Staying at Zero with Affine Processes

An Application to Term Structure Modelling

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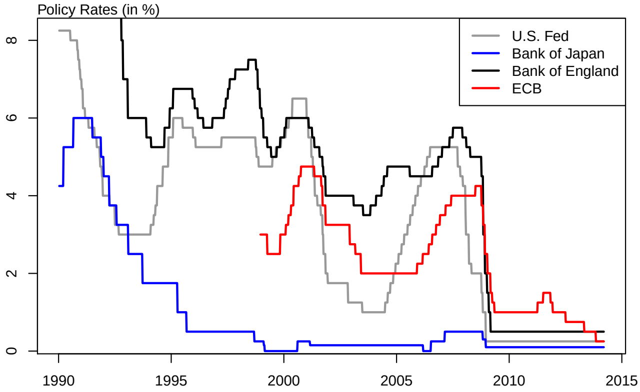
Contents

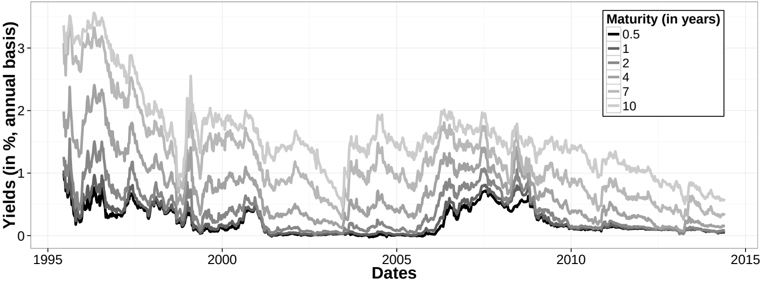
Zero lower bound (ZLB)

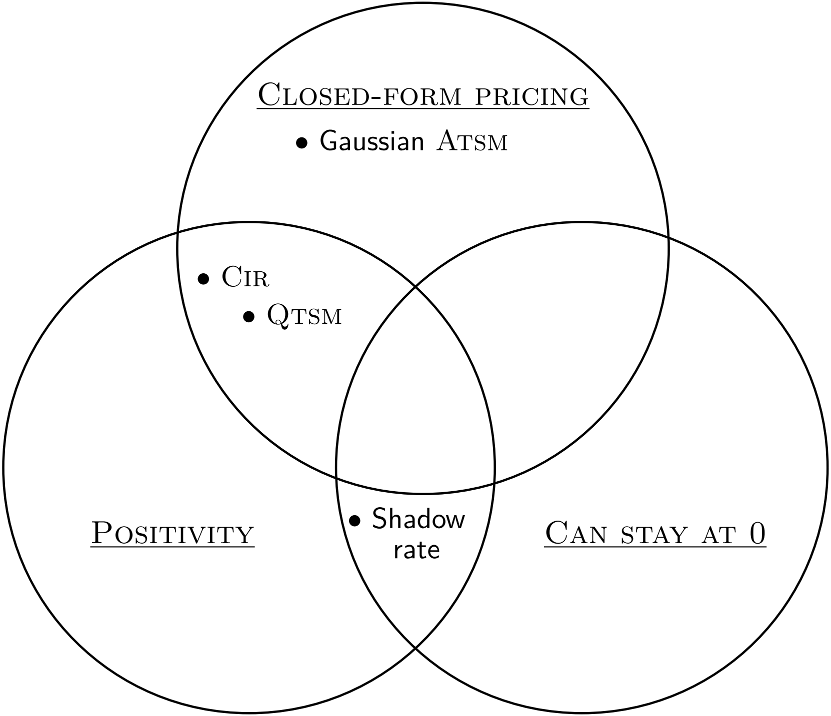
Several of the major central banks now face the ZLB

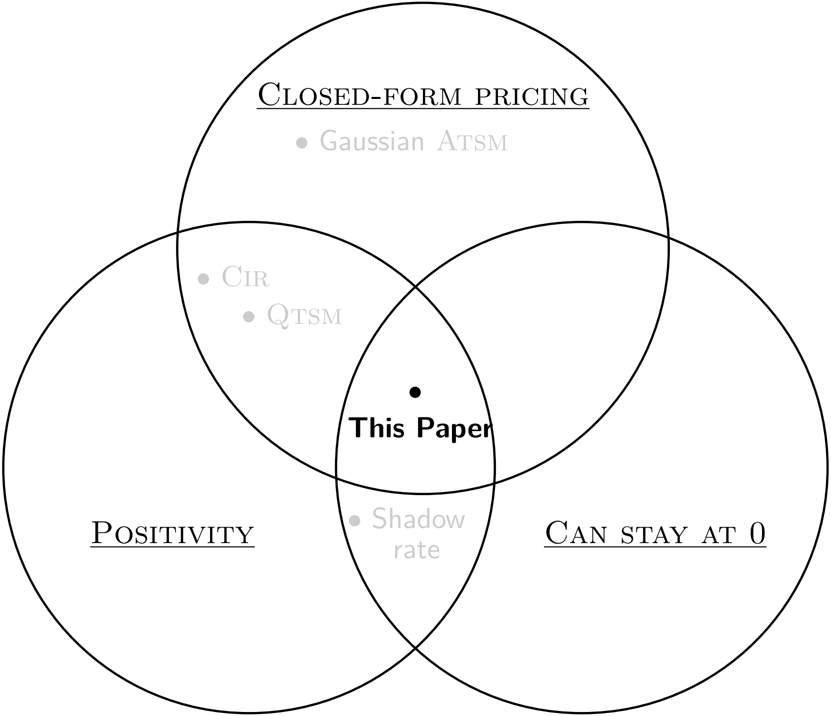


Stylized facts to match

- The short-term nominal rate can stay at the ZLB for several periods.
- In the meantime, longer-term yields can show substantial fluctuations [JGB yields from June 1995 to May 2014]




Closed-form pricing
• Gaussian Atsm
• Cir
• Qtsm
Positivity
• Shadow rate
Can stay at 0


Closed-form pricing
Gaussian Atsm
Cir
Qtsm
This Paper
Positivity
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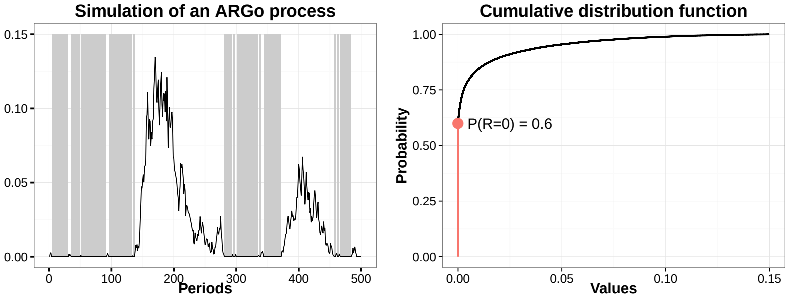
Our ZLB model: a primer

$\longrightarrow$ We introduce a new affine process:



What we do in this paper

- We derive affine non-negative processes staying at 0 (ARG_0 processes) to build a Term Structure Model which is:
  - providing positive yields for all maturities;
  - consistent with the ZLB with a short-rate experiencing prolonged periods at 0 *while* long-term rates still fluctuates;
  - affine: thus closed-form formulas for bond-pricing and lift-off probabilities are available.
- Empirical assessment on JGB yields (June 1995 to May 2014). Good performance of our model in terms of:
  - fitting yield levels and conditional variances;
  - calculating Risk-Neutral *and* Historical lift-off probabilities.

Related literature

- Term structure models at the ZLB: Black (1995), Ichiue & Ueno (2007), Kim & Singleton (2012), Krippner (2012), Renne (2012), Kim & Priebsch (2013), Wu & Xia (2013), Bauer & Rudebusch (2013), Christensen & Rudebusch (2013).
- Conditional volatilities of yields: Almeida *et al.* (2011), Bikbov & Chernov (2011), Filipovic, Larsson & Trolle (2013), Creal & Wu (2014), Christensen *et al.* (2014).
- Affine and Autoregressive Gamma processes: Darolles *et al.* (2006), Gourieroux & Jasiak (2006), Dai, Le & Singleton (2010), Creal & Wu (2013)
- Lift-off probabilities: Bauer & Rudebusch (2013), Swanson & Williams (2013)

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Defining the Gamma-Zero distribution

We construct a new distribution in two steps:

- $Z \sim \mathcal{P}(\lambda) \Longrightarrow Z(\omega) \in \{\mathbf{0}, 1, 2, \ldots\}$ and $\mathbb{P}(Z = 0) = \exp(-\lambda)$.
- We define $X|Z \sim \gamma_Z(\mu)$, which implies:
  1. If $Z = 0$, X is a dirac point mass at 0.
  2. If $Z > 0$, X is gamma-distributed (continuous on $\mathbb{R}^+$).

Definition

The non-negative r.v. $X \sim \gamma_{\mathbf{0}}(\lambda, \mu)$, $\lambda > 0$ and $\mu > 0$, if

$$X \,|\, Z \sim \gamma_Z(\mu) \quad \text{with} \quad Z \sim \mathcal{P}(\lambda)$$

$$\Rightarrow \qquad \mathbb{P}(X = 0) = \mathbb{P}(Z = 0) = \exp(-\lambda)\,.$$

A mixture distribution

In other words, $X \sim \gamma_0(\lambda, \mu)$ if its (complicated) p.d.f. is:

$$f_X(x\,;\,\lambda,\mu) = \sum_{z=1}^{+\infty}\left[\frac{\exp(-x/\mu)\,x^{z-1}}{(z-1)!\,\mu^z} \times \frac{\exp(-\lambda)\lambda^z}{z\,!}\right]\mathbb{1}_{\{x>0\}} + \exp(-\lambda)\mathbb{1}_{\{x=0\}}$$

However, simple Laplace transform:

$$\varphi_X(u\,;\,\lambda,\mu) := \mathbb{E}\left[\exp(uX)\right] = \exp\left[\lambda\frac{u\mu}{(1-u\mu)}\right] \quad \text{for} \quad u < \frac{1}{\mu}.$$

$\Longrightarrow$ Exponential-affine in λ.

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$\Longrightarrow$ Exponential-affine in λ.

Introducing dynamics: the ARG_0 process

Main goal: Build a dynamic affine process with zero point mass.

Definition

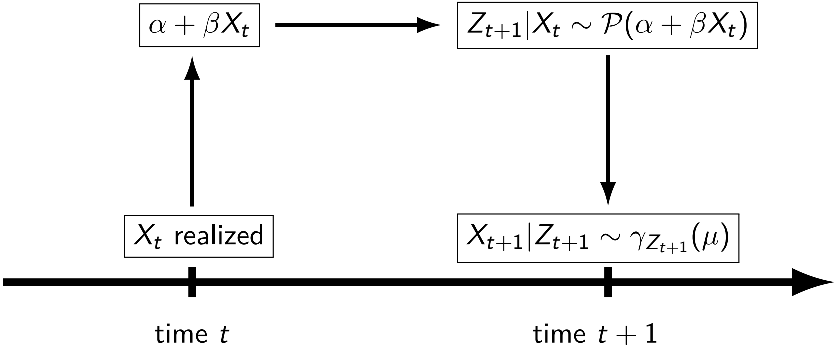
(X_t) is a $\text{ARG}_0(\alpha, \beta, \mu)$ if $(X_{t+1}|\underline{X_t})$ is Gamma-zero distributed:

$$(X_{t+1}|\underline{X_t}) \sim \gamma_0(\alpha + \beta X_t, \mu) \quad \text{for} \quad \alpha \geq 0,\ \mu > 0,\ \beta > 0\,.$$

Again, simple conditional LT, exponential-affine in X_t:

$$\begin{aligned} \varphi_{X,t}(u\,;\,\alpha, \beta, \mu) &:= \mathbb{E}_t\left[\exp(uX_{t+1})\right] \\ &= \exp\left[\frac{u\mu}{1-u\mu}(\alpha + \beta\,X_t)\right]\,, \quad \text{for} \quad u < \frac{1}{\mu}\,. \end{aligned}$$

Summary


$\alpha + \beta X_t$
$Z_{t+1}|X_t \sim \mathcal{P}(\alpha + \beta X_t)$
X_t realized
$X_{t+1}|Z_{t+1} \sim \gamma_{Z_{t+1}}(\mu)$
time t
time $t+1$

Interesting features and properties

Key properties:

- Non-negative and affine process
- Staying at zero with probability:

$$\mathbb{P}(X_{t+1} = 0 | X_t = 0) = \exp(-\alpha) \neq 0.$$

  - $\alpha \neq 0 \implies$ zero is not absorbing.
  - The probability is TV in the multivariate setting.
- Affine first two conditional moments.

Multivariate case

A multivariate VARG process can be obtained easily stacking together conditionally independent ARG processes.

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Risk-neutral dynamics

- The state of the economy is defined by a n-dimensional vector X_t. These factors follow a *VARG* process under $\mathbb{Q}$.

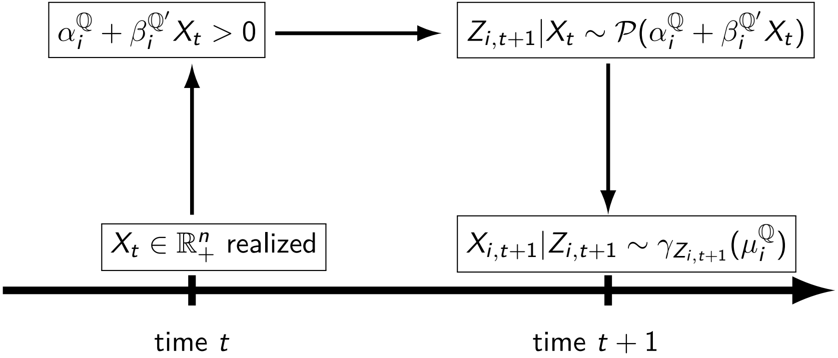
$VARG_\nu$ processes

X_t follows a $\text{VARG}_0(\alpha^{\mathbb{Q}}, \beta^{\mathbb{Q}}, \mu^{\mathbb{Q}})$ if, $\forall t$, $\forall i$:

- $Z_{i,t+1}|X_t \sim \mathcal{P}(\alpha_i^{\mathbb{Q}} + \beta_i^{\mathbb{Q}'} X_t)$.
- $X_{i,t+1}|Z_{i,t+1} \sim \gamma_{Z_{i,t+1}}(\mu_i^{\mathbb{Q}})$ cond. indep across i.

- Each $X_{i,t}$ has a zero point mass.
- X_t has closed-form affine first two moments.

Summary


$\alpha_i^{\mathbb{Q}} + \beta_i^{\mathbb{Q}\prime} X_t > 0$
$Z_{i,t+1} | X_t \sim \mathcal{P}(\alpha_i^{\mathbb{Q}} + \beta_i^{\mathbb{Q}\prime} X_t)$
$X_t \in \mathbb{R}_+^n$ realized
$X_{i,t+1} | Z_{i,t+1} \sim \gamma_{Z_{i,t+1}}(\mu_i^{\mathbb{Q}})$
time t
time $t+1$

Short-rate specification

- The vector of factors X_t is split into two: $X_t = (X_t^{(1)'}, X_t^{(2)'})'$
- The following structure is imposed:

$$\begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix} = \text{constant} + \begin{pmatrix} \beta_{11}^{\mathbb{Q}} & \beta_{12}^{\mathbb{Q}} \\ 0 & \beta_{22}^{\mathbb{Q}} \end{pmatrix} \begin{pmatrix} X_{t-1}^{(1)} \\ X_{t-1}^{(2)} \end{pmatrix} + \xi_t^{\mathbb{Q}}$$

- The short-term rate r_t is given by: $\boxed{r_t = \delta_1' X_t^{(1)}}$

Key Properties

- r_t has a zero point mass.
- $X_t^{(2)}$ appears in $\mathbb{Q}$-expectations of future r_t.

$\Longrightarrow$ In the ZLB, $X_t^{(1)} = 0$ but long-term yields move with $X_t^{(2)}$.

Pricing Formulas

The model belongs to the class of ATSM:

- Yields are affine in the factors for all maturities:

$$R_t(h) = -\frac{1}{h}(A_h{}' X_t + B_h) = \overline{A}'_h X_t + \overline{B}_h.$$

- Recursive closed-form loadings formulas. loadings recursions

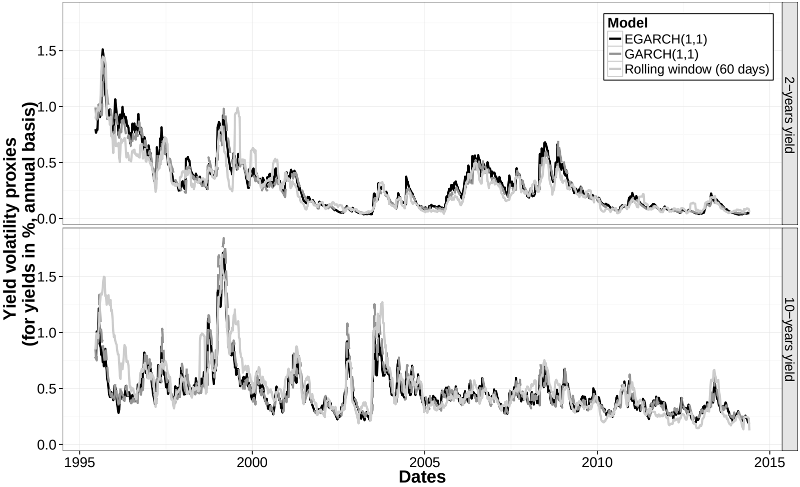
Physical $\mathbb{P}$-dynamics

The SDF is exp-affine with market price of risk vector θ, providing VARG $\mathbb{P}$-dynamics with explicit parameters. $\mathbb{P}$-parameters

$$X_t | \underline{X_{t-1}} \sim \mathrm{VARG}_0(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}}, \mu^{\mathbb{P}})$$

Stylized facts to match (2)

Conditional volatilities: time-varying and maturity dependent.



How to treat it

- Conditional variance of yields:

$$\begin{aligned} & \mathbb{V}_t^{\mathbb{P}}\left[R_{t+1}(h)\right] \\ = \;& \overline{A}_h' \mathbb{V}_t^{\mathbb{P}}(X_{t+1}) \overline{A}_h \\ = \;& \overline{A}_h{}' \left\{ \mathrm{diag}\left[\mu^{\mathbb{P}} \odot \mu^{\mathbb{P}} \odot \left(2\alpha^{\mathbb{P}} + 2\beta^{\mathbb{P}\prime} X_t \right) \right] \right\} \overline{A}_h \end{aligned}$$

- Time-varying and maturity-dependent.

Advantages of an affine framework

NATSM properties

- Yields $R_t(h)$ are non-negative;
- Long-term yields can move while $r_t = 0$ for several periods;
- Unconditional first two moments are available in closed-form;
- Conditional first two moments of yields are affine in X_t;
- Yields forecasts are explicitly affine in X_t;

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Observable variables

State vector $Y_t = (R_t', V_t', S_t')'$ affine in X_t:

R_t: yield levels (6 maturities);

V_t: 2- and 10-y yield conditional (EGARCH) variance;

S_t: SPF for 3-m and 1-y ahead 10-y yield;

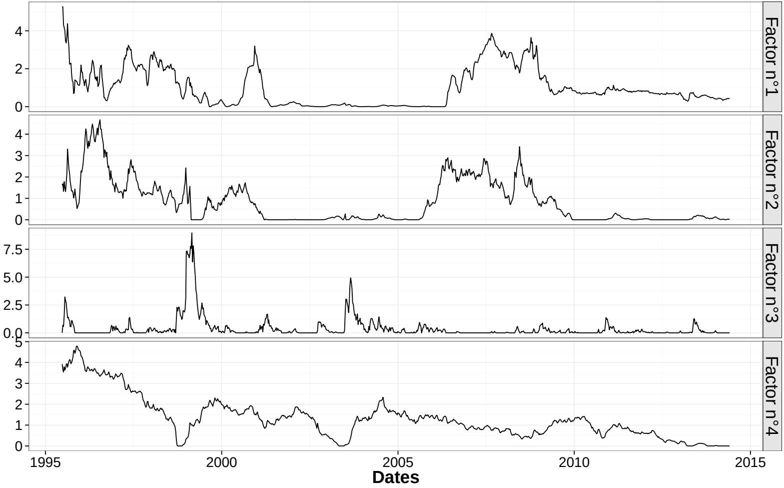
- $\dim(X_t^{(1)}) = 1$, $\dim(X_t^{(2)}) = 3$ and $\nu = 0$;

Estimation technique

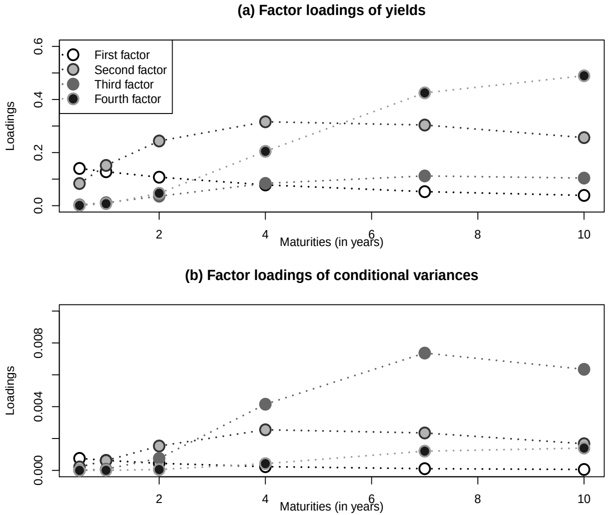
Affine $\mathbb{P}$-dynamics + affine observable variables.

$\Longrightarrow$ Linear Kalman-filter-based QML.

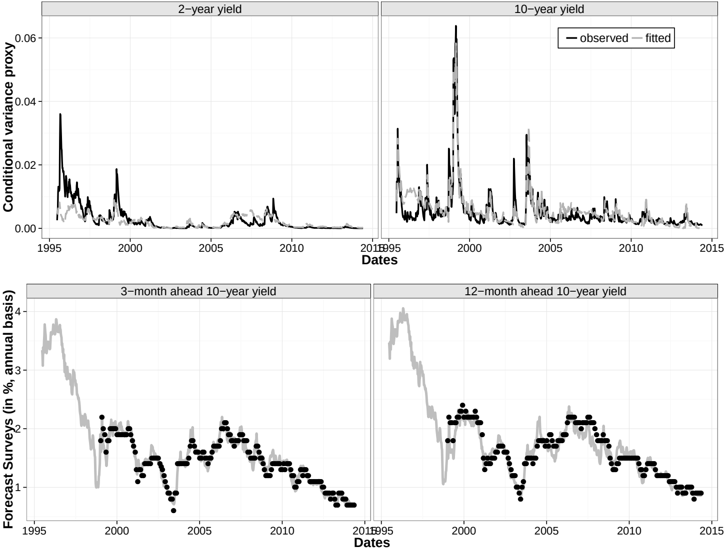
Filtered factors



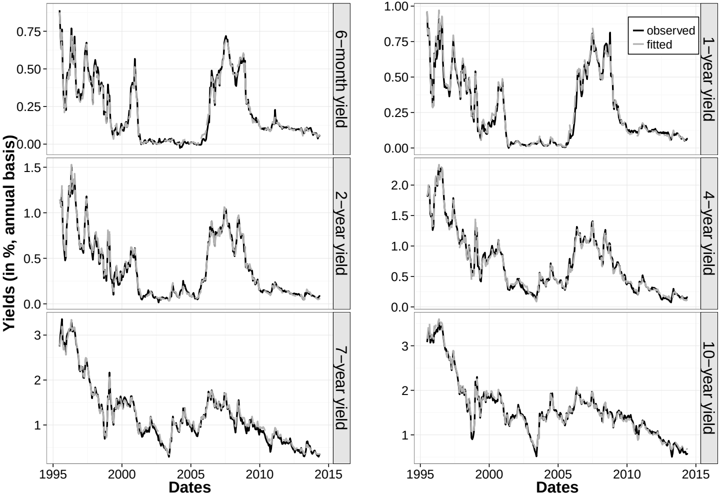
Factor loadings of yields and conditional variances



Fit of Conditional Variances and SPFs



Fit of Yields



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Lift-off probability dates under $\mathbb{P}$ *and* $\mathbb{Q}$

We calculate the following probabilities:

- $\mathbb{P}(r_{t+k} = 0 \,|\, \underline{X_t})$ and $\mathbb{Q}(r_{t+k} = 0 \,|\, \underline{X_t})$;
- $\mathbb{P}(r_{t+k} < 25\, bps. \,|\, \underline{X_t})$ and $\mathbb{Q}(r_{t+k} < 25\, bps. \,|\, \underline{X_t})$.

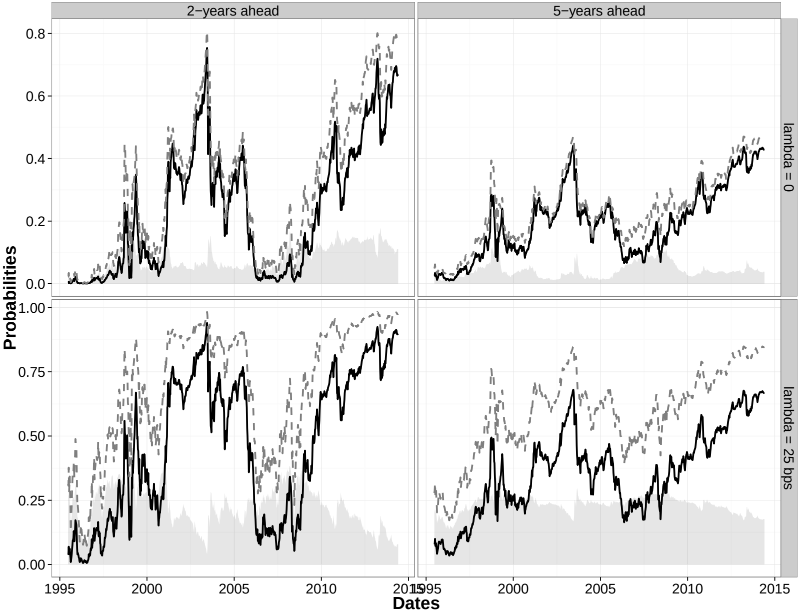
Useful formula

If $z \in \mathbb{R}^+$ and $\varphi_z(u)$ its Laplace transform.

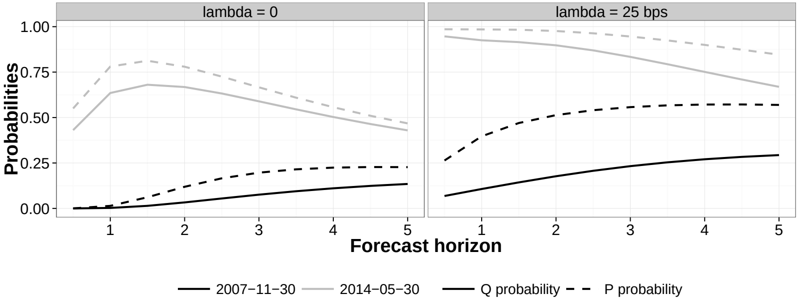
$$\mathbb{P}(z = 0) = \lim_{u \to -\infty} \varphi_z(u)\,.$$

Next two plots:

- *Time-series dimension*: t varies (k = 2yrs and 5yrs).
- *Horizon dimension*: k varies (t = 11/30/07 and 05/30/14).


2−years ahead
5−years ahead
lambda = 0
lambda = 25 bps
Probabilities
Dates
0.0
0.2
0.4
0.6
0.8
0.00
0.25
0.50
0.75
1.00
1995
2000
2005
2010
2015

Horizon dimension of probabilities



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Summary and further research

We have derived **affine non-negative processes staying at 0** and built an affine term-structure model (**NATSM**) gathering:

- a **short-rate consistent with the ZLB** experiencing periods at 0 while **long-run rates still fluctuates**;
- **closed-form formulas** for bond-pricing and lift-off probabilities.

An empirical assessment showed performance of our model for:

- **fitting yield levels and conditional variances**;
- calculating risk-neutral *and* historical **lift-off probabilities**.

Further research: Empirical comparison of NATSMs, derivatives pricing.

Thank you for your attention.

Contents

- The loadings recursions are given by:

$$
\begin{aligned}
R_t(h) &= -\frac{1}{h}(A_h' X_t + B_h) \\
A_h &= -\delta + \beta^{\mathbb{Q}} \left(\frac{A_{h-1} \odot \mu^{\mathbb{Q}}}{1 - A_{h-1} \odot \mu^{\mathbb{Q}}} \right) \\
B_h &= B_{h-1} + \alpha^{\mathbb{Q}\prime} \left(\frac{A_{h-1} \odot \mu^{\mathbb{Q}}}{1 - A_{h-1} \odot \mu^{\mathbb{Q}}} \right)
\end{aligned}
$$

- $\odot$ is the element-wise product.

back

The historical dynamics

- The SDF is exp-affine with market price of risk vector θ:

$$\frac{d\mathbb{P}_{t,t+1}}{d\mathbb{Q}_{t,t+1}} = \exp\left[\theta' X_{t+1} - \psi_t^{\mathbb{Q}}(\theta)\right]$$

Change of measure property

X_t follows a VARG$_\nu(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}}, \mu^{\mathbb{P}})$ process under the historical measure $\mathbb{P}$.

$$\alpha_j^{\mathbb{P}} = \frac{\alpha_j^{\mathbb{Q}}}{1 - \theta_j\, \mu_j^{\mathbb{Q}}}, \qquad \beta_j^{\mathbb{P}} = \frac{1}{1 - \theta_j\, \mu_j^{\mathbb{Q}}}\, \beta_j^{\mathbb{Q}}, \qquad \mu_j^{\mathbb{P}} = \frac{\mu_j^{\mathbb{Q}}}{1 - \theta_j\, \mu_j^{\mathbb{Q}}}.$$

Rk: ν is the same under both measures. back

Table : Parameter estimates

	$\mathbb{P}$-parameters			$\mathbb{Q}$-parameters	
	Estimates	Std.		Estimates	Std.
α_4	3.2455	0.1118		3.2347	0.1113
$\beta_{1,1}$	0.9663	0.0078		0.9794	0.0042
$\beta_{2,2}$	0.9978	0.0005		0.9957	0.0006
$\beta_{3,3}$	0.9486	0.0044		0.9705	0.0023
$\beta_{4,4}$	0.9967	0.0005		0.9933	0.0003
$\beta_{2,1}$	0.0308	0.0041		0.0308	0.0041
$\beta_{3,2}$	0.1094	0.0059		0.1120	0.0061
$\beta_{4,3}$	$3.88{\cdot}10^{-4}$	$2.28{\cdot}10^{-5}$		$3.87{\cdot}10^{-4}$	$2.27{\cdot}10^{-5}$
μ_1	1	–		1.0135	0.0040
μ_2	1	–		0.9980	0.0005
μ_3	1	–		1.0231	0.0023
μ_4	1	–		0.9967	0.0003
Other Parameters					
δ_1	0.0030	0.0003			
θ_1	-0.0133	0.0039	θ_2	0.0020	0.0005
θ_3	-0.0226	0.0022	θ_4	0.0033	0.0003
σ_R	0.0407	0.0003			
σ_V	$3 \cdot 10^{-3}$	–	σ_S	0.15	–

Univariate case: lift-offs formulas

- $Z \in \mathbb{R}^+$ and $\varphi_Z(u)$ its Laplace transform.

$$\mathbb{P}_Z\{0\} = \lim_{u \to -\infty} \varphi_Z(u) \, .$$

- Lift-off probabilities: $(X_t) \sim \text{ARG}_0(\alpha, \beta, \mu)$ and $\varphi_{t,h}(u_1, \ldots, u_h)$ its multi-horizon conditional Laplace transform.
  - $\mathbb{P}(X_{t+h} = 0 \,|\, X_t) = \lim_{u \to -\infty} \varphi_{t,h}(0, \ldots, 0, u)$
  - $\mathbb{P}\left[X_{t+1} = 0, \ldots, X_{t+h} = 0 \,\middle|\, X_t\right] = \lim_{u \to -\infty} \varphi_{t,h}(u, \ldots, u)$
    $= \exp(-\alpha\, h - \beta\, X_t) \, ,$
  - $\mathbb{P}\left[X_{t+1} = 0, \ldots, X_{t+h} = 0, X_{t+h+1} > 0 \,\middle|\, X_t)\right]$
    $= \exp\left[-\alpha\, h - \beta\, X_t\right] \left[1 - \exp(-\alpha)\right] \, , \quad h > 1.$

Multivariate Case

- $Z \in \mathbb{R}_+^n$ and $\varphi_Z(u_1, \ldots, u_n)$ its Laplace transform.

$$\mathbb{P}_Z\{0, \ldots, 0\} = \lim_{u \to -\infty} \varphi_Z(u, \ldots, u)\,.$$

- Notations: Multi-horizon conditional LT.

$$\begin{aligned}
\varphi_{t,k}^{\mathbb{P}}(u_1, \ldots, u_k) &= \mathbb{E}^{\mathbb{P}}\left[\exp\left(u_1^{'} X_{t+1} + \ldots + u_k^{'} X_{t+k}\right) \,\middle|\, X_t\right] \\
&= \exp\left[\mathcal{A}_k^{'} X_t + \mathcal{B}_k\right] \\
\varphi_{R,t,k}^{(h)\mathbb{P}}(v_1, \ldots, v_k) &= \mathbb{E}\left[\exp\left(v_1 R_{t+1}(h) + \ldots + v_k R_{t+k}(h)\right) \mid X_t\right]
\end{aligned}$$

Lift-offs

- $\mathbb{P}\left[r_{t+k} = 0 \,|\, X_t\right] = \lim_{u \to -\infty} \varphi^{(1)\mathbb{P}}_{R,t,k}(0, \ldots, 0, u)$
- $$\begin{aligned}\mathbb{P}\left[r_{t+1} = 0, \ldots, r_{t+k} = 0 \,|\, X_t\right] \\ = \lim_{u \to -\infty} \varphi^{(1)\mathbb{P}}_{R,t,k}(u, \ldots, u) = p_{r,t,k} \quad \text{(say)}\end{aligned}$$
- $\mathbb{P}\left[r_{t+1} = 0, \ldots, r_{t+k-1} = 0, r_{t+k} > 0 \,|\, X_t\right] = p_{r,t,k-1} - p_{r,t,k}$
- $$\begin{aligned}\mathbb{P}\left[v' R^{(t+k)}_{t+1}(h) > \lambda \,|\, X_t\right] \\ = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \frac{Im\left[\varphi^{(h)\mathbb{P}}_{R,t,k}(i\,v\,x)\exp(-i\,\lambda\,x)\right]}{x}\,dx\end{aligned}$$