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Earnings Management and Earnings Quality: Theory and Evidence

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ABSTRACT: We study a model of earnings management and provide predictions about the time-series properties of earnings quality and reporting bias. We estimate the model to empirically separate two components of investor uncertainty: fundamental economic uncertainty, and information asymmetry between the manager and investors due to reporting noise. We find that (1) the null hypothesis of zero reporting bias is rejected; (2) the ratio of the variance of the noise introduced by the reporting process to the variance of earnings shocks is, on average, 45 percent; (3) the reporting noise plays a significantly less prominent role in valuation, due to the persistence of shocks to economic earnings; (4) the magnitude of investors' uncertainty created by reporting noise about firms' assets in place and about future earnings is similar; and (5) ignoring the possibility of reporting distortions would bias the estimates of variance and persistence of economic earnings.

Keywords: dynamic earnings management; earnings quality; structural estimation.

I. INTRODUCTION

Earnings management and earnings quality are central topics in theoretical and empirical research in accounting. The existing theoretical literature on earnings management and earnings quality has predominantly focused on settings in which firms make a single reporting decision. However, the corporate disclosure environment is dynamic and characterized by repeated reporting decisions. In the existing empirical literature, measures of earnings quality are, as Gerakos (2012) put it, "typically estimated in the cross section and therefore do not take into account the fact that earnings are best described by a dynamic process. These techniques have not significantly changed in over 20 years."

We study a dynamic model of earnings management in which firms take into account both long- and short-term considerations when reporting earnings. The model, which is characterized by persistent uncertainty about balance sheet and future earnings, offers a distinction between two components of investor uncertainty: (1) fundamental economic uncertainty, defined as the uncertainty of the manager about the firm's earnings, and (2) information asymmetry between the manager and investors due to reporting or accounting distortions. We believe that this distinction is key to understanding the notion of earnings/accounting quality. We structurally estimate our model to empirically parse out these two components of investor uncertainty in an attempt to address the concern that "archival research cannot satisfactorily parse out the portion of managed earnings from the one resulting from the fundamental earnings process" (Dichev, Graham, Harvey, and Rajgopal 2013, 1; Dechow, Ge, and Schrand 2010). Our structural approach is not immune to measurement error and correlated omitted variables due to model misspecification, but it has the potential to complement the insights of the so-called "reduced-form" approach in this research area.

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Our contribution is twofold. First, we offer a novel dynamic earnings management model that features persistent uncertainty about both the firm's balance sheet and future earnings and persistent information asymmetry between investors and managers. Second, based on the data-generating process of book values and stock prices that our model predicts, we empirically estimate the model's parameters. We use the resulting estimates in order to: quantify the average bias in reported equity; analyze the breakdown of investor uncertainty into the part that is caused by uncertainty about fundamentals and the part that is due to reporting and accounting distortions;[1] and estimate our suggested earnings quality measure.

In our model, the firm's manager privately observes the realization of the firm's earnings in each period and then issues a report to the capital market. Our model is characterized by four key aspects. First, the manager is long-lived, and his incentives are tied to the firm's stock price during his tenure with the firm.[2] Second, when issuing a report, the manager is not confined to report truthfully and can misreport earnings at a cost. The cost of misreporting in a given period depends on the manager's history of misreporting. Earnings management that increases reported earnings today will tend to reverse at some future point in time. Consequently, bias in reported earnings today increases the cost of optimistically biasing reported future earnings, capturing the dynamic nature of the manager's reporting decisions. Third, investors are not perfectly informed about the manager's incentives, resulting in an equilibrium that is characterized by long-lasting information asymmetry between the manager and the capital market.[3] Fourth, earnings are serially correlated. This implies that as the manager releases new reports, investors simultaneously update their beliefs about the firm's existing assets value and future earnings. As a result, in our setting, investors experience long-lasting uncertainty not only about future earnings, but also about the firm's assets in place. To the extent that earnings are noisy in equilibrium, due to the presence of reporting or accounting distortions, balance sheets are noisy, too. To the best of our knowledge, this is the first paper to capture this aspect of the dynamic reporting environment.

The combination of these features provides a parsimonious model of earnings management in a multi-period setting that contributes to the theoretical literature by capturing the dynamic aspect of financial reporting. The dynamic nature of the model is important conceptually, and is critical to the empirical estimation of the model.

We first consider a setting where the manager's horizon is finite and then, using the insights from the finite horizon setting, we focus on the steady-state equilibrium of an infinite horizon setting. In contrast to the finite horizon setting, the steady-state of an infinite horizon setting is characterized by time-invariant price response coefficients, reporting bias, and information asymmetry. In this setting, we define earnings quality as investors' additional uncertainty about the firm's value relative to the manager's own uncertainty. Earnings quality, thus, is here a measure of the extent of the information asymmetry between the manager and investors caused by reporting or accounting distortions.

We then structurally estimate the model's unobservable parameters, such as the distribution of fundamental earnings and the amount of noise that reporting distortions add to financial reports. We use U.S. Compustat annual data for 1990–2016 and estimate the steady-state equilibrium by Maximum Likelihood. In particular, we maximize the joint log-likelihood of reported earnings and stock prices. Since our model relies extensively on variances, we expect our parameter estimates to be sensitive to variations in firm size. To account for the likely cross-sectional variation in estimates, we, therefore, partition the sample into three groups based on market value of equity into small, medium, and large firms.

We first test the null hypothesis of no accounting or reporting noise, and our results strongly reject it. Specifically, the estimate of the variance of noise introduced by the reporting process is significantly positive for all three size groups. Second, we find that the ratio of the variance of the noise introduced by the reporting process to the variance of earnings shocks is, on average, 45 percent, suggesting that the noise added by the reporting process significantly contributes to investor uncertainty. This leads, in our equilibrium, to an earnings response coefficient (ERC) estimate of 3.5, which is significantly lower than the ERC that would prevail absent any reporting or accounting distortions. Also, consistent with the presence of significant noise, we show that given a \$1 increase in reported earnings, investors update their beliefs about economic earnings by only 60 cents rather than \$1, which would have been the case in the absence of reporting or accounting distortions. These findings are broadly consistent with those in Dichev et al. (2013), who report that the chief financial officers in their survey estimate that 50 percent of the earnings quality is due to innate factors (non-discretionary factors), whereas the other 50 percent is due to managerial reporting decisions. However, we also find that, from a valuation perspective, reporting noise plays a significantly lesser role due to the persistence of shocks to economic earnings. Specifically, the variance of reporting noise is only about 5 percent of

[1] We use "reporting distortions" interchangeably with "earnings management." In contrast, "accounting distortions" refer to the loss of information due to accounting standards that prevent managers from fully reflecting the underlying economics of the firm in its financial statements.

[2] For parsimony, we do not consider the possibility that the manager has other incentives, such as beating targets, smoothing earnings, or "taking a big bath." Our analysis, thus, applies to settings in which the manager's primary concern is the performance of the firm's stock price.

[3] We assume that in each period, the manager's cost of manipulating a reported firm's equity is not only a function of the magnitude of the bias, but also depends on an additional factor that is privately observed by the manager. This factor reflects idiosyncratic circumstances in each period that affect the manager's misreporting costs (Dye and Sridhar 2008). For a related assumption, see Fischer and Verrecchia (2000).

the variance of shocks to economic earnings when the latter is scaled by persistence. Fourth, our reporting noise creates uncertainty for investors not only about firms' current economic earnings, but also about firms' assets in place. Our estimates suggest that the two are of similar magnitude. Finally, our estimates establish that ignoring the possibility of reporting distortions would bias the estimates of variance and persistence of economic earnings. For example, we find that the persistence of economic earnings innovations is, on average, 0.6, with smaller firms having a lower persistence and larger firms having a higher persistence. In contrast, when we estimate the model ignoring misreporting, we obtain an average estimate of 0.3, consistent with evidence in prior empirical literature that does not distinguish between economic earnings and misreporting. Similarly, our analysis, thus, suggests that ignoring the presence of reporting distortions would lead researchers to significantly underestimate the persistence of economic earnings (by about 50 percent) and to overestimate the volatility of economic earnings (also by about 50 percent).

The paper proceeds as follows. Section II discusses the related literature. Section III lays out the setting of our model. In Section IV, we derive and analyze the equilibrium of our model. We start with the equilibrium of the finite horizon setting, and then derive and analyze the steady-state equilibrium of the setting in which the manager's tenure with the firm is infinite. In Section V, we present the structural estimation of our model. In Section VI, we extend the model to allow for the inter-temporal correlation of reporting noise. Section VII concludes.

II. RELATED LITERATURE

Although earnings management has been studied extensively in the theoretical literature, two key features that characterize corporate reporting environments have been largely overlooked: the dynamic nature of the firms' earnings process, and the persistent information asymmetry between managers and investors. By including these features, our model complements the existing theoretical literature on earnings management and earnings quality, a literature that has largely focused on two modeling approaches. The first approach studies single-period settings with no inter-temporal consideration (e.g., Fischer and Verrecchia 2000; Guttman, Kadan, and Kandel 2006). The second approach studies two-period settings and assumes that once the manager determines the bias in the report of the first period, the first period's bias must fully reverse in the second period (e.g., Sankar and Subramanyam 2001; Kirschenheiter and Melumad 2002; Ewert and Wagenhofer 2011).[4] In such settings, the manager has no reporting discretion in the second period and, thus, these models do not capture the dynamic nature of reporting decisions in practice. We extend the theoretical literature on earnings management by studying what we believe to be a more realistic setting. In particular, we study a multi-period (both finite and infinite) setting without imposing any exogenously assumed reversal of the reporting bias. Instead, we assume that the manager's cost of biasing the report in a given period is a function of the cumulative bias in reported earnings over the years. In other words, the manager's cost of biasing his report increases in the magnitude of the bias in reported equity. This assumption is consistent with the evidence in Barton and Simko (2002) of previous optimism in financial reporting reducing managers' ability to optimistically bias earnings in the future.

Zakolyukina (2018) is a closely related paper. Zakolyukina (2018) estimates a model of earnings manipulation using restatements data. Its main goal is to assess the frequency of generally accepted accounting principles (GAAP) violations and the price consequences of such violations. In our model, investors set prices rationally, so prices are unbiased. Also, we focus on within-GAAP earnings management aiming to disentangle uncertainty due to reporting and accounting distortions from that due to economic fundamentals.

In addition to the theoretical literature, there exists a large body of empirical literature on earnings quality. However, the existing measures of earnings quality have been critiqued for their inability to distinguish between the uncertainty in financial reports due to the nature of the business fundamentals (nondiscretionary component) and the noise due to reporting or accounting distortions (e.g., Dechow et al. 2010; Dichev et al. 2013). For example, Gerakos (2012) asserts that these measures "suffer from measurement error and correlated omitted variables, which lead to Type 1 errors and Type 2 errors." We structurally estimate our model to try addressing these concerns and empirically separate the portion of investor uncertainty due to reporting or accounting distortions (earnings quality) from the portion of investor uncertainty due to the firms' economic earnings process.

III. SETTING

This is a stylized model of earnings management with the following three elements: (1) A long-lived manager with capital market incentives: In each period, the manager observes the firm's economic earnings and then issues a report about the

[4] This literature focuses on the valuation role of financial reporting in which the manager's compensation is exogenous. A different stream of literature takes a principal-agent approach and focuses on the stewardship role of financial reporting, in which the manager's compensation is endogenously set by the principal (see, for example, Beyer, Guttman, and Marinovic 2014).

earnings to investors;[5] (2) Reporting discretion: When issuing a report, the manager is not confined to reporting earnings truthfully. However, misreporting earnings is personally costly to the manager; (3) Information asymmetry: Investors are not perfectly informed about the manager's incentives, resulting in an equilibrium that is characterized by *ex post* information asymmetry between the manager and the capital market. Next, we provide more details about the setting.

We consider a multi-period setting in which a firm generates economic earnings e_t in every period $t \in \{1, 2, ..., T\}$. To capture the fact that the firm's earnings can be correlated over time, and following Ohlson (1995), we assume that economic earnings are characterized by the following AR1 process:

$$e_t = \rho e_{t-1} + v_t, \tag{1}$$

where $\rho \in [0, 1)$, $v_t \sim N\left(0, \sigma_v^2\right)$, and v_i is independent of v_j for any $i \neq j$. The AR1 specification, while parsimonious, allows us to capture persistence and mean reversion, both of which are important properties of earnings (Gerakos and Kovrijnykh 2013). The earnings innovation v_t represents changes in the firm's economic performance, arising, for instance, from fluctuations in the firm's input prices and changes in revenues due to fluctuations in demand. To maintain the model's tractability and generality, we do not model the accrual and cash flow component of earnings individually.

We assume the firm does not make any payouts to shareholders (such as dividends). Thus, aggregate earnings, which we denote by θ_t, is the sum of earnings e_t over the years, and captures the firm's equity at time t. The firm's equity at time t is given by:

$$\theta_t = \theta_{t-1} + e_t. \tag{2}$$

In every period, the firm's manager privately learns the firm's earnings and issues a report about the firm's equity, r_t, to the market. The manager can manipulate the report, but he bears personal costs of doing so. In particular, we assume that the manager's biasing costs in a given period are:

$$\frac{c}{2}(r_t - \theta_t - \eta_t)^2,$$

where $\eta_t \sim N\left(0, \sigma_\eta^2\right)$. We refer to η_t as "reporting noise." The realization of η_t is privately observed by the manager before he makes his reporting decision, and may reflect—among other things—distortions driven by accounting rules, and idiosyncratic circumstances that affect the manager's misreporting incentives, as in Dye and Sridhar (2008). The fact that misreporting costs are volatile is a natural assumption in settings where managers must confront the possibly noisy views of other parties—such as the CFO, the board, and the auditor—before releasing a financial report, or where the strength of the firm's internal control system fluctuates. In the base model, we assume that η_t is independent over time, whereas in Section VI, we consider the case when η_t is persistent.

We do not impose the restriction that the bias in reported earnings must reverse at some fixed (but arbitrarily chosen) point in time. Instead, the model can accommodate any mechanical reversal of discretionary accruals in subsequent periods.[6] However, because the cost of misreporting is convex in the cumulative bias, $r_t - \theta_t$, a manager who engaged in more aggressive earnings misreporting in the past will face stronger incentives to reverse such manipulations in the future. The manager's misreporting cost depends on the difference between reported equity r_t and the (true) value of equity, θ_t, although the firm's equity is only a fraction of firm value as the equity does not include expected future earnings. This feature captures the notion that the manager is penalized for manipulating the accounting reports that are often not meant to be forward-looking. This assumption is also consistent with Barton and Simko (2002), who predict and find that: "Managers' ability to optimistically bias earnings decreases with the extent to which the balance sheet overstates net assets relative to a neutral application of GAAP."

In addition to the biasing costs, the manager's incentives are tied to the firm's stock price. We denote the firm's stock price at time t by p_t. The manager's payoff in each period t is assumed to be:

$$p_t - \frac{c}{2}(r_t - \theta_t - \eta_t)^2.$$

In each period, the manager maximizes the present value of his future payoffs, as in Acharya, DeMarzo, and Kremer (2011). In particular, in any period t, the manager chooses r_t to maximize:

[5] Alternatively, one can assume that the manager observes the true earnings with noise, which will not qualitatively affect the results.
[6] The full reversal feature of the bias at the end of the manager's tenure can easily be incorporated into this model by assuming that the parameter c depends on t and letting $\lim_{t \to T} c_t = \infty$.

$$U_t = \mathbb{E}\left[\sum_{k=t}^{T} \delta^{k-t}\left(\tilde{p}_k - \frac{c}{2}\left(r_k - \tilde{\theta}_k - \tilde{\eta}_k\right)^2\right)|\theta_t, e_t, \eta_t\right], \quad (3)$$

where $\delta \in [0, 1)$ is the manager's discount factor and captures the extent of the manager's myopia (we use $\tilde{x}$ to denote a random variable and x its realization).[7] This description of managers' incentives is by no means exhaustive. For example, it ignores the possibility that managers are directly concerned with the objective of beating thresholds (as in Burgstahler and Dichev 1997) or smoothing earnings (as described by Graham, Harvey, and Rajgopal [2005]). However, this description captures in a parsimonious manner the role of capital market concerns as a driver of earnings management. In addition, our characterization assumes that the manager's sole action is to report the firm's financial performance. Thus, we do not consider the possibility of real earnings management, as in Stein (1989).[8]

The firm's price in every period t is set by risk-neutral investors based on all publicly available information, which consists solely of the history of reports, $h_t \equiv \{r_1, r_2, ..., r_t\}$. The stock price in period t is set equal to investors' expectation of the firm's equity value plus the present value of future earnings. For simplicity, and without loss of generality, we assume that the interest rate used by investors in pricing future earnings is zero. The firm's price at time t as a function of prior reports is:

$$p_t = \mathbb{E}_t\left[\tilde{\theta}_t + \sum_{k=t+1}^{\infty} \tilde{e}_k\right], \quad (4)$$

which, given the AR1 structure of earnings, yields:

$$p_t = \mathbb{E}_t[\tilde{z}_t], \quad (5)$$

where $\mathbb{E}_t$ is the investors' expectation given the history of reports, h_t, and:

$$z_t \equiv \theta_t + \frac{\rho}{1-\rho} e_t, \quad (6)$$

is the manager's own assessment of the firm value in period t based on the history of economic earnings. To see this, notice that since the manager has more information than investors, by the law of iterated expectations, we have that:[9]

$$p_t = \mathbb{E}_t\left[\tilde{\theta}_t + \sum_{k=1}^{\infty} \tilde{e}_{t+k}\right] = \mathbb{E}_t\left[\mathbb{E}\left[\tilde{\theta}_t + \sum_{k=1}^{\infty} \tilde{e}_{t+k}\middle|\theta_t, e_t\right]\right] = \mathbb{E}_t\left[\tilde{\theta}_t + \frac{\rho}{1-\rho}\tilde{e}_t\right] = \mathbb{E}_t[\tilde{z}_t].$$

Equation (6) shows that we can think of the stock price as being the sum of two components: an estimate of the firm's equity, θ_t, and an estimate of the firm's future earnings $\frac{\rho}{1-\rho} e_t$. As we demonstrate below, investors will remain uncertain about both components of firm value throughout the manager's tenure. The model's parameters and the manager's payoff structure are assumed to be common knowledge.

IV. EQUILIBRIUM

We focus on equilibria in which the price is a linear function of all prior reports. In particular, we consider equilibria in which the price function takes the following form:

$$p_t = p_0 + \sum_{j=1}^{t} \alpha_j^t \left(r_j - \mu_j\right), \quad (7)$$

for some equilibrium parameters $\{\alpha_j^t, \mu_j\}$, where α_j^t reflects the sensitivity of the price at time t to a report issued at time j, and μ_j is the (unconditional) expected period j report.

[7] For simplicity, we assume that the manager's stock-based incentive is constant over time. However, this need not be constant over time, or even does not need to have a constant mean over time.

[8] In our view, this is not a significant limitation. The analysis would be qualitatively similar if the manager had to exert productive effort a_t and report the value of equity in every period. One could assume that earnings are: $e_t = a_t + \rho e_{t-1} + v_t$, where a_t is effort in period t, and the manager's payoffs are: $\mathbb{E}_t\left[\sum_{k=t}^{T} \delta^{k-t}\left(\tilde{p}_k - \frac{c}{2}\left(r_k - \tilde{\theta}_k - \tilde{\eta}_k\right)^2 - \frac{a_t^2}{2}\right)\right]$. In this model, the manager can affect the reports via effort and bias. For a given report, the manager chooses the mix effort/bias based on the relative marginal cost of each action. In a linear equilibrium, both actions—effort and bias—will be proportional to the sensitivity of price to future reports. However, the report's noise will not be affected by the presence of hidden effort. This is consistent with a contracting model in which the manager incentives are endogenous if one restricts attention to linear contracts.

[9] If the interest rate $r > 0$ were positive, then the value of the firm would be $z_t = \theta_t + \frac{\rho}{1+r-\rho} e_t$, and our empirical estimates $\hat{\rho}$ measures $\frac{\rho}{1+r}$, rather than ρ. In that way, the empirical estimates can be regarded as a lower bound for estimates of the persistence of economic earnings ρ.

First, we analyze a setting in which the manager has a finite horizon with the firm. Then, we analyze the infinite horizon setting as the limit case of the finite horizon setting, and study the steady-state equilibrium that arises when the manager has been with the firm—and will remain with the firm—for sufficiently long, such that his horizon does not affect the manager's reporting choices or investors' valuation of the firm. We structurally estimate the infinite horizon model in Section V.

Manager with a Finite Horizon

In this section, we assume that the manager's tenure is finite, i.e., $T < \infty$. When deciding about the bias in the report of the firm's equity, the manager takes into account the effect of his current report on the current biasing cost and on the trajectory of future prices and biasing costs. As time goes by, both the information environment (as reflected in investor uncertainty about firm value) and the manager's horizon change. This results in a reporting strategy that changes over the manager's tenure with the firm. We start by deriving the manager's equilibrium reporting strategy and the market pricing function. We then discuss the time-series properties of the reported earnings and the book-to-market ratios.

Reporting Strategy and Market Prices

Given the finite nature of the manager's optimization problem, we use backward induction to solve for the equilibrium. In the manager's last period with the firm, $t = T$, the manager's optimization problem (3) yields the optimal report:

$$r_T = \theta_T + \eta_T + \frac{\alpha_T^T}{c}.$$

The manager's utility at $t = T$ is, thus, given by:

$$U_T = p_0 + \sum_{j=1}^{T} \alpha_j^T \left(r_j - \mu_j\right) - \frac{1}{2c}\left(\alpha_T^T\right)^2.$$

Going backward one period, at $t = T - 1$, the manager maximizes:

$$U_{T-1} = p_{T-1} - \frac{c}{2}\left(r_{T-1} - \theta_{T-1} - \eta_{T-1}\right)^2 + \delta\mathbb{E}(U_T),$$

which yields an optimal report of:

$$r_{T-1} = \theta_{T-1} + \eta_{T-1} + \frac{\alpha_{T-1}^{T-1} + \delta\alpha_{T-1}^{T}}{c}.$$

By induction, one can see that for any $t \leq T$, the manager's reporting strategy is given by:

$$r_t = \theta_t + \eta_t + A_t^T, \tag{8}$$

where the bias in the reported equity (or, equivalently, the cumulative bias in the reported earnings), A_t^T, is given by:

$$A_t^T = \frac{\sum_{k=0}^{T-t} \delta^k \alpha_t^{t+k}}{c}. \tag{9}$$

Note that, so far, we have not used the properties of the distribution of $\{\theta_t, \eta_t\}$ in the derivation of the reporting strategy; thus, the structure of this reporting strategy is independent of the details of the distribution of $\{\theta_t, \eta_t\}$.[10]

From Equation (8), we can see that reported equity is distorted by two factors. First, reported equity is distorted by a random factor with mean zero, η_t, representing accounting and reporting distortions that are unknown to investors. The presence of this component results in investors not being able to perfectly infer the true value of the firm's equity, θ_t, or the current period's earnings, e_t. Second, the reported equity is distorted relative to the true equity by a deterministic component, A_t^T, which depends on both the manager's tenure with the firm, t, and his remaining horizon, $T - t$ (see Gerakos and Kovrijnykh [2013] for a similar result). In equilibrium, investors correctly anticipate A_t^T. The deterministic component, A_t^T, can be interpreted as the expected cumulative discretionary accruals in period t. The equilibrium bias A_t^T is the result of the manager trading off the benefits from influencing the current and subsequent stock prices against his current biasing costs. As expected, the higher the sensitivity of subsequent prices to the current report r_t, the larger the bias A_t^T.

[10] Of course, the values of α_t^{t+k} and, hence, of A_t^T are affected by the distribution of $\{\theta_t, \eta_t\}$.

Before concluding this section, we study the evolution of stock prices. Given risk-neutral pricing, the price function is a martingale (i.e., in period t, the best predictor of p_{t+1} is p_t) and can, thus, be expressed as:

$$p_t = p_{t-1} + \beta_t(r_t - \mathbb{E}_{t-1}(r_t)), \tag{10}$$

where:

$$\beta_t = \frac{Cov_{t-1}(r_t, z_t)}{Var_{t-1}(r_t)}.$$

The subscript in Cov_{t-1} and Var_{t-1} indicates that the variance and covariance are conditioned on the history of reports at time $t - 1$, i.e., $\{r_1, r_2, ..., r_{t-1}\}$. The parameter $\beta_t > 0$ is reminiscent of an earnings response coefficient (ERC) and changes over the manager's tenure. This confirms that the ERC β_t is not constant in an environment characterized by uncertainty and learning when the manager's horizon is finite.[11]

Earnings Quality

The model provides an opportunity to shed light on the notion of earnings quality and, thus, bridge (part of) the gap that separates theoretical and empirical research on earnings quality. Demski's (1973) analysis implies that a general definition of earnings quality is—to put it mildly—elusive. However, under some conditions, a higher informativeness is a desirable property of earnings reports.[12] Here, we adopt a definition of earnings quality that emphasizes the informativeness of reported earnings relative to the manager's information set. Formally, we define earnings quality as:

$$EQ_t \equiv -Var_t(z_t), \tag{11}$$

where z_t is the firm's economic value (see Equation (6)), which is known to the manager; and $Var_t\ (z_t)$ is the variance of z_t conditional on the history of reports available to investors at time t, $\{r_1, r_2, ..., r_t\}$.[13] $Var_t\ (z_t)$, therefore, measures the information asymmetry that has built up over time between the manager and investors due to the reporting process. Absent reporting noise, investors would perfectly infer the unmanaged earnings and equity, and hence, $Var_t\ (z_t)$ would be zero for all t. This would have resulted in a stock price that equals the firm's economic value, z_t.

EQ_t captures the informativeness of the reports by measuring the extent of uncertainty about the manager's information that investors are left with after they observe the history of reports. Of course, the amount of uncertainty investors are left with critically depends on the amount of uncertainty they faced before the reports were released. In that sense, EQ_t is a well-defined measure of informativeness only if the researcher is able to identify investors' prior uncertainty. Given the earnings process, investors' prior uncertainty is determined by the standard deviation of the firm's earnings innovations and the persistence of such innovations. Formally, this implies that we must control for $\{\sigma_v, \rho\}$ if EQ_t is to provide a meaningful measure of earnings quality. Using Equation (10), we can express earnings quality as:

$$EQ_t = -\left[Var_{t-1}(z_t) - \beta_t^2 Var_{t-1}(r_t)\right]. \tag{12}$$

EQ_t will evolve deterministically throughout the manager's tenure until the steady-state is reached. Figure 1 shows that the asymmetry of information between the manager and shareholders increases over time, at a decreasing pace, ultimately converging to the steady-state level. Information asymmetry increases over time, despite the presence of learning, because the firm's economic value z_t accumulates earnings innovations v_t over time. In the absence of reports, the overall investor uncertainty about the firm's value would explode. The presence of financial reporting, however, ensures that investor uncertainty stays finite.

Steady-State: A Basis for Empirical Analysis

We now turn to the analysis of the infinite horizon case; i.e., $T \to \infty$. We focus on the steady-state equilibrium, attained once the manager has stayed with the firm sufficiently long; i.e., $t \to \infty$. The infinite horizon setting is particularly amenable for empirical analysis because neither the manager's horizon nor the time he has been with the firm affects the joint distribution of reports and prices. Below, we derive and analyze the equilibrium of the steady-state. In Section V, we discuss our structural estimation of the steady-state and report our findings.

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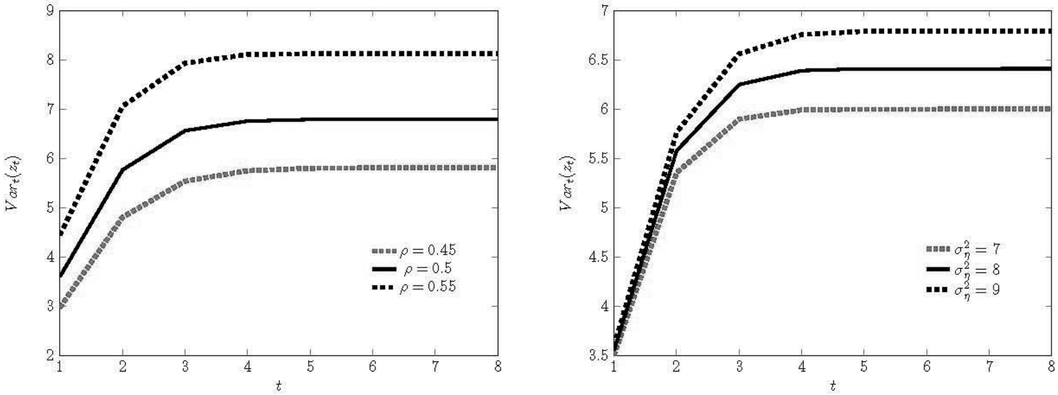
[11] This may explain perhaps why, empirically, the ERC has proven unstable (see, e.g., Collins and Kothari 1989). Similarly, note that the book-to-market ratio $\frac{r_t}{p_t}$ evolves throughout the manager's tenure.

[12] In fact, Blackwell's (1953) informativeness is the only generally desirable property of accounting systems in the setting of Demski (1973).

[13] Recall that z_t is the value of the firm, as perceived by the manager, given the firm's equity, θ_t, and persistence of earnings, ρ.

FIGURE 1
The Effect of $\{\rho, \sigma_\eta\}$ on the Evolution of Information Asymmetry Over Time



(a) $(\sigma_v = 1, \sigma_\eta = 1)$ (b) $(\sigma_v = 1, \rho = 0.5)$

The residual variance (i.e., the reciprocal of earnings quality) increases over time, but it is bounded from above by the steady-state level. The serial correlation of earnings ρ and the volatility of reporting noise σ_η increase the residual variance of firm value, thus reducing earnings quality.

Let $\boldsymbol{\omega}_t \equiv \{\theta_t, e_t, \eta_t\}$ be the random vector containing the firm's fundamentals $\{\theta_t, e_t\}$ and the reporting noise η_t. The vector $\boldsymbol{\omega}_t$ is the state of the world about which investors are uncertain. The firm's financial reporting contributes to resolving part of that uncertainty, but does so imperfectly because of investor uncertainty about reporting distortions.

In general, the evolution of investor uncertainty, as measured by $Var_t\,(\boldsymbol{\omega}_t)$, evolves according to the formula:

$$Var_t(\boldsymbol{\omega}_t) = Var_{t-1}(\boldsymbol{\omega}_t) - \frac{Cov_{t-1}(\boldsymbol{\omega}_t, r_t)' Cov_{t-1}(\boldsymbol{\omega}_t, r_t)}{Var_{t-1}(r_t)}. \tag{13}$$

Investor uncertainty depends on the time t and on the model's primitive parameters, $\boldsymbol{\varphi} \equiv \left(\sigma_v^2, \sigma_\eta^2, \rho\right)$. However, in steady-state, the variance and the correlation structure of investors' beliefs are time-invariant. In this case, investor uncertainty $Var_t\,(\boldsymbol{\omega}_t)$ can be represented by five numbers, as reflected in the following matrix:

$$Var_t(\boldsymbol{\omega}_t) = \begin{bmatrix} \bar{\sigma}_\theta^2 & \bar{\sigma}_{\theta e} & \bar{\sigma}_{\theta\eta} \\ \cdot & \bar{\sigma}_e^2 & \bar{\sigma}_{\theta e} \\ \cdot & \cdot & \bar{\sigma}_\eta \end{bmatrix}$$

that depend on the model's primitive parameters $\boldsymbol{\varphi}$. (Notice that we use $\bar{\sigma}^2$ for the posterior steady-state variance and $\bar{\sigma}^2$ for the unconditional variance.)

The following lemma characterizes the ergodic distribution of investors' beliefs.

Lemma 1: In steady-state, the variance-covariance matrix of the firm's fundamentals satisfies:

$$-\bar{\sigma}_{\theta\eta} = \bar{\sigma}_\eta^2 = \bar{\sigma}_\theta^2 \tag{14}$$

$$-\bar{\sigma}_{e\eta} = \bar{\sigma}_{e\theta}. \tag{15}$$

Thus, investor uncertainty $Var_t\,(\boldsymbol{\omega}_t)$ can be characterized by $\left\{\bar{\sigma}_\theta^2, \bar{\sigma}_e^2, \bar{\sigma}_{\theta e}\right\}$ that solve the following system of equations:

$$\bar{\sigma}_\theta^2 = \bar{\sigma}_\theta^2 + \rho^2\bar{\sigma}_e^2 + 2\rho\bar{\sigma}_{\theta e} + \sigma_v^2 - \frac{\left(\bar{\sigma}_\theta^2 + 2\rho\bar{\sigma}_{\theta e} + \rho^2\bar{\sigma}_e^2 + \sigma_v^2\right)^2}{\bar{\sigma}_\theta^2 + \rho^2\bar{\sigma}_e^2 + 2\rho\bar{\sigma}_{\theta e} + \sigma_v^2 + \sigma_\eta^2} \tag{16}$$

$$\bar{\sigma}_e^2 = \rho^2\bar{\sigma}_e^2 + \sigma_v^2 - \frac{\left(\rho\bar{\sigma}_{\theta e} + \rho^2\bar{\sigma}_e^2 + \sigma_v^2\right)^2}{\bar{\sigma}_\theta^2 + \rho^2\bar{\sigma}_e^2 + 2\rho\bar{\sigma}_{\theta e} + \sigma_v^2 + \sigma_\eta^2} \tag{17}$$

$$\bar{\sigma}_{\theta e} = \rho^2\bar{\sigma}_e^2 + \rho\bar{\sigma}_{\theta e} + \sigma_v^2 - \frac{\left(\bar{\sigma}_\theta^2 + 2\rho\bar{\sigma}_{\theta e} + \rho^2\bar{\sigma}_e^2 + \sigma_v^2\right)\left(\rho\bar{\sigma}_{\theta e} + \rho^2\bar{\sigma}_e^2 + \sigma_v^2\right)}{\bar{\sigma}_\theta^2 + \rho^2\bar{\sigma}_e^2 + 2\rho\bar{\sigma}_{\theta e} + \sigma_v^2 + \sigma_\eta^2}. \tag{18}$$

The equalities in (14) and (15) show that in steady-state, investors have extracted all possible information from priors, such that the investors are equally uncertain about fundamentals and about the reporting noise. This somewhat counterintuitive result is required for the stability of the steady-state: If this condition were not satisfied, then the posterior variance of the fundamentals and the noise would change over time (although in opposite directions). Notice again that investor uncertainty about fundamentals is determined endogenously based on the history of reports, whereas investor uncertainty about the reporting noise is exogenous, due to the *iid* nature of η_t (we relax this assumption in Section VI).

Given the complexity of the expressions in Lemma 1, we proceed by presenting, as an example, the case when earnings display no persistence; i.e., when ρ is zero.

Example 1: Consider the case when earnings are *iid*, $\rho = 0$. Then, investors' uncertainty about the following period's report is the sum of their uncertainty about the current book value θ, next-period's earnings v, and next-period's reporting noise η. That is:

$$Var_{t-1}(r_t) = \bar{\sigma}_\theta^2 + \sigma_v^2 + \sigma_\eta^2,$$

and investor uncertainty about the state of the world $\boldsymbol{\omega}_t$, $Var_t(\boldsymbol{\omega}_t)$, is characterized by:

$$\bar{\sigma}_\theta^2 = \frac{\left(\sqrt{\sigma_v^2\left(\sigma_v^2 + 4\sigma_\eta^2\right)} - \sigma_v^2\right)}{2},$$

$$\bar{\sigma}_e^2 = \left(1 - \frac{\sigma_v^2}{Var_{t-1}(r_t)}\right)\sigma_v^2,$$

$$\bar{\sigma}_{\theta e} = \frac{\sigma_\eta^2}{Var_{t-1}(r_t)}.$$

The steady-state earnings quality EQ_∞ is simply $Var_t(z_t) = \bar{\sigma}_\theta^2$.

We now return to the general case in which ρ is not zero. The steady-state analysis has two main implications. First, denoting the steady-state bias by $A_\infty \equiv \lim_{t\to\infty} A_t^{t+T}$, the manager's report in period t can be written as:

$$r_t = A_\infty + \theta_t + \eta_t. \tag{19}$$

Second, in steady-state, the coefficients of the price function are independent of time; therefore, the change in price can be written as a linear function of the surprise in reported earnings:

$$\Delta p_t = \beta(r_t - \mathbb{E}_{t-1}(r_t)), \tag{20}$$

where $\beta \equiv \frac{Cov_{t-1}(z_t, r_t)}{Var_{t-1}(r_t)}$. On the surface, the above price Equation (20) might seem trivial, but the expected report, $\mathbb{E}_{t-1}(r_t)$, depends on the entire history of reports, given the persistence of earnings. However, some algebra reveals that the price dynamics can be written in a compact manner as a function of the reported earnings and the market values of equity in the previous two periods. Denoting by $\gamma \equiv \frac{\partial \mathbb{E}_t(e_t)}{\partial r_t}$ the sensitivity of investors' expectations about the true earnings with respect to r_t, we obtain the following result:

Lemma 2: In steady-state, the price change is characterized by the following difference equation:

$$\Delta p_t = \beta(r_t - p_{t-1}) - \beta\rho(r_{t-1} - p_{t-2}) + \rho\left(1 + \gamma\frac{\rho}{1-\rho}\right)\Delta p_{t-1}. \tag{21}$$

In steady-state, the price change depends upon two lags of the difference between book and market value of equity and a lag of the price change (which has a momentum flavor to it). In the absence of persistence ($\rho = 0$), only the first term would survive. This term captures the report's surprise relative to market expectations. The other two terms in the above equation arise because of persistence, and they capture the effect of the report on the market's assessment of the firm's discounted cash flows.

The price function is an important ingredient of the empirical analysis that can be exploited to obtain information about some of the model's primitive parameters. Moreover, the linearity of the price equation means we could, in principle, estimate it by ordinary least squares (OLS).

In addition to revising their expectations of z_t and e_t, investors also revise their expectations about the reporting noise η_t when observing a report r_t. We denote the sensitivity of investors' expectations about the reporting noise by $\kappa \equiv \frac{\partial \mathbb{E}_t(\eta_t)}{\partial r_t}$. An analysis of the steady-state distribution of prices and reports reveals that the response coefficients $\{\beta, \gamma, \kappa\}$ are linked to the model parameters $\left\{\sigma_v^2, \sigma_\eta^2, \rho\right\}$ via the following equations.

Lemma 3: In steady-state, the response coefficients $\{\beta, \gamma, \kappa\}$ are given by:

$$\beta = \frac{\bar{\sigma}_\theta^2 + \rho^2\bar{\sigma}_e^2 + 2\rho\bar{\sigma}_{\theta e} + \sigma_v^2 + \frac{\rho\left(\rho^2\bar{\sigma}_e^2 + \rho\bar{\sigma}_{\theta e} + \sigma_v^2\right)}{1-\rho}}{Var_{t-1}(r_t)},$$

$$\gamma = \frac{\rho^2\bar{\sigma}_e^2 + \rho\bar{\sigma}_{\theta e} + \sigma_v^2}{Var_{t-1}(r_t)},$$

$$\kappa = \frac{\sigma_\eta^2}{Var_{t-1}(r_t)},$$

where:

$$Var_{t-1}(r_t) = \bar{\sigma}_\theta^2 + \rho^2\bar{\sigma}_e^2 + 2\rho\bar{\sigma}_{\theta e} + \sigma_v^2 + \sigma_\eta^2.$$

and $\left\{\bar{\sigma}_\theta^2, \bar{\sigma}_e^2, \bar{\sigma}_{\theta e}\right\}$ are given in Lemma 1.

These parameters characterize the sensitivity of investors' beliefs to a report r_t. β is particularly significant for accounting research, as it represents an ERC, namely, the sensitivity of stock returns to reported earnings. Given the above analysis of the steady-state equilibrium, we can now proceed with the empirical analysis.

V. EMPIRICAL ANALYSIS

Estimation Method and Identification

In this section, we estimate the steady-state equilibrium described in Section IV. In particular, we estimate the model parameters—including the persistence of earnings, ρ, the variance of earnings innovations, σ_v^2, and the variance of reporting noise, σ_η^2—based on the joint distribution of book value of equity, r_t, and stock price, p_t. By contrast, c cannot be identified based on stock prices and reported earnings data. Since c only affects the mean of reported earnings, we cannot separate it from the mean of economic earnings.

The model predicts a deterministic relation between the history of reported earnings and the stock price. In practice, prices are noisy and they move for reasons that are independent of reports, such as liquidity shocks, noise trading, or market sentiments. To accommodate this aspect of the data, we assume that the stock price is affected by random *iid* shocks, such as liquidity trades. This unobserved random variable, denoted by $\zeta_t \sim N\left(0, \sigma_\zeta^2\right)$, will represent the error term. We denote the actual price (i.e., the price observed by the econometrician) by $\tilde{p}_t$, where:

$$\tilde{p}_t = p_t + \zeta_t.$$

Accordingly, the change in the actual price is denoted by:

$$\Delta\tilde{p}_t = \Delta p_t + \Delta\zeta_t.$$

Substituting these expressions into the price Equation (21) yields our econometric model:

$$\Delta\tilde{p}_t = \beta(r_t - \tilde{p}_{t-1}) - \beta\rho(r_{t-1} - \tilde{p}_{t-2}) + \rho\left(1 + \frac{\rho\gamma}{1-\rho}\right)\Delta\tilde{p}_{t-1} + \eta_t, \tag{22}$$

where the error term n_t is a moving average process including two lags of the price noise:

$$n_t = \zeta_t\left(\beta - 1 - \frac{\gamma\rho^2}{1-\rho} - \rho\right)\zeta_{t-1} + \left((1-\beta)\rho + \frac{\rho^2\gamma}{1-\rho}\right)\zeta_{t-2}. \tag{23}$$

In the presence of the noise in price ζ, the price equation becomes an autoregressive-moving average model with exogenous inputs, or ARMAX(1,2,2).

Formally, we estimate the model by Maximum Likelihood (MLE) and maximize the joint likelihood of the time-series of observed prices and reports, $\{\tilde{p}_t, r_t\}_{t=1}^{T}$, with respect to the model's parameters, including the variance of the noise in prices, σ_ζ^2. The log-likelihood, for an arbitrary firm i is:

$$\mathcal{L}_i = \log f\left(\tilde{\mathbf{p}}_i, \mathbf{r}_i | \rho, \sigma_v^2, \sigma_\eta^2, \sigma_\zeta^2\right).$$

With slight abuse of notation, we write the likelihood as the product of the density of prices conditional on reports and the marginal density of reports:

$$\mathcal{L}_i = \log\left[f_{p|r}\left(\tilde{\mathbf{p}}_i |\ \mathbf{r}_i; \rho, \sigma_v^2, \sigma_\eta^2, \sigma_\zeta^2\right) \cdot f_r\left(\mathbf{r}_i; \rho, \sigma_v^2, \sigma_\eta^2\right)\right].$$

We partition the sample into three groups based on size (see more details below). We estimate the model at the group level, hence, the MLE estimator for a group size $j \in \{1, 2, 3\}$ is defined as:

$$\boldsymbol{\varphi}_j^{MLE} \equiv \arg\max \sum_{i=1}^{I_j} \mathcal{L}_i\left(\varphi_j\right),$$

where I_j denotes the number of firms in group j. In essence, our model is a two-dimensional autoregressive-moving average model, VARMA, represented by the vector $(\tilde{p}_t, r_t)$.

Identification Strategy

The likelihood of $(\tilde{p}_t, r_t)$ depends on the likelihoods of two independent variables (m_t, n_t), where n_t is computed based on Equation (22) as:

$$\begin{aligned} n_t &= \Delta\tilde{p} - \left(\beta(r_t - \tilde{p}_{t-1}) - \beta\rho(r_{t-1} - \tilde{p}_{t-2}) + \rho\left(1 + \frac{\rho\gamma}{1-\rho}\right)\Delta\tilde{p}_{t-1}\right) \\ &= \zeta_t + \left(\beta - 1 - \frac{\gamma\rho^2}{1-\rho} - \rho\right)\zeta_{t-1} + \left((1-\beta)\rho + \frac{\rho^2\gamma}{1-\rho}\right)\zeta_{t-2}, \end{aligned}$$

and m_t can be constructed as follows:[14]

$$m_t \equiv \Delta r_t - \rho\Delta r_{t-1} = v_t + \eta_t - (\rho + 1)\eta_{t-1} + \rho\eta_{t-2}.$$

This equation reveals that the identification of the level of the variances $\left(\sigma_v^2, \sigma_\eta^2, \rho\right)$ can, in principle, be obtained by looking at the dynamics of reported earnings Δr_t. Indeed, our model assumptions imply that reported earnings Δr_t is an ARMA process, where the autoregressive part, $\rho\Delta r_{t-1}$, is driven by the persistence of economic earnings shocks, ρ, and the moving average part, $(\eta_t - (\rho - 1)\ \eta_{t-1} + \rho\eta_{t-2})$, is driven by the reversal of the reporting bias (such reversal is caused by the assumption that the cost of misreporting depends on the cumulative bias in reported equity).

The variance-covariance matrix of $\{m_t\}_{t=3}^{T}$ is fully described by *Var* (m_t), *Cov* (m_t, m_{t-1}), and *Cov* (m_t, m_{t-2}). All other elements are zero, since m_t only includes elements up to a lag of two. Specifically, the variance-covariance matrix of the entire vector of firm i's adjusted reports, $\{m_t\}_{t=1}^{T}$, takes the following form:

$$\sum(m) = \begin{bmatrix} s_0 & s_1 & s_2 & & 0 & 0 \\ s_1 & \ddots & \ddots & \ddots & & 0 \\ s_2 & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & s_2 \\ 0 & & \ddots & \ddots & \ddots & s_1 \\ 0 & 0 & & s_2 & s_1 & s_0 \end{bmatrix} \tag{24}$$

where:

[14] In the estimation, we allow $E\zeta_t$ to be different from zero to avoid a trivial misspecification. This is consistent with the model (in fact, it is a slight generalization of the model), given that the reporting strategy of the manager does not depend on the mean of the reporting noise.

$$s_0 \equiv Var(m_t) = \sigma_v^2 + 2\left(\rho^2 + \rho + 1\right)\sigma_\eta^2;$$

$$s_1 \equiv Cov(m_t, m_{t-1}) = -(\rho + 1)^2 \sigma_\eta^2; \text{ and}$$

$$s_2 \equiv Cov(m_t, m_{t-2}) = \rho\sigma_\eta^2.$$

Similarly, the variance-covariance matrix of $\{n_t\}_{t=1}^T$ is fully described by $Var\ (n_t)$, $Cov\ (n_t, n_{t-1})$, and $Cov\ (n_t, n_{t-2})$. All other elements are again zero, since n_t also only includes elements up to a lag of two. In particular, from the expression of n_t in Equation (23), the variance-covariance matrix of $\{n_t\}_{t=1}^T$ is given by:

$$\sum(n) = \begin{bmatrix} u_0 & u_1 & u_2 & & 0 & 0 \\ u_1 & \ddots & \ddots & \ddots & & 0 \\ u_2 & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & u_2 \\ 0 & & \ddots & \ddots & \ddots & u_1 \\ 0 & 0 & & u_2 & u_1 & u_0 \end{bmatrix} \tag{25}$$

$$u_0 \equiv Var(n_t) = \left(1 + \left(\beta - 1 - \frac{\gamma\rho^2}{1-\rho} - \rho\right)^2 + \left((1-\beta)\rho + \frac{\gamma\rho^2}{1-\rho}\right)^2\right)\sigma_\zeta^2;$$

$$u_1 \equiv Cov(n_t, n_{t-1}) = \left(\beta - 1 - \frac{\gamma\rho^2}{1-\rho} - \rho\right)\left(1 + (1-\beta)\rho + \frac{\gamma\rho^2}{1-\rho}\right)\sigma_\zeta^2; \text{ and}$$

$$u_2 \equiv Cov(n_t, n_{t-2}) = \left((1-\beta)\rho + \frac{\gamma\rho^2}{1-\rho}\right)\sigma_\zeta^2.$$

The estimation technique exploits the structure of the variance-covariance matrices in (24) and (25). MLE will attempt to find values for the model's parameters that make the sample covariances for lags of three or more as close to zero as possible.

While the model predicts that the mean of (m_t, n_t) is zero, we allow for non-zero means in the estimation procedure to increase the flexibility of the empirical model. Observe that, qualitatively, the predictions of the model do not change when one allows for non-zero $\mathbb{E}(\eta_t)$ and $\mathbb{E}(\zeta_t)$. Solving the model with $\mathbb{E}(\zeta_t) \neq 0$ generates the same likelihood that we actually used in the estimation.[15]

We compute the negative log-likelihoods of the multi-variate normal of (m_t, n_t) using MATLAB's ecmnobj function. Since m_t and n_t are distributed independently of each other, the log-likelihoods can be computed separately and summed up.

The log-likelihood function of firm i can be written as:

$$\mathcal{L}_i = -\frac{T-2}{2}\log\left(\sqrt{\left|\sum(\mathbf{m})\right|}\right) - \frac{T-2}{2}\log\left(\sqrt{\left|\sum(\mathbf{n})\right|}\right) - \frac{(\mathbf{m}_i - \boldsymbol{\mu}_m)' \sum(\mathbf{m})^{-1}(\mathbf{m}_i - \boldsymbol{\mu}_m)}{2} - \frac{(\mathbf{n}_i - \boldsymbol{\mu}_n)' \sum(\mathbf{n})^{-1}(\mathbf{n}_i - \boldsymbol{\mu}_n)}{2}.$$

If MATLAB's Cholesky decomposition chol of either $\Sigma(\mathbf{m})$ or $\Sigma(\mathbf{n})$ indicates that one of the matrices is not positive definite or the parameter estimates are not admissible (i.e., negative variances or $|\rho| \geq 1$), we set the value of the negative log-likelihood to infinity. Using MATLAB's global optimization routine particleswarm, we obtain estimates for the model's parameters $\sigma_v^2, \sigma_\eta^2, \rho, \sigma_\zeta^2$. In order to compute asymptotic standard errors, we calculate the inverse of the Hessian of the log-likelihood evaluated at the parameter estimates using MATLAB's hessian function, and use the Delta Method to compute the standard errors of functions of the underlying model parameters, such as the response coefficients and our earnings quality metric.

[15] Estimating a slightly more general model that nests the original model as a special case allows one to avoid trivial misspecification errors, and offers the possibility to test some of the model's assumptions.

As Box, Jenkins, Reinsel, and Ljung (2015) note, the identification of VARMA models can be a delicate issue: There may exist multiple VARMA representations with different parameters that give rise to the same coefficients in the infinite moving average representation, thus being informationally equivalent (see Hannan and Deistler 1988).[16] However, the identification of our model is relatively straightforward because the adjusted report process m_t is independent of the process of n_t and can, therefore, be treated as a univariate ARMA process.[17]

This estimation technique allows us to parse out two components of investor uncertainty: (1) fundamental economic uncertainty, defined as the uncertainty of the manager about the firm's future earnings, and (2) information asymmetry between the manager and investors due to reporting or accounting distortions. We will also be able to compute our measure of earnings quality and an estimated level of bias. However, our structural estimation is not immune to measurement error and correlated omitted variables due to model misspecification. If the model omits first-order effects on the variance-covariance of economic earnings, reported earnings, and prices, the estimates will be invalid. For example, the model includes neither managers' incentives to manipulate earnings to reach certain targets nor information sources other than the manager's earnings report. If such incentives or alternative information sources significantly alter the reporting behavior of the manager and the variance-covariance structure of the book value and market value of equity, the model will be misspecified. Even though it is unlikely that the model literally describes all factors influencing this variance-covariance structure, we think it has the potential to complement the insights of the so-called "reduced-form" approach in this area and provide a starting point to studying the relations between financial reporting and market prices using a structural approach.

Identification Based on Reduced-Form Price Equation

In the following paragraph, we discuss the potential and limitation of identifying the model parameters based on the reduced-form price equation in Lemma 2. As shown in Equations (21) and (22), the price equation is linear in two lags of the difference between book and market value of equity and a lag of the price change. We wish to estimate the parameters $\left\{\sigma_v^2, \sigma_\eta^2, \rho\right\}$ and, based on these estimates, compute the associated response coefficients $\{\beta, \gamma, \kappa\}$. By estimating the price equation:

$$\Delta\tilde{p}_t = b_1(r_t - \tilde{p}_{t-1}) + b_2(r_{t-1} - \tilde{p}_{t-2}) + b_3\Delta\tilde{p}_{t-1} + n_t, \tag{26}$$

one can obtain estimates of the reduced-form parameters, denoted $\mathbf{b} \equiv \{b_1, b_2, b_3\}$. There are several ways to estimate the price equation in (26). A simple, but naïve, approach is to estimate it via OLS and use the resulting estimates of $\mathbf{b}$ to back out estimates of the response coefficients $\left\{\hat{\beta}, \hat{\gamma}\right\}$ and the serial correlation coefficient $\hat{\rho}$. Unfortunately, this approach is flawed given our econometric model, where the observed price contains variation that is unexplained by the model, ζ_t. The reason is that it ignores the fact that some explanatory variables in the regression, such as $\Delta\tilde{p}_{t-1}$, are correlated with the error term n_t, as in the classic omitted correlated variables problem. As is well known, this issue renders the OLS estimates of (26) inconsistent. Alternatively, one could estimate the price using an iterative method, such as two-stage least squares (2SLS) in the spirit of Durbin (1960). These iterative methods are consistent, but they are not efficient. Alternatively, Maximum Likelihood Estimation is both efficient and allows the researcher to explicitly handle the error term structure in (23). Based on the reduced-form estimates $\{b_1, b_2, b_3\}$, one can infer some of the model's parameters. Specifically, one can invert the reduced-form coefficients to obtain estimates for the persistence ρ and the response coefficients β, γ, and κ by solving:

$$\hat{\beta} = \hat{b}_1;\ \hat{\rho} = -\frac{\hat{b}_2}{\hat{b}_1};\ \hat{\gamma} = \frac{\left(\hat{b}_3 - \hat{\rho}\right)(1-\hat{\rho})}{\hat{\rho}^2};\ \hat{\kappa} = 1 - \hat{\beta} + \frac{\hat{\rho}}{1-\hat{\rho}}\hat{\gamma}. \tag{27}$$

We still need to find the variances σ_v^2 and σ_η^2. Now, according to Lemma 3, the response coefficients β and γ are themselves functions of the two variances σ_η^2 and σ_v^2, so the price function does provide information about these parameters. Unfortunately, the price equation only allows us to identify the ratio of the variances $\frac{\sigma_v^2}{\sigma_\eta^2}$, but not the variances themselves. The reason is that β and γ depend upon the signal-to-noise ratio $\frac{\sigma_v^2}{\sigma_\eta^2}$, but are independent of the level of these variances. Intuitively, the stock price could be relatively insensitive to reports for two reasons: either there is little uncertainty about fundamentals to start with, or the report is too noisy to resolve that uncertainty. Hence, based on the dynamic behavior of prices, we cannot distinguish between these two hypotheses. To distinguish σ_v^2 from σ_η^2, we need more information than that provided by the evolution of prices. This

[16] For identifiability of the parameters, the matrix operators of the VARMA(p,q) representation $\Phi\ (B)\ \mathbf{Z}_t = \Theta\ (B)a_t$ should be left-coprime and satisfy a rank condition (see Box et al. 2015, 528). They should also satisfy the classic stationarity and invertibility conditions.

[17] We verify the identification and the speed of convergence of MLE with a Monte Carlo Simulation. We use 100 replications of a single firm with 25, 50, 75, and 100 observations. The speed of convergence is reasonable, with the Root Mean Squared Error being less than 0.1 for all variables for 50 observations, and less than 0.05 for 100 observations when the model's standard deviations are set equal to 1. Details including the Root Mean Square Errors based on the simulation are available upon request.

TABLE 1

Descriptive Statistics (n = 101, 393)

	Mean	Median	Std. Dev.	Min.	Max.
Small Firms (No. Firms = 2,114; No. Obs. = 33,775)					
p_t	63	38	77	0	1,481
r_t	44	28	48	0	561
MB-Ratio	2.03	1.31	2.48	0.00	29.70
Medium Firms (No. Firms = 1,925; No. Obs. = 33,824)					
p_t	565	378	592	0	11,229
r_t	327	218	341	0	6,731
MB-Ratio	2.41	1.77	2.30	0.00	29.72
Large Firms (No. Firms = 1,671; No. Obs. = 33,794)					
p_t	15,074	4,124	36,610	0	651,834
r_t	8,137	2,125	21,567	0	350,938
MB-Ratio	2.90	2.16	2.55	0.01	29.91

r_t and p_t denote reported equity and market value of equity, respectively, in \$ millions. The market value of equity is calculated as the closing price at the end of the fiscal period times the number of shares outstanding. Both book value and market value of equity have been adjusted by adding the cumulative dividends declared in the sample period 1990–2016. Market-to-book (*MB-Ratio*) is calculated as the market value of equity divided by the book value of equity without adjusting for dividends. We partition the sample based on average market value of equity, p_t, during the sample period. Small (medium, large) firms comprise firms whose average market value of equity is in the bottom (medium, top) 33 percentiles.

extra information can be gleaned from the time-series of reports $\{r_t\}_{t=1}^{T}$ and requires moments beyond those captured in Equation (26).

Data

The model requires data on the book value of equity and stock price. We use annual data and delete firms with a negative market value or book value of equity and a market-to-book ratio above 30 during the sample period. In order to accommodate the model's assumption of no dividend payments, we add any dividend payment in the sample period to both the book value and market value of equity. Data on stock prices, number of shares outstanding, common stockholders' equity, and dividends are obtained from Compustat for the period from 1990 to 2016.

Our model relies extensively on estimates of variances. As a result, we expect our parameter estimates to be sensitive to variations in firm size. To account for this likely variation in estimates, we partition the sample based on average market value of equity adjusted for dividend payments, p_t, during the sample period. Small (medium, large) firms comprise firms whose average market value of equity is in the bottom (medium, top) 33 percentiles. Table 1 provides descriptive statistics for the data used to estimate the model. Each of the three size categories has around 33,800 annual observations, with numbers varying slightly across the three groups as a result of the entire time-series of a firm being allocated to one size group. The number of firms is smallest for the group of large firms, as larger firms tend to be part of the sample for a longer period.

Since we also expect some heterogeneity among industries due to different technologies, investment opportunities, and competitive situations, we also require the SIC code to classify firms following the Fama-French 12 industry classification. For the estimation, we demean both the dependent and independent variables, Δr_t and Δp_t, based on the Fama-French 12 industry classification (see, e.g., Gormley and Matsa 2014). In addition, our estimation allows all random variables to have non-zero mean.

Findings

Table 2 provides the MLE parameter estimates of the model's four parameters for small, medium, and large firms: the standard deviation of economic earnings innovations, σ_v; the persistence of economic earnings, ρ; the standard deviation of reporting noise, σ_η; and the standard deviation of the unexplained price variations, σ_ζ. All parameters are significant at the 1 percent level.

As expected, the variances in Table 2 are monotonically increasing in size. The standard deviation of economic earnings innovations σ_v is of the same order of magnitude as the standard deviation of reporting noise σ_η, but, depending on firm size, an order of magnitude smaller than the standard deviation of unexplained price variations σ_ζ. This suggests two things. First, our results are consistent with longstanding empirical evidence that the volatility of prices is excessive relative to the volatility of

TABLE 2

MLE Estimates of Model Parameters

	Small	**Medium**	**Large**
σ_v	0.122	0.764	16.106
	(0.001)	(0.008)	(0.155)
σ_η	0.062	0.489	13.255
	(0.001)	(0.005)	(0.104)
ρ	0.495	0.585	0.818
	(0.008)	(0.007)	(0.002)
σ_ζ	0.509	3.908	194.143
	(0.002)	(0.016)	(0.786)

We report the persistence of economic earnings, ρ, the standard deviation of economic earnings shocks, σ_v, and the standard deviation of reporting noise, σ_η, as well as the standard deviation of unexplained price variations, σ_ζ. σ_v, σ_η, and σ_ζ are in \$100 millions. Asymptotic Standard Errors (in parentheses) are computed based on the expected inverse of the Information Matrix.

fundamentals or the available information about such fundamentals (e.g., Shiller 1981; LeRoy and Porter 1981). Second, and more central to this study, it conveys that there is significant reporting noise in reported earnings and equity.

The average persistence of economic earnings ρ across size categories in Table 2 is approximately 0.6, suggesting that earnings are fairly sticky even on an annual basis. Earnings shocks are persistent, particularly among large firms, whose serial correlation coefficient is 0.818 versus 0.495 for small firms.

A large literature in accounting has focused on the time-series properties of reported earnings, as opposed to economic earnings (see, e.g., Ball and Watts 1972; Albrecht, Lookabill, and McKeown 1977). This literature has documented that reported earnings follow a random walk with a trend. Using an ARMA model, Gerakos and Kovrijnykh (2013) estimate the serial autocorrelation of reported earnings to be 0.362. This (compared to our estimates) relatively low value can be reconciled with our findings when we ignore the possibility of accounting and reporting distortions. Specifically, when we estimate the model imposing $\sigma_\eta = 0$, we obtain an estimate of ρ equal to 0.2 for small and medium firms and 0.5 for large firms (see Table 4), consistent with evidence in prior empirical literature that does not distinguish between economic earnings and misreporting.[18]

Our estimates strongly reject the null hypothesis of zero reporting noise ($\sigma_\eta = 0$) for all three size categories (see Table 3). If we ignored the possibility of accounting and reporting distortions, not only would our estimates of economic earnings persistence be downward-biased, but our estimates of the variance of economic earnings innovations would also be upward-biased. Specifically, when we estimate the model imposing $\sigma_\eta = 0$, we obtain estimates of σ_v that are between 30 percent and 80 percent larger than the estimates of the unrestricted model (see Table 4).[19] These results illustrate the importance of considering the implications of misreporting before making inferences about economic earnings based on financial statement data.

Based on the parameter estimates in Table 2, we can estimate the price's reduced-form coefficients, as reported in Table 5. The model predicts that the change in price is a function of the book value and market value of equity in the previous two years, as described in Equation (26). The estimates of b_1, b_2, and b_3 have the expected sign and are significant at the 1 percent level for all three size categories. One might think that the reduced-form price coefficients can be obtained directly from an OLS estimation of the price equation. Unfortunately, such estimates are inconsistent, because the residual of the price equation is correlated with the covariates.

Next, we calculate estimates of investors' response coefficients. When observing a report r_t, investors revise their expectations about firm value z_t, this period's economic earnings e_t, and reporting noise ζ_t. Specifically, β is the sensitivity of stock price to reported equity or, equivalently, earnings; γ is the sensitivity of investors' expectation of this period's economic earnings to reports; and κ captures the sensitivity of investors' expectations of reporting noise to reports.

18 In the estimation, we allow n_t and m_t to have non-zero means. Our estimate for $\mathbb{E}[\tilde{n}_t] = \beta(1-\rho)\mathbb{E}\left[\tilde{\zeta}_t\right]$ is statistically significant for all three size groups. This result can be due to the fact that some of our pricing assumptions are too restrictive (risk neutrality, market efficiency, rationality, etc.). The estimates (standard errors) are 0.121 (0.0026) for small firms, 1.497 (0.020) for medium firms, and 45.193 (0.722) for large firms. In contrast, our estimate for $\mathbb{E}[\tilde{m}_t] = \mathbb{E}[\tilde{v}_t]$ is statistically insignificant for all three size groups. The estimates (standard errors) are 0.000 (0.001) for small firms, 0.005 (0.005) for medium firms, and 0.163 (0.097) for large firms.

19 When we impose restriction $\sigma_\eta = 0$, our estimate of $\mathbb{E}[\tilde{n}_t] = \beta(1-\rho)\mathbb{E}\left[\tilde{\zeta}_t\right]$ is statistically significant for all three size groups. The estimates (standard errors) are 0.159 (0.003) for small firms, 2.037 (0.024) for medium firms, and 66.331 (1.198) for large firms. In contrast, our estimate of $\mathbb{E}[\tilde{m}_t] = \mathbb{E}[\tilde{v}_t]$ is again statistically insignificant for all three size groups. The estimates (standard errors) are 0.000 (0.001) for small firms, 0.006 (0.007) for medium firms, and 0.309 (0.169) for large firms.

TABLE 3

Likelihood Ratio Test (H_0: $\sigma_\eta = 0$)

	Log-lik	Log-lik ($\sigma_\eta = 0$)	p-value
Small	−140,730	−141,080	0.000
Medium	−125,380	−125,970	0.000
Large	−209,130	−211,600	0.000

Estimates of investors' response coefficients are again consistent with the presence of reporting noise. In a world where earnings were reported truthfully (e.g., Ohlson 1995), the earnings response coefficient β would equal $1 + \frac{\rho}{1-\rho}$, which is equal to approximately, 2.0, 2.4, and 5.5, for small, medium, and large firms, respectively, given our estimates of ρ in Table 2. In contrast, the estimates of the earnings response coefficient β varies from 1.5 for small firms to 3.1 for large firms and hence is, on average, 30 percent smaller than the theoretical values that would arise if earnings were reported truthfully. Our estimate for β is similar to that found in previous studies. Kothari (1992), for example, using firm-specific time-series price-earnings regressions over one-year windows, estimates earnings response coefficients with a mean of 2.61 and a median of 2.00 using earnings levels scaled by price, and a mean of 3.31 and a median of 1.82 using earnings changes scaled by price.

The fact that investors perceive reports to be noisy is also reflected in the estimates of the response coefficients γ and κ. In a world where earnings were reported truthfully, γ would equal 1 and κ would equal 0. In contrast, Table 6 shows that γ is about 0.6 and κ is significantly positive—consistent with investors attributing a significant fraction of any report to reporting noise.

Reporting noise creates uncertainty for investors not only about firms' current economic earnings, but also about firms' assets in place. Consequently, investors update their expectations not only about this period's economic earnings e_t when observing a report r_t, but also about the firm's assets in place θ_{t-1}. Since $r_t = A_\infty + \theta_{t-1} + e_t + \zeta_t$, where A_∞ is a constant, our estimates suggest that for large firms, investors update their expectations about this period's economic earnings e_t by 52 cents (from $\gamma = 0.519$; see Table 6) and assets in place θ_{t-1} by 26 cents (from $1 - \gamma - \kappa = 0.263$) per dollar of reported equity or earnings. The remainder of 22 cents (from $\kappa = 0.218$) is attributed to reporting noise. For small and medium-sized firms, investors' general inferences are similar. However, investors' beliefs about assets in place are somewhat less sensitive to reported equity or earnings than for large firms, while investors' beliefs about current-period earnings are somewhat more sensitive. In summary, investors use reports to update their beliefs about both components of firm value—earnings and assets in place. Across all size groups, their beliefs about assets in place (respectively, earnings) are more (respectively, less) sensitive to reports than they would be in a world without misreporting.

In order to shed light on the extent to which firms' current economic earnings versus firms' assets in place contribute to misreporting-induced uncertainty faced by investors, we estimate the individual components of investor uncertainty about the firm's economic value.

By the definition of z_t in Equation (6), we have that in steady-state:

$$Var_t(z_t) = \bar{\sigma}_\theta^2 + \left(\frac{\rho}{1-\rho}\right)^2 \bar{\sigma}_e^2 + \frac{\rho}{1-\rho}\bar{\sigma}_{\theta e}. \tag{28}$$

TABLE 4

MLE Estimates of Model Parameters with No Misreporting ($\sigma_\eta = 0$)

	Small	Medium	Large
σ_v	0.158	1.116	29.029
	(0.001)	(0.005)	(0.124)
ρ	0.211	0.206	0.503
	(0.007)	(0.007)	(0.007)
σ_ζ	0.523	4.053	210.199
	(0.002)	(0.017)	(0.874)

We report the persistence of earnings, ρ, the standard deviation of earnings shocks, σ_v, and the standard deviation of unexplained variations in price, σ_ζ, under the assumption of no misreporting, i.e., by imposing $\sigma_\eta = 0$. σ_v and σ_ζ are in \$100 millions. Asymptotic Standard Errors (in parentheses) are computed based on the expected inverse of the Information Matrix.

TABLE 5

Reduced-Form Price Coefficients

	Small	Medium	Large
b_1	1.512	1.640	3.120
	(0.017)	(0.018)	(0.027)
b_2	−0.748	−0.959	−2.553
	(0.020)	(0.020)	(0.026)
b_3	0.822	1.070	2.731
	(0.012)	(0.010)	(0.015)

The table summarizes the estimates of the coefficients in Equation (26): $\Delta\tilde{p}_t = b_1(r_t - \tilde{p}_{t-1}) + b_2(r_{t-1} - \tilde{p}_{t-2}) + b_3\Delta\tilde{p}_{t-1} + n_t$. Asymptotic Standard Errors (in parentheses) are computed based on the expected inverse of the Information Matrix and the Delta Method.

Recall that $Var_t\,(z_t)$ captures only the uncertainty about the firm's economic value that is due to the information asymmetry between the manager and the investors. That is, absent information asymmetry (or, equivalently, absent reporting bias), this variance would have been zero.

Equation (28) shows that investor uncertainty about the firm's current economic value has two sources: (1) uncertainty about the balance sheet, θ_t, which captures the value of assets in place; and (2) uncertainty about the stream of expected future earnings, $\frac{\rho}{1-\rho}e_t$, based on the persistence, ρ, that is due to the uncertainty about the current earnings. In addition, these two sources of uncertainty are correlated. We provide estimates for $\bar{\sigma}_\theta$, $\bar{\sigma}_e$, and $\sqrt{\bar{\sigma}_{\theta e}}$ in Table 7. These estimates suggest that reporting noise gives rise to investor uncertainty associated with assets in place, $\bar{\sigma}_\theta$, that is similar in magnitude to investors' uncertainty associated with current-period economic earnings, $\bar{\sigma}_e$. However, because of earnings persistence, investors' uncertainty about future earnings due to reporting noise in current earnings is higher, and given by $\left(\frac{\rho}{1-\rho}\right)\bar{\sigma}_e$. While the persistence factor $\frac{\rho}{1-\rho}$ for small firms is roughly 1 (0.98), for large firms, the uncertainty about future earnings, $\left(\frac{\rho}{1-\rho}\right)\bar{\sigma}_e \approx 4.5 * \sigma_e$, is roughly 4.3 times larger than the uncertainty about the value of assets in place, as given by $\bar{\sigma}_\theta$. For medium-sized firms, the uncertainty about future earnings is about 50 percent larger than the uncertainty about the value of assets in place. In summary, reporting noise generates investor uncertainty about both assets in place and future earnings. Due to the high persistence of earnings for large firms ($\rho \simeq 0.8$), the uncertainty created by reporting noise about future periods' economic earnings is amplified and hence, it is particularly detrimental to the predictability of firm values.

To examine the relative magnitude of reporting noise, we estimate the ratio of the variance of reporting noise σ_η^2 to the variance of economic earnings innovations σ_v^2. Table 8 provides estimates of the ratio for each size category. The average estimate of this ratio is 0.45 across the three size groups. This suggests that a significant portion of investor uncertainty about firm values is caused by reporting noise rather than variation in fundamentals.

This is largely consistent with the empirical literature that documents that reporting noise plays a significant role for investor uncertainty. Dichev et al. (2013) report that in their survey, CFOs estimate that 50 percent of the value uncertainty is due to innate factors (non-discretionary factors), whereas the other 50 percent is due to managerial reporting decisions.

TABLE 6

Belief Response Coefficients

	Small	Medium	Large
β	1.512	1.640	3.120
	(0.017)	(0.018)	(0.027)
γ	0.674	0.590	0.519
	(0.008)	(0.007)	(0.005)
κ	0.149	0.190	0.218
	(0.004)	(0.004)	(0.003)

β is the sensitivity of investors' expectation about firms' value, z_t, or stock price to reported equity, r_t, or, equivalently, reported earnings; γ is the sensitivity of investors' expectation of this period's economic earnings, e_t, to reports; and κ captures the sensitivity of investors' expectations of reporting noise, ζ_t, to reports. Asymptotic Standard Errors (in parentheses) are computed based on the expected inverse of the Information Matrix and the Delta Method.

TABLE 7

Estimates of Sources of Uncertainty

	Small	Medium	Large
$\bar{\sigma}_\theta$	0.057	0.440	11.720
	(0.001)	(0.006)	(0.019)
$\bar{\sigma}_e$	0.066	0.471	11.259
	(0.000)	(0.000)	(0.001)
$\sqrt{\bar{\sigma}_{\theta e}}$	0.051	0.375	9.552
	(0.000)	(0.000)	(0.007)

$\bar{\sigma}_\theta$ represents the steady-state standard deviation of economic equity, θ_t, conditional on the history of reports, and $\bar{\sigma}_e$ represents the steady-state standard deviation of economic earnings, e_t, conditional on the history of reports. $\bar{\sigma}_{\theta e}$ is the covariance between equity and earnings, conditional on the history of reports. All variables are in \$100 millions.

A mitigating factor, however, is that in our model, earnings innovations v_t are persistent, unlike reporting noise η_t, which is assumed to be *iid*.[20] Therefore, from a valuation perspective, the variability of v_t is a more relevant determinant of investor uncertainty than the variability of reporting noise. Specifically, the ratio of the variance of reporting noise σ_η^2 to the variance of economic earnings scaled by persistence $\left(1+\frac{\rho}{1-\rho}\right)^2\sigma_v^2$ yields an average estimate of 5 percent across size groups (6.5 percent, 7.1 percent, and 2.2 percent for small, medium, and large firms, respectively). Hence, our estimates suggest that from a valuation perspective, reporting and accounting distortions contribute less to investor uncertainty than innate factors, in particular, for large firms.

Another measure that can help us assess the effect of misreporting on investor uncertainty is the measure of earnings quality EQ_∞. Table 8 provides estimates of EQ_∞. Recall that earnings quality EQ_∞ measures investors' total uncertainty about firm value caused by reporting and accounting distortions, i.e., above and beyond the manager's own uncertainty. Specifically, we have defined $EQ_t \equiv -Var_t\,(z_t)$, where $Var_t\,(z_t)$ is the residual variance of the firm's economic value at time t, z_t, conditional on the history of reports available to investors at time t, namely, $\{r_1, r_2, ..., r_t\}$.

If there were no information asymmetry between the manager and investors, this measure would be zero. Nevertheless, a lower EQ_∞ should not be viewed *per se* as a reporting failure because this measure reflects uncertainty caused jointly by fundamentals and by reporting noise. In short: low earnings quality can be caused by a high persistence of innovations ρ, high variance of earnings innovations σ_v^2, and high variance of reporting noise σ_η^2. Moreover, this measure is not scale-invariant as total uncertainty scales up with firm size and, consistently, our estimates of EQ_∞ are monotonically increasing in firm size (see Table 8). Hence, we cannot make direct comparisons across size categories. As for the variance of reporting noise, we can compare the earnings quality metric to the standard deviation of economic earnings scaled by persistence $\left(1+\frac{\rho}{1-\rho}\right)^2\sigma_v^2$ in order to get a sense of earnings quality as it pertains to a valuation perspective. The thus-scaled earnings quality measure equals –0.170, –0.246, and –0.288 for small, medium, and large firms, respectively. From a valuation perspective, the earnings quality metric is, thus, of the same magnitude for all size groups and hence does not suggest a systematic variation of earnings quality across firm size. On average, it suggests that investor uncertainty about firm value due to accounting and reporting distortions equals around 20–25 percent of the uncertainty about firm value created by a shock to current economic earnings.

Finally, in order to further assess the effect of reporting noise on earnings quality, we consider the counterfactual effect of decreasing the variance of reporting noise σ_η^2 by 10 percent. Naturally, a decrease in the variability of reporting noise increases earnings quality. Specifically, Table 8 shows that one can expect an improvement of earnings quality by about 6 percent if, for example, policy initiatives or revisions of accounting standards were to limit reporting and accounting distortions such that their variance decreases by 10 percent. Overall, our analysis illustrates the importance of considering the underlying economics when evaluating the improvements one may expect from policy changes aimed at improving earnings quality.

VI. PERSISTENT NOISE

Our baseline estimation assumes that reporting noise is *iid* across periods. However, it is possible that reporting noise is serially correlated. For example, a positive shock to the manager's misreporting incentives due to liquidity needs may be followed by similar shocks in subsequent periods if the manager's liquidity needs are sticky. To allow for the possibility of persistent noise, we assume that η_t follows an AR1 stationary process:

[20] The analysis allowing for serially correlated reporting noise is provided in Section VI.

TABLE 8

Signal-to-Noise Ratio and Earnings Quality

	Small	Medium	Large
$\frac{\sigma_\eta^2}{\sigma_v^2}$	0.256	0.409	0.677
	(0.012)	(0.016)	(0.021)
EQ_∞	−0.990	-0.830×10^2	-22.612×10^4
	(0.048)	(0.034×10^2)	(0.733×10^4)
$EQ_\infty^{10\%}$	−0.916	-0.774×10^2	-21.730×10^4
	(0.044)	(0.031×10^2)	(0.680×10^4)

$EQ_\infty^{10\%}$ denotes a counterfactual measure of earnings quality assuming that the variance of reporting and accounting distortions, σ_η^2, decreases by 10 percent. EQ_∞ and $EQ_\infty^{10\%}$ are in $ millions.

$$\eta_t = \phi\eta_{t-1} + \epsilon_t,$$

where ϵ_t is white noise, with $\epsilon_t \sim N\left(0, \sigma_\epsilon^2\right)$ and $|\varphi| < 1$. The presence of persistent noise significantly complicates the dynamics of prices. The price function is now characterized as follows.

Lemma 4: In steady-state, the price change is characterized by the following difference equation:

$$\Delta\tilde{p}_t = \beta\Delta r_t - \beta(\rho + \phi)\Delta r_{t-1} + \beta\phi\rho\Delta r_{t-2} + (\rho + \kappa + \phi(1 - \kappa) - \rho\gamma)\Delta\tilde{p}_{t-1} + \rho(\phi\kappa + \gamma\phi - \phi - \kappa)\Delta\tilde{p}_{t-2} + n_t,$$

where:

$$n_t = \Delta\zeta_t - (\rho + \kappa + \phi - \phi\kappa - \rho\gamma)\Delta\zeta_{t-1} - \rho(\phi\kappa + \gamma\phi - \phi - \kappa)\Delta\zeta_{t-2}.$$

When the reporting noise is persistent ($\phi > 0$), the price is an ARMAX(2,3,3) instead of an ARMAX(1,2,2). As before, the price change is a function of lags of the book value and the price change. In turn, the error term n_t follows a moving average MA(3) process, and its variance-covariance matrix depends now on the persistence of earnings, ρ, and noise, ϕ, as well as the response coefficients β, γ, and κ. Inspection of the price equation suggests that identification of ϕ and ρ may be a problem when, for instance, the persistence of earnings and reporting noise are the same.

The dynamics of reports also become more complex once we allow for non-zero ϕ. We compute a measure of adjusted reports m_t that eliminates the effect of persistence of economic earnings:

$$m_t = v_t - \phi v_{t-1} + \epsilon_t - (\rho + 1)\epsilon_{t-1} + \rho\epsilon_{t-2}.$$

This completes the econometric specification of the model with persistent noise.

As before, we estimate the model by MLE. Specifically, we maximize the joint likelihood of $\{r_t, \tilde{p}_t\}_{t=1}^T$ with respect to the model's parameters. Table 9 and Table 10 summarize our findings. All parameters are significant at the 1 percent level except the (newly introduced) persistence of reporting noise, ϕ, for medium-sized firms.

The most significant finding of this extension is the relative robustness of all model estimates when compared to our estimates in Section V. Estimates of the persistence of economic earnings remain virtually the same (decrease by 4 percent for small firms, no change for medium-sized firms, and an increase of 2 percent for large firms). Similarly, estimates of the standard deviation of economic earnings change by only 4 percent for small firms, 0 percent for medium-sized firms, and 13 percent for large firms. As expected, the standard deviation of the reporting noise is most impacted by the introduction of persistent shocks to the reporting noise. While it remains virtually unchanged for medium-sized firms, the standard deviation of reporting noise decreases by 18 percent for small firms and increases by 19 percent for large firms.

The average estimate of the persistence of reporting noise, ϕ, is close to zero, specifically, 0.014. But the estimate varies across size categories in Table 9, going from −0.219 for small firms to 0.268 for large firms. In summary, there is no consistent takeaway in terms of the persistence of reporting noise across size groups. However, the estimates of the baseline model in Section V seem to be relatively robust to the introduction of persistent (versus *iid*) reporting noise.[21]

[21] Our estimate of $\mathbb{E}[n_t] = \beta(1-\rho)\mathbb{E}[\zeta_t]$ is statistically significant for all three size groups. The estimates (standard errors) are 0.126 (0.003) for small firms, 1.502 (0.021) for medium firms, and 41.047 (0.692) for large firms. Our estimate of $\mathbb{E}[m_t] = \mathbb{E}[v_t]$ is statistically insignificant for all three size groups. The estimates (standard errors) are 0.000 (0.001) for small firms, 0.005 (0.005) for medium firms, and 0.124 (0.064) for large firms.

TABLE 9

MLE Estimates of Model Parameters with Persistent Reporting Noise

	Small	Medium	Large
σ_v	1.619	0.588	1.980
	(0.030)	(0.014)	(0.050)
σ_η	0.256	0.236	2.478
	(0.011)	(0.007)	(0.052)
ρ	0.476	0.583	0.834
	(0.008)	(0.007)	(0.002)
ϕ	−0.219	−0.007	0.268
	(0.016)	(0.015)	(0.014)
σ_ζ	25.931	15.277	371.426
	(0.219)	(0.128)	(3.014)

We report the persistence of earnings, ρ, the persistence of reporting noise, ϕ, the standard deviation of earnings shocks, σ_v, and standard deviation of reporting noise, σ_η, as well as the standard deviation of noise in prices, σ_ζ. σ_v, σ_η, and σ_ζ are in \$100 millions. Asymptotic Standard Errors (in parentheses) are computed based on the expected inverse of the Information Matrix.

Relative to the model without persistence in reporting noise (i.e., $\phi = 0$) in Section V, this model yields a higher ratio of variance of reporting noise to variance of earnings shocks. Specifically, the mean ratio of the variance of reporting noise to the variance of earnings innovations is 0.6 (rather than 0.45). However, adjusting the ratio for persistence in both economic earnings and reporting noise, i.e., scaling $\left(1+\frac{\phi}{1-\phi}\right)^2\sigma_\eta^2$ by $\left(1+\frac{\rho}{1-\rho}\right)^2\sigma_v^2$, yields again the same average estimate of 5 percent as in Section V. The model with persistent reporting noise yields a lower EQ_∞ for large firms due to the positive persistence in reporting noise, but a somewhat higher EQ_∞ for small and medium firms that show a negative persistence in reporting noise.

VII. CONCLUDING REMARKS

We study a dynamic model of earnings management and earnings quality that allows us to partition investor uncertainty about firm value into two distinct components: (1) fundamental uncertainty about firm value, which is shared by managers and investors; and (2) information asymmetry between managers and investors due to reporting or accounting distortions. We analyze a finite horizon setting and the steady-state of an infinite horizon setting. In addition to the theoretical analysis of how the two components of investor uncertainty affect various items of interest to accounting research, we structurally estimate our steady-state model to empirically separate the portion of investor uncertainty due to reporting or accounting distortions from the portion of investor uncertainty due to firms' underlying economic earnings process. Our empirical findings suggest that the reporting noise contributes significantly to investor uncertainty about firm values, and that ignoring the presence of misreporting may lead to inferences that significantly underestimate the persistence of earnings. These findings add to the empirical literature on earnings quality that faces difficulties in parsing managed earnings from fundamental earnings (Dechow et al. 2010; Dichev et al. 2013).

The theoretical model and the corresponding structural estimation can be extended in several directions. First, one could endogenize the managers' payoffs by looking at the optimal contract under moral hazard. This can be accomplished, in a

TABLE 10

Signal-to-Noise Ratio and Earnings Quality with Persistent Reporting Noise

	Small	Medium	Large
$\frac{\sigma_\eta^2}{\sigma_v^2}$	0.158	0.401	1.252
	(0.009)	(0.020)	(0.053)
EQ_∞	−0.741	-0.819×10^2	-40.306×10^4
	(0.040)	(0.037×10^2)	(0.983×10^4)

EQ_∞ in \$ millions.

tractable manner, by restricting attention to linear-exponential-normal settings. Second, one could study the effect of uncertainty about the manager's horizon by explicitly modeling managerial turnover in a search-theoretic framework. In addition, the robustness of our empirical analysis could potentially benefit from generalizing the cost function of manipulation and the time-series of the firm's economic earnings.

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APPENDIX A

Analytical Results

We prove Lemmas 1, 2, and 3 for the general case in which the reporting noise η_t is persistent (i.e., $|\phi| < 1$). Ignoring any constant terms, let the data-generating process be described by three variables:

$$e_t = \rho e_{t-1} + v_t,$$

$$\eta_t = \phi \eta_{t-1} + \epsilon_t,$$

$$r_t = \theta_t + \eta_t,$$

where—given the no dividend and zero interest rate assumption—the value of assets in place follow the following process:

$$\theta_t = \theta_{t-1} + e_t.$$

We assume that the innovations are distributed as $\begin{pmatrix} v_t \\ \epsilon_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_\epsilon^2 \end{pmatrix}\right)$ and that $|\phi| < 1$. Notice that we have omitted the constant A_∞ from the report process to simplify the exposition.

Proof of Lemma 1

The steady-state can be defined as a vector independent of time t, $\left\{\sigma_\theta^2, \sigma_{\theta e}, \sigma_e^2, \bar{\sigma}_\eta^2\right\}$ representing the ergodic distribution of investors' beliefs and, more specifically, the variance-covariance matrix of their beliefs about firm fundamentals and reporting noise:

$$Var_t\begin{pmatrix} \theta_t \\ e_t \\ \eta_t \end{pmatrix} \equiv \begin{pmatrix} \bar{\sigma}_\theta^2 & \bar{\sigma}_{\theta e} & \bar{\sigma}_{\theta\eta} \\ \cdot & \bar{\sigma}_e^2 & \bar{\sigma}_{e\eta} \\ \cdot & \cdot & \bar{\sigma}_\eta^2 \end{pmatrix}.$$

The above variance-covariance matrix satisfies the Kalman filter equation:

$$Var_t\begin{pmatrix} \theta_t \\ e_t \\ \eta_t \end{pmatrix} = Var_{t-1}\begin{pmatrix} \theta_t \\ e_t \\ \eta_t \end{pmatrix} - \frac{Cov_{t-1}\left(\begin{pmatrix} \theta_t \\ e_t \\ \eta_t \end{pmatrix}, r_t\right)^2}{Var_{t-1}(r_t)}.$$

The variance-covariance matrix can, thus, be written as:

$$\begin{pmatrix} \bar{\sigma}_\theta^2 & \bar{\sigma}_{\theta e} & \bar{\sigma}_{\theta\eta} \\ & \bar{\sigma}_e^2 & \bar{\sigma}_{e\eta} \\ & & \bar{\sigma}_\eta^2 \end{pmatrix} = \begin{pmatrix} Var_{t-1}(\theta_t) & Cov_{t-1}(e_t, \theta_t) & Cov_{t-1}(\theta_t, \eta_t) \\ \cdot & Var_{t-1}(e_t) & Cov_{t-1}(e_t, \eta_t) \\ \cdot & \cdot & Var_{t-1}(\eta_t) \end{pmatrix} - \frac{\begin{pmatrix} Cov_{t-1}(r_t, \theta_t) \\ Cov_{t-1}(r_t, e_t) \\ Cov_{t-1}(r_t, \eta_t) \end{pmatrix} \begin{pmatrix} Cov_{t-1}(r_t, \theta_t) \\ Cov_{t-1}(r_t, e_t) \\ Cov_{t-1}(r_t, \eta_t) \end{pmatrix}'}{Var_{t-1}(r_t)} \quad (29)$$

where:

$$Var_{t-1}(\theta_t) = Var_{t-1}(\theta_{t-1} + \rho e_{t-1} + v_t) = \bar{\sigma}_\theta^2 + \rho^2 \bar{\sigma}_e^2 + 2\rho\bar{\sigma}_{\theta e} + 2\rho\bar{\sigma}_{\theta e} + \sigma_v^2,$$

$$Var_{t-1}(e_t) = \rho^2 \bar{\sigma}_e^2 + \sigma_v^2,$$

and:

$$Var_{t-1}(\eta_t) = \phi^2 \bar{\sigma}_\eta^2 + \sigma_\epsilon^2.$$

Therefore:

$$Var_{t-1}(r_t) = Var_{t-1}(\theta_{t-1} + \rho e_{t-1} + v_t + \phi \eta_{t-1} + \epsilon_t) = \bar{\sigma}_\theta^2 + \rho^2 \bar{\sigma}_e^2 + \sigma_v^2 + \phi^2 \bar{\sigma}_\eta^2 + \sigma_\epsilon^2 + 2\left(\rho \bar{\sigma}_{\theta e} + \phi \bar{\sigma}_{\theta\eta} + \phi\rho \bar{\sigma}_{e\eta}\right) \tag{30}$$

One can verify that in steady-state:

$$-\bar{\sigma}_{\theta\eta} = \bar{\sigma}_\theta^2 = \bar{\sigma}_\eta^2$$

$$-\bar{\sigma}_{e\eta} = \bar{\sigma}_{e\theta}.$$

Hence, we only need three equations to characterize the steady-state equilibrium:

$$\bar{\sigma}_\theta^2 = \bar{\sigma}_\theta^2 + \rho^2 \bar{\sigma}_e^2 + 2\rho \bar{\sigma}_{\theta e} + \sigma_v^2 - \frac{\left(\bar{\sigma}_\theta^2 (1-\phi) + (2-\phi)\rho \bar{\sigma}_{\theta e} + \rho^2 \bar{\sigma}_e^2 + \sigma_v^2\right)^2}{\bar{\sigma}_\theta^2 (1-\phi)^2 + \rho^2 \bar{\sigma}_e^2 + 2\rho \bar{\sigma}_{\theta e}(1-\phi) + \sigma_v^2 + \sigma_\epsilon^2} \tag{31}$$

$$\bar{\sigma}_e^2 = \rho^2 \bar{\sigma}_e^2 + \sigma_v^2 - \frac{\left(\rho \bar{\sigma}_{\theta e}(1-\phi) + \rho^2 \bar{\sigma}_e^2 + \sigma_v^2\right)^2}{\bar{\sigma}_\theta^2 (1-\phi)^2 + \rho^2 \bar{\sigma}_e^2 + 2\rho \bar{\sigma}_{\theta e}(1-\phi) + \sigma_v^2 + \sigma_\epsilon^2} \tag{32}$$

$$\bar{\sigma}_{\theta e} = \rho^2 \bar{\sigma}_e^2 + \rho \bar{\sigma}_{\theta e} + \sigma_v^2 - \frac{\left(\bar{\sigma}_\theta^2 (1-\phi) + (2-\phi)\rho \bar{\sigma}_{\theta e} + \rho^2 \bar{\sigma}_e^2 + \sigma_v^2\right)\left(\rho \bar{\sigma}_{\theta e}(1-\phi) + \rho^2 \bar{\sigma}_e^2 + \sigma_v^2\right)}{\bar{\sigma}_\theta^2 (1-\phi)^2 + \rho^2 \bar{\sigma}_e^2 + 2\rho \bar{\sigma}_{\theta e}(1-\phi) + \sigma_v^2 + \sigma_\epsilon^2}. \tag{33}$$

Based on this result, we can compute earnings quality, noting that:

$$Var_t(z_t) = Var_t(\theta_t) + \left(\frac{\rho}{1-\rho}\right)^2 Var_t(e_t) + \left(\frac{\rho}{1-\rho}\right) Cov_t(\theta_t, e_t).$$

■

Proof of Lemma 2

We can now derive the price equation. It is convenient to define the evolution of four processes representing investors' beliefs about fundamentals and reporting noise:

$$p_t = p_{t-1} + \beta(r_t - \mathbb{E}_{t-1}(r_t)) \tag{34}$$

$$\hat{e}_t = \underbrace{\rho \hat{e}_{t-1}}_{\mathbb{E}_{t-1}(e_t)} + \gamma(r_t - \mathbb{E}_{t-1}(r_t)) = \rho \hat{e}_{t-1} + \gamma \frac{\Delta p_t}{\beta} \tag{35}$$

$$\hat{\eta}_t = \underbrace{\phi \hat{\eta}_{t-1}}_{\mathbb{E}_{t-1}(\eta_t)} + \kappa(r_t - \mathbb{E}_{t-1}(r_t)) = \phi \hat{\eta}_{t-1} + \kappa \frac{\Delta p_t}{\beta} \tag{36}$$

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \rho \hat{e}_{t-1} + \iota(r_t - \mathbb{E}_{t-1}(r_t)) = \hat{\theta}_{t-1} + \rho \hat{e}_{t-1} + \iota \frac{\Delta p_t}{\beta} \tag{37}$$

where $\iota = \kappa - 1$ and $\Delta p_t = p_t - \mathbb{E}_{t-1}(p_t) = p_t - p_{t-1}$ is the price change, $\hat{e}_t = \mathbb{E}_t(e_t)$, $\hat{\eta}_t = \mathbb{E}_t(\eta_t)$, $\hat{\theta}_t = \mathbb{E}_t(\theta_t)$.[22] Now, introduce the lag operator L as $Lx_t = x_{t-1}$. Then the previous equations can be written as follows:

[22] Notice that by definition: $p_t = \mathbb{E}_t\left[\theta_t + \frac{\rho}{1-\rho} e_t\right]$. Define $z_t = \theta_t + \frac{\rho}{1-\rho} e_t$. By standard properties of the normal distribution, $\mathbb{E}_t[z_t] = \mathbb{E}_{t-1}[z_t] + \beta (r_t - \mathbb{E}_{t-1}(r_t))$ where $\beta = \frac{Cov_{t-1}(z_t, r_t)}{Var_{t-1}(r_t)}$. On the other hand: $\mathbb{E}_{t-1}[z_t] = \mathbb{E}_{t-1}\left[\theta_{t-1} + e_t + \frac{\rho}{1-\rho} e_t\right] = \mathbb{E}_{t-1}\left[\theta_{t-1} + \frac{e_t}{1-\rho}\right] = \mathbb{E}_{t-1}\left[\theta_{t-1} + \frac{\rho e_{t-1} + v_t}{1-\rho}\right] = \mathbb{E}_{t-1}\left[\theta_{t-1} + \frac{\rho e_{t-1}}{1-\rho}\right] = p_{t-1}$.

$$\hat{e}_t = \rho L\hat{e}_t + \gamma\frac{\Delta p_t}{\beta}$$

$$\hat{\eta}_t = \phi L\hat{\eta}_t + \kappa\frac{\Delta p_t}{\beta},$$

$$\hat{\theta}_t = L\hat{\theta}_t + \rho L\hat{e}_t + \iota\frac{\Delta p_t}{\beta}$$

or equivalently:

$$\hat{e}_t = \frac{\gamma}{\beta}(I-\rho L)^{-1}\Delta p_t \tag{38}$$

$$\hat{\eta}_t = \frac{\kappa}{\beta}(I-\phi L)^{-1}\Delta p_t. \tag{39}$$

$$(I-L)\hat{\theta}_t = \rho L e_t + \frac{\iota}{\beta}\Delta p_t = \rho\frac{\gamma}{\beta}(I-\rho L)^{-1}L\Delta p_t + \frac{\iota}{\beta}\Delta p_t, \tag{40}$$

where I denotes the identity operator (i.e., $Ix_t = x_t$), and $(I - aL)^{-1}$ denotes the inverse operator of $I - aL$. Now observe that:

$$r_t - \mathbb{E}_{t-1}r_t = r_t - \hat{\theta}_{t-1} - \rho\hat{e}_{t-1} - \phi\hat{\eta}_{t-1} = r_t - L\hat{\theta}_t - \frac{\rho\gamma}{\beta}(I-\rho L)^{-1}L\Delta p_t - \frac{\phi\kappa}{\beta}(I-\phi L)^{-1}L\Delta p_t.$$

On the other hand $r_t - \mathbb{E}_{t-1}r_t = \frac{\Delta p_t}{\beta}$, so the last equation can be written as:

$$\frac{\Delta p_t}{\beta} = r_t - L\hat{\theta}_t - \frac{\rho\gamma}{\beta}(I-\rho L)^{-1}L\Delta p_t - \frac{\phi\kappa}{\beta}(I-\phi L)^{-1}L\Delta p_t. \tag{41}$$

Applying operator $\beta\ (I - L)\ (I - \rho L)\ (I - \varphi L)$ to both sides and rearranging terms yields:

$$\Delta p_t = \beta\Delta r_t - \beta(\rho+\phi)\Delta r_{t-1} + \beta\phi\rho\Delta r_{t-2} + (\rho+\kappa+\phi(1-\kappa)-\rho\gamma)\Delta p_{t-1} + \rho(\phi\kappa+\gamma\phi-\phi-\kappa)\Delta p_{t-2} + n_t \tag{42}$$

Now, assume the observed price is noisy and given by:

$$\tilde{p}_t = p_t + \zeta_t$$

where $\zeta_t \sim N\left(0, \sigma_\zeta^2\right)$. Then the observed price change follows:

$$\tilde{\Delta}p_t = \Delta p_t + \Delta\zeta_t$$

The econometrician observes two variables $\{r_t, \tilde{p}_t\}$. Then the econometric model is:

$$\Delta p_t = \beta\Delta r_t - \beta(\rho+\phi)\Delta r_{t-1} + \beta\phi\rho\Delta r_{t-2} + (\rho+\kappa+\phi-\phi\kappa-\rho\gamma)\Delta\tilde{p}_{t-1} + \rho(\phi\kappa+\gamma\phi-\phi-\kappa)\Delta\tilde{p}_{t-2} + n_t$$

where the disturbance term n_t follows a moving average process:

$$n_t = \Delta\zeta_t - (\rho+\kappa+\phi-\phi\kappa-\rho\gamma)\Delta\zeta_{t-1} - \rho(\phi\kappa+\gamma\phi-\phi-\kappa)\Delta\zeta_{t-2}.$$

The price function is, thus, an ARMAX(2,3,3). When $\phi = 0$, so the reporting noise is *iid,* the observed price change can be written as the following ARMAX(1,2,2):

$$\Delta\tilde{p}_t = \beta(r_t - \tilde{p}_{t-1}) - \beta\rho(r_{t-1} - \tilde{p}_{t-2}) + \left(\frac{\rho^2\gamma}{1-\rho} + \rho\right)\Delta\tilde{p}_{t-1} + n_t.$$

where:

$$n_t = \zeta_t + \left(\beta - 1 - \frac{\gamma\rho^2}{1-\rho} - \rho\right)\zeta_{t-1} + \left((1-\beta)\rho + \rho^2\gamma\frac{1}{1-\rho}\right)\zeta_{t-2}.$$

To prove that this representation is indeed equivalent to that found for the case $\phi > 0$, notice that:

$$
\begin{aligned}
p_t &= p_{t-1} + \beta(r_t - \mathbb{E}_{t-1} r_t) = p_{t-1} + \beta(r_t - p_{t-1} - \mathbb{E}_{t-1}[-p_{t-1} + r_t]) \\
&= p_{t-1} + \beta\left(r_t - p_{t-1} - \mathbb{E}_{t-1}\left[-\theta_{t-1} - \frac{\rho e_{t-1}}{1-\rho} + r_t\right]\right) = p_{t-1} + \beta\left(r_t - p_{t-1} - \mathbb{E}_{t-1}\left[-\theta_{t-1} - \frac{\rho e_{t-1}}{1-\rho} + \theta_{t-1} + e_t + \eta_t\right]\right) \\
&= p_{t-1} + \beta\left(r_t - p_{t-1} - \mathbb{E}_{t-1}\left[-\theta_{t-1} - \frac{\rho e_{t-1}}{1-\rho} + \theta_{t-1} + \rho e_{t-1} + \eta_t\right]\right) = p_{t-1} + \beta\left(r_t - p_{t-1} - \mathbb{E}_{t-1}\left(\frac{-\rho^2}{1-\rho} e_{t-1}\right)\right) \\
&= p_{t-1} + \beta\left(r_t - p_{t-1} + \frac{\rho^2}{1-\rho}\hat{e}_{t-1}\right)
\end{aligned}
$$

Plugging Equation (38) and using $\phi = 0$ yields:

$$
\Delta p_t = \beta(r_t - p_{t-1}) + \frac{\rho^2}{1-\rho}\gamma\frac{\Delta p_{t-1}}{I - \rho L}
$$

Multiplying both sides by $(I - \rho L)$ leads to:

$$
\Delta p_t = \beta(r_t - p_{t-1}) - \rho\beta(r_{t-1} - p_{t-2}) + \left(\frac{\rho^2}{1-\rho}\gamma + \rho\right)\Delta p_{t-1},
$$

which is the expression presented in Lemma 2. ■

Proof of Lemma 3

Using the steady-state variances derived in Lemma 1, we can compute the response coefficients as follows:

$$
\beta \equiv \frac{Cov_{t-1}\left(\theta_t + \frac{\rho e_t}{1-\rho}, r_t\right)}{Var_{t-1}(r_t)} = \frac{\bar{\sigma}_\theta^2 + 2\rho\bar{\sigma}_{\theta e} + \rho^2\bar{\sigma}_e^2 + \sigma_v^2 + \phi\bar{\sigma}_{\theta\eta} + \phi\rho\bar{\sigma}_{e\eta} + \frac{\rho}{1-\rho}\left(\rho\bar{\sigma}_{\theta e} + \rho^2\bar{\sigma}_e^2 + \sigma_v^2 + \phi\rho\bar{\sigma}_{e\eta}\right)}{Var_{t-1}(r_t)},
$$

$$
\gamma \equiv \frac{Cov_{t-1}(e_t, r_t)}{Var_{t-1}(r_t)} = \frac{\rho^2\bar{\sigma}_e^2 + \rho\bar{\sigma}_{\theta e} + \sigma_v^2 + \phi\rho\bar{\sigma}_{e\eta}}{Var_{t-1}(r_t)},
$$

$$
\kappa \equiv \frac{Cov_{t-1}(\eta_t, r_t)}{Var_{t-1}(r_t)} = \frac{\phi\bar{\sigma}_{\theta\eta} + \rho\phi\bar{\sigma}_{e\eta} + \phi^2\bar{\sigma}_\eta^2 + \sigma_\epsilon^2}{Var_{t-1}(r_t)}.
$$

■