,t}$ is exponential: $\tau'_{1,t} = \tau'_{1,0}e^{-\delta t}$. Solving the differential equation in Lemma 1 with initial conditions $b_{1.0} = 0$ and $\tau'_{1,0}$ given, we find that

$$\int_t^{\infty} b'_{1,s} ds = \frac{\tau_{1,0}}{\rho + \delta - \frac{1+\phi}{\phi} L \frac{1}{\delta}}\frac{1}{\delta}\left[e^{-\delta t} + \frac{\delta}{\frac{\rho - \sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}}{2}} e^{\frac{\rho - \sqrt{\rho^2 + 4\frac{1+\phi}{\phi}L}}{2} t}\right]$$

It is straightforward that the ratio $\frac{\int_t^{\infty} b'_{1,s} ds}{e^{-\delta t}}$ is increasing with respect to t, wich means that $\int_t^{\infty} b'_{1,s} ds$ is more backloaded than $\tau'_{1,t}$.

D.5 Optimal policy with rigid prices, infinite liquidity trap and exponential natural rate

The problem we want to solve is

$$\min_{\{x_t,x_{1,t},b_{1,t},\tau_{1,t}\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t}\left\{(1+\phi)\,x_t^2+\frac{1+\phi}{\phi}\frac{\chi}{1-\chi}\left[(x_t-x_{1,t})^2+Lb_{1,t}^2\right]\right\}dt$$

$$\begin{aligned}
s.t.\ \dot{x}_t &= -\left(\check{r}_0-\rho\right)e^{-\delta s}+\chi\left(\Lambda_2-\Lambda_1\right)b_{1,t}\\
\dot{x}_{1,t} &= -\left(\check{r}_0-\rho\right)e^{-\delta s}-\Lambda_1 b_{1,t}\\
\dot{b}_{1,t} &= \rho b_{1,t}+\frac{1+\phi}{\phi}\left(x_t-x_{1,t}\right)-\tau_{1,t}\\
&+\ Transversality\ conditions
\end{aligned}$$

Rather than writing the Hamiltonian, I will replace the restrictions into the objective function, so we are left with an unconstrained maximization problem. Integrating the laws of motion for $x_t, x_{1,t}$ we find

$$\begin{aligned}
x_t^2 &= \left[\left(\check{r}_0-\rho\right)\frac{e^{-\gamma t}}{\gamma}-\chi\left(\Lambda_2-\Lambda_1\right)\int_t^{\infty} b_{1,s}ds\right]^2\\
\left(x_t-x_{1,t}\right)^2 &= L^2\left(\int_t^{\infty} b_{1,s}ds\right)^2
\end{aligned}$$

Let us first consider the case with the ad-hoc objective function in which the planner does not care about the within component of heterogeneity. In this case, the problem is

$$\min \int_0^{\infty} e^{-\rho t}\left\{\begin{array}{c}(1+\phi)\left[\left(\check{r}_0-\rho\right)\frac{e^{-\gamma t}}{\gamma}-\chi\left(\Lambda_2-\Lambda_1\right)\beta_{1,t}\right]^2+\\ +\frac{1+\phi}{\phi}\frac{\chi}{1-\chi}L^2\beta_{1,t}^2\end{array}\right\}dt$$

Note that there is no interaction between the different periods, so we can solve this as a period-by-period optimization. Differentiating with respect to $\beta_{1,t}$ we obtain

$$\begin{aligned}
-\chi\left(\Lambda_2-\Lambda_1\right)2\left(1+\phi\right)\left[\left(\check{r}_0-\rho\right)\frac{e^{-\gamma t}}{\gamma}-\chi\left(\Lambda_2-\Lambda_1\right)\beta_{1,t}\right]+2\frac{1+\phi}{\phi}\frac{\chi}{1-\chi}L^2\beta_{1,t} &= 0\\
\Rightarrow \beta_{1,t} = \frac{\chi\left(\Lambda_2-\Lambda_1\right)}{\frac{1}{\phi}\frac{\chi}{1-\chi}L^2+\left[\chi\left(\Lambda_2-\Lambda_1\right)\right]^2}x^{No\ redist}&\\
\Rightarrow \bar{x}_t = \frac{x_t^{NT}}{1+\phi\frac{1-\chi}{\chi}\left[\theta\left(\lambda_1,\lambda_2\right)\right]^2}&
\end{aligned}$$

where $x_t^{NT}\equiv\left(\check{r}_0-\rho\right)\frac{e^{-\gamma t}}{\gamma}$.

Let us now consider the case in which the planner cares about both between and within heterogenities. Then, the problem is (where I am using that $\dot{\beta}_{1,t} = -b_{1,t}$)

$$\min \int_0^{\infty} e^{-\rho t} \left\{ \begin{array}{c} (1+\phi) \left[(\check{r}_0 - \rho) \frac{e^{-\gamma t}}{\gamma} - \chi (\Lambda_2 - \Lambda_1) \beta_{1,t} \right]^2 + \\ + \frac{1+\phi}{\phi} \frac{\chi}{1-\chi} L \left[L \beta_{1,t}^2 + \dot{\beta}_{1,t}^2 \right] \end{array} \right\} dt$$

Define

$$F(t, z_1, z_2) = e^{-\rho t} \left\{ \begin{array}{c} (1+\phi) \left((\check{r}_0 - \rho) \frac{e^{-\gamma t}}{\gamma} - \chi (\Lambda_2 - \Lambda_1) z_1 \right)^2 + \\ + \frac{1+\phi}{\phi} \frac{\chi}{1-\chi} L \left[L (z_1)^2 + (z_2)^2 \right] \end{array} \right\}$$

so the optimization problem can be written as

$$\min \int_0^{\infty} F\left(t, \beta_{1,t}, \dot{\beta}_{1,t}\right) dt$$

Note that $F(t, \cdot, \cdot)$ is a convex function for any fixed value of t, so the first order conditions are sufficient for an optimum. The Euler-Lagrange condition is

$$\frac{\partial F\left(t, \beta_{1,t}, \dot{\beta}_{1,t}\right)}{\partial z_1} = \frac{\partial}{\partial t} \left(\frac{\partial F\left(t, \beta_{1,t}, \dot{\beta}_{1,t}\right)}{\partial z_2} \right)$$

If we calculate these derivatives, we get the Euler-Lagrange condition

$$\left\{ \begin{array}{c} -\phi (1-\chi)(\Lambda_2 - \Lambda_1) \left((\check{r}_0 - \rho) \frac{e^{-\gamma t}}{\gamma} - \chi (\Lambda_2 - \Lambda_1) \beta_{1,t} \right) + \\ + L^2 \beta_{1,t} \end{array} \right\} = L \left\{ -\rho \dot{\beta}_{1,t} + \beta''_{1,t} \right\}$$

$$\Rightarrow \dot{b}_{1,t} = \left\{ \begin{array}{c} \phi (1-\chi)(\Lambda_2 - \Lambda_1) \left((\check{r}_0 - \rho) \frac{e^{-\gamma t}}{\gamma} - \chi (\Lambda_2 - \Lambda_1) \beta_{1,t} \right) + \\ - L^2 \beta_{1,t} \end{array} \right\} \frac{1}{L} + \rho b_{1,t}$$

Define $R_t \equiv (\check{r}_0 - \rho) \frac{e^{-\gamma t}}{\gamma}$, so that $\dot{R}_t = -\gamma R_t$. In matrix form

$$\begin{bmatrix} \dot{b}_{1,t} \\ \dot{\beta}_{1,t} \\ \dot{R}_t \end{bmatrix} = \begin{bmatrix} \rho & -\left(\frac{\phi(1-\chi)\chi(\Lambda_2-\Lambda_1)^2}{L} + L \right) & \frac{\phi(1-\chi)(\Lambda_2-\Lambda_1)}{L} \\ -1 & 0 & 0 \\ 0 & 0 & -\gamma \end{bmatrix} \begin{bmatrix} b_{1,t} \\ \beta_{1,t} \\ R_t \end{bmatrix}$$

I solve this system numerically, using only the two negative eigenvalues and initial conditions $b_{1,0} = 0$, $R_0 \equiv \frac{(\check{r}_0 - \rho)}{\gamma}$.

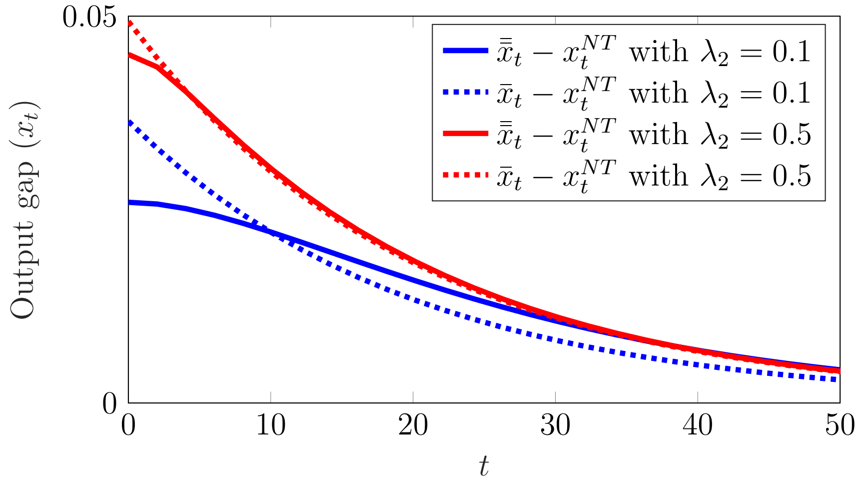

Figure 15 – Optimal output gap (expressed as a difference with respect to the output gap without transfers) with the ad-hoc loss function ($\{\bar{x}_t\}$) and with the actual loss function ($\{\bar{\bar{x}}_t\}$). The exercise is repeated for two values of λ_2.

Finally, I solve numerically for $\alpha_{-\gamma}, \alpha_\vartheta$ and obtain the paths for $x_t, x_{1,t}$ as

$$x_t = R_t - \chi\left(\Lambda_2 - \Lambda_1\right)\beta_{1,t}$$
$$x_{1,t} = x_t + L\int_t^\infty b_{1,s}ds$$
$$= x_t + L\beta_{1,t}$$

Figure 15 displays the optimal output gap (expressed as a difference with respect to the output gap without transfers) with the ad-hoc loss function and with the actual loss function (i.e., in the former case, we ignore the within component of heterogeneity). In both cases, we can observe that with a higher λ_2 it is optimal to obtain more macro stabilization.

E Proofs for Section 6

Let us first show how to derive the linearized Euler equation. The non-linear Euler is

$$Y_{t+1} = \beta_t\left(\left\{B_0^h\right\}_h\right)R_tC_t + G_{t+1}$$

Since in steady state we have (by assumption) $Y^* = 1,\ G^* = 0:$

$$y_{t+1} = \frac{\beta_t\left(\left\{B_0^h\right\}_h\right)R_tC_t - Y^*}{Y^*} + g_{t+1}$$

where we define $g_t \equiv \frac{G_t}{Y^*}$. To a first order, we have that

$$\begin{aligned}\frac{\beta_t\left(\left\{B_0^h\right\}_h\right) R_t C_t - Y^*}{Y^*} &= \ln\left(\frac{\beta_t\left(\left\{B_0^h\right\}_h\right) R_t C_t}{C^*}\right)\\ &= r_t - \varrho_t\left(\left\{B_0^h\right\}_h\right) + c_t\\ &= r_t - \varrho_t\left(\left\{B_0^h\right\}_h\right) + y_t - g_t\end{aligned}$$

so replacing above we obtain the linearized Euler equation

$$y_{t+1} - y_t = r_t - \varrho_t\left(\left\{B_0^h\right\}_h\right) + g_{t+1} - g_t$$

Let us now show that we can decompose the effect of a transfer as proportional to a weighted average of government purchases multipliers.

Suppose first that we had a redistribution, but government purchases are permanently set to zero (i.e. $g_t = 0\ \forall t$). Then, the equilibrium is characterized by

$$\begin{cases} y_{t+1} - y_t = r_t - \varrho_t\left(\left\{B_0^h\right\}_h\right)\\ r_t = r^* + \kappa_{y,t} y_t\end{cases}$$

If instead we have government purchases but no redistribution, the equilibrium is characterized by

$$\begin{cases} y_{t+1} - y_t = r_t - \varrho^* + g_{t+1} - g_t\\ r_t = r^* + \kappa_{y,t} y_t\end{cases}$$

The two systems will be equivalent (i.e. they will generate the same path for $\{y_t, r_t\}_{t\geq 0}$) if we set

$$g_{t+1} - g_t = \varrho^* - \varrho_t\left(\left\{B_0^h\right\}_h\right)\ \forall t$$

Iterating this expression, we get that

$$g_t = g_{t+T} - \sum_{s=0}^{T}\left(\varrho^* - \varrho_{t+s}\left(\left\{B_0^h\right\}_h\right)\right)$$

If we take the limit $T \to \infty$, we find

$$g_t = -\sum_{s=0}^{\infty}\left(\varrho^* - \varrho_{t+s}\left(\left\{B_0^h\right\}_h\right)\right)$$

Since the system with the redistribution and no government purchases is equivalent to the system with no redistribution and government purchases, their cumuluative effect on output must be the same:

$$
\begin{aligned}
Transfer\ Multiplier &= \frac{1}{\int T_0^h dF_h^*} \sum_{t=0}^{\infty} \frac{1}{(R^*)^t} (Y_t - 1) \cong \frac{1}{\int T_0^h dF_h^*} \sum_{t=0}^{\infty} \frac{1}{(R^*)^t} y_t \\
&= \frac{1}{\int T_0^h dF_h^*} \sum_{t=0}^{\infty} \left(\sum_{s=t}^{\infty} \frac{1}{(R^*)^t} (\varrho_s - \varrho^*) \right) m_t^G \\
&= \theta \sum_{t=0}^{\infty} \omega_t m_t^G
\end{aligned}
$$

where

$$
\begin{aligned}
\theta &\equiv \frac{1}{\int_h T_0^h dF_h^*} \sum_{t=0}^{\infty} \sum_{s=t}^{\infty} \frac{1}{(R^*)^t} (\varrho_s - \varrho^*) \\
\omega_t &\equiv \frac{\frac{1}{(R^*)^t} \sum_{s=t}^{\infty} (\varrho_s - \varrho^*)}{\sum_{t=0}^{\infty} \sum_{s=t}^{\infty} \frac{1}{(R^*)^t} (\varrho_s - \varrho^*)}
\end{aligned}
$$

Let us now derive the government expenditure multipliers corresponding to a Taylor rule (that only reacts to the output gap, since inflation is assumed to be zero). Replacing the Taylor rule into the Euler equation, we obtain

$$
y_{t+1} = (1 + \kappa_y) y_t + g_{t+1} - g_t
$$

Iterating this expression we find

$$
y_t = (1 + \kappa_y)^{-T} y_{t+T} - \sum_{s=1}^{T} (1 + \kappa_y)^{-s} (g_{t+s} - g_{t+s-1})
$$

If $T \to \infty$

$$
y_t = g_t (1 + \kappa_y)^{-1} - \sum_{s=1}^{\infty} g_{t+s} \frac{\kappa_y}{(1 + \kappa_y)^{1+s}}
$$

Therefore, the cumulative effect on output is

$$\begin{aligned}
\sum_{t=0}^{\infty}\frac{1}{R^t}y_t &= \sum_{t=0}^{\infty}\frac{1}{R^t}g_t\left(1+\kappa_y\right)^{-1} - \sum_{t=0}^{\infty}\sum_{s=1}^{\infty}\frac{1}{R^t}g_{t+s}\frac{\kappa_y}{\left(1+\kappa_y\right)^{1+s}} \\
&= \sum_{t=0}^{\infty}\frac{1}{R^t}g_t\left(1+\kappa_y\right)^{-1} - \sum_{t=0}^{\infty}\sum_{s=1}^{\infty}\frac{\kappa_y}{\left(1+\kappa_y\right)^2}\left\{\begin{array}{c} g_1 + g_2\left[\frac{1}{1+\kappa_y}+\frac{1}{R^t}\right]+ \\ +g_3\left[\frac{1}{(1+\kappa_y)^2}+\frac{1}{R^1}\frac{1}{1+\kappa_y}+\frac{1}{R^2}\right]+\ldots \end{array}\right\} \\
&= \sum_{t=0}^{\infty}\frac{1}{R^t}g_t\left(1+\kappa_y\right)^{-1} - \sum_{t=1}^{\infty}\frac{1}{R^t}g_t\frac{\kappa_y}{\left(1+\kappa_y\right)^{1+t}}R\frac{R^t-\left(1+\kappa_y\right)^t}{R-\left(1+\kappa_y\right)} \\
&= \left(1+\kappa_y\right)^{-1}g_0 + \sum_{t=1}^{\infty}\frac{1}{R^t}g_t\left[\left(1+\kappa_y\right)^{-1} - \frac{\kappa_y}{\left(1+\kappa_y\right)^{1+t}}R\frac{R^t-\left(1+\kappa_y\right)^t}{R-\left(1+\kappa_y\right)}\right]
\end{aligned}$$

Then, the multipliers are

$$m_t^G = \left\{\begin{array}{cc} \left(1+\kappa_y\right)^{-1} & if\ t=0 \\ \left(1+\kappa_y\right)^{-1} - \frac{\kappa_y}{\left(1+\kappa_y\right)^{1+t}}R\frac{R^t-(1+\kappa_y)^t}{R-(1+\kappa_y)} & if\ t>0 \end{array}\right.$$