Just-in-Time Portfolio Insurance

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Abstract

Conventional portfolio insurance places a floor on the value of a portfolio relative to its value at inception. Just-in-time portfolio insurance instead places a floor on the value of a portfolio relative to its value on the previous day. The owner of a just-in-time insured portfolio can floor any daily price relative just after it is realized. The owner can also choose not to floor any daily price relatives and hence receive the value appreciation over the term of the contract. Dynamic programming (DP) can be used to value the Bermudan optionality embedded in a just-in-time insured portfolio. We apply DP using both the benchmark Black Scholes model and a new approach based on abstract algebra. The latter approach leads to a simple arbitrage-free closed-form valuation formula for our Bermudan-style insurance.

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1 Introduction to Just-in-Time Portfolio Insurance

> We must recognize the possibility that a system of measurement may be arbitrary otherwise than in the choice of unit; there may be arbitrariness in the choice of the process of addition – Norman Robert Campbell, Foundations of Science, 1957.

Consider the payoff arising from investing \$1 in a long-only non-dividend paying portfolio at $t = 0$ and liquidating at some fixed future time $T > 0$. Let $S_0 > 0$ denote the initial spot value of the portfolio at time and let $S_T \geq 0$ denote its value at some fixed liquidation time T. The investor receives the random amount $\frac{S_T}{S_0} \geq 0$ at time $t = T$. This non-negative random payoff can realize between zero and one, resulting in a loss, which can be as high as the original \$1 investment.

Portfolio insurance is a well known financial product designed to limit the losses that can arise on an uninsured portfolio. There are several variations of portfolio insurance, but one of the simplest involves European-style equity put options. Suppose that the investor not only invests \$1 in the risky portfolio, but also invests in a European put written on the portfolio, with unit dollar notional, maturity date T, and strike rate $k > 0$. The put position pays off $0 \vee \left(k - \frac{S_T}{S_0}\right)$ at T. If $\frac{S_T}{S_0} \leq k$ at time $t = T$, the investor will exercise the put and hence place a floor of k on the price relative $\frac{S_T}{S_0}$. If $P_0(k, T)$ is the initial price of the put, the price relative changes from $\frac{S_T}{S_0} \geq 0$ on the uninsured portfolio to $\frac{(S_T/S_0) \vee k}{1+P_0(k,T)} \geq \frac{k}{1+P_0(k,T)}$ on the put protected portfolio. Hence, this well known form of conventional portfolio insurance introduces a strictly positive floor on the wealth relative. When a stock is protected by a put written on it, the combined position is alternatively called a married put, a covered put, or a protective put. We will refer to a conventionally insured portfolio as a married put.

Just after a crash, the volatility of the asset tends to be both higher and have more volatility itself. Unless the crash happens to occur right at maturity, a conventionally insured portfolio will have remaining convexity in both the value of the underlying portfolio and in its volatility. The combination of higher volatilities and positive convexity leads a convexity benefit induced by the crash. A valuation model consistent with volatility clustering will price this convexity benefit into the volatility value of a conventionally insured portfolio. Hence, the buyer of conventional portfolio insurance is paying for more than just protection from the initial crash.

In response to this phenomena, the financial industry has introduced over-the-counter contracts called either gap options[42] or crash cliquets[12]. Gap options/crash cliquets trade over the counter (OTC) on SPX and SX5E presently. They are often written by hedge funds to banks who mainly buy them as a hedge against providing CPPI (Constant Proportion Portfolio Insurance). A long position in a crash cliquet also has a regulatory benefit to large banks who must pass stress tests such as CCAR. The market prices of crash cliquets are observable by banks from services like Markit/Totem and also from a few specialist brokers. When hedge funds write crash cliquets, they typically do not hedge them. Optimal dynamic hedging of cliquets, including crash cliquets, is discussed in Petrelli et. al.[35].

Gap options/crash cliquets pay off and expire when a daily return is sufficiently negative. The cessation of the contract implies that the post-crash convexity benefit in conventional portfolio insurance is not priced in. Insuring a portfolio via a crash cliquet is similar to protecting a corporate or sovereign bond from default via a credit default swap (CDS). However, for a gap option/crash cliquet, exercise is forced by a negative daily return realizing in absolute value above some positive threshold, eg. the gap option must be exercised if its underlying asset loses more than 10% over a day. In contrast, a CDS need not be exercised on a default. The decision to exercise or not is endogenous.

This suggests an alternative form of portfolio insurance, called just-in-time (JIT) portfolio insurance, which we focus on in this paper. Like gap options/crash cliquets, the value of a reference portfolio is monitored periodically, eg. daily, and for some fixed horizon, eg. a year. Also like gap options/crash cliquets, the benefit arises from the possibility of capping a daily percentage loss, or equivalently flooring a daily percentage gain. For a JIT insured portfolio, the owner can apply a one-time floor to any daily return just after the return is realized. If and when the floor is applied, the contract terminates and the balance is paid out. Many investors have a strong desire to exit the market after large price drops or crashes. For these investors, the cash payout arising from flooring can be kept in some safe asset. Of course, the JIT insured investor who floors and exits can re-enter the market at any later time, possibly on an insured basis.

The concept of JIT insurance has been debated in the actuarial literature. A Minneapolis digital healthcare startup called Bind Benefits currently offers consumers the ability to add "just in time" healthcare coverage. In an insurance blog, Glenn McGillivray, a Managing Director of the Institute for Catastrophic Loss Reduction, argues against JIT insurance in the context of catastrophe insurance. In response to this opinion, this paper considers both the pricing and hedging of a JIT insured portfolio. One can think of JIT portfolio insurance as an alternative to a crash cliquet. The CPPI hedging and regulatory benefit would be similar.

The payoffs from holding an uninsured portfolio and a JIT insured portfolio can both be described mathematically in a continuous-time setting as follows. Let $S_t \geq 0$ denote the spot value of a reference uninsured portfolio at time $t \in (0, T]$. Suppose that the investor monitors the value of this portfolio N times over $[0, T]$, where N is a positive integer. Let $\triangle t = T/N$ be the length of the time interval between each assessment of the portfolio value. For every integer $M = 1, \ldots, N-1$, an uninsured investor can decide to liquidate the portfolio early at time $M\triangle t$. For each $M = 1, \ldots, N-1$, the payoff arising from investing \$1 in the uninsured portfolio at $t = 0$ and liquidating at time $M\triangle t$ is $\frac{S_{M\triangle t}}{S_0} = \frac{S_{(M-1)\triangle t}}{S_0} \frac{S_{M\triangle t}}{S_{(M-1)\triangle t}}$ at time $t = M\triangle t$.

The holder of a JIT insured portfolio can also exit the market at a time $M = 1, \ldots, N$ of their choosing, but moreover can replace the most recent price relative $\frac{S_{M\triangle t}}{S_{(M-1)\triangle t}}$ with a contractually specified constant $\underline{R} > 0$ called a floor. A portfolio protected by just-in-time insurance maturing at time $N > 0$ can be liquidated at any time $M = 1, \ldots, N$, for $\frac{S_{(M-1)\triangle t}}{S_0}\underline{R}$, or held to the insurance contract's fixed maturity date $T - N\triangle t$ and then liquidated for $\frac{S_{N\triangle t}}{S_0} = \frac{S_T}{S_0}$.

Just-in-time portfolio insurance differs from conventional portfolio insurance because only the most recent price relative can be floored, as opposed to the performance from inception to expiry. JIT portfolio insurance differs from a crash cliquet because the flooring decision in a JIT insured portfolio is endogenous. The holder of a JIT insured portfolio must decide daily whether to floor yesterday's return or not. This flooring decision can take into account more than just the magnitude of yesterday's return. For example, the amount of time left on the JIT insurance contract and an assessment of future volatility over this period are additional state variables that can affect the flooring strategy. The optionality embedded in just-in-time portfolio insurance resembles that of a crash cliquet, but differs because exercise is not forced. Instead, the optionality embedded in just-in-time portfolio insurance has a Bermudan-style exercise feature, allowing its holder to optimize over allowed exercise strategies, allowing tailoring to individual circumstances, eg. other portfolio holdings or tax implications. In most models, one would expect that a JIT insured portfolio has a value which is greater than that of portfolio protected by a crash cliquet, but which is lower than that of a conventionally insured portfolio.

Of course, the benefit arising from the option to one-time floor a daily return does not come for free. In the contract we explore, the holder of a JIT insured portfolio surrenders dividends from the underlying in full or partial payment for the optionality received. When the reference portfolio has an initial value of one dollar, surrendering dividends lowers the initial value below \$1 and adding optionality raises it back up. One of the main objectives of this paper is to determine the net change from surrendering dividends in return for optionality. As part of this valuation objective, we also determine the optimal flooring strategy. The other main objective is to determine a hedging strategy for the issuer of a JIT insured portfolio.

In this paper, we develop a recursion that governs arbitrage-free values of a JIT insured portfolio when the instantaneous returns of the underlying portfolio depend deterministically on its price relative since the last close and on time. This continuous-time process interpolates a discrete-time process for which the log price of the underlying uninsured portfolio follows a random walk. The volatility clustering that motivates JIT portfolio insurance and crash cliquets as alternatives to conventional portfolio insurance can be captured interperiod by this specification, but not across periods. While the independence of the discrete-time increments of the underlying log price process may or may not be empirically supported, these simple discrete-time dynamics allow the payoff from a JIT insured portfolio to be replicated by a novel strategy which involves rolling over short-dated European put options. The issuer of a JIT insured portfolio can hedge their liability via a semi-static trading strategy in short-dated European puts and their underlying asset. The hedger trades only at times when flooring can occur, eg. daily. When each share of an uninsured portfolio is combined with one European put written on it, the combination is called a married put. Between trading times, the hedger holds the right number of short-dated married puts. At trading times, the expiring puts are exercised if the daily return is floored, otherwise they expire worthless. In the latter case, the share position that the puts were married to is revised downwards. The revenue from selling some of the shares is used to buy European puts maturing at the next flooring time in a quantity which is one to one with each unsold share. Our discrete-time dynamics allow this trade to be self-financing no matter what the underlying price level is. In this way, the hedger rolls over married puts until they are finally exercised, if ever.

The continuous time dynamics that we assume contain the benchmark Black Merton Scholes (BMS) model as a special case. As a result, we first illustrate how this recursion is solved in the BMS model to obtain a unique initial value for a JIT insured portfolio We find that while it is straightforward to numerically value a JIT insured portfolio in this context, there is no explicit valuation formula. Moreover the computational cost is proportional to the number of opportunities to apply a one-time floor.

To address these deficiencies, we provide a solution to the recursion using a new approach, based on the quote at the beginning of this paper. It is widely appreciated in asset pricing theory that the choice of a numeraire is arbitrary. By changing the numeraire, a problem can simplify via a reduction of dimension. However, it is less widely appreciated that the choice of the process of addition is arbitrary as well. Standard axiomatizations of addition in abstract algebra regard addition as any binary operation that is closed, commutative, associative, and possesses an identity. In this paper, we propose a new form of addition which will be particularly convenient for valuing an insured portfolio containing either European or Bermudan options. By changing the addition from ordinary addition to our alternative, complicated recursive problems involving compound optionality reduce to the bond math encountered in an introductory finance course.

It is well known that an option's value is increasing in the volatility of its underlying risky asset. As a result, the arbitrage-free value of an option on a risky asset must be above its intrinsic value obtained by ruling out the possibility that a presently in-the money option goes out of-the-money or vice versa. Consider the value before maturity of a married put giving its owner the right to choose at maturity between a riskless asset paying the strike price at maturity and a risky asset with finite volatility. The absence of arbitrage implies that this married put must not only be currently more valuable than either of its two underlying assets, but it must also currently be less valuable than the *sum* of the current values of its two underlying assets. In other words, the value of the married put is not only an enhanced maximum, but it is also a diminished addition.

Based on this observation, our idea is to treat the value of a married put as a diminished addition of the present value of the floor with the current value of the underlying uninsured portfolio. Our diminished addition is actually a generalization of ordinary addition that takes the friction induced by the finite volatility of the underlying risky asset into account. When the volatility of the underlying risky asset becomes infinite, our generalized addition simplifies into ordinary addition. When this volatility is subsequently lowered to some finite value, the married put's value becomes a diminished addition of the two underlying asset values, which has all of the mathematical properties of an ordinary addition of two non-negative numbers.

To illustrate this idea in the simplest possible context, suppose that an investor owns a zero coupon bond paying \$3 in 1 year, as well as a portfolio of risky assets, currently valued at \$4. Suppose that interest rates and dividends are both zero for the next year. Then the current value is obviously $3 + 4 = 7$. This valuation is model-free. The payoff if held for 1 year and liquidated is the ordinary sum of \$3 and the risky portfolio value in 1 year.

Now suppose that the payoff is not this sum, but is instead the payoff that arises if the investor must choose between the bond paying \$3 and the risky portfolio in 1 year. The new payoff arises by marrying a European put struck at \$3 and maturing in 1 year to the risky portfolio, currently worth \$4. The current value of the married put must be at least \$4, since throwing away the bond leads to this value. This arbitrage-free lower bound of \$4 is attained if the volatility of the risky portfolio is zero. The arbitrage-free current value of the choice between the bond and the risky portfolio must also be less than \$7, since the payoff from holding both of the limited liability assets to maturity exceeds the payoff from just the more valuable of the two assets at maturity.

This upper bound of \$7 on the married put value is attained if the volatility of the underlying risky portfolio is infinite. In a model such as Black Scholes, the risk-neutral probability density of the risky portfolio value acts like a degenerate Bernoulli distribution. As volatility approaches infinity, the probability of S_1 being below any finite positive level such as 3 approaches one. The current value of the risky portfolio remains 4 due to a vanishingly small probability of an arbitrarily large payoff. When the payoff S_1 is replaced by $3 \vee S_1$, the payoff on the lower probability one branch of the effective Bernoulli distributed S_1 improves from 0 to 3, so the current value improves from 4 to 7. In other words, the opportunity to receive the more valuable of two assets at some future time has the same value today as the opportunity to receive both, when one is riskless and the other has infinite risk.

When the risky portfolio has strictly positive but finite volatility, an arbitrage-free value of the married put written on it lies somewhere between \$4 and \$7. Our model will value the married put at $3 \oplus^p 4$, where $g_1 \oplus^p g_2 \equiv (g_1^p + g_2^p)^{\frac{1}{p}}, g_1, g_2 \geq 0, p \geq 1$. For example, if $p = 2$, our model values the married put at $3 \oplus^2 4 = \sqrt{3^2 + 4^2} = 5$, which is between 4 and 7. In short, we think of the ordinary addition of the value of a riskless asset with the value of a risky asset as valuing the choice between them under infinite volatility of the risky asset. We find a model in which valuation under finite volatility merely diminishes this ordinary addition. Our diminished addition $\oplus^p$ does not lose the mathematical properties that ordinary addition has over non-negative reals, such as being increasing, commutative, and associative. Moreover the risk-neutral distribution underlying our model resembles the lognormal distribution, but has heavier tails. Recall that our model values the married put paying $3 \vee S_1$ at 5, when $S_0 = 4$ and the model parameter $p = 2$. The risk-neutral probability that $S_1 < 3$ is 3/5. The share measure probability of the same event (absolute put delta) is 1/5, the lower value arising due to a higher mean of the underlying.

To illustrate the usefulness of an associative valuation operator, consider the simplest multi-period context. Suppose that an investor wants to convert a non-dividend paying stock into a stock paying quarterly proportional dividends for the next 6 months. Just after the second quarterly dividend is paid, the investor wants to liquidate the entire position by selling one share in 6 months. Suppose that the investor's desired dollar payout each quarter is 1/3 of the cum price. If the investor buys 5/3 shares initially, then after the first payout, the shareholdings drop to 4/3. After the second payout, the shareholdings drop from 4/3 to one and then this share is sold. The initial dollar cost of creating this desired stream of cash flows is obviously $(5/3)S_0$. More generally, suppose that the desired dollar payout is the proportion $\underline{R}$ of the cum price. Then the cost of synthesizing two quarterly proportional payouts is $(2\underline{R} + 1)S_0$, where $\underline{R}$ is the desired proportion. This valuation is model-free.

Next consider the JIT insured portfolio written on one share of a non dividend paying asset and with the opportunity to apply a one time floor of 1/3 on the quarterly price relative for each of the next two quarters. If the floor is not applied in either quarter, the entire position is to be liquidated by selling one share in 6 months. The JIT insured portfolio has two quarterly flooring opportunities just as owning 5/3 shares leads to two quarterly proportional dividends. Suppose that the riskfree rate is zero. Using an arbitrage-free model, we will value the JIT insured portfolio as $(\underline{R} \oplus^p \underline{R} \oplus^p 1)S_0$, for some $p > 1$. This value is less than the ordinary sum $(2\underline{R} + 1)S_0$, since the sum $g_1 \oplus^p g_2 \equiv (g_1^p + g_2^p)^{\frac{1}{p}}, g_1, g_2 \geq 0, p \geq 1$ is an ordinary sum for $p = 1$ and is decreasing in p. Our JIT insured portfolio pricing formula simplifies to $(\underline{R} \oplus^p \underline{R} \oplus^p 1)S_0 = ((2^{\frac{1}{p}}\underline{R}) \oplus^p 1)S_0 = ((2\underline{R}^p) + 1)^p S_0$. For example, if $\underline{R} = 1/3$ and $p = 2$, then we value the JIT insured portfolio at $\sqrt{(2/9) + 1S_0}$, which is less than $(5/3)S_0$. However if the volatility is infinite, then our model value rises to $(5/3)S_0$. Notice that our valuation at $((2^{\frac{1}{p}}\underline{R}) \oplus^p 1)S_0$ bundles the two quarterly opportunities to floor at $\underline{R}$ into $2^{\frac{1}{p}}\underline{R}$ and then gives the same value as a single opportunity to floor at $2^{\frac{1}{p}}\underline{R}$ i one quarter. This is analogous to valuing one share with two quarterly dividends with proportion $\underline{R}$ as having the same value $(2\underline{R}+1)S_0$ as one share paying one dividend in one quarter with proportion $2\underline{R}$. In both cases, associativity is used implicitly to first value the dividends/optionality.

Our diminished addition interacts with ordinary multiplication in almost exactly the same way that ordinary addition does. The sole exception is that ordinary multiplication is not a repetition of our diminished addition. The interaction of our diminished addition with ordinary multiplication becomes relevant when interest rates and dividends are nonzero, leading to multiplication by present value factors. Under constant carrying costs, we treat the value of a conventionally insured portfolio as a diminished sum of the present value of the floor with the current value of the underlying uninsured and non-dividend paying portfolio. As a result of our valuation formula being a diminished sum, the two underlying asset values appear only once in our married European put formula, whereas each appears three times in the BMS formula. The value of the married put will be affine in the present value of the floor in our new algebra. In other words, our algebraic approach leads to a value function for the married put which we call pseudo-affine in the floor. As we move from conventional portfolio insurance to JIT portfolio insurance with Bermudan optionality, the underlying risky asset becomes another right to choose between two assets. The pseudo-affinity of our value function becomes important because the composition of two pseudo-affine functions is itself pseudo-affine. As a result, this new approach is able to explicitly determine the fair value of our Bermudan-style JIT insured portfolio in closed-form for any given floor $\underline{R} > 0$ and term $T > 0$. We call our formula closed-form because in contrast to the Black Scholes model valuation, the computational cost is invariant to the number of opportunities to apply a one-time floor.

Our derivation uses a sub-field of abstract algebra called pseudo-analysis, which was developed after the initial applications of conventional portfolio insurance in the 1980's. Pseudo-analysis considers the possibility of allowing addition, subtraction, multiplication, and division to be replaced by new binary operations. In our application of pseudo-analysis to the valuation of conventionally and JIT insured portfolios, we generalize ordinary addition $+$ and subtraction $-$ into new addition and subtraction operations $\oplus^p$ and $\ominus^p$, which we call power-plus and power-minus respectively. Multiplication and division are left unchanged. We show that the value of the option to choose between a risky asset and a riskless one is linear in the current value of the two assets in our algebra. For a JIT insured portfolio, the repeated option to floor or continue is valued by composing a pseudo-linear function with a pseudo-linear function. Since the resulting composition is itself pseudo-linear, we are ultimately able to determine the initial fair value of our Bermudan-style JIT insured portfolio as a pseudo-linear function of the two underlying asset values. The coefficient on the riskless asset turns out to be a finite geometric pseudo-series that simplifies when the dividend yield on the underlying portfolio is a positive constant $q > 0$. The result is a remarkably simple closed-form formula for the initial value of a JIT insured portfolio.

We emphasize that our final formula's computational cost is invariant to the number of opportunities to apply a one-time floor. As a result, our abstract algebra approach can theoretically also be applied to the valuation of American-style JIT portfolio insurance. As the time $\triangle t$ between flooring opportunities tends to zero, one must investigate whether or not our finite geometric pseudo-series converges to a non-trivial pseudo-analytic integral. As this is wholly a mathematical question rather than a financial one, we defer this investigation to a subsequent paper.

Our valuation approach complements probability theory with tools from abstract algebra. For both conventionally and JIT insured portfolios, the valuation methodology that we propose is semi-robust, in that it is consistent with multiple supporting dynamics for the underlying uninsured portfolio value, which are all arbitrage-free alternatives to the geometric Brownian motion (GBM) used by Black Scholes. In contrast to GBM, these alternative dynamics can lead to realistic implied volatility surfaces.

Our valuation approach not only determines a smile-consistent initial value for a JIT insured portfolio, but also determines the optimal strategy for deciding when to apply the floor in a JIT insurance contract. We show that there is positive probability of applying the floor in a JIT insurance contract before maturity. The optimal strategy is to apply the floor the first time that the daily price relative realizes below a deterministically time-varying critical price ratio, which we determine in closed-form.

This paper has three primary contributions. First, we propose a novel form of portfolio insurance which is a hybrid of conventional portfolio insurance based on marrying a European equity put with its underlying and crash-cliquet based portfolio insurance, which is an equity analog of default protection. Assuming that the discrete-time risk-neutral dynamics of the underlying uninsured portfolio value are those of the exponential of a random walk, we develop a semi-static and self-financing hedging strategy which rolls over short-dated married puts, and hece hedges against jumps. Second, we introduce the use of pseudo-analysis in mathematical finance, and emphasize that it allows insights from linear algebra to be applied to some non-linear problems such as Bermudan option valuation. We remark that the special case of pseudo-analysis called max-plus algebra has been similarly applied to conventional portfolio insurance in El Karoui and Meziou[18]. We use a different special case of pseudo-analysis to provide simple closed-form formulas for the JIT insured portfolio value, although we also address the valuation of a conventionally insured portfolio. Third, we draw an analogy between valuing a JIT insured portfolio using pseudo-analysis with constant carrying costs and the much simpler task of valuing a discrete coupon bond under constant interest rates. In this analogy, the value of each option to floor a price relative at a discrete time corresponds to the valuation of a positive coupon paid at the same discrete time. Also, the positive dividend yield from the underlying portfolio when valuing each flooring opportunity in a JIT insured portfolio corresponds to the positive interest rate when discounting a coupon payment. This third contribution offers the tantalizing suggestion that the huge literature on coupon bond valuation can be re-applied to the under-explored but financially important area of pricing Bermudan derivative securities.

An overview of this paper is as follows. The next section explores the valuation of a JIT insured portfolio when the discrete-time risk-neutral stochastic process for the log price of the underlying is a random walk. A recursion governing arbitrage-free values of a JIT insured portfolio is developed and is used to develop a hedging strategy which rolls over short-dated married puts. The solution of the recursion is then illustrated using the benchmark Black Scholes model. The next section introduces pseudo-analysis and focusses on the particular type of pseudo-analysis that we apply, which generalizes ordinary addition and subtraction, but not ordinary multiplication and division. To show how pseudo-analysis can be applied to derivatives pricing, we first apply it the valuation of a conventionally insured portfolio. We find a formula which is simpler than the corresponding BMS valuation and which can moreover be treated as pseudo-affine in the floor. The following section then shows that the recursion governing JIT insured portfolio values is linear in our new algebra. This pseudo-linear recursion is solved by a unique pseudo-linear function whose form is inspired by the well known coupon bond pricing formula. We first provide an "open-form" pseudo-linear expression in which a coefficient is a finite geometric pseudo-series, whose computational cost is proportional to the number of opportunities to apply a one-time floor. The following section then shows that by merely requiring that the dividend yield on the underlying portfolio be a positive constant $q > 0$, our JIT insured portfolio valuation formula can also be written in closed-form. In this section, we also determine the first critical price relative. Finally, we determine a closed-form formula for the par floor that equates the value of foregone dividends to the value of JIT portfolio insurance. In the next to penultimate section, we value a European call written on a JIT insured portfolio in closed form. The penultimate section links the pseudo-linear recursion governing convexity measures to pseudo-linear first-order pseudo-difference equations. The final section summarizes the paper and contains suggestions for future research. The paper concludes with four technical appendices.

2 Recursion & Hedging Strategy for JIT Insured Portfolio

In this section, we develop a recursion governing the arbitrage-free values of a JIT insured portfolio. For $T > 0$ and N a fixed positive integer, let $\triangle t \equiv T/N$ be the amount of time between flooring opportunities. At the valuation time 0, the first of the N exercise opportunities is at time $\triangle t$ and the last is at time $T = N\triangle t$. A JIT insured portfolio allows the flooring of any one of the $N \geq 1$ local price relatives $\frac{S_{\triangle t}}{S_0}, \frac{S_{2\triangle t}}{S_{\triangle t}}, \ldots, \frac{S_T}{S_{(N-1)\triangle t}}$ at some positive floor $\underline{R} > 0$. If the owner of the JIT insured portfolio elects to floor a price relative at period $i = 1, \ldots, N$ then the contract stops and the owner receives an immediate cash payment equal to a constant multiple $\underline{R}$ of the price relative at the last flooring opportunity, $\frac{S_{(i-1)\triangle t}}{S_0}$. If flooring never occurs, then the owner receives the price relative $\frac{S_T}{S_0}$ at the maturity date T. We will assume in the next subsection that the underlying portfolio pays divedends. The owner of the JIT insured portfolio does not receive these dividends, just as the owner of an option does not. As a result, the payoffs described above are a complete description of the payoffs arising to the holder of a JIT insured portfolio.

2.1 Assumptions and Notation

We assume that the riskfree interest rate is constant at $r \in \mathbb{R}$, and that the dividend yield on the underlying portfolio is constant at $q \in \mathbb{R}$. We assume no arbitrage and hence the existence of an equivalent martingale measure $\mathbb{Q}_-$. Under $\mathbb{Q}_-$, the futures price is a positive martingale and hence the log of the futures price has negative drift. We assume that the underlying spot price process S is a continuous-time strictly positive $\mathbb{Q}$ semi-martingale over a finite time interval $[0, T]$. We allow jumps in S and assume that its sample paths are RCLL. We also have further restrictions on S

which we now describe.

For some fixed positive integer N, suppose that we split the time interval $[0,T]$ into N non-overlapping sub-intervals $[0,\triangle t),[\triangle t,2\triangle t),\ldots,[(N-1)\triangle t,N\triangle t]$ where $N\triangle t=T$. Consider a time $t\in[(i-1)\triangle t,i\triangle t)$ in the i-th sub-interval. At such a time, we require that the risk-neutral dynamics of S depend only on S, t, and the price relative $\frac{S_{t-}}{S_{(i-1)\triangle t}}$. Hence, S is a time-inhomogeneous Markov process in itself and a historical price relative $\frac{S_{t-}}{S_{(i-1)\triangle t}}$, where $S_{(i-1)\triangle t}$ is the pre-jump value at time $(i-1)\triangle t$. To formalize this statement, let W be a $\mathbb{Q}_{-}$ standard Brownian motion. Let $\mu_t^{\mathbb{Q}_-}(dx)$ be a counting measure for the jumps in $\ln S$ and let $\nu^{\mathbb{Q}_-}\left(\frac{S_{t-}}{S_{(i-1)\triangle t}},t,dx\right)$ be its risk-neutral compensator at time t.

Consider the following stochastic differential equation $\frac{dS_t}{S_{t-}}=$:

$$(r-q)dt+\sigma\left(\frac{S_{t-}}{S_{(i-1)\triangle t}},t-(i-1)\triangle t\right)dW_t+\int\limits_{\mathbb{R}-\{0\}}(e^x-1)\left[\mu_t^{\mathbb{Q}_-}(dx)-\nu^{\mathbb{Q}_-}\left(\frac{S_{t-}}{S_{(i-1)\triangle t}},t-(i-1)\triangle t,dx\right)\right], \tag{1}$$

for $t\in[(i-1)\triangle t,i\triangle t],i=1,\ldots,N$, and where $S_0>0$ is a known constant. We assume that the constants $r,q\in\mathbb{R}$, the function $\sigma(R,\tau):\mathbb{R}^+\times[0,\triangle t]\mapsto\mathbb{R}^+$ and the compensating measure $\nu^{\mathbb{Q}_-}(R,\tau,dx):\mathbb{R}^+\times[0,\triangle t]\times\mathbb{R}-\{0\}\mapsto\mathbb{R}^+$ are all known at time 0. We require σ to be a bounded function and the integral $\int\limits_{\mathbb{R}-\{0\}}(e^x-1)\nu^{\mathbb{Q}_-}(R,\tau,dx)$ to be finite. As a result, S is positive always.

Notice that σ and $\nu^{\mathbb{Q}_-}$ do not depend on i. Hence, if we examine the RHS of (149) only at the N discrete times $0,\triangle t,\ldots,(N-1)\triangle t$, then at such times, the three coefficients governing $\frac{dS_t}{S_{t-}}$ are all $(r-q,\sigma(1,0),\nu^{\mathbb{Q}_-}(1,0,dx))$, independent of i. The S process starts anew at each of the N discrete times $0,\triangle t,\ldots,(N-1)\triangle t$, and evolves until the next discrete time in a manner that is independent of the paths of the drivers W and μ before the earlier discrete time. Hence in discrete time, the spot price S is the exponential of a random walk. As a result, payoffs that are linearly homogeneous in a strike price and a risky asset price will be valued using a linearly homogeneous value function, As a result, floating-strike European put options have the same value during the floating period as a static position in their underlying asset. This property will later be used to develop a replicating strategy for a JIT insured portfolio, which succeeds even under jumps.

We describe a JIT insured portfolio value at some future calendar date $i\triangle t,i=1,\ldots,N$ assuming that no price relative has been floored earlier at $\triangle t,\ldots,n\triangle t$. We can capture the assumption of no prior flooring by interpreting the future value as one of an insured portfolio whose coverage period begins on the future valuation date. As we roll back the valuation time, we increase the coverage period, and hence add value to the insured portfolio by increasing the amount of optionality its owner has. In the last half of the paper, we will be introducing a new type of addition which captures the added optionality directly.

The JIT insured portfolio has positive probability of the floor being applied early in our model simply because the last price relative can be arbitrarily close to zero. At each possible flooring opportunity $i\triangle t,i=1,\ldots,N$, there exists a strictly positive critical price relative R_i^* such that the price relative $\frac{S_i}{S_{i-1}}$ is floored whenever it realizes below R^*.

For $n=1,2,\ldots,N$, let $J_t^{(n)}$ be the (spot) value of the JIT insured portfolio at time $t\in$

$[T - n\triangle t, T-(n-1)\triangle t]$ with n exercise opportunities remaining. For brevity, our notation $J_t^{(n)}$ suppresses the dependence on the 2 contractual parameters, namely the floor $\underline{R} > 0$ and the maturity date $T \geq t$, which are both constants. Similarly, our notation $J_t^{(n)}$ suppresses the dependence on the 2 environmental parameters, namely the riskfree rate and the dividend yield which are both constant at $r \in \mathbb{R}$ and $q \in \mathbb{R}$ respectively.

2.2 Analysis

Given no prior flooring, the payoff at T of a JIT insured portfolio is:

$$J_T^{(1)} = \frac{S_{T-\triangle t}}{S_0}\underline{R} \vee \frac{S_T}{S_0}. \tag{2}$$

As a result, an arbitrage-free value a period earlier, conditional on no prior flooring, is given by the probabilistic representation:

$$\begin{aligned} J_{T-\triangle t}^{(1)} &= E_{T-\triangle t}^{\mathbb{Q}-} e^{-r\triangle t}\left(\frac{S_{T-\triangle t}}{S_0}\underline{R} \vee \frac{S_T}{S_0}\right) \\ &= \left[E_{T-\triangle t}^{\mathbb{Q}-} e^{-r\triangle t}\left(\underline{R} \vee \frac{S_T}{S_{T-\triangle t}}\right)\right]\frac{S_{T-\triangle t}}{S_0} \\ &= m_1 \frac{S_{T-\triangle t}}{S_0} \end{aligned} \tag{3}$$

where:

$$m_1 = E_{T-\triangle t}^{\mathbb{Q}-} e^{-r\triangle t}\left(\underline{R} \vee \frac{S_T}{S_{T-\triangle t}}\right). \tag{4}$$

From (3), m_1 is a multiplier, while from (4) is a (conditional) mean. Viewed at time $T - \triangle t$, this mean is also an arbitrage-free value of a married put position which combines a \$1 investment in the risky asset with a European put with one dollar notional, strike notional $\underline{R}$, and maturity T. Since the constants $r, q \in \mathbb{R}$, the function $\sigma(R,\tau) : \mathbb{R}^+ \times [0, \triangle t] \mapsto \mathbb{R}^+$ and the compensating measure $\nu^{\mathbb{Q}-}(R,\tau,dx) : \mathbb{R}^+ \times [0,\triangle t] \times \mathbb{R} - \{0\} \mapsto \mathbb{R}^+$ are all known at time 0, the value m_1 can be calculated from just this information. In other words, m_1 is deterministic.

The update condition at $T - \triangle t$ is:

$$J_{T-\triangle t}^{(2)} = \frac{S_{T-2\triangle t}}{S_0}\underline{R} \vee m_1 \frac{S_{T-\triangle t}}{S_0}. \tag{5}$$

At time $T - 2\triangle t$, $\frac{S_{T-2\triangle t}}{S_0}$ is known while $\frac{S_{T-\triangle t}}{S_0}$ is random. If $J_{T-\triangle t}^{(2)}$ is graphed against $\frac{S_{T-\triangle t}}{S_0}$, then (5) implies that m_1 is the change in slope at the single kink. Hence, m_1 is not only a multiplier and a mean, but it is also a measure of the convexity in the payoff $J_{T-\triangle t}^{(2)}$. We think of m_1 as an optionality measure, which will evolve as we recurse backwards.

Equation (2) can be re-written as:

$$J_T^{(1)} = \frac{S_{T-\triangle t}}{S_0}\underline{R} \vee m_0 \frac{S_T}{S_0}, \tag{6}$$

where the initial optionality measure is $m_0 = 1$. Notice that (5) is just a lagged version of (6). It follows from (3) and (4) that an arbitrage-free value a period earlier is given by the probabilistic representation:

$$J^{(2)}_{T-2\triangle t} = m_2 \frac{S_{T-2\triangle t}}{S_0}, \tag{7}$$

where:

$$m_2 = E^{\mathbb{Q}-}_{T-2\triangle t} e^{-r\triangle t} \left(\underline{R} \vee m_1 \frac{S_{T-\triangle t}}{S_{T-2\triangle t}} \right). \tag{8}$$

Since the constants $r, q \in \mathbb{R}$, the function $\sigma(R,\tau) : \mathbb{R}^+ \times [0, \triangle t] \mapsto \mathbb{R}^+$ and the compensating measure $\nu^{\mathbb{Q}-}(R,\tau,dx) : \mathbb{R}^+ \times [0,\triangle t] \times \mathbb{R} - \{0\} \mapsto \mathbb{R}^+$ are all known at time 0, the value m_2 can be calculated from just this information and m_1. In other words, m_2 is also deterministic.

The update condition at $T - 2\triangle t$ is:

$$J^{(3)}_{T-2\triangle t} = \frac{S_{T-3\triangle t}}{S_0} \underline{R} \vee m_2 \frac{S_{T-2\triangle t}}{S_0}, \tag{9}$$

which is a lagged version of (5). More generally, the update condition at $T - n\triangle t$ is:

$$J^{(n+1)}_{T-n\triangle t} = \frac{S_{T-(n+1)\triangle t}}{S_0} \underline{R} \vee m_n \frac{S_{T-n\triangle t}}{S_0} = \frac{S_{T-(n+1)\triangle t}}{S_0} \left(\underline{R} \vee m_n \frac{S_{T-n\triangle t}}{S_{T-(n+1)\triangle t}} \right). \tag{10}$$

Evolving back one period, we have

$$J^{(n+1)}_{T-(n+1)\triangle t} = \frac{S_{T-(n+1)\triangle t}}{S_0} E^{\mathbb{Q}-}_{T-(n+1)\triangle t} e^{-r\triangle t} \left(\underline{R} \vee m_n \frac{S_{T-n\triangle t}}{S_{T-(n+1)\triangle t}} \right) = \frac{S_{T-(n+1)\triangle t}}{S_0} m_{n+1}. \tag{11}$$

Evolving back to time 0 by setting $n = N - 1$, every initial arbitrage-free value of the JIT insured portfolio has the probabilistic representation:

$$J^{(N)}_0 = m_N, \tag{12}$$

where m_N is the deterministic value at $n = N - 1$ of the solution m_{n+1} to the recursion:

$$m_{n+1} = E^{\mathbb{Q}-}_{T-(n+1)\triangle t} e^{-r\triangle t} \left(\underline{R} \vee m_n \frac{S_{T-n\triangle t}}{S_{T-(n+1)\triangle t}} \right), \tag{13}$$

subject to $m_0 = 1$. Since each m_n is a measure of the optionality in a JIT insured portfolio, we refer to (13) as the optionality update recursion.

Under zero or negative dividend yield $q \leq 0$, note that the solution m_n is increasing in n by Jensen's inequality. For any dividend yield, there is tension in (13) between it and the floor $\underline{R}$. Raising the dividend yield lowers the value of m_{n+1} while raising the floor raises the value. For any given dividend yield and risk neutral dynamics, the initial par floor is defined as the unique value of $\underline{R}$ which makes $m_N = 1$.

2.3 Critical Price Relatives

At the n-th step of the recursion, $n = 0, 1 \ldots N-1$, the update condition is:

$$J^{(n+1)}_{T-n\triangle t} = \frac{S_{T-(n+1)\triangle t}}{S_0}\underline{R} \vee m_n \frac{S_{T-n\triangle t}}{S_0} = \frac{S_{T-(n+1)\triangle t}}{S_0}\left(\underline{R} \vee m_n \frac{S_{T-n\triangle t}}{S_{T-(n+1)\triangle t}}\right). \tag{14}$$

At time $T - n\triangle t$, the owner of the JIT insured portfolio will floor if and only if the maximum in brackets in (14) is $\underline{R}$. If the payoff is floored, the cash payment can either be invested in a riskless asset earning the riskless rate or used to buy $\frac{\underline{R}}{S_0}$ shares of the dividend-paying underlying asset whose risk-neutral expected return is again the riskfree rate. In contrast, if kept alive, no cash flows are received. If kept alive, the near term option is written on a wasting asset, due to both the ultimate underlying price dropping as dividends are paid and the time decay experienced in the underlying options. Balancing these effects are the optionality at the next exercise time.

The critical daily price relative $R^*_{T-n\triangle t}$ is defined as the unique value of the current daily price relative $\frac{S_{T-n\triangle t}}{S_{T-(n+1)\triangle t}}$ which equates the two arguments of the maximum:

$$m_n R^*_{T-n\triangle t} = \underline{R}. \tag{15}$$

Once the solution m_n to the recursion is known, the critical price relatives $R^*_{T-n\triangle t}$ also become known:

$$R^*_{T-n\triangle t} = \frac{\underline{R}}{m_n}, \qquad n = 0, 1 \ldots N-1. \tag{16}$$

The JIT insured portfolio should be floored at time $T - n\triangle t$ if and only if the daily price relative is below $R^*_{T-n\triangle t}$.

Recall that under zero dividends, m_n increases with n. As a result, the critical daily price relatives decrease with n. As a function of calendar time, the exercise boundary rises over time and reaches $\underline{R}$ at maturity. Suppose that in the first few days of the contract, some daily price relative realizing below $\underline{R}$ leads to no flooring. The same price relative can lead to optimal flooring if it occurs later. This feature distinguishes JIT portfolio insurance from a crash cliquet, which has a constant, contractually-specified exercise boundary.

2.4 Hedging a JIT Insured Portfolio by Rolling Married Puts

In this section, the solution of the recursion is used to delineate a hedging strategy for the issuer of a JIT insured portfolio. The hedging strategy will use European puts along with their underlying risky asset. In practice, put premia are quoted either on a per share notional basis or on a dollar notional basis. When quoted on a per share basis, the payoff from one put with strike K and maturity date T is $(K - S_T)^+$, which is the textbook payoff. When quoted on a dollar notional basis instead, the payoff at T from one put written on one dollar notional at time 0 is $\left(k - \frac{S_T}{S_0}\right)^+$, where $k > 0$ is called the strike rate (per dollar of notional). By varying the number of puts held, one varies the initial notional controlled by the put position. Let $\check{\$}_t$ be the initial notional controlled by the put position at time $t \in [0, T]$. If held static to its maturity date T, this position pays off

$\check{\$}_t \left(k - \frac{S_T}{S_0}\right)^+$ then. We will be specifying the initial dollar notional $\check{\$}_t$ to control via puts in the hedging strategy. By varying the number of shares held at time t, one also varies the initial notional controlled by an equity position. We will also be specifying the initial dollar notional $\acute{\$}_t$ to control via equity in the hedging strategy. If the dividends received from the risky asset between t and T are reinvested back in the risky asset, then the payoff at T from liquidating this equity investment then is $\acute{\$}_t e^{q(T-t)} \frac{S_T}{S_0}$.

The hedging strategy for a JIT insured portfolio consists of rolling over short-dated married puts. A married put is a long position in a European put and its underlying asset such that the initial notional controlled by the equity position with dividends reinvested to the put maturity matches the initial notional controlled by the put. For a married put position with strike rate k and maturity date T, $\acute{\$}_t = \check{\$}_t e^{-q(T-t)}$ and the dollar payoff at T is $\check{\$}_t \left(k - \frac{S_T}{S_0}\right)^+ + \check{\$}_t e^{-q(T-t)} e^{q(T-t)} \frac{S_T}{S_0} = \check{\$}_t \left(k \vee \frac{S_T}{S_0}\right)$.

Assuming that the risk-neutral discrete-time dynamics of the log price of the uninsured portfolio are those of a known random walk, the issuer of a JIT insured portfolio can completely hedge their liability by owning the right notional of short dated-married puts at the right strike rate. The solution of the optionality update recursion turns out to be the right notional of married puts. The strike rate depends on this notional as well as the uninsured portfolio value. At each flooring opportunity, the expiring puts in the hedge are exercised if flooring occurs and expire worthless otherwise. In the latter case, the share position that the expiring puts are married to is partially sold off. The revenue from the share sale is used to buy the right notional of puts at the right strike rate and maturing at the next flooring opportunity. The notional in puts purchased matches the dividend adjusted notional of shares kept so that the hedge continues to be a married put position. The strike is chosen to provide the proper floor on the price relative should the puts purchased be exercised when they mature. The dynamics ensure that the semi-static trading strategy is self-financing no matter what the underlying asset spot price level is. The use of puts as hedge instruments ensures that the rolling married put hedge succeeds even if the underlying portfolio value jumps, assuming of course that the put writer does not default.

We simulate the hedging strategy for the first 2 periods and then give the general formula at the i-th time step. From (12), the arbitrage-free initial value of the JIT insured portfolio can be split into a dollar investment in puts maturing at $\triangle t$ and a dollar investment $\acute{\$}_0^{(N)}$ in stock:

$$J_0^{(N)} = m_N = m_N - m_{N-1} e^{-q\triangle t} + \acute{\$}_0^{(N)},$$

where the dollar investment in stock is:

$$\acute{\$}_0^{(N)} \equiv m_{N-1} e^{-q\triangle t}.$$

To find the dollar notional $\check{\$}_0^{(N)}$ and the strike rate $K^{(N)}$ of the put position, notice from the definition (13) of m_n with $n = N$ that the dollar investment in puts is given by:

$$\begin{aligned} m_N - m_{N-1}e^{-q\triangle t} &= E_0^{\mathbb{Q}-} e^{-r\triangle t} \left(\underline{R} \vee m_{N-1} \frac{S_{\triangle t}}{S_0} \right) - m_{N-1}e^{-q\triangle t} \\ &= E_0^{\mathbb{Q}-} e^{-r\triangle t} \left[\left(\underline{R} \vee m_{N-1} \frac{S_{\triangle t}}{S_0} \right) - m_{N-1} \frac{S_{\triangle t}}{S_0} \right] \\ &= m_{N-1} E_0^{\mathbb{Q}-} e^{-r\triangle t} \left[\left(\frac{\underline{R}}{m_{N-1}} - \frac{S_{\triangle t}}{S_0} \right) \vee 0 \right]. \end{aligned} \tag{17}$$

As a result, the initial dollar notional in puts is $\check{\$}_0^{(N)} = m_{N-1}$, the strike rate is $K^{(N)} = \frac{\underline{R}}{m_{N-1}}$, and the maturity date is $T^{(N)} = \triangle t$. Notice that since $\check{\$}_0^{(N)} = m_{N-1}$ and $\acute{\$}_0^{(N)} \equiv m_{N-1}e^{-q\triangle t}$, the put and the stock are married. Since m_{N-1} is known at time zero once the recursion is solved, the married put notional m_{N-1} and the strike rate $\frac{\underline{R}}{m_{N-1}}$ are not anticipating.

For $t \in [0, T]$, let $p_t(k, T)$ denote the value of a European put written on \$1 of initial notional paying $\left(k - \frac{S_T}{S_0}\right)^+$ at its maturity date T. Let V_t denote the value of the married put hedge at time $t \in [0, \triangle t]$:

$$V_t = m_{N-1} p_t \left(\frac{\underline{R}}{m_{N-1}}, \triangle t \right) + m_{N-1} e^{-q(\triangle t - t)} \frac{S_t}{S_0}.$$

At $\triangle t$, the value of the married put hedge is:

$$V_{\triangle t} = m_{N-1} \left(\frac{\underline{R}}{m_{N-1}} - \frac{S_{\triangle t}}{S_0} \right)^+ + m_{N-1} \frac{S_{\triangle t}}{S_0} = \underline{R} \vee m_{N-1} \frac{S_{\triangle t}}{S_0}. \tag{18}$$

If $\frac{S_{\triangle t}}{S_0} \leq R^*_{\triangle t} \equiv \frac{\underline{R}}{m_{N-1}}$, then the JIT insured portfolio should be floored. As a result, the European put in the married put hedge is exercised and the payoff from the married put hedge is $\underline{R}$. If If $\frac{S_{\triangle t}}{S_0} > R^*_{\triangle t} \equiv \frac{\underline{R}}{m_{N-1}}$, then the JIT insured portfolio should not be floored at $\triangle t$. In this case, the $\triangle t$ maturity puts will expire worthless and the share position is:

$$J_{\triangle t}^{(N-1)} = m_{N-1} \frac{S_{\triangle t}}{S_0} = (m_{N_1} - m_{N-2} e^{-q\triangle t}) \frac{S_{\triangle t}}{S_0} + \acute{\$}_{\triangle t}^{(N-1)} \frac{S_{\triangle t}}{S_0},$$

where the initial dollar notional controlled by the new equity position is:

$$\acute{\$}_{\triangle t}^{(N-1)} \equiv m_{N-2} e^{-q\triangle t}.$$

To find the dollar notional $\check{\$}_0^{(N-1)}$ and the strike rate $K^{(N-1)}$ of the new position in puts maturing at $2\triangle t$, notice from the definition (13) of m_n with $n = N - 1$ that the dollars invested in $2\triangle t$

maturity puts at $\triangle t$ is:

$$\begin{aligned}(m_{N-1}-m_{N-2}e^{-q\triangle t})\frac{S_{\triangle t}}{S_0} &= \left(E^{\mathbb{Q}-}_{\triangle t}e^{-r\triangle t}\left\{\underline{R}\vee m_{N-1}\frac{S_{\triangle t}}{S_0}\right) - m_{N-2}e^{-q\triangle t}\right\}\frac{S_{\triangle t}}{S_0}\\ &= \left\{E^{\mathbb{Q}-}_{\triangle t}e^{-r\triangle t}\left[\left(\underline{R}\vee m_{N-1}\frac{S_{2\triangle t}}{S_{\triangle t}}\right) - m_{N-2}\frac{S_{2\triangle t}}{S_{\triangle t}}\right]\right\}\frac{S_{\triangle t}}{S_0}\\ &= E^{\mathbb{Q}-}_{\triangle t}e^{-r\triangle t}\left[\left(\frac{S_{\triangle t}}{S_0}\underline{R}\vee m_{N-2}\frac{S_{2\triangle t}}{S_0}\right) - m_{N-2}\frac{S_{2\triangle t}}{S_0}\right]\\ &= m_{N-2}E^{\mathbb{Q}-}_{\triangle t}e^{-r\triangle t}\left[\left(\frac{S_{\triangle t}}{S_0}\frac{R}{m_{N-2}} - \frac{S_{2\triangle t}}{S_0}\right)\vee 0\right]. \end{aligned} \tag{19}$$

As a result, the initial dollar notional controlled by the position in puts at $\triangle t$ is $\check{\$}^{(N-1)}_{\triangle t} = m_{N-2}$, the strike rate is $K^{(N-1)} = \frac{S_{\triangle t}}{S_0}\frac{R}{m_{N-2}}$, and the maturity date is $T^{(N-1)} = 2\triangle t$. Notice that since $\check{\$}^{(N-1)}_{\triangle t} = m_{N-2}$ and $\acute{\$}^{(N-1)}_{\triangle t} \equiv m_{N-2}e^{-q\triangle t}$, the put and the stock are again married.

This self-financing hedging strategy in married puts can be rolled forward to the earlier of optimal flooring and maturity. Assuming no prior optimal flooring, we have at the i-th time step that $\check{\$}^{(N-i)}_{i\triangle t} = m_{N-i-1}$, $K^{(N-i)} = \frac{S_{i\triangle t}}{S_0}\frac{R}{m_{N-i-1}}$, $T^{(N-i)} = (i+1)\triangle t$, and $\acute{\$}^{(N-i)}_{i\triangle t} \equiv m_{N-i-1}e^{-q\triangle t}$, for $i = 0, 1, \ldots, N-1$. This rolling over strategy in short-term married puts is non-anticipating, self-financing, and payoff replicating, even in the presence of jumps.

A necessary condition for getting a unique valuation of the JIT insured portfolio and hence exercise boundary and hedging strategy is to specify the conditional expectation operator in (13). This is equivalent to specifying the distribution governing the IID increments of the discrete-time random walk governing $\ln S$. Once this distribution is specified, it is not necessary to specify the details of the continuous-time process for ln S in order to obtain a unique valuation of the JIT insured portfolio. However, a sufficient condition for getting a unique valuation of the JIT insured portfolio is to uniquely specify the function $\sigma(R,\tau) : \mathbb{R}^+ \times [0,\triangle t] \mapsto \mathbb{R}^+$ and the compensating measure $\nu^{\mathbb{Q}-}(R,\tau,dx) : \mathbb{R}^+ \times [0,\triangle t] \times \mathbb{R} - \{0\} \mapsto \mathbb{R}^+$. We illustrate a unique valuation for the JIT insured portfolio via this approach in the next section.

3 Numerical Solution in the Black Scholes Model

The Black Scholes model with constant volatility σ can be used to obtain a unique initial value for the JIT insured portfolio. In this model, the positive martingale M started at one is the exponential of a Brownian motion:

$$S_t = S_0 e^{\sigma W_t + (r-q-\sigma^2/2)t}, \qquad t \in [0,T],$$

where σ is a known nonzero constant. This arises from our SDE by zeroing out jumps ($\mu^{\mathbb{Q}-}_t = \nu^{\mathbb{Q}-}_t = 0$) and moreover by setting the volatility function $\sigma(R,\tau)$ to a constant, also called σ.

Recall from (4):

$$\begin{aligned} m_1 &= E^{\mathbb{Q}-}_{T-\triangle t} e^{-r\triangle t}\left(\underline{R} \vee \frac{S_T}{T_{T-\triangle t}}\right) \\ &= \underline{R} e^{-r\triangle t} N\left(\frac{\ln \underline{R} - (r-q-\sigma^2/2)\triangle t}{\sigma\sqrt{\triangle t}}\right) + e^{-q\triangle t} N\left(\frac{-\ln \underline{R} + (r-q+\sigma^2/2)\triangle t}{\sigma\sqrt{\triangle t}}\right), \end{aligned} \tag{20}$$

from well-known manipulations. Recall from (8):

$$\begin{aligned} m_2 &= E^{\mathbb{Q}-}_{T-2\triangle t} e^{-r\triangle t}\left(\underline{R} \vee m_1 \frac{S_{T-\triangle t}}{S_{T-2\triangle t}}\right) \qquad (21) \\ &= \underline{R} e^{-r\triangle t} N\left(\frac{\ln(\underline{R}/m_1) - (r-q-\sigma^2/2)\triangle t}{\sigma\sqrt{\triangle t}}\right) + m_1 e^{-q\triangle t} N\left(\frac{\ln(m_1/\underline{R}) + (r-q+\sigma^2/2)\triangle t}{\sigma\sqrt{\triangle t}}\right). \end{aligned}$$

While we could substitute (20) in the 3 places that m_1 appears in the last line of (21), the resulting expression would become unwieldy. However, it is straightforward and computationally efficient to numerically evaluate the recursion (13) using the Black Scholes model:

$$\begin{aligned} m_{n+1} &= E^{\mathbb{Q}-}_{T-(n+1)\triangle t} e^{-r\triangle t}\left(\underline{R} \vee m_n \frac{S_{T-n\triangle t}}{S_{T-(n+1)\triangle t}}\right) \qquad (22) \\ &= \underline{R} e^{-r\triangle t} N\left(\frac{\ln(\underline{R}/m_n) - (r-q-\sigma^2/2)\triangle t}{\sigma\sqrt{\triangle t}}\right) + m_n e^{-q\triangle t} N\left(\frac{\ln(m_n/\underline{R}) + (r-q+\sigma^2/2)\triangle t}{\sigma\sqrt{\triangle t}}\right), \end{aligned}$$

for $n = 0, 1, \ldots, N-1$, subject to $m_0 = 1$. The solution for m_{n+1} when $n = N-1$, i.e. m_N, is the desired initial value of the JIT insured portfolio in the Black Scholes model. In the remainder of this paper, we explore an alternative valuation methodology, which allows us to solve the recursion in (22) explicitly via a simple closed-form formula. As mentioned in the introduction, this alternative approach is based on pseudo-analysis, which we introduce in the next section.

4 Pseudo-Analysis with Power-Plus

4.1 Introduction to Pseudo-Analysis

Pseudo-analysis considers the possibility of using non-standard binary operations to replace the usual addition and/or multiplication operations. A special case of pseudo-analysis called tropical arithmetic was introduced by the Brazilian computer scientist Imre Simon. In tropical arithmetic, ordinary addition is typically replaced by an idempotent binary operation such as the maximum $\vee$ or the minimum $\wedge$. Both operations are called idempotent, since $g \vee g = g$ and $g \wedge g = g$. A sum based on idempotent addition is weakly monotonic in its two operands, but is not strictly monotonic. An example of tropical arithmetic is the max-times algebra, which uses the maximum as the new non-standard and idempotent addition, while retaining ordinary multiplication as the new multiplication.

Pseudo-analysis has been popularized by Pap[34] based on earlier work by Aczel[1, 2] and Maslov[29, 30]. In general, pseudo-analysis maps ordinary addition and multiplication to two new binary operations called pseudo-addition and pseudo-multiplication respectively. Pseudo-addition is a binary operation which is closed, commutative, non-decreasing, associative, and has an identity. The prefix pseudo indicates that the binary operation need not be literally addition. The inverse of pseudo-addition need not exist, but is called pseudo-subtraction when it does. In pseudo-analysis, one also works with pseudo-multiplication, which is also closed, commutative, non-decreasing, associative, and has an identity. The pseudo-multiplication also need not be ordinary multiplication and pseudo-division need not exist. Pseudo-multiplication is required to distribute over pseudo-addition. Furthermore, the pseudo-product of the pseudo-additive identity and any element in the set must be the pseudo-additive identity. Pseudo-analysis is therefore conducted over a semi-ring.

In a special case of pseudo-analysis, this map employs a strictly monotonic and continuous function called a generator. This strict monotonicity allows inversion of pseudo-addition and pseudo-multiplication, which is a form of subtraction and division called pseudo-subtraction and pseudo-division. Pseudo-subtraction and pseudo-division in turn allow pseudo-difference ratios. By taking limits one can develop pseudo-derivatives and their inverses called pseudo-integrals. This so-called g calculus Pap[34] can be applied to solve particular non-linear differential equations using methods such as the pseudo-superposition principle[38], which is the analog of the superposition principle applying only to linear differential equations.

4.2 The Power-Plus Prod Semi-field

The generator that we employ in this paper is a power function defined over non-negative reals. For this very special case of pseudo analysis, pseudo-multiplication reduces to ordinary multiplication so that only the ordinary addition binary operation is changed. One can show that the only generator for which pseudo-multiplication over non-negative reals reduces to ordinary multiplication is the power function. For an introduction to pseudo-addition and subtraction which uses a strictly increasing power function as the generator, see Mesiar and Rybárik[32].

In this paper, we restrict the power parameter p used in the power function to the interval $p \in [1, \infty]$. When the power parameter p is restricted in this way, the power function becomes both strictly increasing and weakly convex. For $p \in [1, \infty]$, we refer to the new binary operation playing the role of addition as power plus. When the power parameter is one, the power plus binary operation reduces to ordinary addition. At the other extreme, sending the power parameter to infinity sends the power plus operation to the ordinary maximum. For each fixed value of our parameter, our power plus and the ordinary multiplication combine over the non-negative reals to create an algebraic structure called a commutative semi-field that we can use to value options. Any option pricing function can be regarded as a function of the risk-neutral mean of the underlying uncertainty and its fixed strike price or floor. The unique output of any function of two arguments can in turn be regarded as the result of applying a binary operation to the two inputs. The idea of using our power plus as a replacement for ordinary addition is especially computationally useful when one must maximize over many alternatives over many dates, eg. for a Bermudan basket option or for the cheapest to deliver option with optionality over both timing and quality. In this paper, we will have only two alternatives at each decision time, but we will have an arbitrarily large number of decision times. We use this idea in this paper to value both a portfolio whose global return $\frac{S_T}{S_0}$ can be floored at $\underline{R} > 0$, or a portfolio allowing the flooring of any one of the N local price relatives $\frac{S_{\triangle t}}{S_0}, \frac{S_{2\triangle t}}{S_{\triangle t}}, \ldots, \frac{S_T}{S_{(N-1)\triangle t}}$ at $\underline{R} > 0$. In both cases, the insured portfolio is valued in an arbitrage-free manner and in closed-form. By closed form, we mean that the computational cost of the formula is invariant to the number N of local price relatives, out of which one can be floored. We will be valuing an insured portfolio with a finite number of local returns to which a floor can be applied, say one million. The approach can also be tried for an American-style insured portfolio but we do not address this extension in this paper.

Ordinary addition and ordinary multiplication of real numbers occur in an algebraic structure called a commutative field $(\mathbb{R}; +, 0; \times, 1)$. Here, $(\mathbb{R}; +, 0)$ is called the additive structure, while $(\mathbb{R}/0; \times, 1)$ is called the multiplicative structure. Both of these algebraic structures are commutative groups and multiplication distributes over addition. Multiplication of any number by zero yields zero. These are the defining properties of a commutative field.

Since the multiplicative structure $(\mathbb{R}/0; \times, 1)$ is a group, for every real number $a_1 \neq 0$, there is another real number a_2 such that $a_1 \times a_2 = 1$. As a result, we can introduce a third binary operation called division $\div$, defined by $a_2 = 1 \div a_1$. Similarly, since the additive structure $(\mathbb{R}; +, 0)$ is a group, for every real number a_1, there is another real number a_2 such that $a_1 + a_2 = 0$. Hence, we can introduce a fourth binary operation called the the minus operation $-$, defined by $a_2 = -a_1$. Here, $a_2 = -a_1 \in \mathbb{R}$ is called the additive inverse of $a_1 \in \mathbb{R}$.

When the real numbers $\mathbb{R}$ are replaced by the non-negative numbers $\mathbb{R}^+$, the lack of an additive inverse in $\mathbb{R}^+$ implies that $(\mathbb{R}^+; +, 0; \times, 1)$ is no longer a commutative field. However, $(\mathbb{R}^+; +, 0; \times, 1)$ can be regarded as the canonical example of a commutative semi-field. A commutative semi-field arises when the additive structure of a commutative field is just required to be a monoid (semi-group with identity). Due to its importance, the commutative semi-field $(\mathbb{R}^+; +, 0; \times, 1)$ is called the probability semi-field, see Lothaire[28], or the semi-field of non-negative reals. In this commutative semi-field, every division other than by zero maintains non-negativity. In contrast, there are subtractions that exit the non-negative reals, so subtraction as a binary operation for *arbitrary* elements is not allowed. However, one can subtract a smaller non-negative number from a larger non-negative number or subtract a non-negative number from itself.

In this paper, we consider an algebraic structure which replaces the ordinary addition in the commutative semi-field $(\mathbb{R}^+; +, 0; \times, 1)$ with a new addition operation called power plus and denoted by $\oplus^p$. We continue to use ordinary multiplication, and hence ordinary division. Hence, we employ the algebraic structure $(\mathbb{R}^+; \oplus^p, 0; \times, 1)$, where for arbitrary elements $g_1 \in \mathbb{R}^+, g_2 \in \mathbb{R}^+$:

$$g_1 \oplus^p g_2 \equiv (g_1^p + g_2^p)^{\frac{1}{p}}, \tag{23}$$

where $p \in [1, \infty]$. This algebraic structure is recognized as a commutative semi-field in Valverde-Albacete and Peláez-Moreno[43], who study its application to Renyi entropy. For completeness, Appendix 1 of this paper directly proves that for any fixed $p \in [1, \infty]$, the algebraic structure $(\mathbb{R}^+; \oplus^p, 0; \times, 1)$ is another commutative semi-field. We refer to $(\mathbb{R}^+; \oplus^p, 0; \times, 1)$ as the power-plus-prod semi-field. When $p = \infty$, the power-plus-prod semi-field reduces to the max-prod semi-field

$(\mathbb{R}^+; \vee, 0; \times, 1)$. When $p = 1$, the power-plus-prod semi-field reduces to the plus-prod semi-field $(\mathbb{R}^+; +, 0; \times, 1)$. Hence, as p varies through the interval $[1, \infty]$, we sweep out a family of commutative semi-fields defined over $\mathbb{R}^+$, containing the probability semi-field at one extreme ($p = 1$) and an idempotent semi-field at the other extreme ($p = \infty$). As we vary p from 1 to ∞, the algebraic structure of a commutative semi-field is being deformed. The general notion of deformation of algebraic structure was introduced by Gerstenhaber[19]. The particular deformation that we use is closely related to a deformation introduced by Maslov[29], which he calls dequantization.

5 First Applications of Power-Plus Prod Pricing

The assumption of no arbitrage is not sufficient in itself to uniquely value either a conventional or a JIT insured portfolio. In this section, we show how pseudo-analysis conducted in the power-plus prod semi-field can be used to find a unique value function for a conventionally insured portfolio. We solve the more complicated problem of uniquely pricing a JIT insured portfolio in the next section.

5.1 A Pseudo-Linear Pricing Principle

Arbitrage pricing theory (APT) is fundamentally a linear pricing theory. To illustrate this foundational consideration, suppose that at some time $t \geq 0$, one wishes to value a payoff at time $t + \triangle t$ of the form $A(t) + B(t)S_{t+\triangle t}$, where $A(t)$ and $B(t)$ only depend on the history up to time t. This payoff function is affine in $S_{t+\triangle t}$. Then the arbitrage-free value of this payoff at time t is:

$$E_t^{\mathbb{Q}-} e^{-r\triangle t}[A(t) + B(t)S_{t+\triangle t}] = A(t)e^{-r\triangle t} + B(t)e^{-q\triangle t}S_t, \tag{24}$$

due to the linearity of the conditional expectation operator $E_t^{\mathbb{Q}-}$. The value function on the RHS of (24) is the ordinary sum of the current values, $A(t)e^{-r\triangle t}$, of the bond paying \$A(t) at T, and $e^{-q\triangle t}B(t)S_t$, which is the dollar value of a claim to $B(t)$ shares at T. The value function on the RHS of (24) is also affine in S_t. Hence under APT, a future payoff that is affine in a future price $S_{t+\triangle t}$ has an arbitrage-free current value that is affine in the current price $S_{t+\triangle t}$. We refer to this important result as the linear pricing principle.

By using power-plus prod pricing, this principle can be adapted to the pricing of some non-linear payoffs. Suppose that at some time $t \geq 0$, one wishes to value a payoff at time $t + \triangle t$ of the form $A(t) \oplus^\infty B(t)S_{t+\triangle t} = A(t) \vee B(t)S_{t+\triangle t}$, where $A(t)$ and $B(t)$ again only depend on the history up to time t. We call the claim with this payoff a chooser, since its owner can choose at $t + \triangle t$ whether to have $A(t)$ bonds or $B(t)$ shares. The chooser's payoff function $A(t) \vee B(t)S_{t+\triangle t}$ is conventionally non-linear in the payoffs $A(t)$ and $B(t)S_{t+\triangle t}$ arising from holding $A(t)$ zero coupon bonds and $B(t)e^{-q(T-t)}$ dividend-paying shares to $t + \triangle t$. However, the payoff function $A(t) \vee B(t)S_{t+\triangle t}$ is the pseudo-sum of $A(t)$ and $B(t)S_{t+\triangle t}$ when $\vee$ is treated as the pseudo-addition $\oplus^\infty$. Thus, the payoff function $A(t) \vee B(t)S_{t+\triangle t}$ is conventionally non-affine in $S_{t+\triangle t}$, but it is max-prod affine in $S_{t+\triangle t}$.

The power-plus prod value of the chooser at time t is defined to be:

$$E_t^{\mathbb{Q}-} e^{-r\triangle t}[A(t) \vee B(t)S_{t+\triangle t}] = A(t)e^{-r\triangle t} \oplus^{p(\triangle t)} B(t)e^{-q\triangle t}S_t \quad (25)$$

$$= \left(\left(A(t)e^{-r\triangle t}\right)^{p(\triangle t)} + \left(B(t)e^{-q\triangle t}S_t\right)^{p(\triangle t)}\right)^{\frac{1}{p(\triangle t)}}, \quad (26)$$

where $p(\tau) : [0, \infty] \mapsto [1, \infty]$ is required to be a declining function of its argument τ with boundary conditions $p(0) = \infty$ and $p(\infty) = 1$. Examples of p functions meeting our requirements are $p(\tau) = e^{\frac{1}{\sigma^2\tau}}$ or $p(\tau) = \frac{1}{\sigma^2\tau} + 1$, where σ^2 is a measure of the variance rate of S. Since the variance of $\ln S_\tau$ in the BMS model is $\sigma^2\tau$, the power p is the precision of $\ln S$ plus one when $p(\tau) = \frac{1}{\sigma^2\tau} + 1$.

(25) states that the value at t of the option to choose at $t + \triangle t$ between a riskless asset paying $A(t)$ and a risky asset paying $B(t)S_{t+\triangle t}$ is a power-plus sum of the current values of the two assets, namely $A(t)e^{-r\triangle t}$ and $B(t)e^{-q\triangle t}S_t$. If we regard the payoff and value of the chooser as functions of the value S of one share of its underlying risky asset, then (25) implies that a max-prod affine payoff in $S_{t+\triangle t}$ leads to a power-plus prod affine payoff in S_t. Notice that the standard linear pricing principle arises as a special case of this pseudo-linearity pricing principle (25) when $p(\triangle t) = 1$. We will use this pseudo-linearity pricing principle for $p(\triangle t) \in [1, \infty]$ repeatedly in the remainder of the paper.

The power-plus prod value of the chooser in (26) is a Schatten power norm of the two vector $\begin{bmatrix} A(t)e^{-r\triangle t} \\ B(t)e^{-q\triangle t}S_t \end{bmatrix}$ containing the two pseudo-summands in (25). Notice that the power $p \in [1, \infty]$ in this power norm has become a function of the time to maturity of the chooser $\triangle t$. Under no arbitrage and no carrying costs, $r = q = 0$, the value of a chooser must be increasing in the time to maturity $\triangle t$. Appendix 2 proves that the Schatten power norm is declining in p. We require that $p(\triangle t)$ be declining in $\triangle t \geq 0$ so that the chooser's value is increasing in $\triangle t$ for $r = q = 0$.

Returning to the case of constant carrying costs $r, q \in \mathbb{R}$, the restriction of the power parameter p to the interval $[1, \infty]$ is required for (26) to be a norm. Every norm is convex in its elements and hence the chooser's value is convex in its two underlying asset values $A(t)e^{-r\triangle t}$ and $B(t)e^{-q\triangle t}S_t$. Since the chooser's payoff $A(t) \vee B(t)S_{t+\triangle t}$, is also convex in its two underlying asset values $A(t)$ and $B(t)S_{t+\triangle t}$, we say that the value function is convexity-preserving. Similarly, the payoff $A(t) \vee B(t)S_{t+\triangle t}$, is increasing in its two underlying asset values $A(t)$ and $B(t)S_{t+\triangle t}$, while the value function in (25) is increasing in its two underlying asset values $A(t)e^{-r\triangle t}$ and $B(t)e^{-q\triangle t}S_t$. As a result, we say that the power plus prod value function is also monotonicity-preserving. We conclude that our value function is monotonicity and convexity-preserving in $\begin{bmatrix} A(t)e^{-r\triangle t} \\ B(t)e^{-q\triangle t}S_t \end{bmatrix}$.

Recall that examples of a p function meeting our requirements were $p(\tau) = e^{\frac{1}{\sigma^2\tau}}$ or $p(\tau) = \frac{1}{\sigma^2\tau} + 1$. The arbitrage-free lower bound on a chooser's valuation is obtained by the limit as $\sigma \downarrow 0$:

$$E_t^{\mathbb{Q}-} e^{-r\triangle t}[A(t) \vee B(t)S_{t+\triangle t}] \geq A(t)e^{-r\triangle t} \vee B(t)e^{-q\triangle t}S_t. \quad (27)$$

The arbitrage-free upper bound on a chooser's valuation is obtained by the limit as $\sigma \uparrow \infty$:

$$E_t^{\mathbb{Q}-} e^{-r\triangle t}[A(t) \vee B(t)S_{t+\triangle t}] \leq A(t)e^{-r\triangle t} + B(t)e^{-q\triangle t}S_t. \quad (28)$$

For $\sigma \in (0, \infty)$, the chooser's value function is between its arbitrage-free bounds and in fact is arbitrage-free.

It is not that difficult to directly specify an arbitrage-free value function for a chooser, but it is harder to be roughly consistent with the market's pricing of the option that it contains. The power-plus prod approach produces a reasonable implied volatility surface for this option. See Figure 1 below for which $p(\tau) = 1 + \frac{10}{\tau}, \tau \geq 0$.

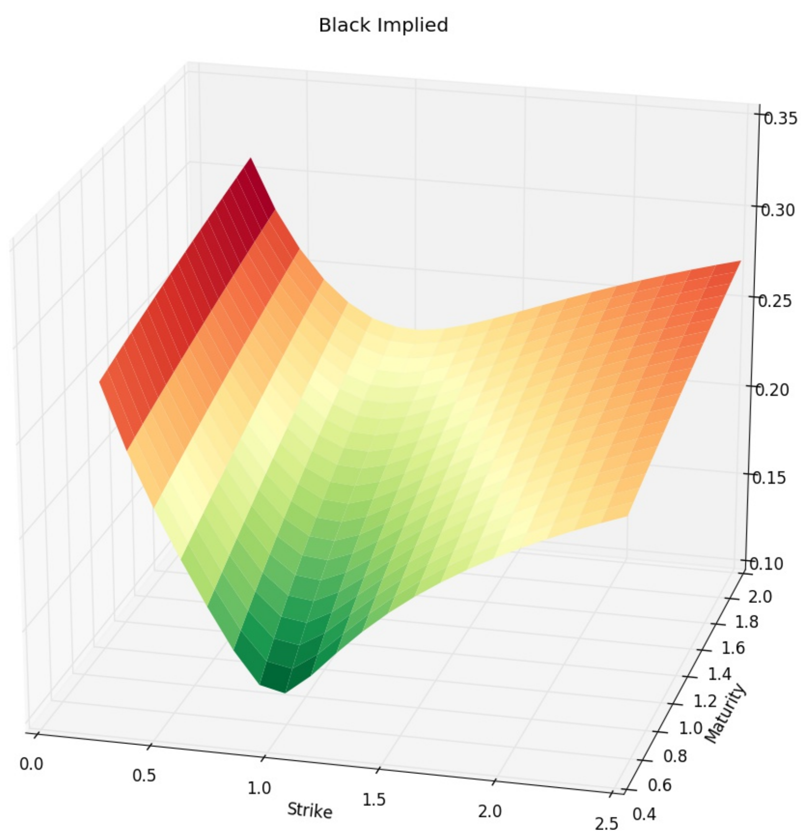


Figure 1: Black Implied Volatility vs. Strike Price and Maturity

Qin and Todorov [36] show how to recover a Lévy density such that the Lévy process followed by the log price, $\ln S$, is consistent with a smooth given European option pricing function such as the one implicit in (25). Appendix 3 also provides an alternative construction of a consistent arbitrage-free continuous-time stochastic process for the underlying portfolio value. This well known construction is based on the additional assumption of S being a time-inhomogeneous Markov martingale in just itself and the last closing price, and also having sample path continuity.

Whether the underlying spot price process S has continuous sample paths or not, the resulting risk-neutral marginal of S is the same. Suppose we zero out rates and dividends, $r = q = 0$, and set $A(t) = K, B(t) = 1$ in the chooser's value function, (26). For $K > 0$ and $S > 0$, let

$$V_p(S, K) = (K^p + S^p)^{\frac{1}{p}} \tag{29}$$

be the initial power-plus prod value of a claim paying $K \vee S_T$ at T and for $p(T) = p \in [1, \infty]$. Here, S is the initial value of the underlying risky asset. Let $V_b(S, K)$ be the corresponding BMS value of a claim paying $K \vee S_T$ at T. For this common payoff, both models enjoy the following discrete symmetry:

$$V_p(S, K) = V_p(K, S) \qquad V_b(S, K) = V_b(K, S). \tag{30}$$

The valuation formula for V_p in (29) also has the following scaling symmetry group:

$$\bar{S} = \lambda S, \bar{K} = \lambda K, \bar{V} = \lambda V, \bar{p} = p, \qquad \lambda > 0. \tag{31}$$

In other words, $\bar{V}$ is the same function of $\bar{S}, \bar{K}, \bar{p}$ as V is of S, K, p. The BMS value of a married put, $V_b(S, K)$, also enjoys this scaling symmetry group (with $\bar{\tau} = \tau$ replacing $\bar{p} = p$). The fact that both models support the discrete symmetry (30) and the scaling symmetry (31) imply that geometric put call symmetry holds for both models. The proof goes as follows. Let $C(S, K)$ and $P(S, K)$ denote European call and put values. By put call parity:

$$C(S, K_c) + K_c = V(S, K_c) = P(S, K_c) + S. \tag{32}$$

Since $V(S, K_c) = V(K_c, S)$, we can also write

$$C(S, K_c) + K_c = V(K_c, S) = P(K_c, S) + K_c, \tag{33}$$

from the second equality in (32). Subtracting K_c from (33):

$$C(S, K_c) = P(K_c, S) = \frac{1}{\lambda} P(\lambda K_c, \lambda S), \tag{34}$$

by the linear homogeneity of the put value function in its two arguments. Set $\lambda = \frac{S}{K_c}$ in (34):

$$C(S, K_c) = \frac{K_c}{S} P\left(S, \frac{S^2}{K_c}\right).$$

Letting $K_p \equiv \frac{S^2}{K_c}$, we have geometric put call symmetry:

$$\frac{C(S, K_c)}{\sqrt{K_c}} = \frac{P(S, K_p)}{\sqrt{K_p}}, \tag{35}$$

where $\sqrt{K_c K_p} = S$. Hence, geometric put call symmetry (35) holds for both the BMS and the PPP model.

The valuation formula for V_p in (29) also has a second symmetry group based on exponentiation:

$$\bar{S} = S^\lambda, \bar{K} = K^\lambda, \bar{V} = V^\lambda, \bar{p} = p/\lambda, \qquad \lambda > 0. \tag{36}$$

There is no way to transform $\sigma^2 T$ so that the BMS value of a married put, $V_b(S, K)$, enjoys this second symmetry group based on exponentiation. The reader can check that the BMS PDE:

$$\frac{\partial}{\partial \tau} V(S, \tau) = \frac{\sigma^2}{2} S^2 \frac{\partial^2}{\partial S^2} V(S, \tau) \tag{37}$$

has the symmetry group:

$$\bar{S} = S^\lambda e^{-\frac{\sigma^2}{2}\lambda(\lambda-1)}, \bar{V} = V, \bar{\tau} = \lambda^2 \tau, \qquad \lambda > 0.$$

However, the initial condition $V(S, 0) = S \vee K$ is incompatible with this transformation group even if we set $\bar{K} = K^\lambda$. Suppose that we require a solution V_p of the BMS PDE (37) to be a power payoff at zero time to maturity, instead of a married put payoff:

$$V_p(S, 0) = \left(\frac{S}{K}\right)^p. \tag{38}$$

Then the following symmetry group is present:

$$\bar{S} = S^\lambda, \bar{K} = K^\lambda, \bar{V} = V, \bar{p} = p/\lambda, \qquad \lambda > 0. \tag{39}$$

This can be regarded as an exponential analog of the Brownian scaling property holding for the heat equation. We refer to it as exponential self similarity. This symmetry group (39) holding for power claim values in BMS is similar (but not identical) to the one (36) holding for the married put value under the power plus prod (PPP) pricing model.

Let k be a positive integer. Suppose S_T is real-valued with mean S_0. Exponentiating the algebra implies that the analogs of affine payoffs $e^k(S_T - S_0), k = 1, 2, \ldots$ are the positive integer power payoffs $\left(\frac{S_T}{S_0}\right)^k, k = 1, 2 \ldots$, where now S_T is positive with mean S_0. For zero drift GBM with deterministic instantaneous variance rate $\sigma^2(t)$, integer moment calculations are straightforward:

$$E^{\mathbb{Q}_b} \left(\frac{S_T}{S_0}\right)^k = e^{\frac{\bar{\sigma}^2 T}{2} k(k-1)},$$

where:

$$\bar{\sigma}^2(T) \equiv \frac{1}{T} \int_0^T \sigma^2(t) dt.$$

For zero drift GBM with deterministic instantaneous variance rate $\sigma^2(t)$, the max payoff can only be expressed in terms of the special function $N(z) \equiv \int_{-\infty}^z \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$:

$$E^{\mathbb{Q}_b} (K \vee S_T | S_t = S) = K N\left(\frac{\ln(K/S) + \bar{\sigma}^2 T/2}{\bar{\sigma}\sqrt{T}}\right) + S N\left(\frac{\ln(S/K) + \bar{\sigma}^2 T/2}{\bar{\sigma}\sqrt{T}}\right). \tag{40}$$

For a centered symmetric Dagum random variable, the situation is exactly the opposite. The expectation of the max payoff is so simple that it can be regarded as an addition of strike and spot:

$$E^{\mathbb{Q}_d}\left(K \vee S_T | S_0 = S\right) = \left(K^{p(T)} + S^{p(T)}\right)^{\frac{1}{p(T)}} \equiv K \oplus^{p(T)} S. \tag{41}$$

In contrast, second and higher integer moment calculations require a special function called the beta function:

$$E^{\mathbb{Q}_d}\left(\frac{S_T}{S_0}\right)^k = \frac{p(T)-1}{p(T)}\beta\left(1 - \frac{k}{p(T)}, 1 + \frac{k-1}{p(T)}\right), \qquad k = 2, 3 \ldots,$$

where:

$$\beta(a,b) \equiv \int_0^\infty g^{a-1}(1+g)^{-a-b} dg.$$

If markets traded only power payoffs, it would be hard to favor the Dagum distribution over the lognormal distribution on an aesthetic basis. However since the market actually trades max payoffs such as those from a married put, it becomes hard to favor the lognormal distribution over the Dagum distribution on an aesthetic basis. The heavier tail of the Dagum also favors its use in financial applications.

These symmetry based observations have an important implication. In the BMS model, M/S is a function of two variables K/S and $\sigma^2 T$, as is well known. However, in the PPP model, $(M/S)^{p(T)}$ is a function of just one variable $(K/S)^{p(T)}$:

$$(M/S)^{p(T)} = 1 + (K/S)^{p(T)}. \tag{42}$$

One can think of a random variable $M_T = K \oplus^{p(T)} S_T$ with mean M as just a pseudo-shifted version of the Dagum random variable S_T with mean S. Recall that dividing by the mean and raising to the power p is the Dagum analog of subtracting the mean and dividing by the standard deviation. Hence (42) indicates that for the normalized variables, optionality of S_T with K is captured by ordinary addition with 1.

The information content of option prices is described by a precision curve $p(\tau)$ in the PPP type model, rather than by an implied volatility surface using the BMS language. The dimension reduction from two to one corresponds to call valuation in the Bachelier model for which the ratio of a call price to an ATM call price depends only on scaled moneyness. The corresponding dimension reduction from two to one in BMS holds only for the value of a claim with a power payoff, not for a claim payoff capturing optionality per se. This suggests that the plethora of results for leveraged ETF's available under the BMS model or more generally for characteristic functions of log price in exponential Lévy model have option based counterparts in the PPP setting.

The PPP valuation formula (29) for a married put is an ℓ^p norm of the 2 vector $\begin{bmatrix} K \\ S \end{bmatrix}$. Consider the dual norm defined by:

$$\bar{V}\left(\begin{bmatrix} \bar{K} \\ \bar{S} \end{bmatrix}\right) \equiv_{S,K \geq 0, S^p + K^p \leq 1}^{\sup} \{S\bar{S} + K\bar{K}\}. \tag{43}$$

By a well known result, the optimization delivers the $\ell^{\bar{p}}$ norm of the 2 vector $\begin{bmatrix}\bar{K}\\ \bar{S}\end{bmatrix}$, where $\frac{1}{\bar{p}}+\frac{1}{p}=1$:

$$\bar{V}\left(\begin{bmatrix}\bar{K}\\ \bar{S}\end{bmatrix}\right)=\left(\bar{K}^{\bar{p}}+\bar{S}^{\bar{p}}\right)^{\frac{1}{\bar{p}}}.$$

Solving for $\bar{p}$ when $\frac{1}{\bar{p}}+\frac{1}{p}=1$ implies:

$$\bar{p}=\frac{p}{p-1}\text{for } p>1. \tag{44}$$

It follows that the change of variables (43) and (44) is a discrete symmetry enjoyed by $V_p(K,S)$. This discrete symmetry does not hold for the BMS model value of a married put, $V_b(K,S)$. Hence when comparing $V_p(K,S)$ to $V_b(K,S)$, $V_p(K,S)$ has one more discrete symmetry and one more continuous symmetry group, (36), i.e. exponential self-similarity.

The married put value $V_p(K,S)$ is convex individually in K for S a parameter and individually in S for K a parameter. In either case, $V_p(K,S)$ is the Fenchel Legendre transform of its convex conjugate. WLOG, suppose that we conjugate out S in favor of the partial derivative/model delta V_S and then conjugate out V_S is favor of S. Since our valuation formula in (29) is obtained by an optimization over the first partial derivative in S, the envelope theorem suggests that differentiation w.r.t. parameter K will simplify. Indeed, differentiating our married put valuation formula (29) w.r.t. K gives the following simple risk-neutral cumulative distribution function (CDF) of the price relative $\frac{S_T}{S}$:

$$\mathbb{Q}_{-}\left\{\frac{S_T}{S}<\frac{K}{S}\right\}=\left(1+\left(\frac{K}{S}\right)^{-p}\right)^{-\alpha}, \tag{45}$$

where $\alpha\equiv\frac{p-1}{p}>0$. The RHS of (45) describes the CDF of a Burr[9] type III distribution. A positive random variable X has a Burr type III distribution if its CDF is:

$$F(x)=(1+x^{-p})^{-\alpha}, \tag{46}$$

for $x>0, p>0, \alpha>0$. Here p and α are two independent shape parameters. Burr[9] showed that the CDF in (46) solves the non-linear ordinary differential equation (ODE):

$$\frac{d\ln F(x)}{d\ln x}=p\alpha[1-F^{\frac{1}{\alpha}}(x)], \qquad x>0, p>0, \alpha>0, \tag{47}$$

subject to the mean one constraint $\int_0^\infty x dF(x)=1$. This ODE is a Bernoulli ODE. As a result, it can be transformed into a linear ODE via the power map $u(\ln x)=F(x)^{1-\frac{1}{\alpha}}$.

Besides the two independent shape parameters $p>0$ and $\alpha>0$ in (46), one can introduce location μ and scale b parameters via $X=\frac{Y-\mu}{b}$. However setting $\mu=0$ changes the support so if we demand support on $(0,\infty)$, then we can only scale $X=\frac{Y}{b}$. The CDF in (46) describes a positive random variable with a Burr type III distribution normalized to have a mean of one. Scaling is the used to achieve any desired mean. The standard deviation relative to a given mean can be determined but the spread of outcomes about the mean is actually governed by the parameter p. The parameter α renders some control on higher moments but in in our application, the parameter α is linked to the parameter $p \in [1, \infty]$ via $\alpha \equiv \frac{p-1}{p}$. As a result, there is only one free parameter p in (45) which controls the shape of the CDF about the mean.

Champernowne[14] motivates our restriction $\alpha \equiv \frac{p-1}{p}$ in (46) by observing that the resulting one parameter CDF:

$$F(x) = (1 + x^{-p})^{\frac{1-p}{p}}, \qquad p > 1, \tag{48}$$

leads to a symmetric *Lorenz curve*. In general, the Lorenz curve of a non-negative random variable with finite mean $\mu > 0$ is generated via its CDF. For a once differentiable CDF $F(x)$ with probability density function (PDF) $f(x) = F'(x)$, the Lorenz curve $\Lambda(q) : [0, 1] \mapsto [0, 1]$ is defined by:

$$\Lambda(q) = \Lambda(F(x)) = \frac{\int_0^x u\, f(u)\, du}{\int_0^\infty u\, f(u)\, du} = \frac{\int_0^x u\, f(u)\, du}{\mu}.$$

In financial terms, the Lorenz curve $\Lambda_-(q) : [0, 1] \mapsto [0, 1]$ relates the share measure distribution function $F_+(K) \equiv \mathbb{Q}_+\{S_T < K\}$ to the risk-neutral measure distribution function $F_-(K) \equiv \mathbb{Q}_-\{S_T < K\}$:

$$F_+(K) = \Lambda_\pm(F_-(x)) = \frac{\int_0^x u\, f_-(u)\, du}{\int_0^\infty u\, f_-(u)\, du} = \frac{\int_0^x u\, f_-(u)\, du}{S}.$$

A Lorenz curve is convex and in fact the Lorenz curve $\Lambda_\pm(q)$ generated by $F_-(K) \equiv \mathbb{Q}_-\{S_T < K\}$ is just the convex conjugate of the convex increasing function linking the normalized European put value $\pi(k) \equiv P(K)/S$ to a normalized strike $k \equiv K/S$. As a result, the inverse of the Lorenz curve is an increasing *concave* distortion function $q_- = \Lambda_\pm^{-1}(q_+)$. To restore convexity while maintaining an increasing relationship, define a complementary Lorenz curve $\Lambda_\mp : [0, 1] \mapsto [0, 1]$ relating the risk-neutral survival probability $1 - q_- = 1 - F_-(K) = \mathbb{Q}_-\{S_T > K\}$ to the share-measure survival probability $1 - \Lambda_\mp = 1 - F_+(K) = \mathbb{Q}_+\{S_T > K\}$:

$$\mathbb{Q}_-\{S_T > K\} = \Lambda_\mp(\mathbb{Q}_+\{S_T > K\}), \tag{49}$$

or equivalently:

$$1 - q_- = \Lambda_\mp(1 - \Lambda_\pm(q_-)). \tag{50}$$

The Lorenz curve $\Lambda_\pm(q)$ generated by $F_-(K) \equiv \mathbb{Q}_-\{S_T < K\}$ is said to be *symmetric* if:

$$\Lambda_\mp(q) = \Lambda_\pm(q), \text{ for all } q \in [0, 1]. \tag{51}$$

Substituting (51) in (50):

$$1 - q_- = \Lambda_\pm(1 - \Lambda_\pm(q_-)). \tag{52}$$

This matches the definition of Lorenz curve symmetry given in Taguchi[41]:

$$\Lambda(1 - \Lambda(q)) = 1 - q, \text{ for all } q \in [0, 1]. \tag{53}$$

Taguchi[41] mentions that the lognormal distribution also has a symmetric Lorentz curve. Following a suggestion by Champernowne[14], Taguchi[41] proves that a given PDF $f(x)$ with finite mean $\mu > 0$ generates a symmetric Lorenz curve if and only if:

$$f(x) = \left(\frac{\mu}{x}\right)^3 f\left(\frac{\mu^2}{x}\right), \qquad x > 0.$$

This is equivalent to:

$$\frac{x}{\mu} f(x) = \left(\frac{\mu}{x}\right)^2 f\left(\frac{\mu^2}{x}\right), \qquad x > 0.$$

Integrating x from some positive level $K > 0$ to infinity quickly leads to the following gap-call binary-put symmetry:

$$\frac{1}{\mu}\int_K^\infty x f(x) dx = \int_K^\infty \left(\frac{\mu}{x}\right)^2 f\left(\frac{\mu^2}{x}\right) dx = \int_0^{\mu^2/K} f(r) dr, \tag{54}$$

where $r = \mu^2/x$. When f is a risk-neutral density so that $\mu = S$, the LHS can be interpreted as the share measure probability that a stock price exceeds a strike price K. It is also the value of a gap call paying the price relative if a stock price exceeds K. The RHS is the value of a binary put whose strike, μ^2/K, is such that its geometric mean with the gap call strike K is the current stock price $\mu = S$. Financially, we have the following binary-put gap-call symmetry:

$$\mathbb{Q}_-\{S_T < K_p | S_t = S\} = \mathbb{Q}_+\{S_T > K_c | S_t = S\}, \qquad \text{for } K_c > 0, K_p > 0, s.t. \sqrt{K_c K_p} = S. \tag{55}$$

Rewriting in terms of European put $P(K_p, S)$ and call $C(K_c, S)$ value functions:

$$\frac{\partial}{\partial K_p} P(K_p, S) = \frac{\partial}{\partial S} C(K_c, S), \qquad \text{for } K_c > 0, K_p > 0, s.t. \sqrt{K_c K_p} = S. \tag{56}$$

Using the linear homogeneity of C in K_c and S, Euler's theorem implies that the RHS is:

$$\frac{\partial}{\partial K_p} P(K_p, S) = \frac{1}{S}\left[C(K_c, S) - K_c \frac{\partial}{\partial K_c} C(K_c, S)\right], \qquad \text{for } K_c > 0, K_p > 0, s.t. \sqrt{K_c K_p} = S. \tag{57}$$

Distributing $\frac{1}{S}$ and substituting $K_c = S^2/K_p$:

$$\begin{aligned} \frac{\partial}{\partial K} P(K, S) &= \frac{1}{S} C(K, S) - \frac{S}{K}\frac{\partial}{\partial K} C\left(\frac{S^2}{K}, S\right) \\ &= \frac{\partial}{\partial K}\left[\frac{K}{S} C\left(\frac{S^2}{K}, S\right)\right], \qquad K > 0. \end{aligned} \tag{58}$$

Integrating on K from $K = 0$ to $K = K_p$ implies:

$$P(K_p, S) = \frac{K_p}{S} C\left(\frac{S^2}{K_p}, S\right). \tag{59}$$

Recalling $K_c = \frac{S^2}{K_p}$ leads to the following Geometric Put Call Symmetry (GPCS):

$$\frac{P(K_p, S)}{\sqrt{K_p}} = \frac{C(K_c, S)}{\sqrt{K_c}}, \qquad \text{for } K_p > 0, K_c > 0, s.t. \sqrt{K_p K_c} = S, \tag{60}$$

first formulated in Bates[5] for the Black[6] model. It follows that GPCS (60) also holds for European options valued via the restricted Burr type III distribution with risk-neutral CDF for $\frac{S_T}{S}$ given in (48). Carr and Lee[13] give equivalent characterizations of GPCS in terms of a symmetric Black implied volatility curve and symmetry of the PDF of $\ln(S_T/S)$ under a hybrid measure $\mathbb{H}$ for which $\ln(S_t/S)$ is a martingale. From Champernowne[14], (60) is equivalent to symmetry of the Lorenz curve, (52). Finally, from Candel Ato and Barnardic[10], (60) is also equivalent to:

$$E^{\mathbb{Q}_-}[S_T | S_t = S, K_p < S_T < K_c] = S, \quad \text{for } K_c > 0, K_p > 0, s.t. \sqrt{K_c K_p} = S. \tag{61}$$

When GPCS holds, semi-static hedging of barrier and lookback options becomes feasible when their underlying price process is max or min continuous. Notice that the parameter $p > 1$ in the risk-neutral CDF (48) does not appear in the relationship (60) between European put and call values. This implies that p can be independently randomized in an unknown way and the static hedge will nonetheless succeed. The nature of the process can also change from p randomized Burr to σ randomized geometric Brownian motion and the semi-static hedge will still succeed. The lookback hedge just requires max or min continuity and that GPCS holds when the relevant extrema changes. For single or double barrier options, the second condition can be relaxed to GPCS holding when the underlying is at the barrier. Note that violations of GPCS amount to removing commutativity when treating the value of a married put as a sum of its strike price and its underlying futures price.

When the CDF has the simple form $F(x) = (1 + x^{-p})^{\frac{1-p}{p}}, p > 1$, in (48), the parameter p has a couple of probabilistic interpretations. First, $p - 1$ governs the tail behavior in both the left and right tails. Consider first the $\mathbb{Q}_-$ CDF of the continuously compounded return $\ln(S_T/S)$. Recall:

$$F_-(\ell) \equiv \mathbb{Q}_- \left\{ \ln\left(\frac{S_T}{S}\right) < \ell \right\} = (1 + e^{-p\ell})^{\frac{1-p}{p}}, \qquad p > 1, \tag{62}$$

where $\ell \equiv \ln\left(\frac{K}{S}\right)$. When ℓ is very negative, $e^{-p\ell}$ dwarfs 1 so it can be ignored. As a result, the left tail of the $\mathbb{Q}_-$ CDF has behavior $F_-(\ell) \sim e^{(p-1)\ell}$ for ℓ very negative. We can also consider how the $\mathbb{Q}_+$ survival function approaches zero as ℓ approaches infinity. Recall:

$$F_+(\ell) \equiv \mathbb{Q}_+ \left\{ \ln\left(\frac{S_T}{S}\right) > \ell \right\} = (1 + e^{p\ell})^{\frac{1-p}{p}}, \qquad p > 1. \tag{63}$$

When ℓ is very positive, $e^{p\ell}$ dwarfs 1 so it can be ignored. As a result, the right tail of the $\mathbb{Q}_+$ survival function has behavior $F_+(\ell) \sim e^{(1-p)\ell}$ for ℓ very positive. Since both tails are exponential, they are heavier than the Gaussian tails governing continuously compounded returns in the BMS model.

When the CDF has the simple form $F(x) = (1 + x^{-p})^{\frac{1-p}{p}}, p > 1$, in (48), the parameter p has a second probabilistic interpretation. For our restricted Burr III, $1/p$ plays the same role as standard deviation does for a Gaussian. Since a restricted Burr III is a positive random variable, say $\frac{S_T}{S}$, raising $\frac{S_T}{S}$ to the power p is analogous to dividing a zero mean Gaussian random variable by its standard deviation. While standard deviation is defined via the second moment, it also is relevant in the financial context when convexity is induced by the max function, rather than by squaring. Consider the Bachelier model where the underlying asset's instantaneous normal volatility, $a(t)$, is a deterministic function of time:

$$dS_t = a(t)dW_t, \qquad t \in [0, T].$$

Next consider the initial value of an ATM chooser paying $S_T \vee S$ at T:

$$E^{\mathbb{Q}}(S_T \vee S) \equiv V_a = S + \frac{1}{\sqrt{2\pi}}\text{Std}^{S_T|S}(T) \tag{64}$$

where:

$$\text{Std}^{S_T|S}(T) \equiv \sqrt{\int_0^T a^2(t)dt}$$

is the standard deviation of $S_T|S$. V_a exceeds S due to Jensen's inequality applied to the max function. Solving (64) for the standard deviation gives:

$$\text{Std}^{S_T|S}(T) = \sqrt{2\pi}(V_a - S). \tag{65}$$

Next consider the PPP model value of an ATM chooser also paying $S_T \vee S_0$ at T:

$$E^{\mathbb{Q}}(S_T \vee S) \equiv V_p = S2^{\frac{1}{p(T)}}. \tag{66}$$

V_p again exceeds S due to Jensen's inequality applied to the max function. Solving (66) for $\frac{1}{p(T)}$:

$$\frac{1}{p(T)} = \frac{\ln(V_p/S)}{\ln 2}. \tag{67}$$

Comparing (65) with (67) motivates our interpretation of $1/p(T)$ playing the same role when valuing an ATM chooser as the standard deviation of S_T/S does in the Bachelier model with deterministic volatility.

The parameter p can also be understood in the context of away-from-the -money options. The value function for a married put, $V_p(K, S) = (K^p + S^p)^{\frac{1}{p}}$ is linearly homogeneous in K and S for each $p \in [1, \infty]$. Hence, by Euler's theorem:

$$V_p(K, S) = K\frac{\partial}{\partial K}V_p(K, S) + S\frac{\partial}{\partial S}V_p(K, S). \tag{68}$$

From standard probabilistic arguments, we have under zero carrying costs that:

$$V_p(K, S) = E^{\mathbb{Q}_-}[K \vee S_T|S_t = S] = K\mathbb{Q}_-\left\{\frac{S_T}{S} < \frac{K}{S}\right\} + S\mathbb{Q}_+\left\{\frac{S_T}{S} > \frac{K}{S}\right\}, \tag{69}$$

where $\mathbb{Q}_+$ denotes share measure. Suppressing the dependence of p on $\tau \equiv T - t$, differentiating (69) implies from (29) that:

$$\begin{aligned}
\frac{\partial}{\partial K} V_p(K,S) &= \mathbb{Q}_- \left\{ \frac{S_T}{S} < \frac{K}{S} \right\} = \left(1 + \left(\frac{K}{S}\right)^{-p}\right)^{\frac{1-p}{p}} = \left(\frac{K^p}{K^p + S^p}\right)^{\frac{p-1}{p}} \\
\frac{\partial}{\partial S} V_p(K,S) &= \mathbb{Q}_+ \left\{ \frac{S_T}{S} > \frac{K}{S} \right\} = \left(1 + \left(\frac{K}{S}\right)^{p}\right)^{\frac{1-p}{p}} = \left(\frac{S^p}{K^p + S^p}\right)^{\frac{p-1}{p}}. \qquad (70)
\end{aligned}$$

Notice from the explicit expressions for the $\mathbb{Q}_-$ CDF and the $\mathbb{Q}_+$ survival function that the following symmetry relation is holding when $\frac{S_T}{S}$ has $\mathbb{Q}_-$ CDF (48):

$$\mathbb{Q}_+ \left\{ \frac{S_T}{S} > \frac{K}{S} \right\} = \mathbb{Q}_- \left\{ \frac{S_T}{S} < \frac{S}{K} \right\}. \qquad (71)$$

This also follows from (55).

The ratio of the two probabilities in (70) is just a power of K/S:

$$\mathbb{Q}_- \left\{ \frac{S_T}{S} < \frac{K}{S} \right\} / \mathbb{Q}_+ \left\{ \frac{S_T}{S} > \frac{K}{S} \right\} = \left(\frac{K}{S}\right)^{p-1}. \qquad (72)$$

Consider how the fraction on the LHS changes for small changes in K/S. As K/S goes up, the numerator rises, the denominator falls, and hence the ratio rises. By considering infinitessimal percentage changes in both the fraction and K/S, it is easy to see from (72) that this rise is constant:

$$\frac{d \ln}{d \ln(K/S)} \left(\mathbb{Q}_- \left\{ \frac{S_T}{S} < \frac{K}{S} \right\} / \mathbb{Q}_+ \left\{ \frac{S_T}{S} > \frac{K}{S} \right\} \right) = p - 1.$$

In other words, the elasticity of substitution between K and S is the same positive constant $p-1$ in our model across all $K > 0$ and $S > 0$.

Removing our restriction that $\alpha = \frac{p-1}{p}$, a random variable with a two parameter Burr type III distribution arises as the reciprocal of a random variable with the more widely studied Burr type XII distribution. The latter is sometimes simply called the Burr distribution. As a result, the Burr type III distribution is sometimes called "reciprocal Burr" (see Tadikmalla[40]) or "inverse Burr" (see the wiki entry on Burr distributions).

In the discussion of the Burr III distribution thus far, the risk-neutral mean of $\frac{S_T}{S}$ has been one. When we allow non-zero carrying costs, we will want the risk-neutral mean of $\frac{S_T}{S}$ to be $e^{(r-q)T}$, which is a positive constant instead. A generalization of the Burr III distribution which allows any positive mean is the Type I Dagum[15] distribution, which describes a non-negative random variable and has 3 free parameters, $b > 0$, $p > 0$, and $\alpha > 0$. A positive random variable X has a Type I Dagum distribution if its CDF is:

$$F(x) = \left(1 + \left(\frac{x}{b}\right)^{-p}\right)^{-\alpha}, \qquad (73)$$

for $x > 0, b > 0, p > 0, \alpha > 0$. The Dagum CDF also solves the Bernoulli ODE 47, but generalizes the mean constraint to $\int_0^\infty x dF(x) = b$ for $b > 0$. For a guide to the Dagum distributions, see the survey paper by Klieber[24]. One genesis of a Dagum distributed random variable is to start with an inverse Weibull distributed random variable with scale parameter θ. One then randomizes θ via a transformed gamma distribution. As shown in Appendix 4, the resulting PDF is a Type I Dagum distribution.

The CDF in (73) is sometimes also called Burr type III. Sherrick, Garcia, and Tirupattur[39] use market prices of options on soybean futures to infer the risk-neutral distribution of the terminal futures price. They test the 3 parameter Burr type III against the standard two parameter lognormal distribution. Their motivation for using Burr Type III as opposed to some other 3 parameter distribution is that the Burr type III distribution covers all of the space regions in the skewness kurtosis plane. They find that "in depicting ex ante price variability, the Burr III substantially outperforms the lognormal."

The Dagum/Burr III distribution is related to other 3 parameter distributions. For example, the reciprocal of a Dagum distributed random variable has a Singh Maddala distribution, see Klieber[24]. The Dagum distribution is a generalization of the log logistic distribution. It follows that the log of a Dagum distributed random variable has a generalized logistic distribution. Let $Y_T \equiv p \ln\left(\frac{S_T}{S}\right)$ and $y \equiv p \ln\left(\frac{K}{S}\right)$. Then:

$$\mathbb{Q}_-\left\{\ln\left(\frac{S_T}{S}\right) < \ln\left(\frac{K}{S}\right)\right\} = \mathbb{Q}_-\{Y_T < y\} = \frac{1}{(1+e^{-y})^\alpha}, \qquad y \in \mathbb{R}, \alpha > 0.$$

Hence, $Y_T = p \ln \frac{S_T}{S}$ has a type 1 generalized logistic distribution, see Johnson, Kotz, and Balakrishnan[23] page 140.

Levy and Duchin[26] find empirical support for the logistic distribution in describing stock returns over horizons of one day to one year. In our application to valuing a JIT insured portfolio, the log of the daily price relative has a generalized logistic distribution. Note that Levy and Levy[27] consider the pricing of European stock options when the underlying stock price has a logistic distribution. When we value the European option embedded in conventional portfolio insurance, we are instead assuming that the *log* of the price relative has a generalized logistic distribution.

Differentiating (45) w.r.t. K leads to the risk-neutral transition probability density function (PDF) of S_T:

$$\mathbb{Q}_-\{S_T \in dK | S_0 = S\} = K\left(S^p + K^p\right)^{\frac{1}{p}-2}\left(\frac{S}{K}\right)^p dK. \tag{74}$$

Figure 2 shows both the lognormal PDF in dashed orange and this power plus PDF in solid blue. As is evident from the implied volatility surface plot in Figure 1, the power plus PDF has heavier tails. Figure 3 focusses on the tails.

Notice that the Dagum PDF (74) explicitly integrates into the following simple conditional CDF:

$$\mathbb{Q}_-\{S_T < K | S_0 = S\} = \left(1 + \left(\frac{K}{S}\right)^{-p}\right)^{\frac{1-p}{p}} = \left(\frac{K^p}{S^p + K^p}\right)^{\frac{p-1}{p}}, p \geq 1. \tag{75}$$

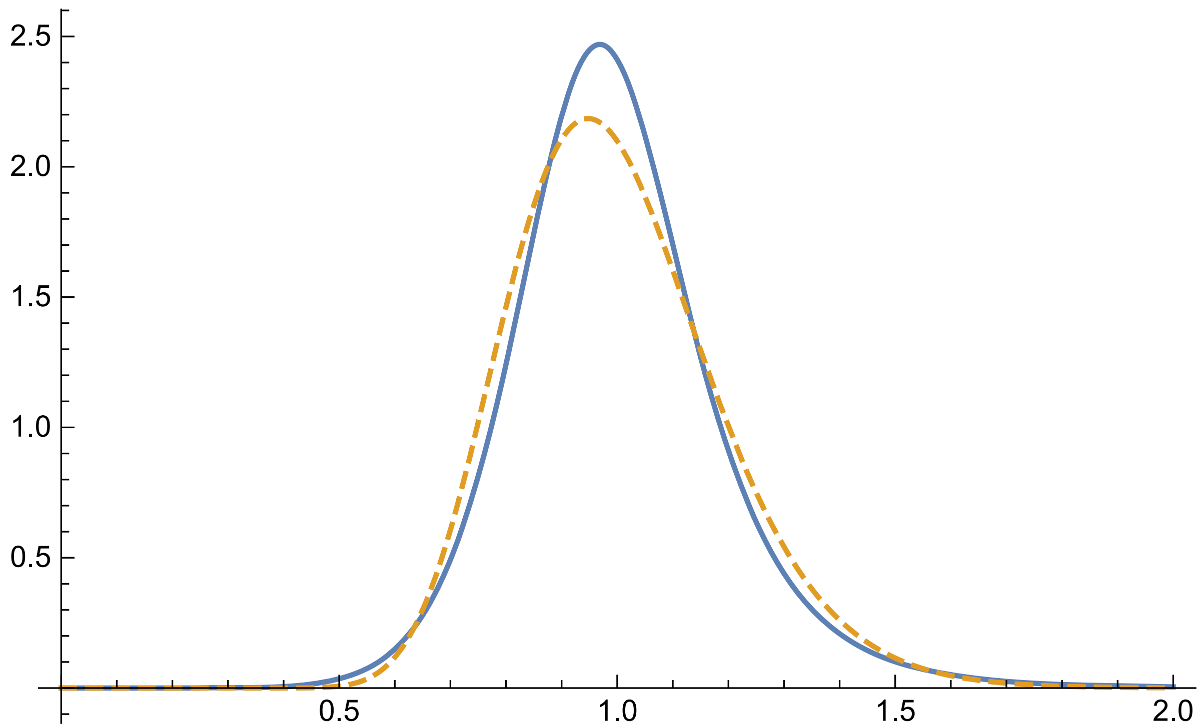


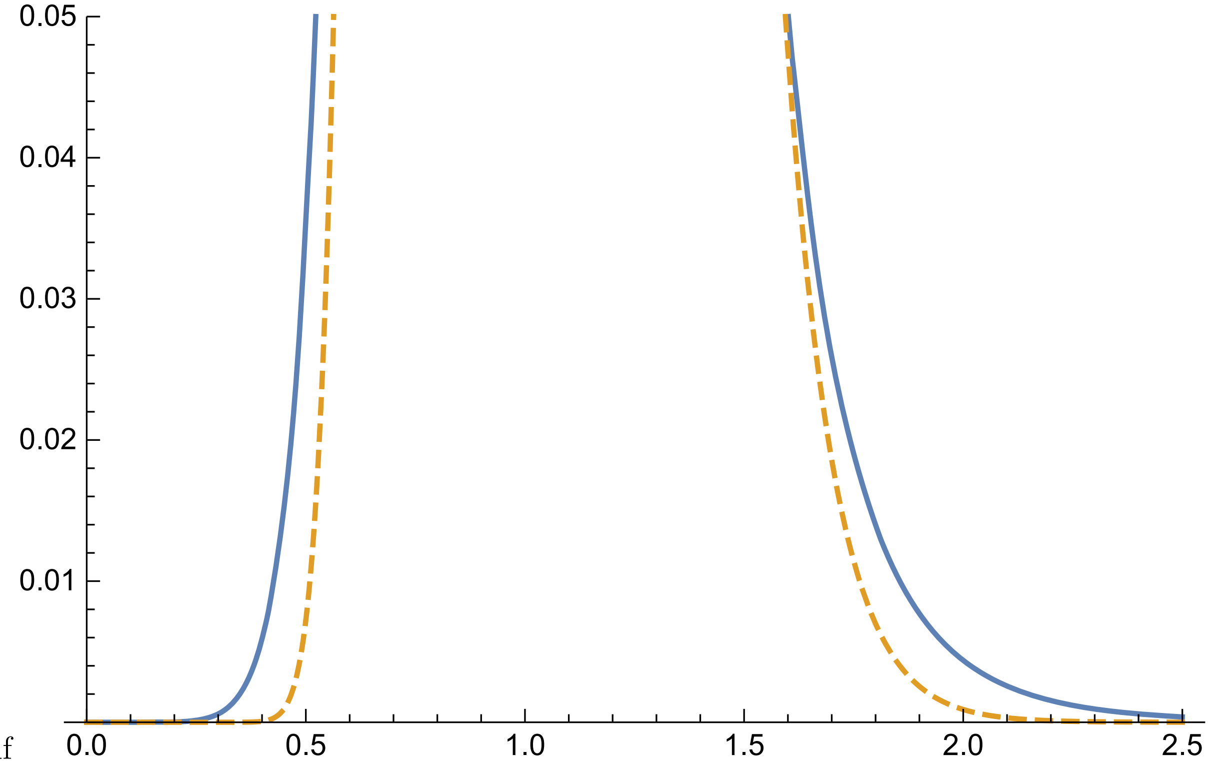
Figure 2: Lognormal PDF (Dashed) and Power Plus PDF (Solid)

The first equality is useful for inversion in K, while the second equality has a nice geometric interpretation when $p = 2$:

$$\mathbb{Q}_{-}\{S_T < K | S_0 = S\} = \frac{K}{\sqrt{S^2 + K^2}}, \qquad K > 0, S > 0. \tag{76}$$

To illustrate, the probability that a stock worth 4 will realize below a strike of 3 is $\frac{3}{\sqrt{4^2+3^2}} = 3/5$. When $p = 2$, the restricted Dagum CDF in (76) is a cosine restricted to an angle domain $(-\pi, 2, 0)$. For any $p \geq 1$, the explicit CDF in (75) also explicitly integrates into the chooser value function $(K^p + S^p)^{\frac{1}{p}}$. Contrast this with the lognormal PDF where the CDF and chooser value function must both be expressed in terms of an integral called the standard normal CDF. Moreover, the Dagum CDF and chooser value functions are both explicitly invertible. The explicit invertibility of the Dagum CDF means the quantile function is explicit, so simulation is straightforward. The explicit invertibility of the power-plus model chooser value function means power-subtraction is as explicit as power-addition, when power-subtraction is well-defined.

The 3 parameter Dagum distribution is a parametric special case of the family of 4 parameter distributions called generalized beta of the second kind (GB2). McDonald and Bookstaber[31] consider European option pricing when the terminal value of the underlying asset is distributed in this more general class. We present the Dagum special case with $\alpha \equiv \frac{p-1}{p}$ in the next subsection.



density2.pdf

Figure 3: Tails of Lognormal PDF (Dashed) and of Power Plus PDF (Solid)

5.2 Power-Plus Prod Pricing of a Conventionally Insured Portfolio

In this subsection, we consider the pricing of a conventionally insured portfolio flooring the random global price relative $\frac{S_T}{S_0}$ at the non-random floor $\underline{R} > 0$. The portfolio value at its maturity date T is:

$$C_T(\underline{R}, T) = \underline{R} \vee \frac{S_T}{S_0}. \tag{77}$$

Notice that the payoff in (77) is max-prod affine in S_T. By the pseudo-linearity pricing principle of the last subsection, the power-plus prod value at any prior time $t \in [0, T]$ of this conventionally insured portfolio is:

$$E_t^{\mathbb{Q}^p_-} e^{-r(T-t)} \left(\underline{R} \vee \frac{S_T}{S_0} \right) = \underline{R} e^{-r(T-t)} \oplus^{p(T-t)} \frac{S_t}{S_0} e^{-q(T-t)} \tag{78}$$

$$= \left[\left(\underline{R} e^{-r(T-t)} \right)^{p(T-t)} + \left(\frac{S_t}{S_0} e^{-q(T-t)} \right)^{p(T-t)} \right]^{\frac{1}{p(T-t)}}, \tag{79}$$

where recall that $p(\tau) : \mathbb{R}^+ \mapsto [1, \infty]$ is required to be declining in $\tau \equiv T - t$ with $p(0) = \infty$ and $p(\infty) = 1$.

To enhance comparability with the benchmark BMS model, we introduce a volatility parameter, $\sigma > 0$, and a function of it, $I \equiv \sigma^2 (T - t)$, called integrated variance. As in the BMS model, we assume that the power-plus-prod model value of any max-prod affine payoff depends on $\sigma > 0$ only through $I \equiv \sigma^2(T-t)$. Under this assumption, the conventionally insured portfolio is valued as:

$$E_t^{\mathbb{Q}_-^p} e^{-r(T-t)} \left(\underline{R} \vee \frac{S_T}{S_0} \right) = \underline{R} e^{-r(T-t)} \oplus^{\tilde{p}(I)} \frac{S_t}{S_0} e^{-q(T-t)} \tag{80}$$

$$= \left[\left(\underline{R} e^{-r(T-t)} \right)^{\tilde{p}(I)} + \left(\frac{S_t}{S_0} e^{-q(T-t)} \right)^{\tilde{p}(I)} \right]^{\frac{1}{\tilde{p}(I)}}, \tag{81}$$

where $\tilde{p}(I)\mathbb{R}^+ \mapsto [1, \infty]$ is required to be declining in $I \equiv \sigma^2(T-t)$ with $p(0) = \infty$ and $p(\infty) = 1$.

5.3 BMS Pricing of a Conventionally Insured Portfolio

It is interesting to compare (81) with the BMS model valuation of a conventionally insured portfolio:

$$E_t^{\mathbb{Q}_-^{bs}} e^{-r(T-t)} \left(\underline{R} \vee \frac{S_T}{S_0} \right) = \underline{R} e^{-r(T-t)} N \left(\frac{\ln\left(\frac{S_0 \underline{R}}{S_t}\right) - (r - q - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right)$$

$$+ \frac{S_t}{S_0} e^{-q(T-t)} N \left(\frac{\ln\left(\frac{S_t}{S_0 \underline{R}}\right) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right). \tag{82}$$

This value function is also monotonicity and convexity-preserving in $\left[\dfrac{\underline{R}}{S_t/S_0}\right]$.

Figures 4 and 5 at the end of the paper compare the two model values of a conventionally insured portfolio . Figures 4 plots both model values against the floor $\underline{R}$. In this plot, the BMS variance rate was set to 1, while the power-plus variance rate was set at $\sigma^2 = 0.875$. The two values are virtually identical. Figure 5 also at the end of the paper is an error plot.

Recall that $\tilde{p}(I)$ is required to be declining in integrated variance $I \equiv \sigma^2(T-t)$ with $\tilde{p}(0) = \infty$ and $\tilde{p}(\infty) = 1$. To obtain how $\tilde{p}(I)$ depends on I, one can use moment matching, which involves equating the expected value of a convex function of S_T/S_0 in the power-plus-prod model with the BMS model value of the same payoff. Standard choices for the convex function of S_T/S_0 are $(S_T/S_0)^2$ or $(\ln(S_T/S_0))^2$ or $-2\ln(S_T/S_0)$. Since the payoff $\underline{R} \vee (S_T/S_0)$ is also convex in S_T/S_0, the two model values of the conventionally insured portfolio can also be used to determine how $\tilde{p}(I)$ depends on I.

Consider the special case $t = 0, \underline{R} = S_0 = 1$, and $r = q = 0$. The power-plus-prod model value (81) of the conventionally insured portfolio simplifies to $E_t^{\mathbb{Q}_-^p}(1 \vee S_T) = 2^{1/\tilde{p}(I)}$, while the BMS model value (82) simplifies to $E_t^{\mathbb{Q}_-^{bs}}(1 \vee S_T) = 2N\left(\frac{\sqrt{I}}{2}\right)$. Equating the two expressions and taking logs:

$$\frac{1}{\tilde{p}(I)} \ln 2 = \ln\left(2N\left(\frac{\sqrt{I}}{2}\right)\right). \tag{83}$$

Solving for $\tilde{p}(I)$:

$$\tilde{p}(I) = \frac{\ln 2}{\ln\left(2N\left(\frac{\sqrt{I}}{2}\right)\right)}. \tag{84}$$

Notice that the specification for $\tilde{p}(I)$ in (84) is declining in integrated variance $I \equiv \sigma^2(T-t)$ with $\tilde{p}(0) = \infty$ and $\tilde{p}(\infty) = 1$. When compared to previous feasible suggestions such as $\tilde{p}(I) = e^{\frac{1}{\sigma^2\tau}}$ or $\tilde{p}(I) = 1 + \frac{1}{\sigma^2\tau}$, (84) is a slightly more complicated specification, but has the virtue of treating σ identically across models in the special case of this paragraph. We will use the power function specification in (84) as our default choice going forward, even for payoffs very different from $1 \vee S_T$.

Recall that the power-plus-prod approach treats the current value of a conventionally insured portfolio as a pseudo-sum of the current value of its two underlying assets, $\underline{R}$ and $\frac{S_t}{S_0}$. We can also try to treat the BMS model value, given by the RHS of (82), as an alternative addition of $\underline{R}$ and $\frac{S_t}{S_0}$. This alternative addition will be closed and commutative but unfortunately, this alternative addition is not associative. As a result, pseudo-analysis is not applicable for the BMS model. Notice that the current price relative $\frac{S_t}{S_0}$ appears just once in (81), while it appears thrice in (82). As observed previously, this makes repeated optionality unwieldy. Moreover, (80) makes clear that the value of the European option to choose between a floor and a risky asset value is power-plus prod affine in the floor $\underline{R}$. the current price relative $\frac{S_t}{S_0}$. In the next subsection, we will show the advantages that a pseudo-affine value function brings, when we apply power-plus prod pricing to the repeated optionality present in a JIT insured portfolio.

6 Power-Plus Prod Pricing of a JIT Insured Portfolio

In this section, we show that the recursion (13) governing arbitrage-free values of a JIT insured portfolio is linear in the power-plus prod semi-field. We develop an analogy to the linear recursion governing the value of a bond paying N discrete coupons. This analogy is used to conjecture a formula for the value of a JIT insured portfolio giving N opportunities to one-time floor a daily price relative. This conjecture is verified in Appendix 4. We call our valuation formula for the JIT insured portfolio "open-form", because the number of terms in the formula grows by one each time that a new opportunity to one-time floor is added. The following two sections develop a simple closed-form valuation formula by simply assuming a positive dividend yield from the underlying portfolio, i.e. $q > 0$. In contrast to the open-form formula of this section, the computational complexity of our closed-form pricing formula for the JIT insured portfolio will be invariant to the number of flooring opportunities.

6.1 Analysis

Consider dividing the pseudo-linear pricing principle (25) by S_t:

$$E_t^{\mathbb{Q}-} e^{-r\triangle t}\left[\frac{A(t)}{S_t} \vee B(t)\frac{S_{t+\triangle t}}{S_t}\right] = E_t^{\mathbb{Q}-}\left[\frac{A(t)}{S_t}e^{-r\triangle t} \vee B(t)\frac{S_{t+\triangle t}}{S_t}e^{-r\triangle t}\right] = \alpha(t)e^{-r\triangle t} \oplus^{p(\triangle t)} B(t)e^{-q\triangle t}, \tag{85}$$

where $\alpha(t) \equiv \frac{A(t)}{S_t}$ and recall $p(\triangle t)$ is required to be declining in $\triangle t \geq 0$ with $p(0) = \infty$ and $p(\infty) = 1$.

Recall the optionality update recursion (13):

$$m_{n+1} = E^{\mathbb{Q}-}_{T-(n+1)\triangle t} e^{-r\triangle t} \left(\underline{R} \vee m_n \frac{S_{T-n\triangle t}}{S_{T-(n+1)\triangle t}} \right), \tag{86}$$

for $n = 0, 1, \ldots, N-1$, subject to $m_0 = 1$. Evaluating (85) at $t = T - (n+1)\triangle t$, $\alpha(t) = \underline{R}$, and $B(t) = m_n$, the pseudo-linear pricing principle in (85) implies that the recursion (86) is pseudo-linear:

$$m_{n+1} = \underline{R} e^{-r\triangle t} \oplus^{p(\triangle t)} m_n e^{-q\triangle t}, \tag{87}$$

subject to $m_0 = 1$.

Consider the valuation of a coupon bond paying a coupon of $\underline{R}$ each period for N periods and a final principal payment of 1 when the riskfree rate is constant at r. Let $B_t^{(n)}$ be the coupon bond price at time $t \in [0, T]$ when n coupons are remaining. This bond price is the unique solution to the linear recursion:

$$B^{(n+1)}_{T-(n+1)\triangle t} = \underline{R} e^{-r\triangle t} + B^{(n)}_{T-n\triangle t} e^{-r\triangle t}, \tag{88}$$

for $n = 0, 1, \ldots, N-1$ subject to $B_T^{(0)} = 1$. Comparing (87) with (122), the two problems become identical if we set $\oplus^{p(\triangle t)} = +$ and $q = r$ in (87).

The recursion (122) is linear in B and its solution will be linear in the forcing term $\underline{R} e^{-r\triangle t}$. The well-known bond pricing formula motivates the following conjecture for the solution of the pseudo-linear recursion (87):

$$m_n = \underline{R} e^{-r\triangle t} A_n \oplus^{p(\triangle t)} e^{-qn\triangle t}, \tag{89}$$

for $n = 0, 1, \ldots, N-1$, where A_n is defined by the "open-form" formula:

$$A_n \equiv 1 \oplus^{p(\triangle t)} e^{-q\triangle t} \oplus^{p(\triangle t)} e^{-2q\triangle t} \oplus^{p(\triangle t)} \cdots \oplus^{p(\triangle t)} e^{-(n-1)q\triangle t}. \tag{90}$$

Notice from (89) that the conjectured value for m_n is pseudo-affine in the present value of the floor, $\underline{R} e^{-r\triangle t}$ Appendix 5 uses induction to prove that (89) solves the pseudo-linear recursion (87).

The initial value of the JIT insured portfolio allowing any one of the N price relatives $\frac{S_{\triangle t}}{S_0}, \frac{S_{2\triangle t}}{S_{\triangle t}}, \ldots, \frac{S_T}{S_{(N-1)\triangle t}}$ to be floored at $\underline{R} > 0$ is given by:

$$J_0^{(N)} = m_N = \underline{R} e^{-r\triangle t} A_N \oplus^{p(\triangle t)} e^{-qT}, \tag{91}$$

since $T = n\triangle t$. Notice that the initial coupon bond value $B_0^{(N)}$ arises if we set $q = r$ and $\sigma = \infty$ in (91). Hence, we can treat coupon bond valuation as a special case of power-plus pricing of a JIT insured portfolio.

Setting $n = N$ in (90) and substituting in (91) implies $J_0^{(N)} =$:

$$\underline{R} e^{-r\triangle t} \oplus^{p(\triangle t)} \underline{R} e^{-r\triangle t - q\triangle t} \oplus^{p(\triangle t)} \underline{R} e^{-r\triangle t - 2q\triangle t} \oplus^{p(\triangle t)} \cdots \oplus^{p(\triangle t)} \underline{R} e^{-r\triangle t-(n-1)q\triangle t} \oplus^{p(\triangle t)} e^{-qT}. \tag{92}$$

Each term in (92) has a financial interpretation as the contribution to value due to optionality at the associated time. The first term, $\underline{R} e^{-r\triangle t}$, is the contribution to value due to the option to floor at $\triangle t$. The second term, $\underline{R}e^{-r\triangle t-q\triangle t}$, is the contribution to value due to the option to floor at $2\triangle t$. The last two terms, $\underline{R}e^{-r\triangle t-(n-1)q\triangle t} \oplus^{p(\triangle t)} e^{-qT}$, capture the contribution to value due to the European option to floor or not floor at T.

To illustrate the associativity of our valuation operation, consider the special case $N = 2$ and using commutativity, write the initial valuation formula as:

$$J_0^{(2)} = \underline{R}e^{-r\triangle t} \oplus^{p(\triangle t)} e^{-qT} \oplus^{p(\triangle t)} \underline{R}e^{-r\triangle t-q\triangle t}. \tag{93}$$

The two pseudo-additions in (93) can be done in either order:

$$(\underline{R}e^{-r\triangle t} \oplus^{p(\triangle t)} e^{-qT}) \oplus^{p(\triangle t)} \underline{R}e^{-r\triangle t-q\triangle t} = \underline{R}e^{-r\triangle t} \oplus^{p(\triangle t)} (e^{-qT} \oplus^{p(\triangle t)} \underline{R}e^{-r\triangle t-q\triangle t}). \tag{94}$$

The bracketed pseudo sum on the LHS of (94) is the initial value of a restricted JIT insured portfolio when only the first price relative $\frac{S_1}{S_0}$ can be floored at $\underline{R}$ at time $\triangle t$. Pseudo adding $\underline{R}e^{-r\triangle t-q\triangle t}$ to this bracketed pseudo sum captures the initial value impact of the additional optionality at $T = 2\triangle t$. The bracketed pseudo sum on the RHS of (94) is the initial value of a restricted JIT insured portfolio when only the second price relative $\frac{S_2}{S_1}$ can be floored at $\underline{R}$ at time $T = 2\triangle t$. Pseudo adding $\underline{R}e^{-r\triangle t}$ to this bracketed pseudo sum captures the initial value impact of the additional optionality at $\triangle t$. The LHS of (94) is a forward recursion capturing the initial value impact of adding optionality as maturity runs forward. The RHS is a backward recursion capturing the initial value impact of moving back the first date at which flooring becomes possible. In the BMS model, only this latter backward recursion is available. If the BMS analog of the LHS bracketed term in (94) is first calculated, then it is of no use when valuing the JIT insured portfolio with two opportunities to one time floor a price relative.

Note that if we were using ordinary addition $+$ instead of power-plus-addition $\oplus^{p(\triangle t)}$ in (90), there would be a simplification of the resulting finite geometric series:

$$\tilde{A}_n \equiv 1 + e^{-q\triangle t} + e^{-2q\triangle t} + \cdots + e^{-(n-1)q\triangle t} = \frac{1 - e^{-qn\triangle t}}{1 - e^{-q\triangle t}}, \tag{95}$$

valid for all $q = 0$. The next section shows that our finite geometric pseudo-series A_N defined by (90) with $n = N$ can be analogously simplified provided that $q > 0$.

7 Closed-form Formula for Finite Geometric Pseudo-Series

The valuation equation (89) holds for all $q \in \mathbb{R}$. In this section, we show that for $q > 0$, we can simplify the finite geometric pseudo-series:

$$A_N \equiv 1 \oplus^p e^{-q\triangle t} \oplus^p e^{-2q\triangle t} \oplus^p \cdots \oplus^p e^{-(N-1)q\triangle t} \equiv \bigoplus_{n=0}^{N-1^p} e^{-qn\triangle t}, \tag{96}$$

where recall N is a strictly positive integer. The simplification will be shown to be the one expected by replacing the ordinary subtraction operation $-$ with a pseudo-subtraction operation that we call power minus $\ominus^p$, i.e.

$$A_N = \frac{1 \ominus^p e^{-qT}}{1 \ominus^p e^{-q\triangle t}}, \qquad p \in [1, \infty], \tag{97}$$

where for $g_1 \geq g_2 \geq 0$, $g_1 \ominus^p g_2 \equiv (g_1^p - g_2^p)^{\frac{1}{p}}$ and $T = N\triangle t$.

Consider first the commutative semi-field $(\mathbb{R}^+; +, 0; \times, 1)$ which is the special case when $p = 1$ of the commutative semi-field $(\mathbb{R}^+; \oplus^p, 0; \times, 1), p \in [1, \infty]$. For an arbitrary element $g_2 \in \mathbb{R}^+$, there exists an element $1/g_2 \in \mathbb{R}^+$ such that $g_2 \times (1/g_2) = 1$. As a result, for any $g_1 \in \mathbb{R}^+$, $g_2 \in \mathbb{R}^+$, the binary operation $\div$ called division and defined by $g_1 \div g_2 \equiv g_1 \times (1/g_2)$ is well-defined. However, for an arbitrary element $g \in \mathbb{R}^+$, there does not necessarily exist an element $-g \in \mathbb{R}^+$ such that $g + (-g) = 0$. As a result, for any $g_1 \in \mathbb{R}^+$, $g_2 \in \mathbb{R}^+$, the binary operation $-$ defined by $g_1 - g_2 \equiv g_1 + (-g_2)$ is not well defined. Hence $(\mathbb{R}^+; +, 0; \times, 1)$ is not a field and so is called a proper commutative semi-field. Note however that for arbitrary elements $g_1 \in \mathbb{R}^+$ and $g_2 \in [0, g_1]$, the quantity $g_1 - g_2$ is a non-negative real. Motivated by this observation, suppose that for any $g_1 \in \mathbb{R}^+$, and $g_2 \in [0, g_1]$, we define a binary operation $-$ called subtraction by $g_1 - g_2 \equiv g_1 + (-g_2)$. Since the RHS is non-negative, subtraction and is a well-defined binary operation on our proper semi-field $(\mathbb{R}^+; +, 0; \times, 1)$ provided that the restriction $g_1 \geq g_2 \geq 0$ is holding, i.e the non-negative real being subtracted is smaller than or equal to the non-negative real being subtracting from. In particular, when the two non-negative reals are equal, we have:

$$g - g = 0.$$

Now consider the more general commutative semi-field $(\mathbb{R}^+; \oplus^p, 0; \times, 1), p \in [1, \infty]$. Suppose that for arbitrary elements $g_1 \in \mathbb{R}^+, g_2 \in [0, g_1]$, we define the power subtraction operation $\ominus^p, p \in [1, \infty]$ by:

$$g_1 \ominus^p g_2 \equiv (g_1^p - g_2^p)^{\frac{1}{p}}, \tag{98}$$

for $p \in [1, \infty]$. In particular when $p = \infty$, $g_1 \ominus^\infty g_2 = 0$ if $g_2 = g_1$ and $g_1 \ominus^\infty g_2 = g_1$ if $g_2 \in [0, g_1)$.

For $p \geq 0$, and $g_1 \geq g_2 \geq 0$, we have $g_1^p \geq g_2^p$, and hence $g_1 \ominus^p g_2 \in \mathbb{R}^+$. As a result, the power subtraction operation $\ominus^p, p \in [1, \infty]$ is still well-defined in our more general commutative semi-field $(\mathbb{R}^+; \oplus^p, 0; \times, 1), p \in [1, \infty]$, so long as the non-negative real being subtracted is smaller than or equal to the non-negative real being subtracting from. In particular, we have:

$$g \ominus^p g = 0, \qquad p \in [1, \infty]. \tag{99}$$

For a fixed positive integer N, the pseudo summation $\bigoplus_{n=0}^{N-1^p}$ in (96) is defined as:

$$A_N \equiv \bigoplus_{n=0}^{N-1^p} e^{-qn\triangle t} \equiv 1 \oplus^p e^{-q\triangle t} \oplus^p \ldots \oplus^p e^{-q(N-1)\triangle t}. \tag{100}$$

Multiplying both sides by $e^{-q\triangle t}$ yields:

$$e^{-q\triangle t} A_N = e^{-q\triangle t} \oplus^p \ldots \oplus^p e^{-q(N-1)\triangle t} \oplus^p e^{-qN\triangle t}. \tag{101}$$

Since $q > 0$, we have $e^{-q\triangle t} \in (0, 1)$, and hence the LHS of (101) is less than the LHS of (100). This allows us to power subtract (101) from (100):

$$A_N \ominus^p e^{-q\triangle t} A_N = 1 \ominus^p e^{-qN\triangle t}, \tag{102}$$

using (99) $N-1$ times. Factoring A_N out on the LHS, we can divide both sides by $1 \ominus^p e^{-q\triangle t} \in (0,1)$:

$$A_N = \frac{1 \ominus^p e^{-qT}}{1 \ominus^p e^{-q\triangle t}} \qquad p \in [1, \infty], \tag{103}$$

since $T = N\triangle t$, where recall from (98) that for $g_1 \geq g_2 \geq 0$:

$$g_1 \ominus^p g_2 \equiv (g_1^p - g_2^p)^{\frac{1}{p}} . \tag{104}$$

(103) is our closed-form formula for a finite geometric pseudo-series. Using the definition (110) of $\ominus^p$ in (103), the simplified formula for the finite geometric pseudo-series expressed using the ordinary subtraction operation $-$ is given by:

$$A_N = \left(\frac{1 - e^{-pqT}}{1 - e^{-pq\triangle t}} \right)^{\frac{1}{p}}, \qquad p \in [1, \infty]. \tag{105}$$

8 Closed-Form Pricing of Just-in-Time Insured Portfolio

Substituting (97) with $p = p(\triangle t)$ in (91), the initial value of our JIT insured portfolio simplifies to the following closed-form formula:

$$J_0^{(N)} = \underline{R} e^{-r\triangle t} \frac{1 \ominus^{p(\triangle t)} e^{-qT}}{1 \ominus^{p(\triangle t)} e^{-q\triangle t}} \oplus^{p(\triangle t)} e^{-qT}, \tag{106}$$

where eg. $p(\triangle t) = 1 + \frac{1}{\sigma^2 \triangle t}$ or $p(\triangle t) = \frac{\ln 2}{\ln\left(2N\left(\frac{\sigma\sqrt{\triangle t}}{2}\right)\right)}$.

Consider (106) as $\sigma \uparrow \infty$ so that $p \downarrow 1$:

$$\lim_{p \downarrow 1} J_0^{(N)} = \underline{R} e^{-r\triangle t} \frac{1 - e^{-qT}}{1 - e^{-q\triangle t}} + e^{-qT}, \tag{107}$$

This is the value of a portfolio where continuously paid proportional dividends have been swapped for discretely paid lagged proportional dividends. More precisely, the dollar size of the discrete dividend paid at the discrete time $i\triangle t$ is a constant fraction $\underline{R} > 0$ of the last period price relative $\frac{S_{(i-1)\triangle t}}{S_0}$.

By changing the numeraire from par bonds worth 1 to dividend-paying shares also worth 1, the valuation of the lagged proportional dividend-paying portfolio reduces to valuing a discretely-paying coupon bond under a constant discount rate. By changing both the numeraire from bonds to shares and the addition operation from $+$ to $\oplus^p$, the valuation of the JIT insured portfolio also reduces to valuing a discretely-paying coupon bond under a constant discount rate.

Recall that we arrived at (106) by solving a recursion arising by thinking of a JIT insured portfolio as a single period married put written on a deferred JIT insured portfolio:

$$J_0^{(N)} = E_0^{\mathbb{Q}-} e^{-r\triangle t} [\underline{R} \vee J_{\triangle t}^{(N-1)}]. \tag{108}$$

However, (106) suggests a different interpretation most easily seen if it is rewritten as:

$$J_0^{(N)} = \underline{R} e^{-r\triangle t} \frac{1 \ominus^{p(\triangle t)} e^{-qT}}{1 \ominus^{p(\triangle t)} e^{-q\triangle t}} \oplus^{p(\triangle t)} e^{-q(T-\triangle t)} e^{-q\triangle t}. \tag{109}$$

(109) implies that the JIT insured portfolio with N opportunities to one-time floor each periodic price relative at $\underline{R}$ has the same initial value as a single period married put struck at $\underline{R}\frac{1\ominus^{p(\triangle t)} e^{-qT}}{1\ominus^{p(\triangle t)} e^{-q\triangle t}}$ and written on $e^{-q(T-\triangle t}$ units of the price relative:

$$E_0^{\mathbb{Q}-} e^{-r\triangle t} \left[\underline{R} \frac{1 \ominus^{p(\triangle t)} e^{-qT}}{1 \ominus^{p(\triangle t)} e^{-q\triangle t}} \vee e^{-q(T-\triangle t)} \frac{S_{\triangle t}}{S_0} \right] = J_0^{(N)}.$$

The associativity property of our valuation operator is allowing this dual view of a single period married put with two different strikes, notionals, and underlyings. The latter view describes the initial cost of the rolling near-term married put replicating strategy for a JIT insured portfolio. Recall that this replicating strategy holds more generally, but in general requires solving a nonlinear recursion numerically. In contrast, the dynamical restrictions of this section render the non-linear recursion pseudo-affine and hence explicitly solvable. As a result, the married put holdings and strike price can be written in closed form.

Recall the definitions of $\oplus^p$ and $\ominus^p$ for generic elements $g_1 \in \mathbb{R}^+$ and $g_2 \in \mathbb{R}^+$:

$$\begin{aligned} g_1 \quad \oplus^p \quad g_2 &\equiv (g_1^p + g_2^p)^{\frac{1}{p}} \\ g_1 \quad \ominus^p \quad g_2 &\equiv (g_1^p - g_2^p)^{\frac{1}{p}}, g_1 \geq g_2 \geq 0. \end{aligned} \tag{110}$$

Using these definitions of $\oplus^p$ and $\ominus^p$ in (124), our closed-form formula for the initial value of the insured portfolio expressed using the ordinary addition operation $+$ and the ordinary subtraction operation $-$ becomes:

$$J_0^{(N)} = \left(\underline{R}^{p(\triangle t)} e^{-rp(\triangle t)\triangle t} \frac{1 - e^{-p(\triangle t)qT}}{1 - e^{-p(\triangle t)q\triangle t}} + e^{-qp(\triangle t)T} \right)^{\frac{1}{p(\triangle t)}}, \tag{111}$$

where eg. $p(\triangle t) = 1 + 1/(\sigma^2 \triangle t)$ or $p(\triangle t) = \frac{\ln 2}{\ln\left(2N\left(\frac{\sigma\sqrt{\triangle t}}{2}\right)\right)}$. We are assuming in (111) that $\underline{R} > 0$, $r \in \mathbb{R}, q > 0$, and $\triangle t = T/N$ for $T > 0, N = 1, 2 \ldots$ Formula (111) is the main result of this paper. It is a simple exact closed-form arbitrage-free formula for the initial value of the JIT insured portfolio, where any one of N price relatives can be floored just after it is realized. Its computational effort is invariant to the number N of opportunities to one-time floor a price relative.

Perpetual just-in-time portfolio insurance arises from its finite-lived counterpart by letting $T \uparrow \infty$, holding $\triangle t$ fixed. Since $\lim_{T\uparrow\infty} e^{-qT} = 0$ for $q > 0$, the initial value of a perpetual JIT insured portfolio is:

$$J_0^{(\infty)} = \frac{\underline{R} e^{-r\triangle t}}{1 \ominus^{p(\triangle t)} e^{-q\triangle t}} = \frac{\underline{R} e^{-r\triangle t}}{(1 - e^{-p(\triangle t)q\triangle t})^{\frac{1}{p(\triangle t)}}}. \tag{112}$$

Returning to the finite-lived case, our arbitrage-free model can be used to determine *critical daily price relatives*, as well as the JIT insured portfolio value. In our pseudo-analytic approach, all of the $N-1$ critical daily price relatives can be explicitly determined in closed form as functions of the inputs. Substituting (89) in (16)), we have:

$$R^*_{T-n\triangle t} = \frac{\underline{R}}{m_n} = \frac{\underline{R}}{\underline{R}e^{-r\triangle t}A_n \oplus^{p(\triangle t)} e^{-qn\triangle t}}, \tag{113}$$

where $A_n = \frac{1\ominus^{p(\triangle t)}e^{-qn\triangle t}}{1\ominus^{p(\triangle t)}e^{-q\triangle t}} = \left(\frac{1-e^{-p(\triangle t)qn\triangle t}}{1-e^{-p(\triangle t)q\triangle t}}\right)^{\frac{1}{p(\triangle t)}}$ for $n = 0, 1 \ldots N-1$.

The initial par floor, $\underline{R}^*$ is the unique value of the floor $\underline{R}$ which causes the initial value of the JIT insured portfolio to equal one. Setting $J_0^{(N)}$ in (111) equal to 1 and replacing $\underline{R}$ with $\underline{R}^*$ implies:

$$\left(\underline{R}^* e^{-r\triangle t}\right)^{p(\triangle t)} \frac{1-e^{-p(\triangle t)qT}}{1-e^{-p(\triangle t)q\triangle t}} + e^{-qp(\triangle t)T} = 1. \tag{114}$$

Solving for $\underline{R}^*$:

$$\underline{R}^* = e^{r\triangle t}\left(1-e^{-p(\triangle t)q\triangle t}\right)^{\frac{1}{p(\triangle t)}} > 0. \tag{115}$$

When $\underline{R} = \underline{R}^*$, the dividends surrendered from a one dollar investment in a JIT insured portfolio are perfectly compensated for by the option to floor one of the N lagged price relatives. When $\underline{R} = \underline{R}^*$, then $m_n = 1$ for all $n = 1, \ldots N-1$ and hence by (116) the value of the JIT insured portfolio matches the value of the uninsured portfolio at each discrete time, i.e.:

$$J^{(n+1)}_{T-(n+1)\triangle t} = \frac{S_{T-(n+1)\triangle t}}{S_0}. \tag{116}$$

9 Valuing a European Call Written on a JIT Insured Portfolio

Let I index an intermediate integer between 0 and N. Let:

$$J^{(N-I)}_{I\triangle t} = \frac{S_{T-I\triangle t}}{S_0} m_{N-I} \tag{117}$$

be the continuation value of the JIT insured portfolio at period I with $N-I$ flooring opportunities left. Consider the payoff from a European call maturing at period I written on $J^{(N-I)}_{I\triangle t}$ and struck at $K > 0$:

$$C_{I\triangle t} = (J^{(N-I)}_{I\triangle t} - K)^+. \tag{118}$$

Substituting (117) in (118) implies that the call payoff decomposes as:

$$C_{I\triangle t} = \left(\frac{S_{T-I\triangle t}}{S_0} m_{N-I} - K\right)^+ = \left(K \vee \frac{S_{T-I\triangle t}}{S_0} m_{N-I}\right) - K. \tag{119}$$

The initial power-plus-prod value of this call payoff is:

$$C_0 = (Ke^{-rI\triangle t} \oplus^{p(I\triangle t)} m_{N-I}e^{-qI\triangle t}) - Ke^{-rI\triangle t}. \tag{120}$$

Recall that m_{N-I} is essentially a bond value calculated using $p(\triangle t)$. This shows that we can mix two time scales, viz $\triangle t$ and $I\triangle t$. In fact, since the ordinary minus sign in (120) is just $\ominus^{p(\infty)}$, (120) mixes three times scales.

10 Link to Pseudo-Linear Pseudo-Difference Equations

When $\underline{R} > \underline{R}^*$, the convexity measures m_n behave like premium coupon bond values whose value rises with n, the number of remaining coupons. Since $m_{n+1} > m_n$, the pseudo-linear recursion (87) can be re-arranged into the following first order inhomogeneous pseudo-linear ordinary pseudo-difference equation:

$$\left(m_{n+1} \ominus^{p(\triangle t)} m_n\right) \oplus^{p(\triangle t)} \left(1 \ominus^{p(\triangle t)} e^{-q\triangle t}\right) m_n = \underline{R}e^{-r\triangle t}1(n = 0, 1, \ldots, N-1), \tag{121}$$

subject to $m_0 = 1$ and $q > 0$. The RHS in (121) is a Heaviside forcing term.

Recall that the coupon bond price $B^{(n)}$ with n coupons of size $\underline{R} > 0$ remaining solves the linear recursion (122), which can be analogously re-arranged into the following first order inhomogeneous linear ordinary difference equation::

$$B^{(n+1)} - B^{(n)} + (1 - e^{-r\triangle t})B^{(n)} = \underline{R}e^{-r\triangle t}1(n = 0, 1, \ldots, N-1), \tag{122}$$

subject to $B^{(0)} = 1$. This problem can be solved for $B^{(n)}$ by first finding a Green's function $G^{(n,m)}$ and then representing the solution as a sum over m. When a coupon bond value is treated as the solution to a first order inhomogeneous linear ordinary difference equation, the Green's function is just the value of a zero coupon bond. Hence, the theory implies the obvious result that a coupon bond has the same value as a weighted sum of zero coupon bonds.

Suppose we write (121) using conventional binary operations:

$$\left(m_{n+1}^{p(\triangle t)} - m_n^{p(\triangle t)}\right) + \left(1 - e^{-q\triangle t p(\triangle t)}\right) m_n^{p(\triangle t)} = \left(\underline{R}e^{-r\triangle t}1(n = 0, 1, \ldots, N-1)\right)^{p(\triangle t)}. \tag{123}$$

This is a linear first order ordinary difference equation with forcing for $m^{p(\triangle t)}$. As is well known, the general solution is a linear combination of a particular solution and a solution to the complementary homogeneous equation. The solution can be written as

$$m_n = \underline{R}e^{-r\triangle t}\frac{1 \ominus^{p(\triangle t)} e^{-qn\triangle t}}{1 \ominus^{p(\triangle t)} e^{-q\triangle t}} \oplus^{p(\triangle t)} e^{-qn\triangle t}, \tag{124}$$

The last term, $e^{-qn\triangle t}$,which is the PV of the final payoff is the solution to the homogeneous ordinary difference equation. The first term, which captures the flooring options is a particular solution.

Consider the non-linear Bernoulli ODE:

$$y'(x) + q(x)y(x) = f(x)y^{\pi}(x), \tag{125}$$

where the constant power π on the RHS is real. Setting $\pi = 1 - p$ and dividing by $y^{1-p}(x)$:

$$y^{p-1}(x)y'(x) + q(x)y^p(x) = f(x).$$

The first term on the LHS is proportional to $\frac{d}{dx}y^p(x)$. Making this substitution simplifies it to:

$$\frac{d}{dx}y^p(x) + pq(x)y^p(x) = (p^p f^p(x))^{\frac{1}{p}}. \tag{126}$$

Comparing (123) with (126), we see that we have been working with a finite difference approximation of a Bernoulli ODE. Just as the non-linear problem (123) governing m_n can be treated as a pseudo-linear ordinary pseudo-difference equation (121), the classically non-linear Bernoulli ODE (125) governing $y(x)$ can be treated as a pseudo-linear ordinary pseudo-derivative equation in the algebra $(\mathbb{R}^+, \oplus^p, 0; \times, 1)$. Pap's g calculus with generating function $g(x) = x^p$ applies in the latter case. For $y(x)$ having the same monotonicity as x^p, let $\frac{ⓓ^p y}{dx}(x) \equiv \left(\frac{d}{dx} y^p(x)\right)^{\frac{1}{p}}$. Raising both sides of (126) to the power p implies:

$$\frac{ⓓ^p y}{dx}(x) \oplus^p [p^{\frac{1}{p}} q^{\frac{1}{p}}(x) y(x)] = [p f(x)]^p. \tag{127}$$

The Green's function for the Bermudan JIT insured portfolio value arises by replacing the RHS Heaviside function in (121) by a Kronecker delta function. The resulting Green's function solution is the value of a European conventionally insured portfolio embedding a single standardized choice at a specified time. As a result, the solution for the value of the many varying size choices embedded in a Bermudan JIT insured portfolio is a weighted pseudo-sum of European conventionally insured portfolio values. When $n = N$, the solution to the pseudo initial value problem (121) is (92), which is repeated here: $m_N = J_0^{(N)} =:$

$$\underline{R}e^{-r\triangle t} \oplus^{p(\triangle t)} \underline{R}e^{-r\triangle t - q\triangle t} \oplus^{p(\triangle t)} \underline{R}e^{-r\triangle t - 2q\triangle t} \oplus^{p(\triangle t)} \cdots \oplus^{p(\triangle t)} \underline{R}e^{-r\triangle t-(n-1)q\triangle t} \oplus^{p(\triangle t)} e^{-qT}. \tag{128}$$

Note that when volatility is infinite, then $\oplus^{p(\triangle t)} = +$. The JIT insured portfolio has the same value as a fictitious derivative security converting a stock paying constant proportional dividends continuously at rate $q > 0$ into a claim paying discrete proportional dividends every $\triangle t$ with non-annualized dividend yield $\underline{R}e^{-r\triangle t}$ along with a liquidating payout of one share at T. This is similar to a buy and hold in the SPY ETF considering that it pays out the dividends from the 500 stocks in the S&P 500 quarterly. The analogy to this less conventional claim is even more direct than for a coupon bond.

The link in this section to the simplest type of difference equation suggests many avenues for future research. For example, we arrived at the initial value of an N period JIT insured portfolio by propagating calendar time backward. In each time step, the dependent variable was the pseudo sum of all future flooring opportunities and we determined how it changes as we step back one period in calendar time. An alternative approach starts with the initial value of a single period JIT insured portfolio and increases the number of one-time flooring opportunities up from one. Let $m \in [1, 2, \ldots N$ be the maturity date of a JIT insured portfolio. Suppose we know the initial value of an m period JIT insured portfolio written on \$1 notional of some risky portfolio. To increase m by one, the underlying is changed from a fixed number of shares to one married put written on those shares and maturing a period later. The adjoint of the Green's function for the difference equation corresponding to backward recursion is a Green's function for the difference equation corresponding to forward recursion. It can alternatively be used to find a formula for the initial value of an N period JIT insured portfolio. The adjoint problem is also useful for the inverse problem of determining coefficients such as dividend yield from observed market prices of JIT insured portfolios of varying maturities. One can alternatively use root finding to determine the volatility embedded in the power by switching back to the use of ordinary addition and subtraction.

The reason we could find a simple closed-form solution to the ordinary pseudo-linear pseudo-difference equation (121) is that it has constant coefficients. This suggests that we should also be able to find a solution to a *partial* pseudo-linear pseudo-difference equation with constant coefficients. One has to use the same pseudo-subtraction operators to define partial differences in the new space variable(s) as was used to define partial differences in time. As a result, the power p or variance rate σ^2 becomes a macro variable/hyper parameter eg. clock speed, rather than a property of just one of the components.

It would also be straightforward to introduce time-dependent coefficients and a time-dependent forcing term into the backward in calendar time ordinary pseudo-linear pseudo-difference equation (121). One should still be able to express the solution as a pseudo-sum using Green's functions, variation of parameters, or generating functions. The sums arising in other swaps besides interest rate and dividend swaps can be re-interpreted as a series of choices instead of a series of cash flows. Deterministic first order linear difference equations have been extended to stochastic first order linear difference equations, particularly in times series analysis. It would be less straightforward to extend the pseudo-analog of the former to the pseudo-analog of the latter since stochastic first order pseudo-linear pseudo-difference equations are unexplored to this author's knowledge.

11 Summary and Future Research

We used pseudo-analysis to obtain a fully explicit formula for the initial value of a JIT insured portfolio, allowing any one of N future price relatives to be floored. We also obtained fully explicit formulas for the first critical price relative and for the par floor. The formulas are simple and in closed-form, which means the formulas have a computational effort that is invariant to N, the number of lagged price relatives for which one can be floored.

The key idea allowing the development of these closed-form formulas was to treat the optionality in JIT portfolio insurance as a generalized addition. When associativity is required of our new addition, one must depart from the benchmark BMS model, but there do exist supporting arbitrage-free dynamics, which moreover produce a realistic implied volatility surface. The absence of arbitrage implies that option pricing formulas are necessarily non-linear in their strike price, but they are allowed to be pseudo-affine in strike. When they are, an option on an option is valued as a pseudo-affine function of a pseudo-affine function, which is again pseudo-affine. This principle leads to simple closed-form pricing formulas for the Bermudan options embedded in JIT portfolio insurance. More generally, this observation opens up the application of pseudo-linear algebra to dynamic programming problems in finance.

Note that standard lower and upper bounds on Bermudan values arise by setting $p = \infty$ and $p = 1$ respectively. This is analogous to lower and upper bounds on arbitrage-free European option values arising from setting the volatility in the BMS model to 0 and infinity respectively. This observation suggests that just as the BMS model is the canonical language for European-style non-linear payoffs typified by European options, one can treat our pseudo-plus valuation as merely a convenient language for the Bermudan-style derivatives typified by JIT portfolio insurance. Just as stochastic volatility can be introduced to achieve more realistic valuations of European-style non-linear payoffs, one can attempt to relax the independence of discrete-time increments of log price in our power plus prod model.

Obviously, the ideas underlying finite geometric series extend in many ways. By shrinking the time $\triangle t$ between exercise opportunities and the magnitude of each exercise opportunity appropriately, the value of an American-style JIT insured portfolio can potentially be obtained from the value of the Bermudan-style JIT insured portfolio. One must investigate whether the resulting pseudo-integral exists.

In general, a sum product of a difference ratio telescopes into a difference of the end points of the dependent variable. The fundamental theorem of calculus is just a limiting form of this observation, which extends via Green's/Stokes' theorem to multiple dimensions. The pseudo-analytic version of these results replaces the ordinary sum and/or ordinary product with a pseudo-sum and/or pseudo-product. These replacement(s) permit a wider scope of application of the addition and/or multiplication used to define a sum-product. When the discrete fundamental theorem of calculus is applied to the exponential function, the result is the well known geometric series. One can replace difference ratios with other linear operators and replace exponential functions with other eigenfunctions.

Future research should focus on:

- calculating the dynamics in other supporting models besides local vol.
- Our hedging strategy for a JIT insured portfolio involved rolling short-term European-style married puts. One can alternatively explore supplementing or replacing married puts with crash cliquets.
- The main idea of this paper is that European option valuation in the power plus prod framework is analogous to portfolio composition. The initial value given for the right to choose at T between a zero coupon bond paying K and a dividend paying stock currently worth S is $Ke^{-rT} \oplus^p (T) Se^{-qT}$. The initial value if given the sum of the two at T instead is $Ke^{-rT} + Se^{-qT}$. Now suppose that the underlying is an individual stock with some idiosyncratic and diversifiable risk. By writing JIT policies on many names simultaneously, the standard math used to evaluate the effectiveness of diversification becomes available in this options context. JIT insurance considered how to think of multiple options spread out in time, but one can equally consider this analysis in the cross section and over a single period for simplicity. Suppose that we also zero out carrying costs for simplicity. An analog of the variance of return as a measure of the riskiness of return on a single asset in our context is:

$$\ln E^{\mathbb{Q}} \left(1 \vee \frac{S_T}{S_0} \right) = \frac{\ln 2}{p(I)},$$

  where recall I is integrated variance. Since p is declining in I, our proposed risk measure is increasing in it. Suppose that we now replace $\frac{S_T}{S_0}$ with $\frac{1}{2}\frac{S_{1T}}{S_{10}} + \frac{1}{2}\frac{S_{2T}}{S_{20}}$ where $\frac{S_{1T}}{S_{10}}$ and $\frac{S_{2T}}{S_{20}}$ are IID and have the same risk as $\frac{S_T}{S_0}$. An ATM basket option has lower value than a basket of ATM options, so our proposed risk measure declines due to diversification. However, if we replace $\frac{S_T}{S_0}$ with $\frac{1}{2}\frac{S_{1T}}{S_{10}} \vee \frac{1}{2}\frac{S_{2T}}{S_{20}}$, then our proposed risk measure does not decline.

- under zero rates and dividends, our valuation formula for a married put $V_p(K,S) = (K^p + S^p)^{\frac{1}{p}}$, is both an L^p norm of the 2 vector $\begin{bmatrix} K \\ S \end{bmatrix}$ and a pseudo sum $V_p(K,S) = K \oplus^p S$. Its convex conjugate $V_q^*(V_K, V_S)$ is an L^q norm of the 2 vector $\begin{bmatrix} V_K \\ V_S \end{bmatrix}$, where subscripts denote partial derivatives and $\frac{1}{q} + \frac{1}{p} = 1$. Hence, its convex conjugate is also a pseudo-sum, $V_q^*(V_K, V_S) = V_K \oplus^q V_S$. The order reversing property of convex conjugation requires that as we increase τ from 0 to ∞ and correspondingly run p backward from ∞ to 1, we must run q forward from 1 to ∞. The reader can check that if $p(\tau) = 1 + \frac{1}{\sigma^2 \tau}$, i.e. one plus precision, then $q(t) = 1 + \sigma^2 \tau$, i.e. one plus remaining quadratic variation. Some problems may be easier to solve in the dual, eg. optimal stopping for an American option arises when a value slope equals ± 1. Also the physical fact that calendar time runs forward/time to maturity runs backward may be more easily accomodated in the dual than the primal.

- valuing American JIT insured portfolio value by letting $N \uparrow \infty$ for fixed T. One starts with the initial value of the conventionally insured portfolio, replaces it with the initial value of the $N = 2$ JIT insured portfolio, replaces that with the initial value of the $N = 3$ JIT insured portfolio, etc. Each additional flooring opportunity involves one more pseudo-addition, but the quantities being pseudo-added are smaller. Surprisingly, the absence of arbitrage alone is insufficient to guarantee that an $N + 1$ JIT insured portfolio does not have lower value than a co-terminal N JIT insured portfolio. However it is reasonable to conjecture that this result is true under power-plus prod valuation.

  Notice that if $p(\triangle t) = 1$ in the geometric series, then the straight sum of the PV's will blow up as $N \uparrow \infty$. However if $p(\triangle t) = \infty$, then the max of the PV's will not blow up as $N \uparrow \infty$. Hence, the mathematical question is how to let $p(\triangle t)$ depend on $\triangle t$ to prevent blowup as $N \uparrow \infty$.

- The valuation has been Markov in the price S and the lagged price. One can add another state variable, eg. a crash indicator, to understand better how JIT portfolio insurance differs from conventional portfolio insurance.

- Additional spatial state variables can be instantaneous variance rate V or its time integral, the running quadratic variation. We can interpret the current setting as arising in an independent stochastic volatility setting after conditioning on the path of V. Hence, the deterministic function $p(\tau) = 1 + \frac{1}{\int_0^\tau V(t)dt}$ captures how one plus precision depends on the V path after conditioning on $V_t = V(t)$ for $t \in [0, \tau]$. The value of a European married put is then a risk neutral expectation, $E^{\mathbb{Q}-}V_p(S,K)$, where $p(\tau) = 1 + \frac{1}{\int_0^\tau V_t dt}$ is the sole random variable. In an exponential Lévy setting, we can replace the halved instantaneous variance rate $V/2$ by the positive gap between the uncompounded net return $\frac{F}{S_0} - 1$ and the continuously compounded rate of return $\ln\left(\frac{F}{S_0}\right)$, needed to move to a given value F starting from S_0.

- So far, the analysis has involved only one risky asset. Consider a financial product whose payoff is the mathematical product of two married put payoffs:

$$\Pi_T(K_1, K_2) = (K_1 \vee S_{1T})(K_2 \vee S_{2T}).$$

When S_{1T} and S_{2T} are independent random variables, the price of the product is the product of the prices:

$$E^{\mathbb{Q}_-}(K_1 \vee S_{1T})(K_2 \vee S_{2T}) = E^{\mathbb{Q}_-}(K_1 \vee S_{1T})E^{\mathbb{Q}_-}(K_2 \vee S_{2T}).$$

Whether or not S_{1T} and S_{2T} are independent random variables, the cross partial of the product payoff is the following binary payoff:

$$\frac{\partial^2}{\partial K_1 \partial K_2}\Pi_T(K_1, K_2) = 1(S_{1T} < K_1, S_{2T} < K_2)$$

Hence under zero carrying costs, the price of the binary payoff is a bivariate CDF:

$$\frac{\partial^2}{\partial K_1 \partial K_2}\Pi_0 = \frac{\partial^2}{\partial K_1 \partial K_2}E^{\mathbb{Q}_-}(K_1 \vee S_{1T})(K_2 \vee S_{2T}) = \mathbb{Q}_-\{S_{1T} < K_1, S_{2T} < K_2\}.$$

By picking a copula, one can create a bivariate Burr/Dagum CDF with Burr/Dagum marginals. For example, the following bivariate Burr III CDF has attractive properties:

$$\mathbb{Q}_-\left\{\frac{S_{1T}}{S10} < k_1, \frac{S_{2T}}{S_{20}} < k_2\right\} = F_-(k_1, k_2) \equiv \left(1 + k_1^{-p} + k_2^{-p} + ck_1^{-p\alpha}k_2^{-(1-\alpha)p}\right)^{\frac{1-p}{p}},$$

where $c > 0$ and $\alpha \in (0, 1)$. The function $h(k_1, k_2) = k_1^{-p} + k_2^{-p} + ck_1^{-p\alpha}k_2^{-(1-\alpha)p}$ solves the homogeneous linear PDE:

$$\frac{k_1}{p}\frac{\partial h}{\partial k_1} + \frac{k_2}{p}\frac{\partial h}{\partial k_2} + h = 0,$$

which is a key step in showing that $F_-(k_1, k_2)$ solves the bivariate Bernoulli (nonlinear) PDE:

$$\frac{\partial \ln F_-}{\partial \ln k_1} + \frac{\partial \ln F_-}{\partial \ln k_2} = (p-1)(1 - F_-^{\frac{p}{p-1}}).$$

Ideally, one can use this bivariate Burr CDF to eg. value a married put maturing at T_1 whose payoff $K_1 \vee M_1$ has an underlying risky asset value M_1 which is the power plus prod model value at time T_1 of a a married put maturing at $T_2 > T_1$ and paying off $K_2 \vee S_{T_2}$ then.

- One can introduce the notion of matrix multiplication in the power plus prod algebra. This would allow the solution of a system of pseudo linear equations. For example, one can imagine equating continuation value to exercise value for a bivariate twice exerciseable married put to find a pair of critical stock prices.

- We have shown that there is an analogy between the value impact of adding N opportunities to one-time floor the local price relative of a portfolio with continuously paid constant dividend yield and the value impact of adding N *discrete* coupons to a zero coupon bond assuming a constant interest rate. This analogy can be used in at least two ways. First, the coupon bond price solves a first order linear ordinary difference equation in time to maturity with a forcing term and a unit initial condition. We expect its pseudo-analytic analog to be pseudo-linear, and hence a non-linear ordinary difference equation, which we can solve. By letting $\triangle t \downarrow 0$, the pseudo-linear ordinary difference equation should reduce to a pseudo-linear ODE. Second, one can try to exploit to exploit other results from fixed income. For example, pseudo duration should measure dividend yield sensitivity, while pseudo convexity motivates trading butterflies when dividend yield becomes stochastic. We can go beyond bond valuation by changing the payoff, the interest rate dynamics, or both. For example, still assuming a constant interest rate, one can also value a coupon bond in closed form, when the coupons are affine in time rather than constant. The option analog can be explored.

  A caveat for this analogy with fixed income pricing is that in fixed income, the valuation methodology for instruments with both positive and negative coupons does not differ from the valuation methodology for instruments such as bonds with positive coupons only. In contrast, the methodology of this paper may not apply to extensions of portfolio insurance with optionality both added and subtracted and whose magnitudes are unknown ex-ante.

- Contrast the power norm of g_1 and g_2, $(g_1^p + g_2^p)^{\frac{1}{p}}$ with the weighted power mean of g_1 and g_2, $(q_1 g_1^p + q_2 g_2^p)^{\frac{1}{p}}$ where q_1 and q_2 are probabilities, i.e. $q_1 \geq 0, q_2 \geq 0, q_1 + q_2 = 1$. A Kolmogorov mean or generalized mean of a random variable is a generalization of this weighted power mean $g^{(-1)}(Eg(X))$. Now consider the standard deviation of a de-meaned random variable X:

$$\text{Std}(X) = \sqrt{E[X^2]},$$

  or the certainty-equivalent of a random payoff X:

$$CE = U^{-1}(E^{\mathbb{P}}U(X)), \text{ for } U \text{ increasing}$$

  or the yield-to-maturity of a zero coupon bond in a short interest rate model:

$$y_t(T) = -\frac{1}{T-t} \ln E_t^{\mathbb{Q}-} \exp\left(-\int_t^T r_u du\right),$$

  or the (Black Scholes) implied variance rate in an uncorrelated short variance rate model:

$$I_t^2(K) = BS^{-1}\left(E_t^{\mathbb{Q}-} BS\left(\frac{1}{T-t}\int_t^T \sigma_u^2 du\right)\right).$$

- Similar payoffs, eg. compound options, extendible options, convertible bonds.
- Very different payoffs, eg. product of time to maturity and standard BM

- extension to more than one underlying asset, eg. Bermudan basket options or quality and timing options.

- Change the function $p(\tau)$ controlling the term structure of European option prices, eg. to $1 + 1/(e^{\sigma^2\tau} - 1)$.

- Change the generator g away from $g(a) = a^p, p \in [1, \infty]$. This changes the strike structure of European option prices. Starting from the commutative semi-field $(\mathbb{R}^+; +, 0; \times, 1)$, introduce an (ideally explicitly) invertible map $g : \mathbb{R}^+ \mapsto S$. For arbitrary elements $a_1 \in S, a_2 \in S$, define:

$$\begin{aligned} a_1 \oplus_{g^-} a_2 &= g^{-1}(g(a_1) + g(a_2)) \\ a_1 \otimes_{g^{-1}} a_2 &= g^{-1}(g(a_1) \times g(a_2)) \end{aligned} \tag{129}$$

  Then $(S; \oplus_{g^{-1}}, g^{-1}(0); \otimes_{g^{-1}}, g^{-1}(1))$ is a commutative semi-field. When g is not a power function, $\otimes_{g^{-1}}$ will not reduce to ordinary multiplication. There may still be an application of $\otimes_{g^{-1}}$, but it will no longer be repeated ordinary addition, as it is when $g(x) = x^p$ for $x \geq 0$ and $p \in [1, \infty]$.

  Here is a well known example of a generator other than a power function. Let $\mathbb{R}_{-\infty} \equiv \mathbb{R} \cup \{-\infty\}$ be the extended reals. If $g(a) = e^{pa}, p \in [1, \infty]$, then $S = \mathbb{R}_{-\infty}$,

$$\begin{aligned} a_1 \oplus_\ell^p a_2 &\equiv \frac{1}{p} \ln(e^{pa_1} + e^{pa_2}) \\ a_1 \otimes_\ell^p a_2 &\equiv a_1 + a_2 \end{aligned} \tag{130}$$

  Hence, $(\mathbb{R}_{-\infty}; \oplus_\ell^p, -\infty; +, 0)$ is obtained from the positive semi-field $(\mathbb{R}^+; +, 0; \times, 1)$ via the generator $g(a) = e^{pa}, p \in [1, \infty]$. Note that when $p = \infty$, max plus algebra arises.

- For both conventional and JIT portfolio insurance, the semi-field used to describe the terminal payoff was max prod over non-negative reals $(\mathbb{R}^+; \vee, 0; \times, 1)$. Write this as $(\mathbb{R}^+; \oplus, 0; \otimes, 1)$. Suppose we use an increasing and continuous generator g to switch the pair of binary operations. The interesting point is that the binary operation $\vee$ will not change. To illustrate, suppose $g(a) = e^a$. Then

$$\begin{aligned} a_1 \oplus_g a_2 &\equiv \ln(e^{a_1} \vee e^{a_2}) = a_1 \vee a_2 \\ a_1 \otimes_g a_2 &\equiv \ln(e^{a_1} \times e^{a_2}) = a_1 + a_2 \end{aligned} \tag{131}$$

  The new semi-field is $(\mathbb{R}_{-\infty}; \vee, -\infty; +, 0)$. As a result, one could redo the results of this paper when the underlying portfolio value is real-valued instead of non-negative. Switching log and exponential, one can also redo the results of this paper when the underlying portfolio value is always greater than one. Hence the starting portfolio value is mapped from 1 to e, while the "loss interval" $(0, 1)$ is mapped to $(1, e)$. The invariance of max to the use of an increasing generator parallels the invariance of ordinary multiplication of non-negative reals to a power map.

- changing the contract to take advantage of other telescoping sums. For example, let $G_N(R) \equiv \sum_{n=0}^{N} R^n = \frac{1-R^{N+1}}{1-R}$ for $R > 0$. Differentiating w.r.t. $\ln R$:

$$RG'_N(R) \equiv \sum_{n=0}^{N} nR^n = R\frac{1-(N+1)R^N + NR^{N+1}}{(1-R)^2}, \tag{132}$$

  which is a closed-form formula for a mixed arithmetic-geometric series. Suppose that the multipliers of the lagged price are now time-dependent and rising arithmetically through time, i.e $c_1 = 1c, c_2 = 2c, \ldots c_N = Nc$. Then the new BMF put can also be solved in closed form. Differentiating (132) w.r.t. $\ln R$ again:

$$\left(R\frac{d}{dR}\right)^2 G_N(R) \equiv \sum_{n=0}^{N} n^2 R^n = R\frac{1+R-(N+1)^2R^N + (2N^2+2N-1)R^{N+1} - N^2R^{N+2}}{(1-R)^3}.$$

  By repeatedly differentiating or integrating w.r.t. $\ln R$, one can clearly develop a formula for $\sum_{n=0}^{N} n^m R^n$, where m is an integer. See the Wiki entry List of Mathematical Series for closed forms for other finite series eg. Binomial coefficients such as $\sum_{n=0}^{N} \binom{N}{n} = 2^N$. So if $q = 0$ and lagged price multipliers are proportional to binomial coefficients, this well known finite sum can be employed. One can similarly employ Faulhaber's formula, which is a closed form formula for the sum of the p-th powers of the first N positive integers. In general, any formula for a finite Z transform or a finite Laplace transform can be checked to see if it has a tropical counterpart (replacing ordinary summation/integration with tropical summation/integration). For an introductory book emphasizing finite series, see Davis[16].

- telescoping sums generalize to the discrete fundamental theorem of calculus. This further generalizes to Stokes theorem, which further generalizes to the generalized Stokes theorem. There should be a pseudo-analytic version of all of these results. The challenge is to find a financial application.

- In practice, stockholders often pay for put protection by writing calls. European-style collars are usually done at zero cost. Suppose that the one-time flooring of a daily return at the buyer's discretion is financed by the one-time capping of the daily return at the seller's discretion. One can either create the upside counterpart of JIT downside protection, or else bundle downside flooring with upside capping and allow bilateral early exercise. If the interest rate is equal to the dividend yield, GPCS applies so it may be possible to pair the daily return floor $\underline{R}$ with a cap $\bar{R}$ such that the cost and/or vega of the position is neutralized. This Bermudan-style forward-starting risk-reversal becomes a way to bet on the covariation of returns and volatility. Furthermore, if the JIT version of an ATM straddle is introduced, one can also trade variance of volatility via a vega-neutral butterf ly.

- The part of pseudo-analysis applied to math finance has so far been restricted to pseudo-linear algebra. However, under zero rates and dividends consider valuing a payoff at T_2 of:

$$J_T = \left(\frac{S_1}{S_0} \vee \underline{R}\right)\left(\frac{S_2}{S_1} \vee \underline{R}\right)\dots\left(\frac{S_N}{S_{N-1}} \vee \underline{R}\right).$$

For $\underline{R} > 0$, floored gross returns are compounded $N-1$ times. Suppose that the N price ratios (gross returns) are IID. Then $J_0 =$

$$E_0\left[\left(\frac{S_1}{S_0} \vee \underline{R}\right)\left(\frac{S_2}{S_1} \vee \underline{R}\right)\dots\left(\frac{S_N}{S_{N-1}} \vee \underline{R}\right)\right] = E_0\left[\left(\frac{S_1}{S_0} \vee \underline{R}\right)E_0\left(\frac{S_2}{S_1} \vee \underline{R}\right)\dots E_0\left(\frac{S_N}{S_{N-1}} \vee \underline{R}\right)\right],$$

by independence and $= \left[E_0\left(\frac{S_1}{S_0} \vee \underline{R}\right)\right]^N$ by identically distributed price ratios. If we now use pseudo-analysis as well, then $J_0 = \left[1 \oplus^{p(\triangle t)} \underline{R}\right]^N$, which is a pseudo-$N$-th order polynomial. It is also the pseudo-analytic future value formula where the interest rate is $\underline{R} > 0$. So we have yet another analogy to interest rates. One can find the implied Lévy density consistent with our single period European option valuation formula $M_0 = E_0\left(c \vee \frac{S_1}{S_0}\right) = 1 \oplus^{p(\triangle t)} \underline{R}$.

- One can also try to combine the pseudo-linearity of the paper with the pseudo-polynomials in this suggested alternative application of pseudo-analysis. For example, suppose that the lag for a European married FS put is $\triangle t = 1$ year, but now the pair of random variables in its payoff are not the year end stock prices S_{N-1} and S_N, but instead the product of monthly floored returns:

$$MFSP_T = \prod_{i=12(N-2)}^{12(N-1)}\left(\frac{S_i}{S_{i-1}} \vee \underline{R}\right) \vee \prod_{i=12(N-1)}^{12N}\left(\frac{S_i}{S_{i-1}} \vee \underline{R}\right).$$

Let $\pi(\tau)$ be the power function which depends on the (lower) volatility of the compounded floored monthly gross returns $\prod_{i=1}^{12}\left(\frac{S_i}{S_{i-1}} \vee \underline{R}\right)$, as opposed to their unfloored counterpart: $\prod_{i=1}^{12}\left(\frac{S_i}{S_{i-1}}\right) = \frac{S_{12}}{S_0}$. The initial value of each maximand is the pseudo Future-Value formula, $\left[1 \oplus^{p(1/12)} \underline{R}\right]^{12}$ where p depends on the volatility of the monthly gross return of the stock. The initial value of this European payoff is their pseudo-sum using $\oplus^{\pi(1)}$:

$$MFSP_0 = \left[1 \oplus^{p(1/12)} \underline{R}\right]^{12} \oplus^{\pi(1)} \left[1 \oplus^{p(1/12)} \underline{R}\right]^{12} = \left[1 \oplus^{p(1/12)} \underline{R}\right]^{12} (1 \oplus^{\pi(1)} 1) = \left[1 \oplus^{p(1/12)} \underline{R}\right]^{12} 2^{\frac{1}{\pi(1)}}.$$

One can try to relate $\oplus^{\pi(1)}$ to $\oplus^{p(1/12)}$.

- One can try to value a conventional Bermudan married put. It might be easier to consider the exchange option version when the two risky assets involved in the exchange are comonotonic. In this case, the Jamshidian trick implies that the conditional expectation of a maximum distributes over ordinary addition, just as ordinary addition of a constant distributes over the conditional expectation of a maximum. Boolean algebra can be employed to analyze this two way distributivity.

- Suppose one wishes to value a portfolio where the fixed proportional dividends with rate q are swapped away for a possibly different fixed fraction $\underline{R}$ of yesterday's reference value. The arbitrage-free value of the portfolio receiving lagged proportional dividends at rate $\underline{R}$ is uniquely determined in a model-free way by setting $p = 1$ in our pseudo-linear recursion for m_n and solving the now linear recursion. When we return p to some value strictly larger than one and assume that the risk-neutral distribution is Dagum, we replace the repeated ordinary addition of lagged proportional dividends with the repeated option to one-time floor a daily return. This suggests that any valuation result in finance that uses ordinary addition has a counterpart replacing addition with optionality. Ordinary addition can be seen as merely the infinite volatility/$p = 1$ version of optionality. Replacing lagged proportional dividends with JIT insurance is but one instance of a more general principle replacing ordinary addition of values via portfolio formation with valuation after adding optionality. Future research should find other instances.

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Appendix 1:Power-Plus-Prod is a Commutative Semi-field

A semi-ring is an algebraic structure which must satisfy 8 axioms that we list shortly. A semi-field is a semi-ring for which all elements also have a multiplicative inverse. A commutative semi-field is a semi-field for which the multiplication is also commutative. Hence, a commutative semi-field is an algebraic structure which satisfies 10 particular axioms. In this appendix, we show that the algebraic structure called power-plus-prod, and denoted by $(\mathbb{R}^+, \oplus^p, \times)$, satisfies the 10 defining axioms of a commutative semi-field.

We begin by first showing that $(\mathbb{R}^+, \oplus^p, \times)$ is a semi-ring. To be a semi-ring, it is necessary that the algebraic structure $(\mathbb{R}^+, \oplus^p)$ is a commutative monoid. This means that $(\mathbb{R}^+, \oplus^p)$ must satisfy the following three axioms:

1. **Addition is Associative:** for $g_1, g_2, g_3 \in \mathbb{R}^+$:

$$(g_1 \oplus^p g_2) \oplus^p g_3 = g_1 \oplus^p (g_2 \oplus^p g_3).$$

   This holds since:

$$(g_1 \oplus^p g_2) \oplus^p g_3 = \left\{[(g_1^p + g_2^p)^{\frac{1}{p}}]^p + g_3^p\right\}^{\frac{1}{p}} = \left\{g_1^p + [g_2^p + g_3^p]^{\frac{1}{p}}]^p\right\}^{\frac{1}{p}} = g_1 \oplus^p (g_2 \oplus^p g_3).$$

2. **Existence Of Identity Element for Addition:** for all $g \in \mathbb{R}^+$, there must exist an element of the set denoted 0 such that:

$$0 \oplus^p g = g \oplus^p 0 = g$$

   We test whether the ordinary zero can act as the identity element for $\oplus^p$. Since $\mathbb{R}^+$ denotes non-negative reals, ordinary zero is contained in the set. Furthermore, ordinary zero is in fact the identity element for $\oplus^p$ since:

$$(0^p + g^p)^{\frac{1}{p}} = (g^p + 0^p)^{\frac{1}{p}} = g.$$

3. **Addition Commutes:** for $g_1, g_2 \in \mathbb{R}^+$:

$$g_1 \oplus^p g_2 = g_2 \oplus^p g_1.$$

   This holds since:

$$(g_1^p + g_2^p)^{\frac{1}{p}} = (g_2^p + g_1^p)^{\frac{1}{p}}.$$

   Hence, $(\mathbb{R}^+, \oplus^p)$ is a commutative monoid.

   For $(\mathbb{R}^+, \oplus^p, \times)$ to be a semi-ring, a further necessary condition is that the algebraic structure $(\mathbb{R}^+, \times)$ is a monoid. This means that $(\mathbb{R}^+, \times)$ must satisfy the following two axioms:

4. **Multiplication is Associative:** for $g_1, g_2, g_3 \in \mathbb{R}^+$:

$$(g_1 \times g_2) \times g_3 = g_1 \times (g_2 \times g_3).$$

   This obviously holds.

5. **Existence Of Identity Element for Multiplication:** for all $g \in \mathbb{R}^+$, there must exist an element of the set denoted 1 such that

$$1 \times g = g \times 1 = g.$$

The set $\mathbb{R}^+$ contains the ordinary number 1, which obviously acts as the identity element for multiplication. Hence, $(\mathbb{R}^+, \times)$ is a monoid.

For $(\mathbb{R}^+, \oplus^p, \times)$ to be a semi-ring, three more axioms must be satisfied:

6. **Multiplication left distributes over addition:** for $g_1, g_2, g_3 \in \mathbb{R}^+$:

$$g_1 \times (g_2 \oplus^p g_3) = (g_1 \times g_2) \oplus^p (g_1 \times g_3).$$

This holds since:

$$g_1 \times (g_2 \oplus^p g_3) = g_1 \times (g_2^p + g_3^p)^{\frac{1}{p}} = ((g_1 \times g_2)^p + (g_1 \times g_3)^p)^{\frac{1}{p}} = (g_1 \times g_2) \oplus^p (g_1 \times g_3). \quad (133)$$

7. **Multiplication right distributes over addition:** for $g_1, g_2, g_3 \in \mathbb{R}^+$:

$$(g_1 \oplus^p g_2) \times g_3 = (g_1 \times g_3) \oplus^p (g_2 \times g_3).$$

This holds since:

$$(g_1 \oplus^p g_2) \times g_3 = (g_1^p + g_2^p)^{\frac{1}{p}} \times g_3 = ((g_1 \times g_3)^p + (g_2 \times g_3)^p)^{\frac{1}{p}} = (g_1 \times g_3) \oplus^p (g_2 \times g_3). \quad (134)$$

8. **Multiplication by zero annihilates $\mathbb{R}$:** For $g \in \mathbb{R}^+$:

$$0 \times g = g \times 0 = 0.$$

This obviously holds. Hence, $(\mathbb{R}^+, \oplus^p, \times)$ is a semi-ring.

In order that $(\mathbb{R}^+, \oplus^p, \times)$ also be a semi-field, it must satisfy the additional condition:

9. **All elements have a multiplicative inverse in** $\mathbb{R}^+$. For any non-negative real g except 0, the set $\mathbb{R}^+$ also contains its multiplicative inverse $1/g$, so $(\mathbb{R}^+, \oplus^p, \times)$ is also a semi-field.

Finally, a semi-field is commutative if:

10. **Multiplication Commutes:** for $g_1, g_2 \in \mathbb{R}^+$:

$$g_1 \times g_2 = g_2 \times g_1.$$

Since standard multiplication is obviously commutative, we conclude that $(\mathbb{R}^+, \oplus^p, \times)$ is a commutative semi-field.

Appendix 2: Proof that Power Norm is Decreasing in Power

Let $g_1 \geq 0, g_2 \geq 0$, and $p \in [1, \infty]$. In this appendix, we offer two short proofs of the fact that the power norm of the vector $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ is decreasing in p.

Suppose that $q > p \in [1, \infty]$. We want to show that $(g_1^q + g_2^q)^{\frac{1}{q}} \leq (g_1^p + g_2^p)^{\frac{1}{p}}$. Equivalently, we want to show that $\frac{(g_1^q+g_2^q)^{\frac{1}{q}}}{(g_1^p+g_2^p)^{\frac{1}{p}}} \leq 1$. The LHS is:

$$\begin{aligned}
\frac{(g_1^q + g_2^q)^{\frac{1}{q}}}{(g_1^p + g_2^p)^{\frac{1}{p}}} &= \left(\frac{g_1^q}{(g_1^p + g_2^p)^{\frac{q}{p}}} + \frac{g_2^q}{(g_1^p + g_2^p)^{\frac{q}{p}}} \right)^{\frac{1}{q}} \\
&= \left[\left(\frac{g_1^p}{g_1^p + g_2^p} \right)^{\frac{q}{p}} + \left(\frac{g_2^p}{g_1^p + g_2^p} \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&\leq \left[\frac{g_1^p}{g_1^p + g_2^p} + \frac{g_2^p}{g_1^p + g_2^p} \right]^{\frac{1}{q}} = 1. \qquad (135)
\end{aligned}$$

It follows that the power norm is decreasing in p for $p \in [1, \infty]$. The second proof uses calculus. Let:

$$c(g_1, p, g_2) \equiv (g_1^p + g_2^p)^{\frac{1}{p}} = e^{\frac{1}{p}\ln(g_1^p + g_2^p)}, \qquad g_1 > 0, p \in [1, \infty], g_2 > 0 \qquad (136)$$

describe the power model value of the simple CB as a function of p instead of τ. We need to show that for each $g_1 > 0$ and $g_2 > 0$, c is decreasing in p for $p \in [1, \infty]$. Differentiating (136) w.r.t. p implies:

$$\begin{aligned}
\frac{\partial}{\partial p} c(g_1, p, g_2) &= e^{\frac{1}{p}\ln(g_1^p + g_2^p)} \left[\frac{g_1^p \ln g_1 + g_2^p \ln g_2}{p(g_1^p + g_2^p)} - \frac{\ln(g_1^p + g_2^p)}{p^2} \right] \qquad g_1 > 0, \tau \geq 0, g_2 > 0, (137) \\
&= \frac{(g_1^p + g_2^p)^{\frac{1}{p}-1}}{p} \left[g_1^p \ln g_1 + g_2^p \ln g_2 - \frac{1}{p} \ln(g_1^p + g_2^p)(g_1^p + g_2^p) \right]. \qquad (138)
\end{aligned}$$

Now:

$$g_1 \leq (g_1^p + g_2^p)^{\frac{1}{p}} \text{ and } g_2 \leq (g_1^p + g_2^p)^{\frac{1}{p}}, \qquad g_1 > 0, p \in [1, \infty], g_2 > 0. \qquad (139)$$

Taking logs:

$$\ln g_1 \leq \frac{1}{p} \ln(g_1^p + g_2^p) \text{ and } \ln g_2 \leq \frac{1}{p} \ln(g_1^p + g_2^p), \qquad g_1 > 0, p \in [1, \infty], g_2 > 0. \qquad (140)$$

Multiplying the first inequality by g_1^p, the second inequality by g_2^p, and adding:

$$g_1^p \ln g_1 + g_2^p \ln g_2 \leq \frac{1}{p} \ln(g_1^p + g_2^p)(g_1^p + g_2^p), \qquad g_1 > 0, p \in [1, \infty], g_2 > 0. \qquad (141)$$

Thus, the quantity in square brackets in (138) is non-positive and hence so is $\frac{\partial}{\partial p} c(g_1, p, g_2)$ for all $p \in [1, \infty]$. It follows that for each $g_1 > 0$ and $g_2 > 0$, c is decreasing in p for $p \in [1, \infty]$.

Appendix 3: Supporting Continuous Martingale Dynamics

In this appendix, we adapt a well known argument due to Dupire[17]. Assuming zero interest rates and dividends, we want to find continuous martingale dynamics for the underlying portfolio value S which are consistent with our assumed structure of insurance premia. Let $M_0(\underline{R}, t)$ be the initial value of an insured portfolio where the insurance coverage starts at a future time $(i-1)\triangle t$ and ends at a later time $t \in [(i-1)\triangle t, i\triangle t)$. The payoff at time t is $\underline{R} \vee \frac{S_t}{S_{(i-1)\triangle t}}$. The initial value of the portfolio with this deferred insurance is therefore.

$$M_0(\underline{R}, t) = E_0^{\mathbb{Q}_-}\left[\underline{R} \vee \frac{S_t}{S_{(i-1)\triangle t}}\right] = \left[\underline{R}^{p(\tau)} + 1\right]^{\frac{1}{p(\tau)}}, \tag{142}$$

where $\tau = t - (i-1)\triangle t$ and $p(\tau)$ is declining in $\tau \geq 0$ with $p(0) = \infty$ and $p(\infty) = 1$.

Let W be a $\mathbb{Q}_-$ standard Brownian motion. We assume that the continuous martingale dynamics for the underlying portfolio value S are described by:

$$\frac{dS_t}{S_t} = \ell\left(\frac{S_t}{S_{(i-1)\triangle t}}, t - (i-1)\triangle t\right) dW_t, \tag{143}$$

for $i = 1, \ldots, N$ and where $\ell(R, \tau) : \mathbb{R}^+ \times [0, \triangle t) \mapsto \mathbb{R}^+$ is the local lognormal volatility function. The objective is to determine how ℓ must depend on its two arguments, so that (142) holds true.

The Dupire formula for the local variance rate of the continuous martingale S at the space time point $(\underline{R}, t)$ is given by:

$$\ell^2(\underline{R}, t - (i-1)\triangle t) = \frac{2\frac{\partial M_0(\underline{R},t)}{\partial t}}{\underline{R}^2\frac{\partial^2 M_0(\underline{R},t)}{\partial \underline{R}^2}}, \qquad \underline{R} > 0, t \in [(i-1)\triangle t, i\triangle t). \tag{144}$$

We now derive the two partial derivatives $\frac{\partial M_0(\underline{R},t)}{\partial t}$ and $\underline{R}^2\frac{\partial^2 M_0(\underline{R},t)}{\partial \underline{R}^2}$ in our model and hence determine S's local lognormal variance rate function in (144). The reader can verify from (142)) that:

$$\frac{\partial M_0(\underline{R}, T)}{\partial \underline{R}} = \left(1 + \underline{R}^{-p(\tau)}\right)^{\frac{1}{p(\tau)} - 1}. \tag{145}$$

$$\underline{R}^2\frac{\partial^2 M_0(\underline{R}, t)}{\partial \underline{R}^2} = [p(\tau) - 1]\underline{R}^{p(\tau)}\left(1 + \underline{R}^{p(\tau)}\right)^{\frac{1}{p(\tau)} - 2}. \tag{146}$$

$$\frac{\partial M_0(\underline{R}, t)}{\partial t} = \left(1 + \underline{R}^{p(\tau)}\right)^{\frac{1}{p(\tau)}} \frac{p'(\tau)}{p(\tau)}\left\{\frac{\underline{R}^{p(\tau)} \ln \underline{R}}{1 + \underline{R}^{p(\tau)}} - \frac{\ln\left(1 + \underline{R}^{p(\tau)}\right)}{p(\tau)}\right\}. \tag{147}$$

Substituting (146) and (147) into (144):

$$\ell^2(\underline{R}, \tau) = \frac{2p'(\tau)}{p(\tau)[p(\tau) - 1]}\left(1 + \underline{R}^{p(\tau)}\right)^2\left\{\frac{\ln \underline{R}}{1 + \underline{R}^{p(\tau)}} - \frac{\ln\left(1 + \underline{R}^{p(\tau)}\right)}{p(\tau)\underline{R}^{p(\tau)}}\right\}. \tag{148}$$

Thus, a supporting continuous martingale S is the solution to the stochastic differential equation:

$$\frac{dS_t}{S_t} = \left(1 + R_t^{p(\tau)}\right) \sqrt{\frac{2p'(\tau)}{p(\tau)[p(\tau) - 1]} \left\{ \frac{\ln R_t}{1 + R_t^{p(\tau)}} - \frac{\ln\left(1 + R_t^{p(\tau)}\right)}{p(\tau) R_t^{p(\tau)}} \right\}} dW_t, \tag{149}$$

for $R_t = \frac{S_t}{S_{(i-1)\triangle t}}$, $\tau = t - (i-1)\triangle t$, $t \in [(i-1)\triangle t, i\triangle t]$, $i = 1 \ldots N$.

Appendix 4: Genesis of Dagum Distribution

In this appendix, we repeat a derivation of the genesis of a Dagum PDF in Klugman et. al.[25], Example 2.49, page 99, but using our notation. Note that they refer to the Dagum PDF as an inverse Burr PDF. The reciprocal of a Weibull distributed random variable is said to have an inverse Weibull distribution. Consider the inverse Weibull PDF with scale parameter $\theta > 0$ and shape parameter $p > 0$:

$$f(x) = \frac{p}{x}\left(\frac{\theta}{x}\right)^p e^{-\left(\frac{\theta}{x}\right)^p}, \qquad x > 0, \theta > 0, p > 0. \tag{150}$$

A power of a gamma distributed random variable is said to have a transformed gamma distribution. Also consider a transformed gamma PDF with scale parameter $b > 0$ and shape parameters $p > 0$ and $\alpha > 0$:

$$f(x) = p\frac{u^\alpha e^{-u}}{x\Gamma(\alpha)} \text{ where } u = \left(\frac{x}{b}\right)^p, \qquad x > 0, b > 0, p > 0, \alpha > 0. \tag{151}$$

Consider randomizing the scale parameter of the inverse Weibull distributed random variable via a transformed gamma distributed random variable with the same shape parameter $p > 0$. The resulting PDF is:

$$\begin{aligned} f(x) &= \int_0^\infty \frac{p\theta^p}{x^{p+1}} e^{-\left(\frac{\theta}{x}\right)^p} \frac{p\theta^{p\alpha-1}}{b^{p\alpha}\Gamma(\alpha)} e^{-\left(\frac{\theta}{b}\right)^p} d\theta, \\ &= \frac{p^2}{b^{p\alpha}\Gamma(\alpha)x^{p+1}} \int_0^\infty \theta^{p+p\alpha-1} \exp\left[-\theta^p\left(x^{-p}+b^{-p}\right)\right] d\theta, \qquad x > 0, b > 0, p > 0, \alpha > 0 \end{aligned} \tag{152}$$

Let $y = \theta^p\left(x^{-p}+b^{-p}\right)$ replace θ as the integrator:

$$\begin{aligned} f(x) &= \frac{p^2}{b^{p\alpha}\Gamma(\alpha)x^{p+1}} \int_0^\infty \left\{y^{\frac{1}{p}}\left(x^{-p}+b^{-p}\right)^{-\frac{1}{p}}\right\}^{p+p\alpha-1} e^{-y} y^{p^{-1}-1} p^{-1}\left(x^{-p}+b^{-p}\right)^{-\frac{1}{p}} dy \\ &= \frac{p}{b^{p\alpha}\Gamma(\alpha)x^{p+1}\left(x^{-p}+b^{-p}\right)^{\alpha+1}} \int_0^\infty y^\alpha e^{-y} dy \\ &= \frac{p\Gamma(\alpha+1)}{b^{p\alpha}\Gamma(\alpha)x^{p+1}\left(x^{-p}+b^{-p}\right)^{\alpha+1}} \\ &= \frac{p\alpha b^p x^{p\alpha-1}}{\left(b^p+x^p\right)^{\alpha+1}} \text{ since } \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \\ &= p\alpha\frac{\left(\frac{x}{b}\right)^{p\alpha}}{x\left(1+\left(\frac{x}{b}\right)^p\right)^{\alpha+1}} \qquad x > 0, b > 0, p > 0, \alpha > 0. \end{aligned} \tag{153}$$

We recognize this last line as the PDF of a Dagum Type 1 random variable with scale parameter $b > 0$ and shape parameters $p > 0$ and $\alpha > 0$. Under zero carrying costs, the power-plus risk-neutral distribution at time t of the terminal spot value S_T conditional on $S_t = S$ is Dagum distributed with scale parameter (and mean) $b = S$ and shape parameters $p = p(T-t)$ and $\alpha = 1-1/p(T-t)$. Since one can construct both Weibull processes and gamma processes, it is possible that a Dagum process might also be constructed. Since a Weibull process and a Gamma process are both increasing, it is not completely obvious how to construct a Dagum process which is not monotonic.

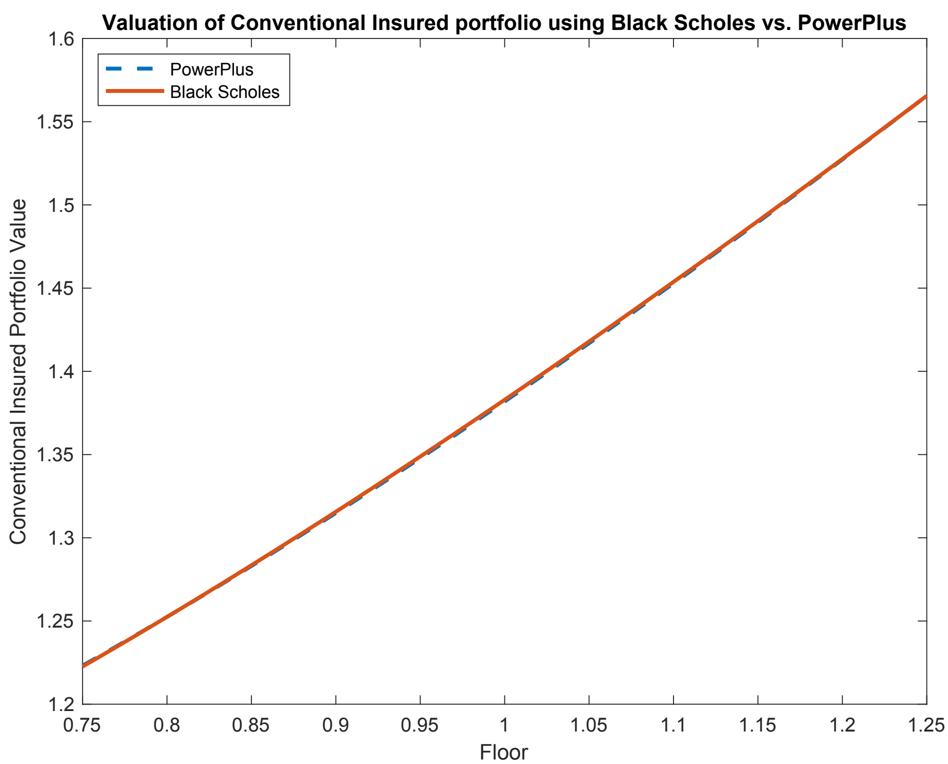
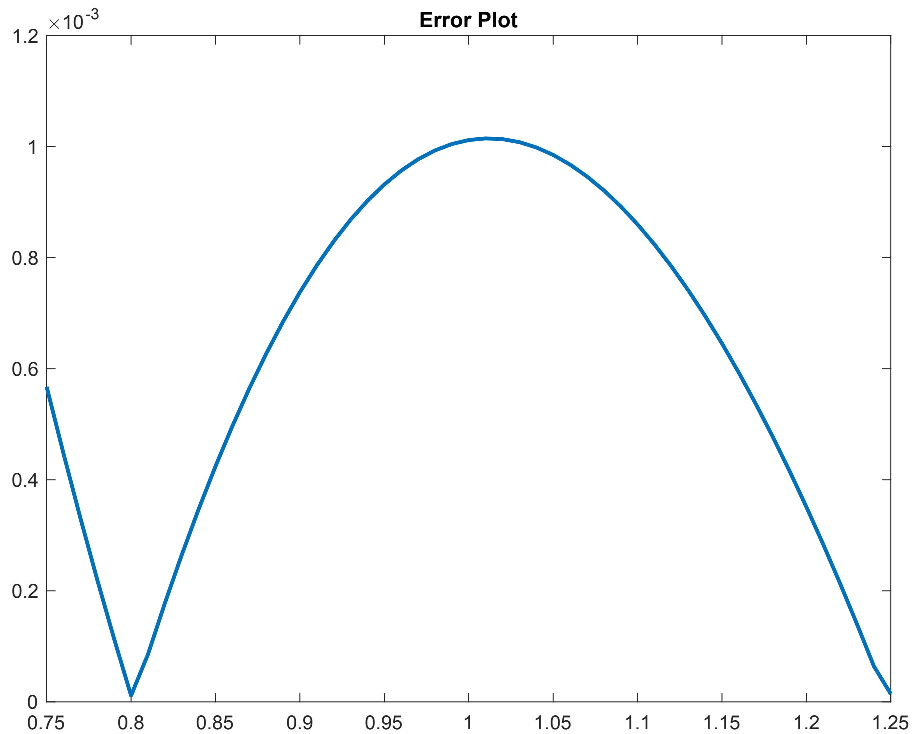


Figure 4: Power Plus and Black Scholes Model Values



Plot.pdf

Figure 5: Difference between Power Plus and Black Scholes Model Values