Cumulative Prospect Theory
and the Representative Investor

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Abstract

Cumulative Prospect Theory has been proposed as an alternative to expected utility theory to explain irregular behavior by economic agents. In particular in Finance it has used to clarify anomalies like the equity premium puzzle. There are certainly hopes and hints that CPT can explain the anomalies, but less attention has been paid to more basic questions. This paper answers some of those. A complete market is not sufficient to guarantee that the market portfolio is efficient so prices may not be determined at the margin by a representative investor. "Over-completion" of the market (the introduction of apparently extraneous derivative assets) can restore efficiency and result in a Pareto efficient allocation of risk. Mutual fund results also obtain only under very restrictive conditions for CPT investors. But mean variance analysis and the resulting CAPM does hold with only minor additional assumptions.

I Introduction

In the 1970s and 1980s, expected utility theory came under increasing question for failing to explain certain irregularities in behavior, and many modifications to the axioms or suggestions for alternate theories were proposed. Prospect theory and its successor, cumulative prospect theory (CPT), are two of the responses that have attracted a good deal of attention. As originally constructed by Kahneman and Tversky (1979) and extended by Tversky and Kahneman (1992), both theories have two component parts, loss-averse utility and a probability weighting function. Together these two features attempt a concise explanation of the major violations of expected utility theory.

CPT has been tested in the laboratory (e.g., List, 2003). The applied literature on CPT has also "tested" it by addressing anomalies to see if loss aversion or probability weighting can explain observed phenomenon. For example Benartzi and Thaler (1995) look at the equity premium puzzle; Giorgi et al. (2005) examine the size and value premium puzzle; Barberis and Xiong, (2006) consider the disposition effect.

This paper takes a much more fundamental approach. It asks the questions: What kinds of portfolios would be formed by investors who are loss adverse or use probability weighting? How do these portfolios differ from those held by risk-averse investors or those who use objective probabilities? Do mutual fund theorems continue to hold? Is the market portfolio efficient in the sense that a representative investor exists?

Section II provides a brief review of CPT and its two component parts. Section III introduces the portfolio maximization problem under CPT. Sections IV and V examine the separate effects of loss aversion and probability weighting. Sections VI and VII scrutinize mutual fund theorems and mean variance analysis under CPT. Section VIII briefly addresses the difficulties in extending CPT to a multiperiod model. Section IX concludes.

II Cumulative Prospect Theory: A Review

For the standard expected utility problem, utility is strictly increasing and concave in wealth or consumption. With loss aversion, utility is still strictly increasing but is reframed to be defined over changes in wealth. It is concave for gains and convex for losses. The resulting S-shaped utility function predicts choices that are risk-averse concerning gains and risk-seeking about losses. Assuming utility is twice differentiable, except possibly at 0, $v'(x) > 0$ for all x, and $x \cdot v''(x) < 0$, $x \neq 0$.

Though choices are risk-seeking over losses, symmetric fair bets are always declined, and the larger the symmetric bet the worse it is. Normalizing the loss-averse function v by setting $v(0) = 0$, this increasing symmetric bet aversion requires that for all positive x, $-v(-x) > v(x)$ and $v'(-x) > v'(x)$.[1]

[1] The risk premium of a simple symmetric bet is $Q(x) \equiv -\frac{1}{2}[v(x) + v(-x)]$. If all symmetric bets are declined, then $Q(x) > 0$ for all $x > 0$ which gives $v(x) < -v(-x)\ \forall x > 0$. If larger bets are worse, then $0 < Q'(x) = -\frac{1}{2}[v'(x) - v'(-x)]$ which gives $v'(-x) > v(x)\ \forall x > 0$.

Tversky and Kahneman (1992, henceforth TK) proposed and estimated a specific loss-averse utility function of the form

$$v(x) = \begin{cases} x^{\alpha} & x \geq 0 \\ -\lambda(-x)^{\beta} & x < 0 \,. \end{cases} \tag{1}$$

Their estimated parameters are α = β = 0.88 and λ = 2.25. Other estimates include α = 0.52 and α = 0.37 by Wu and Gonzalez (1996) using their own and Camerer and Ho's (1994) data.[2] Note that these values of α would seem to indicate only very mild aversion to risk since the relative risk aversion for gains for this function is 1 − α, and of course there is risk seeking for losses. However, the parameter λ also strongly affects the aversion to risk. For example, using TK's parameter values, a fifty-fifty gamble winning $1 or $2 has a certainty equivalent of $1.49 — one cent below the expected value, but an even chance at winning or losing a dollar has a certainty equivalent of −23¢. So while risk aversion over gains is mild, there is a much larger risk premium when both gains and losses are involved.

Instead of using the various outcomes' probabilities directly, CPT uses decision weights, ω, derived from a probability-weighting function. Decision-weighted "expected" utility is computed as

$$\mathrm{E}[v(\tilde{x})] = \sum \omega_i(\boldsymbol{\pi}, \mathbf{x}) v(x_i) \,. \tag{2}$$

Decision weights were originally introduced in Prospect Theory to capture two behavioral effects: (i) the subjective overweighting of rare events which seemed evident in behaviors such as the purchase of lottery tickets and (ii) violations of the independence axiom accounting for the Allais paradox.

As originally proposed, decisions weights could not readily be extended to gambles with more than two outcomes as violations of first-order stochastic dominance might be introduced. Tversky and Kahneman (1992) developed CPT to surmount this problem. They accomplished this by applying weighting functions to the cumulative probability of losses and complementary cumulative probability of gains.

Under CPT, outcomes are first ordered from lowest to highest

$$x_{-n} < x_{-n+1} < \cdots < x_{-1} < x_0 = 0 < x_1 < \cdots x_m \,. \tag{3}$$

Then the cumulative and complementary cumulative probabilities are determined separately for the losses and gains

$$\Pi^{-}_{-i} = \sum_{j=-n}^{-i} \pi_j \qquad \Pi^{+}_{i} = \sum_{j=0}^{m-i} \pi_{m-j} \,. \tag{4}$$

Π^{-}_{-i} are the cumulative probabilities for the loss outcomes, and Π^{+}_{i} are the cumulative complementary probabilities for the gains. Finally, weighting functions, $\Omega^{\pm}$, are applied separately to the

[2]Ho (1994) and Wu and Gonzalez (1996) consider only gains so the parameters β and λ were not estimated.

cumulative and complementary cumulative probabilities, and the decision weights are determined by differencing

$$\omega_{-i} = \Omega^-(\Pi_{-i}^-) - \Omega^-(\Pi_{-i-1}^-) \qquad \omega_i = \Omega^+(\Pi_i^+) - \Omega^+(\Pi_{i+1}^+) \qquad \text{for } i > 0\,. \tag{5}$$

For a continuous distribution, the decision-weight density functions are

$$\omega^-(s) = \frac{d\Omega^-}{ds} = \frac{d\Omega^-(\Pi)}{d\Pi}\pi(s) \qquad \omega^+(s) = -\frac{d\Omega^+}{ds} = -\frac{d\Omega^+(\Pi)}{d\Pi}\pi(s) \tag{6}$$

where $\pi(s) = d\Pi(s)/ds$ is the probability density function over the states.[3]

The overweighting property obtains if the cumulative weighting functions, Ω, have an inverted-S shape. The specific weighting functions proposed by Tversky and Kahneman are[4]

$$\Omega^\pm(\Pi) = \frac{\Pi^{\delta_\pm}}{[\Pi^{\delta_\pm} + (1-\Pi)^{\delta_\pm}]^{1/\delta_\pm}}\,, \tag{7}$$

and they estimated the parameters to be $\delta_- = 0.69$, $\delta_+ = 0.61$. Other estimates for δ_+ reported in Camerer and Ho (1994) range from 0.28 to 1.87.[5] Prelec (1998) proposed the two-parameter weighting function $\Omega^\pm(\Pi) = \exp[-\beta_\pm(-\ell n\,\Pi)^\alpha]$ based on an axiomatic derivation.

The TK weighting function is illustrated in Figure 1 (for the value $\delta = 0.65$). For continuous distributions, the decision weight density for an objective probability density function of $\pi(s)$ is

$$\omega(s) = \Omega'(\Pi)\pi(s) = \frac{\delta\Omega(\Pi)}{\Pi}\left[1 - \frac{1-[(1-\Pi)/\Pi]^{\delta-1}}{\delta\left(1+[(1-\Pi)/\Pi]^{\delta}\right)}\right]\pi(s)\,. \tag{8}$$

For TK's parameters, $\delta_- = 0.69$ and $\delta_+ = 0.61$, the cumulative probability of losses is overweighted for probabilities less than 37.76% and the density is underweighted between the 12.41 and 82.19 percentiles; the gain complementary probability is overweighted for probabilities less than 33.88% and the density is underweighted between the 10.88 and 82.82 complementary

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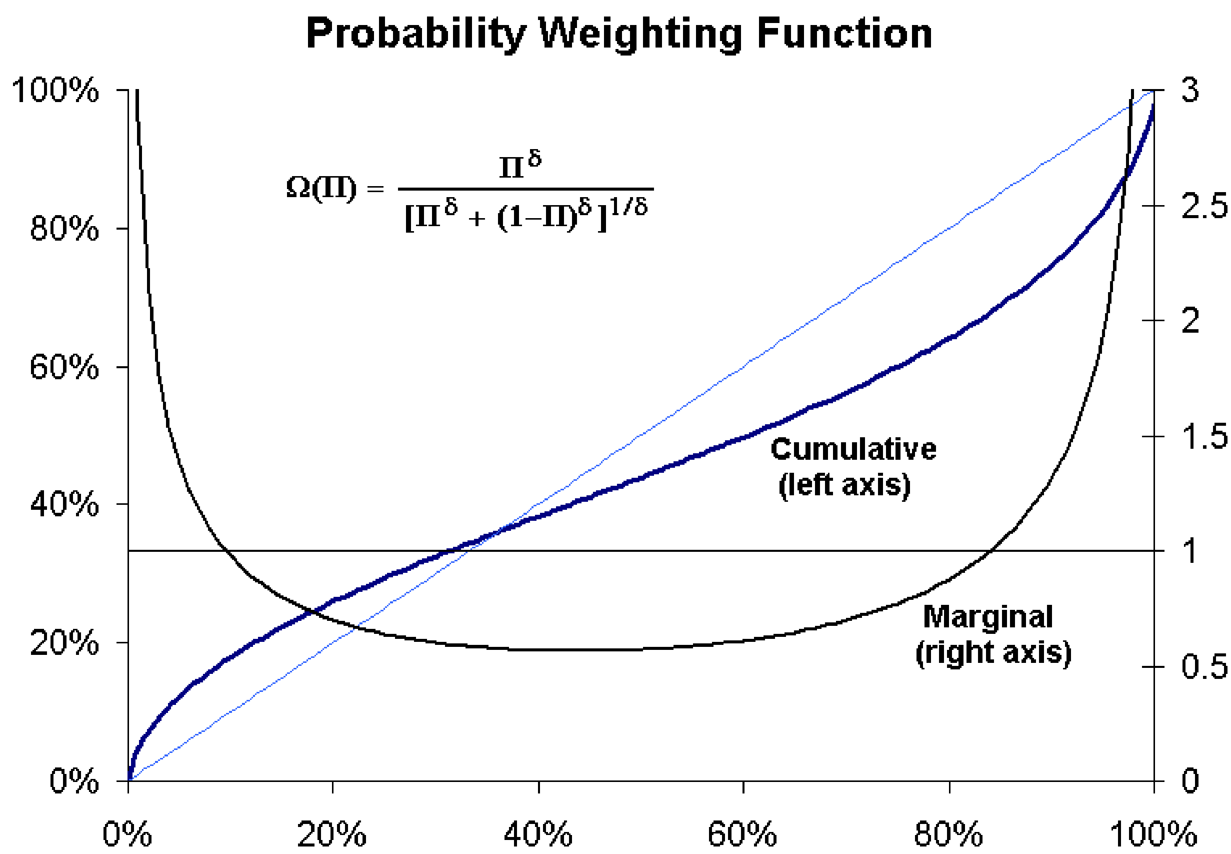
[3] The probability weighting function, Ω, must be strictly increasing. If it is not, it can assign zero or negative decision weight to possible outcomes. If Ω is not continuous it can assign a decision weight atom to an atomless probability distribution. Finally Ω must also be differentiable if a smooth decision weight density function is to be achieved.

[4] It should be noted that the TK weighting function given in (7) is not monotone for all parameter values; therefore, it can assign negative decision weights. For example, for $\delta = 0.25$, $\Omega(\Pi)$ is decreasing over the range of cumulative probabilities $1.56\% < \Pi < 23.62\%$. Negative decision weights would be a severe problem for the interpretation of CPT leading to inconsistencies such as the choice of first-order dominated payoffs. Fortunately, the TK weighting function is monotone for all values of δ greater than the root of the equation $(1-\delta)^{2-\delta} = \delta^{1-2\delta}$. The critical root is $\delta \approx 0.279$, so the function is monotone in the empirically relevant range. See Ingersoll (2008).

[5] Only gains were considered so only that parameter could be estimated. There was only one instance reported with an estimated $\delta > 1$. That was for Gigliotti and Sopher (1993). The next largest estimate was 0.97. For values of δ greater than 1, the decision weights are smaller than the true probabilities for the rare extreme outcomes.

percentiles.



Note that the decision weight, ω_0, for no gain or loss, $x_0 = 0$, is unassigned by equation (5), and the assignments of Ω^- and Ω^+ cannot both be extended to x_0 as they would, in general, give different values. The obvious completion is to set $\omega_0 = 1 - \Omega^-(\Pi_{-1}) - \Omega^+(\Pi_1)$ which makes the weights sum to one. Unfortunately, this will not always be practical. First there may be no outcome of zero, and second, this assignment is not necessarily a positive value.[6] Furthermore, for continuous distributions, a positive or negative atom of probability is often required in an otherwise atomless distribution to match the decision weight densities at $x = 0$.

For CPT, the decision weight applied to a gain of zero might appear to be irrelevant since $v(0) = 0$ so the weight does not affect the computed expected utility. However, when developing pricing results, it is the marginal that matters so the numerical value of ω_0 is important in an equilibrium setting even within CPT. For this reasons, we will require that the decision weights do sum to exactly one for all gambles considered. The only way to assure this property is to use a single weighting function (with $\Omega(1) = 1$) for the entire cumulative distribution ignoring any

[6] For example, using TK's weighting function in (7) with $\delta_- = 0.9$ and $\delta_+ = 0.6$, a gamble with a 10% chance of a loss, an 85% chance of a gain, and a 5% chance of no change has probability weights of 18.80% and 82.25% for the loss and gain which would require $\omega_0 = -1.05\%$. More commonly, however, the CPT decision weights sum to less than one when there is no zero gain outcome. This phenomenon is referred to as subcertainty.

distinction between gains and losses.[7] From Figure 1, the marginal effects are qualitatively similar in both tails so using a single weighting function will only alter the numerical results and not affect their qualitative properties.

A related advantage of using a single weighting function is that the resulting distribution can be treated just like a subjective probability distribution. Standard methods and intuitions like stochastic dominance and Rothschild Stiglitz riskiness can be applied directly to the decision weights. The one caveat is that these subjective distributions belong to the risky prospects and not to any state space in which they are embedded. Two prospects with different orderings for their outcomes can have different subjective distributions even if they are defined on the same state space.

III The Cumulative Prospect Theory Portfolio Problem

CPT was developed in the context of fixed gambles; that is, it was used to evaluate predetermined sets of outcome-probability pairs. However, to analyze portfolio problems we must compare risky prospects whose outcomes are under some control of the decision maker. This leads to two distinct problems. First, in the standard portfolio problem unlimited buying and selling is allowed, and a convex valuation of losses may induce the investor to take unbounded positions. Second, the portfolio is chosen from amongst a set of assets with a known joint probability distribution, but the decision weights used in place of the probabilities cannot be determined until the ordering of the portfolio outcomes across states is known. So as an investor evaluates different portfolios whose returns are not perfectly aligned, a changing set of decision weights must be employed in place of fixed state probabilities.

We will work in a standard single-period market setup. State s occurs with probability π_s and has a strictly positive state price of p_s.[8] Final wealth in state s is $W_s = W_0(1+\hat{x}+x_s)$ where $\hat{x}$, is the reference rate of return assigned zero utility under loss aversion.[9] Most commonly the

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[7] Quiggin (1982) who first proposed the use of a cumulative weighting function applied it to the entire distribution. To see that a single weighting function is required, note that we must have $\Omega^+(1-z) = 1-\Omega^-(z)$ for all z if the decision weights are to sum to one regardless of where in the distribution a zero net gain appears. But when this property holds, the single weighting function $\Omega(z) \equiv \Omega^+(z)$ results in the same mapping of the cumulative distribution as the two separate functions.

[8] The assumption $p_s > 0$ assures there are no arbitrage opportunities. Although state prices can also be represented risk-neutral probabilities, these "probabilities" are not subject to weighting like the actual probabilities since they are constrained by prices and possibilities not likelihoods. In particular, in a complete market, the risk-neutral probabilities, like the state prices, are uniquely determined by the absence of arbitrage. In an incomplete market, the absence of arbitrage also limits the feasible risk-neutral probabilities. These restrictions are identical across investors regardless of how they might weight probabilities.

[9] There is no loss of generality in defining utility in terms of rates of return rather than dollar gains and losses. The zero-utility reference level for dollar gains is $\hat{X} = W_0(1+\hat{x})$, and the dollar gain in excess of the reference level is $X_s = W_0(1+\hat{x}+x_s) - \hat{X} = W_0 x_s$. The utility for dollar gains is then $V(X) = v(X/W_0)$. In particular, for TK loss aversion, utility for dollar gains is $V(X) = X^\alpha$ and for losses is $V(X) = -\Lambda(-X)^\beta$. This can be written equivalently as $v(x) = x^\alpha$ and $-\lambda(-x)^\beta$ for gains and losses, respectively, where $\lambda = \Lambda W_0^{\beta-\alpha}$ after dividing out the arbitrary constant W_0^α. It is only when going to a multi-period setting, in which wealth can change, that using rates of return rather than dollar gains and losses makes a difference due to reframing.

reference level is set to either 0 or the interest rate, but any other value could be used. We will refer to x_s as the excess return and to positive (negative) excess returns as gains (losses) though neither corresponds to the usual usage. The investor's budget constraint is $\sum p_s(1+\hat{x}+x_s)W_0 = W_0$ or, when a risk-free asset with interest rate r_f is available, $\sum p_s x_s = B$ where $B \equiv (r_f - \hat{x})/(1+r_f)$.

In performing the maximization, it must be recognized that decision weights, ω_s, are used in place of objective probabilities, and they are affected by the ordering of the portfolio returns across states. For a fixed ordering of returns, the weights can be considered constant and treated just like the state probabilities in the standard problem. So, with no loss of generality, assuming the ordering of outcomes across states is $x_s \le x_{s+1}$,[10] the investor's partial portfolio problem is

$$\text{Max} \sum \omega_s v(x_s) \qquad \text{subject to} \qquad \sum p_s x_s = B \quad \text{and} \quad x_s \le x_{s+1}\,. \tag{9}$$

Of course, to completely solve the problem, all of the possible outcome orderings would have to be considered and the best portfolio for each compared.

This portfolio problem with no constraints apart from the budget and ordering constraints is a complete markets analysis. An incomplete-market portfolio problem can be analyzed by using additional constraints restricting the feasible set of portfolio returns; i.e., $\mathbf{x} = \mathbf{Y}\mathbf{y}$ where $\mathbf{Y}$ is the $S \times N$ matrix of excess returns on each of the N assets in the S states, and $\mathbf{y}$ is a vector of the allocation to each of the N assets. Short sales restrictions or limited liability can be handled similarly by imposing $\mathbf{y} \ge \mathbf{0}$ or $\mathbf{x} \ge -(1+\hat{x})\mathbf{1}$, respectively.

The Lagrangian for (9) is $L = \sum \omega_s v(x_s) + \eta[B - \sum p_s x_s)] + \sum_{s=1}^{S-1} \kappa_s (x_{s+1} - x_s)$ for a fixed set of ω. The first-order Kuhn-Tucker conditions are

$$\begin{aligned} &0 = \frac{\partial L}{\partial x_s} = \omega_s v'(x_s) - \eta p_s - \kappa_s + \kappa_{s-1} \qquad s = 1,\ldots,S \quad (\kappa_0 \equiv 0) \\ &0 \le \frac{\partial L}{\partial \kappa_s} = x_{s+1} - x_s \qquad 0 = \kappa_s (x_{s+1} - x_s) \qquad s = 1,\ldots,S-1 \\ &0 = \frac{\partial L}{\partial \eta} = B - \sum p_s x_s\,. \end{aligned} \tag{10}$$

When using the objective probabilities, the κ constraints can safely be ignored as the ordering of states is immaterial.

Due to the convexity of the utility function over losses, a maximal loss, $\underline{x}$, is sometimes assumed to keep the optimal portfolio bounded. This might, but need not, be $-(1+\hat{x})$; i.e., total loss of wealth. Of course, a maximum acceptable loss can be imposed as a practical consideration

[10] CPT requires that x_s be strictly less than x_{s+1}. If the two are equal, then a decision weight is assigned to the combined state $\{s, s+1\}$. However, since states s and $s + 1$ are adjacent in the cumulative probabilities, the decision weight for the merged state will equal the sum of the two original decision weights; i.e., $\omega_{\{s,s+1\}} = \omega_s + \omega_{s+1}$. So when $x_s = x_{s+1} = x$, the contribution of these two states to expected (decision-weighted) utility can be expressed as either $\omega_s v(x) + \omega_{s+1} v(x)$ or $\omega_{\{s,\,s+1\}} v(x)$. Consequently, the weak inequality, which is usual for standard optimization, may be assumed.

even when the optimal solution would otherwise still be bounded. Assuming loss-averse utility with a maximum acceptable loss is somewhat similar to adopting a Friedman-Savage utility function that becomes infinitely risk-averse over losses larger in magnitude than $\underline{x}$.[11] We will impose a maximum acceptable loss of $\underline{x}$ and allow $\underline{x} = -\infty$ to cover situations when no explicit maximum loss is to be imposed. If there is a maximal loss, the x_s partial derivatives all are zero when $x_s > \underline{x}$ and are nonpositive when x_s is at its lower bound.[12] Of course, the second order conditions for a maximum may not be satisfied.

To achieve an unconstrained optimum for the portfolio problem, some additional structure must be imposed. One assumption that ensures this even with risk preference over losses is that the utility function satisfies

$$\lim_{x\to\infty} \frac{v(x)}{v(-kx)} = 0 \quad \forall k > 0 \tag{11}$$

which we shall call asymptotic risk avoidance. Because portfolio formation allows only linear trade-offs between outcomes, asymptotic risk avoidance ensures that any "extreme" portfolio will be rejected as sub-optimal. In particular, a very large gain in some state must be financed by proportionally large losses in other states, but asymptotic risk avoidance ensures that when this leverage is sufficiently large, the utility losses will more than offset the utility gains. This notion is made precise in Proposition I below.

The TK loss-averse utility function in (1) has asymptotic risk avoidance if $\alpha < \beta$ since $v(x)/v(-kx) = -k^{-\alpha} x^{\alpha-\beta}/\lambda$. On the other hand, the utility function

$$v(x) = \begin{cases} 1 - e^{-\alpha x} & x \ge 0 \\ -\lambda\left(1 - e^{\beta x}\right) & x < 0 \end{cases} \tag{12}$$

does not have asymptotic risk avoidance for any parameters since $\lim_{x\to\infty} v(x)/v(-kx) = -1/\lambda$.

Proposition I: Bounded Optimal Portfolios with Asymptotic Risk Avoidance. If an investor has asymptotic risk avoidance and a zero-utility reference return less than or equal to the interest rate, then the optimal portfolio has bounded positions in every asset.

Proof: The budget constraint puts an upper bound on the worst return in terms of the best

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[11] Friedman-Savage utility has lower and upper concave portions straddling a convex portion. It need not have its higher inflection point at a gain of zero. In a multi-period problem, the resetting of the anchor point is an added difference between loss-averse and Friedman-Savage utility that must be modeled. The maximum loss can also be made state dependent, and all our results continue to hold, but there is little gain from this generalization and a large cost in complexity.

[12] Formally, the first-order conditions must hold at an optimum even in the convex region. Consider any optimal portfolio and the same portfolio with two returns altered to $x_i \to x_i + p_j\varepsilon$ and $x_j \to x_j - p_i\varepsilon$. This alteration is affordable and the change in expected utility for small ε is $\Delta E[v] = [\pi_i v'(x_i)p_j - \pi_j v'(x_j)p_i]\varepsilon + O(\varepsilon^2)$. If the first-order conditions hold, no small alteration can increase utility. But if the term in brackets is positive (negative), then an increase (decrease) in x_i will increase expected utility and the original portfolio could not have been optimal. We will see, however, there can be at most one state with a realized loss at which the first-order condition is relevant.

return. By definition the smallest and largest excess returns occur in states 1 and S, so $x_1 \leq P(B - p_S x_S)$ where $P^{-1} \equiv \sum_{s=1}^{S-1} p_s$ with the equality holding only when all states but S have this same small excess return. Since utility is increasing and the portfolio outcomes are weakly ordered, the expected utility of this portfolio is

$$\mathrm{E}_{\omega}[v(x)] = \sum_{s=1}^{S} \omega_s v(x_s) \leq \omega_1 v(x_1) + v(x_S) \sum_{s=2}^{S} \omega_s \leq \omega_1 v\big(P(B - p_S x_S)\big) + v(x_S) \sum_{s=2}^{S} \omega_s \, . \tag{13}$$

By the mean-value theorem, $v(P(B - p_S x_S)) = v(-P p_S x_S) + PBv'(aPB - P p_S x_X)$ for some $a \in [0,1]$. For negative outcomes v' is increasing so as x_S gets large, $v(P(B - p_S x_S)) \approx v(-P p_S x_S)$. If the utility function has asymptotic risk avoidance, then for all sufficiently large x_S, $v(x_S)$ will be negligible in size relative to $v(-P p_S x_S)$ and, therefore, $v(P(B - p_S x_S))$, and the right-hand side of (13) will be negative.

Thus for any investor with asymptotic risk avoidance, portfolios with sufficiently large returns must have negative expected utility. Such a portfolio cannot be optimal if positive expected utility portfolios are possible. This must be true if the zero-utility reference return is less than or equal to the interest rate, since the risk-free portfolio has nonnegative utility. Therefore, all optimal portfolios must have bounded positions. +

If asymptotic risk avoidance is severely violated and the limit in (11) is infinite, then similar reasoning shows that unbounded portfolios are always optimal. This is true for the KT loss aversion function if $\alpha > \beta$. If the limit in (11) is finite, then the boundedness of optimal portfolios is indeterminate. Consider an investor with the TK loss averse utility with $\alpha = \beta$ and a reference level of $\hat{x} = 0$ in a two-asset, two-state economy. If the risky asset's expected rate of return exceeds the interest rate, then we know that all risk-averse investors will optimally invest a positive amount in it. This must also be true for a loss-averse investor since for a sufficiently small position, the excess returns in both states will exceed 0, and only the risk-averse portion of the utility function will be applicable. Label the states so that $p_1/\omega_1 < p_2/\omega_2$ and suppose the investor holds a sufficiently large position so that $x_2 < 0 < x_1$, then expected utility is[13]

$$\mathrm{E}[v(x)] = \omega_1 x_1^{\alpha} - \omega_2 \lambda (-x_2)^{\alpha} = \omega_1 x_1^{\alpha} - \omega_2 \lambda [x_1 p_1 - B]^{\alpha} \, p_2^{-\alpha} \, . \tag{14}$$

For extremely large positions, expected utility is $\mathrm{E}[v(x)] \approx x_1^{\alpha}[\omega_1 - \omega_2 \lambda (p_1/p_2)^{\alpha}]$. It is a simple matter to construct examples in which this quantity is positive so infinite leverage is desired or negative so a bounded position is optimal. Note in particular that this example shows that KT's increasing symmetric bet aversion criterion is insufficient to guarantee bounded portfolios.

Given that loss aversion does not automatically lead investors to desire extreme leverage, we can address the problem of characterizing optimal portfolios and answer the related question of how prices are set in equilibrium under CPT. The next section examines optimal portfolios when markets are complete.

[13] The state price to probability ratios cannot be equal or the risky asset would have an expected rate of return equal to the interest rate, and the return realized in the state with the smaller ratio must be positive to hold a portfolio with an expected rate of return in excess of the interest rate.

IV The Loss-Averse Portfolio Problem

To focus on loss aversion, we ignore probability weighting in this section. The objective probabilities are used, and the outcome-ordering conditions are not binding so the Lagrange multipliers, κ_s, may all be set to zero.[14] If there is a maximum acceptable loss, that constraint may be binding so the first-order conditions are

$$0 \geq \frac{\partial L}{\partial x_s} = \pi_s v'(x_s) - \eta p_s \ . \tag{15}$$

This condition holds as an equality except possibly when $x_s = \underline{x}$.

From (15) we see that many of the properties of optimal portfolios are determined by the price-probability ratio, $\theta_s = p_s / \pi_s$, just as under global risk aversion.[15] We will refer to states with low price-probability ratios as good or better states. Since $v''(x) < 0$ for a risk-averse investor and $v'(x_s) = \eta\theta_s$ at the optimum, optimal portfolios for risk-avers investors will have higher returns in better states. For loss-averse investors, states where positive and negative excess rates of return are realized must be considered separately. In addition states where the maximum loss is earned and the first-order condition does not hold must also be considered.

Proposition II: In a complete market, the rates of return realized on all optimal loss-averse portfolios can be characterized by:

(i) Gains are larger in better states; i.e., for x_i, $x_j > 0$, $x_i > x_j$ if and only if $p_i/\pi_i < p_j/\pi_j$.

(ii) If there is no maximum loss ($\underline{x} = -\infty$), then a loss is realized in at most one state. If $\underline{x} > -\infty$, then multiple states can have portfolio losses, but at most one loss is not maximal in size.

(iii) Losses can be earned in better states (lower θ_s) than some of those in which gains are realized, but a state with a loss cannot have both a smaller state price and a higher probability than any state with a gain.

(iv) Risk sharing need not be Pareto efficient when markets are complete even if all investors have identical beliefs and tastes.

Proof: All results follow from the first-order conditions. The separate characterizations are verified and discussed below. +

Amongst states where just gains are earned, the portfolio return will be higher the lower the price-probability ratio just as for a standard investor since the first order conditions hold and margi-

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[14] The analysis here remains valid under probability weighting with the decision weights replacing the objective probabilities provided only one ordering of portfolio outcomes is considered, and the ordering constraints are not binding. See section VI for analysis of the ordering constraints.

[15] With a finite or countable number of states the price-probability ratio is proportional to the ratio of the risk-neutral probabilities, π^*, and true probabilities $\theta_s = (1+r_f)\pi_s^*/\pi_s$. When there is a continuum of states, the price-probability ratio is a state price density.

nal utility is decreasing over gains. This verifies property (i). In some settings an investor's optimal portfolio may have gains in all states. This will occur, for example, if the investor is very risk averse over gains and the market provides little reward for bearing risk (i.e., there is little variation in the price-probability ratio). In such cases, the investor's optimal portfolio will be close to risk-free, $x_s \approx r_f + \hat{x}$ in all states. However, portfolios with only gains can be optimal in other cases as well.[16]

To verify the second property, suppose there are two states, i and j, with losses in the optimal portfolio. The same portfolio with just these two returns altered to $x_i \to x_i - p_j\varepsilon$ and $x_j \to x_j + p_i\varepsilon$ is affordable for any ε. A second-order Taylor expansion for the change in expected utility for this alteration is

$$\Delta \mathrm{E}[v(x)] = \varepsilon[\pi_j p_i v'(x_j) - \pi_i p_j v'(x_i)] + \tfrac{1}{2}\varepsilon^2[\pi_i v''(x_i) p_j^2 + \pi_j v''(x_j) p_i^2] + o(\varepsilon^2)\,. \tag{16}$$

The second term is positive since $v'' > 0$ in each state by assumption. The first term can be made nonnegative by choosing ε to have the same sign as the term in brackets. Therefore, expected utility can be increased by this alteration, and the original allocation could not have been optimal. So a loss that is less than maximal can be realized in at most one state.[17] Of course, if there is no maximum loss ($\underline{x} = -\infty$), then it follows immediately that at most one state can have a loss verifying (ii).

The intuition for this result is that the loss-averse investor will always benefit by increasing a larger loss and reducing a smaller loss if it is possible to do so. Realizing larger losses in one state allows, through the budget constraint, the realization of higher returns in another state. With an ordinary risk-averse utility function, this is not beneficial as the increased loss has a bigger impact on utility. But in the convex portion of a loss-averse utility function, it is better to concentrate any losses in a single state since the marginal utility decreases rather than increases as larger losses are realized. This does not mean, however, that the single loss will be of maximum size, $\underline{x}$, as the example in the next paragraph illustrates. In addition, if there is a maximum loss, then the optimal portfolio may realize losses in two or more states because a single state may not have a large enough state price to finance the gains desired in all the other states.

When an optimal portfolio has a loss, then it need not have its returns inversely related to the price-probability ratio. For example, in the economy $\boldsymbol{\pi} = \{0.75, 0.25\}$, $\mathbf{p} = \{0.8, 0.2\}$, the optimal portfolio for the TK loss-averse utility function with $\alpha = 0.4$, $\beta = 0.6$ and $\lambda = 2$ is $\mathbf{x}^* = \{{}^1/_{64}, {}^{-1}/_{16}\}$. Since the price-probability ratios are $\boldsymbol{\theta} = \{{}^{16}/_{15}, {}^4/_5\}$, the optimal portfolio has its higher return in the worse state. This verifies the first part of property (iii).

Although a loss need not be realized in the worst (highest price-probability ratio) state, it can only be earned in the worst state(s) at or below a given probability. That is, if $\pi_i \le \pi_j$ and $p_j \le p_i$ with at least one equality holding strictly, then $\theta_j < \theta_i$, and the better state j cannot be the only one

[16] For example, consider a two-state economy characterized by $\pi_1 = \pi_2 = ½$, $p_1 = 0.6$, $p_2 = 0.2$. An investor with the TK loss averse function with $\alpha = \beta = ½$ and $\lambda = 2$ with $\hat{x} = 0$ optimally holds the portfolio $x_1 = {}^1/_{12}$ and $x_2 = ¾$.

[17] If $\underline{x} > -\infty$, it is possible to have a maximum loss in one state and a less than maximum loss in another. The proof given is not valid in this case as we cannot choose ε to create a loss of more than the maximum size.

with the negative return. To verify this, note that if some portfolio with $x_j = \ell < 0 < h = x_i$ is affordable, then the portfolio with the rates of return $x_j = h + (p_i - p_j)(h - \ell)/p_j \geq h$, $x_i = \ell$, and identical returns in all other states is also affordable and has higher expected utility since it first order stochastically dominates the former giving the higher of the two returns with at least as high a probability.[18] Therefore, the portfolio with the loss in the better state could not have been optimal. This verifies the second part of property (iii).

This property might seem to be only a curiosity, but that is not the case. When the returns on all optimal portfolios are not monotonically related, the set of efficient portfolios need not be convex, and when the efficient set is not convex, the market portfolio need not be efficient.[19] This in turn means that there may be no representative investor who optimally holds the market portfolio and whose marginal utility can be used to determine prices — and that is the basis for virtually all partial equilibrium models of asset pricing.

More importantly, with loss aversion, a complete market is not the risk-sharing remedy it is under risk-aversion — it does not guarantee a Pareto optimal allocation of risk. Even if Arrow-Debreu securities are available for each state, loss-averse investors can increase their welfare by trading new financial securities they create in zero net supply.

To illustrate consider the two-state economy in the previous example. Suppose a derivative security is created that partitions the second state into two sub-states using exogenous randomization like a coin flip. The sub-states are equally probable and since the randomization is exogenous the risk will be unpriced.[20] The economy will have $\boldsymbol{\pi} = \{0.75, 0.125, 0.125\}$ and $\mathbf{p} = \{0.8, 0.1, 0.1\}$. Two TK loss-averse investors with the same utility function as before can both increase their expected utility from 0.0474 to 0.0913 by swapping $^3/_{32}$ of state securities for states 2a and 2b bringing their holdings to $\{^1/_{64}, ^{-5}/_{32}, ^1/_{32}\}$ and $\{^1/_{64}, ^1/_{32}, ^{-5}/_{32}\}$. Further dividing state two or state one will also increase utility. This example verifies statement (iv).

This example is quite simple and can easily be extended to many market economies with multiple states. In addition to showing that complete markets are insufficient to guarantee Pareto optimality under loss aversion, it also further highlights the problem of a representative investor equilibrium. In virtually all such models, the representative investor is the "average" investor and holds the market portfolio. This example shows that even if all investors are ex ante identical and, a fortiori, representative, none of them may be average and hold the market portfolio.[21] Therefore,

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[18] First-order stochastic dominance is, of course, a valid comparison for lose-averse utility since the function is increasing in the return realized.

[19] See Dybvig and Ross (1982) and Ingersoll (1987) for examples of the nonconvexity of the efficient set when optimal portfolios are not monotonically related.

[20] Whether or not the risk should remain unpriced in the new equilibrium is irrelevant. Any two investors with the assumed utility function could increase their welfare by agreeing to trade at that price so the original allocation of risk clearly is not Pareto efficient. This welfare improvement may not be possible if the investors frame the sub-state trade separately from the portfolio decision so that losses are separately accounted for and not combined. Note also that probability weighting will reduce the advantage of such a trade as each investor would tend to overweight the probability of his own loss, but the advantage need not be eliminated.

[21] See Barberis and Huang (2008) for a more worked-out example with identical CPT investors optimally choosing different portfolios in an incomplete market.

pricing that relies on the marginal properties of the market portfolio, which is assumed to be optimal, may not be valid.

The possibility of using derivative contracts to partition existing states into sub-states leads to further complications as well. In particular even asymptotic risk avoidance is insufficient to guarantee the existence of bounded optimal portfolios so the existence of an equilibrium may be precluded altogether.

To illustrate, consider an economy in which some sub-state is partitioned into two sub-states with probabilities $f\pi$ and $(1-f)\pi$ and state prices fp and $(1-f)p$. Suppose the investor has TK utility with $\beta > \alpha$. In a sequence of such economies characterized by f, the investor sells f^{-b} additional pure f sub-state securities at price fp and purchases $f^{1-b}/(1-f)$ additional pure $1-f$ sub-state securities at price $(1-f)p$. For any f, this alteration is affordable, and the change in expected utility from the original unpartitioned economy is

$$\Delta \mathrm{E}[v(x)] = -f\pi\lambda[(f^{-b}-x)^{\beta}-(-x)^{\beta}]+\pi(1-f)([x+f^{1-b}/(1-f)]^{\alpha}-x^{\alpha}) . \qquad (17)$$

For any positive value of $b < 1/\beta$ the first term vanishes as f goes to zero; while for any $b > 1$ the second term becomes unbounded. Therefore, expected utility can be increased without bound by a sufficiently fine partition of the state.[22] This leads to a fifth portfolio property.

Proposition III: If each investor has a (possibly different) finite maximum acceptable loss and actuarially fair state partitioning is permitted, all optimal loss-averse portfolios will be weakly ordered inversely to the price-probability ratio and all losses will be maximal in size. Further, in many cases, there will be no small gains below some threshold.

Proof: Suppose some optimal portfolio has a higher excess return in a state *i* which is worse than state *j*. Partition the states so that some sub-state of *i* has the same sub-state price as some sub-state of *j*. Since state *j* is better, the sub-state of the better state *j* must now have a higher sub-state probability than the sub-state of *i*. Swapping the returns in these two sub-states is affordable and will create a portfolio that first-order stochastically dominates the assumed optimal portfolio. Therefore, when partitioning is possible, all optimal portfolio returns must be weakly ordered inversely to the price-probability ratio just as they are under universal risk aversion.

Now suppose the excess return in some state is a non-maximal loss, x. Partition the state into two parts with relative sizes f and $1-f$ where $f = x/\underline{x}$. Creat an otherwise identical portfolio with excess returns $\underline{x}$ and 0 in sub-states f and $1-f$. This new portfolio is affordable and because utility is convex over losses, this state now has a large contribution to expected utility than before $\pi[fv(\underline{x}) + (1-f)v(0)] > \pi v(x)$. So again the original portfolio could not have been optimal.

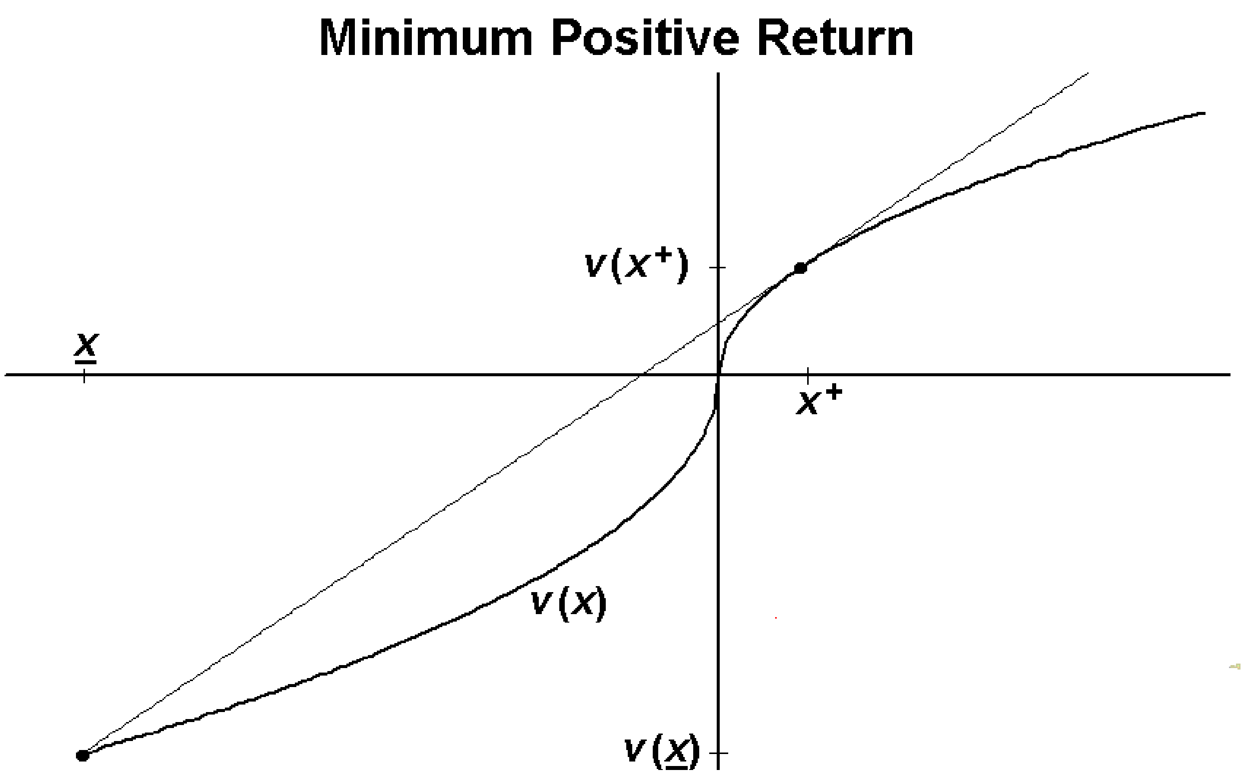
The final part this proposition is best illustrated in Figure 2 below. Partitioning the states by randomizing essentially convexifies the utility function over its lower domain. Define x^+ as the excess return where a line through $v(\underline{x})$ at $\underline{x}$ is tangent to the utility function at a positive return. The investor would always prefer a fair gamble with payoffs $\underline{x}$ and x^+ to any single payoff in the range

[22] This example does not contradict Proposition I. There the states and their probabilities were fixed. Here we are considering creating a sequence of economies where one state's probability is decreasing to vanish in the limit.

($\underline{x}$, x^+). Therefore, no rates of return, except the maximum acceptable loss, will be realized below the return x^+. +

The magnitude of x^+ is very sensitive to the exact utility function assumed, but it can be significantly positive in an economic sense. For example, for the TK loss-averse utility with $\alpha = \beta = 0.88$ and $\lambda = 2.25$, the minimum positive return is 0.04%. However, for $\alpha = \beta = 0.75$, $x^+ = 1.21\%$ and for $\alpha = \beta = 0.5$, $x^+ = 4.5\%$.[23]

It is clear that loss aversion can lead to optimal portfolios that are quite different from those held by risk-averse investors. Indeed, there may be no optimal portfolio at all for loss averse investors unless there are strong restrictions placed on the types of assets permitted. Generally assuming markets are incomplete will not be sufficient. For example loss-averse investors would tend to sell large quantities of deep out-of-the-money puts. Before looking at models with restrictions on the types of assets, we examine in the next sections the second aspect of CPT — probability weighting.



VI The Portfolio Problem under Probability Weighting

In this section, to concentrate on the effects of probability weighting, we first consider some examples using standard risk-averse utility functions. For a strictly concave utility function,

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[23] The minimal positive gain is the solution to $v(x^+) = v(\underline{x}) + (x^+ - \underline{x})v'(x^+)$.

marginal utility is monotonic and invertible so the optimal portfolio given in equation (10) satisfies

$$x_s^* = u'^{-1}\left((\eta p_s + \kappa_s - \kappa_{s-1})/\omega_s\right). \tag{18}$$

If x_s^* is not equal to either of its neighbors, then $\kappa_s = \kappa_{s-1} = 0$, and the optimal return is the inverse marginal utility of some multiple of the price-decision-weight ratio p_s/ω_s. The only difference between this and the standard result is that the decision weight rather than the probability is used in this ratio. If some consecutive states s', $s' + 1$, …, s'' all have the same excess return in the optimal portfolio but different excess returns from those in the other states, then $\kappa_{s'}$ through $\kappa_{s''-1}$ can be positive. In this case,

$$\begin{aligned} x_{s'}^* &= u'^{-1}\left((\eta p_{s'} + \kappa_{s'})/\omega_{s'}\right) \le u'^{-1}(\eta p_{s'}/\omega_{s'}) \\ x_{s''}^* &= u'^{-1}\left((\eta p_{s''} - \kappa_{s''-1})/\omega_{s''}\right) \ge u'^{-1}(\eta p_{s''}/\omega_{s''}). \end{aligned} \tag{19}$$

The inequalities follow because the multipliers are nonnegative and marginal utility is decreasing. The first-order conditions can therefore be summarized as

$$\begin{aligned} x_s^* = u'^{-1}(\eta p_s/\omega_s) \qquad & \text{for} \quad x_{s-1}^* < x_s^* < x_{s+1}^* \\ u'^{-1}(\eta p_{s''}/\omega_{s''}) \le x^* \le u'^{-1}(\eta p_{s'}/\omega_{s'}) \qquad & \text{for} \quad x_{s'}^* = x_{s'+1}^* = \cdots = x_{s''}^* = x^*. \end{aligned} \tag{20}$$

The conditions in the second line of (20) permit the optimal portfolio to allow ties in its outcome rankings across states provided the assumed weak ordering is preserved.

This proposed solution, however, guarantees only an order-constrained optimum. A different ordering of x_s across states might have higher decision-weighted utility. To solve the problem completely, the optimal portfolio for each ordering must be determined from (10) and their maximized decision-weighted utilities, $\sum \omega_s u(x_s^*)$, can then be compared. The optimal portfolio is the constrained portfolio that gives the highest utility.[24]

As an example, consider the simple three-state problem presented in Table I. The interest rate is zero. States *a*, *b*, and *c* have probabilities of 20%, 30% and 50% and state prices of 0.3, 0.3 and 0.4, respectively. State *a* is the most expensive state per unit probability, and state *c* is the least expensive so the optimal portfolio for a risk-averse expected utility maximizer has its returns ordered $x_a < x_b < x_c$.[25] Suppose, instead, the investor has a TK probability-weighting function as given in (7) with a parameter $\delta = 0.7$. If we assume his portfolio returns are also ordered $x_a \le x_b \le x_c$, then the decision weights are 25.6%, 20.13% and 54.26%. The optimal order-unconstrained portfolio (as determined by (18) with $\kappa = 0$) is given in the columns labeled "Assumed" in the middle panel of Table I.

[24] As a practical matter the constrained optimal portfolio need not be determined for all orderings. The constraint(s) that are binding in any one of the optimization problems will often indicate which orderings to try.

[25] See, for example, chapter 8 of Ingersoll (1987) for a proof that all risk-averse investors with state-independent utility and homogeneous beliefs hold portfolios whose returns are identically ordered (and inversely ordered to the price-probability ratio p_s/π_s) in a complete market.

Unfortunately, we see that the order-unconstrained optimal returns are not ordered as assumed, but rather $x_b < x_a < x_c$, and the true decision-weighted utility for this ordering is not 1.065, but only 1.037 as computed with the correct decision weights based on the actual ordering of outcomes as shown in the columns labeled "Corrected." This occurs because under the first ordering, the decision weights overemphasize state *a* and underemphasize state *b* relative to the actual probabilities so the decision-weight ratio, p_s/ω_s, does not align with probability ratio, p_s/π_s.

But the problem does not end with correcting the computation of the decision-weighted utility. This "optimal" solution was determined using a faulty assumption about the ordering of outcomes, but the result indicated that the $x_a \le x_b$ constraint was binding. This suggests that the ordering $x_b \le x_a \le x_c$ be considered. The second panel of Table I shows the calculated optimum under this assumed ordering. Again the order-unconstrained optimal portfolio does not match the assumed order — rather it insists on the originally assumed order of $x_a < x_b < x_c$. Further exploration of all orderings shows that whenever, $x_a < x_b$, increasing x_a and decreasing x_b increases decision-weighted utility and vice versa. Consequently, the optimal portfolio under this probability weighting must have $x_a = x_b$ as shown in the final columns in Table I. In addition we see that the highest return in state *c* is larger under probability weighting than under expected utility maximization.

While this example was obviously created, it was not chosen to achieve unusual results; nor do the results depend on the small number of states or on the existence of a complete market. The flattening of the left tail and the skewing of the right tail is a generic trait of the optimal portfolios for investors who employs probability weighting of the type proposed in CPT. Even when the conversion of probabilities to decision weights is approximately symmetric in the two tails, the effects in the two tails are quire different. The decision weights exceed the true probabilities for both extremes as shown in Figure 1. Since the optimal portfolio's return is decreasing in the ratio p_s/ω_s, the portfolio of an investor using probability weighting has higher returns than that of an expected utility maximizer in both tails where ω_s tends to be greater than π_s;[26] that is, the right tail is longer and the left tail is shorter leading to a right skewing of the optimal portfolio.

In the right tail, this stretching is the only effect. In the left tail, however, the increased return can also alter the outcome ordering which affects the probability weighting as shown in the example in Table I. As in the example, the left tail will often be completely flattened so that the portfolio's return is constant over a range of the low-return states. This is particularly true for portfolios with many outcomes whose probabilities are similar in magnitude.

Flattening need not occur only in the left tail as the example in Table II shows. In this example, the states are ordered *a* to *d* from high to low by their ratios p_s/π_s. However, the decision

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[26] It is possible to construct scenarios where the probability weight for either extreme outcome is less than the associated probability, and, therefore, the probability-weighted optimal portfolio would have a smaller return than the expected-utility maximizing portfolio for the extreme outcome. For example, this would occur in the lower tail for the TK weighting function shown in Figure 1 with $\delta = 0.65$ if the worst state had a probability in excess of 35.87%. Since the best and worst states are likely to be very rare in most applications, right skewing of the optimal portfolio should be a generic result. Right skewing will not occur for an inverted-S shaped weighting function only if an extreme outcome is very common. And the effect must always be present in at least one of the tails if the cumulative weighting function has an inverted S shape with only a single crossing of the 45° line.

weights flip the ordering of the middle two states *b* and *c* for the ratio p_s/ω_s. This alteration would require the optimal portfolio to hold $x_b > x_c$, but this switch in order also alters the decision weights and the ratio — changing them back to the order under the true probabilities.[27] Therefore, the decision-weighted optimal portfolio will hold $x_a < x_b = x_c < x_d$, and the portfolio outcomes have been flattened in the middle of the distribution not the left tail.

In the two examples in Tables I and II, we have seen that the portfolio outcomes are monotone decreasing in the price-probability ratio, p_s/π_s; however, they need not be strictly decreasing as they are for any strictly risk-averse expected utility maximizer. It is also possible to construct examples in which the optimal decision-weighted portfolio's returns are not monotonic in the ratio, p_s/π_s, even for a strictly risk-averse investor.

The inverted S shaped probability weighting function, Ω, increases the importance of both tails of the distribution; therefore, if the ordering of the portfolio's returns moves one state's return from the left to the right tail, the decision weight could remain higher than the probability.

In the example illustrated in Table III, state *c* has a lower ratio, p_s/π_s, than state *b*. Therefore, any risk averse expected utility maximizer would hold a portfolio earning a higher return in state *c* than in state *b*. However, state *b* is less likely than state *c* and is further into the left tail than state *c* is into the right tail. The decision-weighting function, therefore, emphasizes state *b* relative to state *c*, and the decision-weight maximizer might wish to increases the return in state *b* to more than in state *c*. This alters the ordering and affects the decision weights assigned. In this case, however, state *c* has a large probability and state *b* is transferred just as far into the right tail as it was in the left tail so its decision-weight ratio, p_b/ω_b, remains high relative to p_c/ω_c. Therefore, the optimal decision-weighted portfolio has its returns ordered $x_a < x_c < x_b < x_d$ which is not monotonic in the price-probability ratio, p_s/π_s.[28] This will be true for any risk-averse investor; for example, a log utility investor would hold the portfolio $\mathbf{x}' = (0.5, 1.0, 1.33, 2.0)$ using the probabilities, but would hold the portfolio $\mathbf{x}' = (0.75, 0.95, 0.87, 2.51)$ using the decision weights.

The previous three examples illustrate some of the complications of finding an optimal decision-weighted portfolio even for concave utility. In any practical problem the difficulty is multiplied immensely as each possible ordering of the state returns might need to be examined. That is, if there are *n* states, then *n* factorial standard portfolio problems would need to be solved — one for each of the possible orderings. Proposition IV below shows that we can eliminate many of these ordering from consideration.

Proposition IV: Portfolio Ordering. For any two states that are equally probable, the optimal portfolio of a risk-averse or loss-averse investor using decision weights realizes at least as high a return in the state with the smaller state price.

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[27] Any probability weighting function satisfying Ω (0.3) = 0.32, Ω(0.5) = 0.51, Ω(0.7) = 0.68 Ω(1) = 1, will create this example. These conditions are consistent with the inverted-S shape as shown in Figure 1. Note that states *b* and *c* have the same probability so the ordering of their outcomes alone determines which decision weight is assigned to which state.

[28] We must also check the other permutations to verify that this ordering leads to the highest utility. Using proposition IV below, we know that $x_a \le x_b \le x_d$ so only two additional permutations need be considered.

Proof: Consider two states, *i* and *j*, with $\pi_i = \pi_j$. With no loss of generality take $p_i > p_j$. Now assume that the proposition is false and $x_i^* = h > \ell = x_j^*$. The otherwise identical portfolio with $x_i = \ell$ and $x_j = h$ is affordable since $p_i > p_j$. Swapping these two returns will change the order of the outcomes across states. However, since $\pi_i = \pi_j$ and the weighting function depends only on the cumulative probabilities, only the decision weights for states *i* and *j* will be affected and they will simply be swapped. Therefore, the decision-weighted expected utility for the altered portfolio will be equal to that for the originally assumed optimal portfolio. The altered portfolio costs less by $(p_i - p_j)(h - \ell)$, and this extra can be invested in the risk-free asset increasing the return realized in every state. Since this will not alter the ordering of the outcomes, the decision weights will remain the same and this final portfolio will have a higher decision-weighted expected utility than the originally assumed optimal portfolio. Therefore, the original portfolio cannot have been optimal, and we must have $x_i^* \le x_j^*$. +

If we add the assumption that all states are equally likely, this result can be strengthened. Recall that in a complete market, the returns on the optimal portfolios for all risk-averse expected utility maximizers are ordered identically being strictly decreasing in the price-probability ratio. Amongst risk-averse decision-weight maximizers, the optimal portfolios are weakly decreasing in a complete market with equally probable states or a complete market with a continuous, atomless state space.

Proposition V: Weak Monotonicity of Decision-Weighted Portfolio Returns. Assume a complete market with equally likely states. Then the returns on the optimal portfolio of a risk-averse or loss-averse investor using decision weights will be monotone decreasing in the objective price-probability ratio, p/π. For risk-averse investors, the returns will be strictly decreasing over ranges where p/ω is strictly decreasing and constant over ranges where p/ω is increasing or constant.

Proof: Weak monotonicity of the returns follows directly from Proposition IV since all states are equally probable. We need only ascertain when the ordering is strict or not for risk-averse investors.

Consider a range where p/ω is increasing or constant and, contrary to the proposition, that $x_s^* < x_{s+1}^*$. From the first order conditions in (10), the multiplier κ_s must be zero when the portfolio returns differ so using (18)

$$\begin{aligned} x_s^* &= u'^{-1}\left((\eta p_s - \kappa_{s-1})/\omega_s\right) \ge u'^{-1}\left(\eta p_s/\omega_s\right) \\ x_{s+1}^* &= u'^{-1}\left((\eta p_{s+1} + \kappa_{s+1})/\omega_s\right) \le u'^{-1}\left(\eta p_{s+1}/\omega_s\right). \end{aligned} \tag{21}$$

The inequalities follow because the remaining two multipliers are nonnegative and u'^{-1} is a decreasing function. But the monotonicity of u'^{-1} also implies that

$$x_{s+1}^* \le u'^{-1}\left(\eta p_{s+1}/\omega_s\right) \le u'^{-1}\left(\eta p_s/\omega_s\right) \le x_s^* \tag{22}$$

which is a contradiction so $x_s^* = x_{s+1}^*$ when p/ω is increasing or constant.

Now consider a range where p/ω is decreasing and, contrary to the proposition, that $x_s^* = x_{s+1}^*$. Suppose the portfolio is altered by earning ε less in state s and $p_s\varepsilon/p_{s+1}$ more in state s + 1. This altered portfolio has the same cost as the original, and changes expected decision-weighted utility by

$$\begin{aligned} \Delta E_{\omega}[v(x)] &= \omega_s[v(x-\varepsilon)-v(x)] + \omega_{s+1}[v(x+p_s\varepsilon/p_{s+1})-v(x)] \\ &\approx v'(x)p_s\varepsilon\left[\omega_{s+1}/p_{s+1}-\omega_s/p_s\right] > 0 \end{aligned} \tag{23}$$

which is positive since ω/p is increasing. Again this is a contradiction so we must have $x_s^* < x_{s+1}^*$ when p/ω is decreasing. And since the excess returns are ordered, the returns must be as well as they differ by the zero-utility reference return which is constant. +

We have already seen in Table III that this monotonicity result need not obtain in a complete market with states having different probabilities. It can be extended to such markets, however, provided investors can create financial contracts that are fair-value sub-state bets. When such financial contracts can be created, any state with probability, π, and state price, p, can be partitioned into two or more sub-states with proportional probabilities and sub-state prices; i.e., $p'/\pi' = p/\pi$ for all sub-states of the original state. Propositions IV and V can be applied to these equally probable sub-states and then extended back to the original states by aggregation.[29]

Another case of no little interest is a state space with a continuous probability distribution with no atoms. For such a distribution, the state index is somewhat arbitrary and can always be rescaled so that the density function` is constant.[30] Therefore, for continuous-state models, the optimal portfolio for any risk-averse investor using decision weights must have returns that are weakly decreasing in the price-probability ratio p/π with constant or strictly decreasing returns depending on whether p/ω is decreasing or weakly increasing.

Figure 3 below shows a typical example of the three ratios. The ratio p/π is falling (by construction). The ratio ω/π is U-shaped because the decision weights are larger than the actual probabilities for the outcomes in both tails. The ratio p/ω is the quotient of the two and has an inverted U shape. It must be decreasing in the range where ω/π is rising, but is increasing when ω/π is sufficiently steeply declining. The typical case is illustrated with p/ω increasing for the smallest values of the market's return. In such an economy the optimal portfolio of a risk-averse decision-weight maximizer will have the same return in all low outcome states and will have returns increasing with the state, like that of an expected utility maximizer, for higher outcomes.

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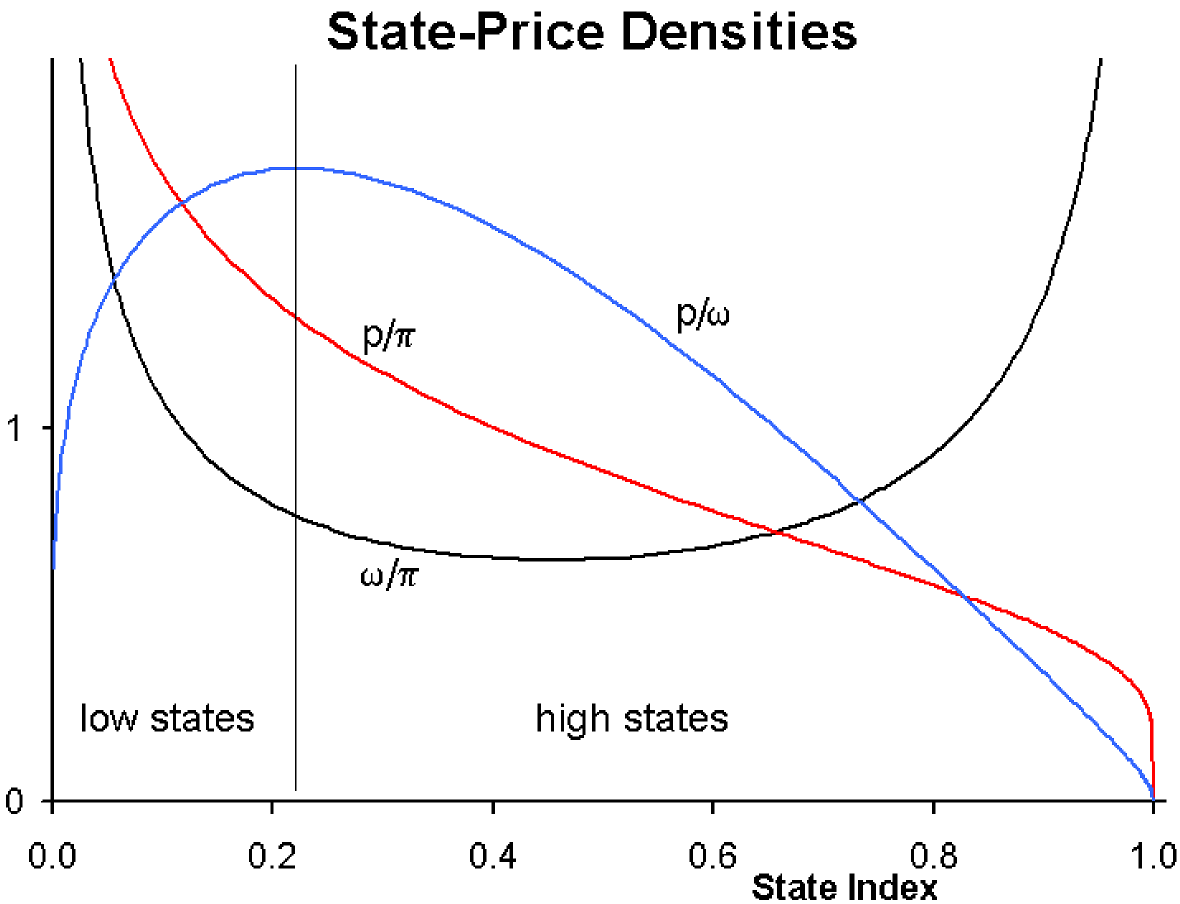
[29] In some cases no equally-probable subdivision of states is possible with a finite number of sub-states; for example when any state has an irrational probability. The equally-probable state partitioning must divide any state into m sub-states each with probability π/m and must create in total n sub-states each with probability $1/n$. But π/m cannot be equal to $1/n$ when π is irrational. Of course, it will always be possible to construct states that are equally likely to any desired accuracy.

[30] Let ξ be any index of the state space, e.g., the return on the market, and let Π(ξ) be its cumulative probability distribution. Then define the new state index $s \equiv \Pi(\xi)$. This index runs from 0 to 1 and its density is clearly constant $\pi(s) = 1$. In the analysis that follows, the index itself is irrelevant as the portfolio properties only depend on ω/π and p/π, which are invariant to the choice of the index.



Another important implication of Proposition V is that, with complete markets and homogeneous objective beliefs, the market portfolio itself will be objectively efficient.[31] This means that a representative investor exists, and all the strong intuitions that follow from this representation will remain available.

Proposition VI: Objective Efficiency of Market Portfolio under Probability Weighting. Assume homogeneous objective beliefs, a complete market with equally likely states (or an atomless continuum of states), all investors are risk-averse or loss averse and use strictly increasing probability weighting functions over the entire objective distribution, and there is at least one risk-averse investor who uses objective probabilities. Then, in equilibrium, the market portfolio's returns will be strictly decreasing in the price-probability ratio and the market portfolio will be objectively risk-averse efficient.

Proof: From proposition V we know that the returns on each investor's optimal portfolio will be weakly ordered inversely to the objective price-probability ratio. Since the market portfolio

[31] A portfolio is efficient if it is optimal for some increasing concave utility function. When markets are complete, $u'(x_s) = \eta p_s/\pi_s$ so marginal utility is proportional to the price-probability ratio. Any portfolio with x_s decreasing in the state price density defines a negative second derivative (or first difference) in the utility function. An increasing, concave utility function can then be constructed up to two constants of integration (which represent the arbitrary level and scale of the cardinal utility function). See Dybvig and Ross (1982) or Ingersoll (1987) for more detail.

is a convex combination of these optimal portfolios, its returns must also be weakly decreasing in the ratio. Assume this monotonicity is not strict; that is, assume there are two states with different price-probability ratios but equal market returns. The risk-averse investor using objective probabilities does hold a strictly monotone portfolio with a higher return in the better state; therefore, if markets are to clear with an equal market return in the two states, some other investor must hold a portfolio with a smaller return in the better state. But this contradicts Proposition V. So in equilibrium, the market portfolio's returns must be strictly decreasing in the objective price-probability ratio and therefore optimal for some strictly risk-averse utility function.[32] +

Proposition VI remains valid even if the market is apparently incomplete provided investors are unconstrained in the types of financial contracts they can introduce. If the introduction of financial assets is unrestricted but the market remains apparently incomplete, the shadow price of any financial asset that have not been created must be the same for all investors. In particular, they must agree on the state prices for all states even if all pure state securities cannot be constructed from the exiting assets. If pure state financial securities were introduced at these prices, the gross demand for them would be zero, and the equilibrium would remain unchanged.[33]

Of course even though the market portfolio is objectively efficient, it is not necessarily true that the resulting equilibrium will be the same that would arise if all investors used objective probabilities. In general the state prices, and therefore the prices of the various assets, will differ. In other words, although the market is objectively efficient in both equilibriums, the representative investor is different.

When the market is not complete, then the inverse ordering between optimal portfolios and the ratio, p/π, need not hold. Of course, that property need not hold amongst investors using the true probabilities either. The portfolio problem in an incomplete market can be analyzed as above by adding constraints, but little can be said in general.

VII Cumulative Prospect Theory and Mutual Fund Separation

Outside of complete markets, the most commonly analyzed market structure is one in which mutual-fund separation holds — in particular the mean-variance model of two-fund separation. Two-fund separation is of considerable interest in finance because it yields strong predictions with sound intuition in a tractable setting. Under two-fund separation,[34] the set of optimal portfolios is spanned by the risk-free asset and a single risky portfolio which is perforce the market portfolio of risky assets. Two-fund separation ensures that there is a representative investor who

[32] Note that the inverse relation need not be strict as it is under homogeneous objective beliefs. It is possible to construct examples with loss-averse investors in which states with the same market return have different price-probability ratios because reducing the return on the portfolio in either state and increasing it in the other will unfavorably alter states' decision weights and decrease expected utility.

[33] The assumption of homogeneous objective beliefs is continued, so we must assume that the introduction of the pure state securities does not change some investors' beliefs.

[34] In the absence of a risk-free asset, two-risky-fund separation can still hold for certain restrictions on utility or probability distributions or both; see Cass and Stiglitz (1970) and Ross (1978). Throughout this paper we shall only consider two-fund money separation when one of the two mutual funds is the risk-free asset.

holds the market portfolio and enables the pricing of all assets based on the first-order conditions of the representative investor applied to the market portfolio. Two-fund separation holds in a market whenever all investors utility functions are from the linear-risk-tolerance (LRT) class with the same cautiousness or all asset distributions come from the separating distributions defined by Ross (1978).

Utility-based two-fund separation will not obtain under CPT. Clearly the S-shaped utility functions are not of the LRT class. Nor, will utility-based two-fund separation hold for LRT investors who use probability weighting unless investors had identical probability weights. While this could be coincidentally true, it would typically only arise if investors had homogeneous objective beliefs and used the same weighting functions. For example, mean-variance analysis remains valid for quadratic utility under probability weighting. All portfolios can be ranked by just knowing their mean and variance, but the means and variances required are those computed using decision weights. Therefore, investors will have different (decision-weighted) mean-variance efficient frontiers if they use different probability weighting functions even if they have homogeneous objective beliefs. Their optimal portfolios will no longer be the same except for leverage and the CAPM equilibrium will not result.

Distributional-based separation also does not hold in general with loss aversion or probability weighting. Ross (1978) has shown that investors will hold combinations of a single risky-asset index portfolio and the risk-free asset if and only if returns are characterized by

$$\begin{aligned} \tilde{r}_i &= r_f + b_i \tilde{y} + \tilde{\varepsilon}_i \\ \text{with} \quad & \mathrm{E}[\tilde{\varepsilon}_i \mid y] = 0 \;\forall i \\ \exists \boldsymbol{\alpha} \text{ such that} \quad & \mathbf{1}'\boldsymbol{\alpha} = 1, \quad \boldsymbol{\alpha}'\tilde{\boldsymbol{\varepsilon}} \equiv 0. \end{aligned} \tag{24}$$

Under the conditions in (24), all risk-averse investors optimally hold some mixture of the risk-free asset and the index portfolio, $\boldsymbol{\alpha}$, with return y which has no residual risk. The optimality of these mixtures follows immediately by second-order stochastic dominance. These conditions are clearly necessary for two-fund separation under CPT,[35] but they are not sufficient.

A loss averse-investor may not choose the stochastically dominating index portfolio if some other portfolio has residual risk that is small and nonzero only when y is sufficiently below its mean. In this case, the two portfolios' gains will have the same risk, but the extra riskiness of the losses on the portfolio with residual risk can lead to its preference by loss-averse investors.

Even if we confine our attention to risk-aversion, (24) is insufficient to guarantee mutual fund separation amongst investors who use probability weighting. Suppose $r_f = 1$, $y = \{-1, 2\}$ with equal probability, and an asset with $b = 1$ has $\varepsilon \equiv 0$ when $y = -1$ and $\varepsilon = \pm 1$ with equal probability when $y = 2$. The residual risk ε is conditionally mean zero, as required, making the asset more risky than the index. Consider a risk-averse investor with a utility function $u(w) = w$ for $w > 0$ and $u(w) = 3w$ for $w \le 0$ and a probability weighting function that assigns $\Omega(0.5) = 0.5$, $\Omega(0.75) = 0.70$. This investor will compute an expected payoff and utility (since all outcomes are positive) of 1.6 for the

[35] They are clearly necessary if we allow an identity weighting function and utility that is linear for losses and concave for gains (though the residual risk free portfolio might only be weakly preferred).

asset. The expected payoff and utility of the index is 1.5. Levering the index down will decrease the expected return and utility. Levering the index up will increase its expected return, but also decrease its expected utility as the worst outcome moves into the high marginal utility region. So this investor's optimal portfolio is not a levered position in the index, and two-fund separation does not hold.

This example illustrates the added complexity of establishing two-fund separation under probability weighting. Recall that two-fund separation requires showing that for any increasing, concave utility function and any portfolio in a large class, there exists a portfolio in a smaller class (those that have no idiosyncratic risk) that gives at least as high expected utility. Using objective probabilities, any portfolio in Ross' class with a given *b* has the same expected return as and is more risky than the levered position in the index with the same *b*. Since this specific levered index portfolio stochastically dominates all assets with the same *b* value, we need not consider each utility function separately — two-fund separation holds trivially. However, with probability weighting, a levered position in the index no longer dominates all other portfolios with the same *b* because the convex portion of an inverted-S-shaped weighting function can increase the subjective mean after an objective-mean-preserving spread. Therefore, risk-neutral investors and those sufficiently close to risk-neutral will prefer the portfolio that is objectively dominated. This does not mean that two-fund separation fails, but it does mean that to verify separation we can no longer just compare portfolios with the same value of *b*. We must potentially compare every portfolio with idiosyncratic risk to all levered index positions with the same or higher subjective mean.

Two related questions immediately arise. Which weighting functions and what restrictions on utility do preserve two-fund separation for Ross' distributions? And can the class of separating distributions be further restricted so that mutual fund separation does hold for some or all loss averse utility functions and inverse-S weighting functions? In fact we have already mostly answered the first questions in our examples. Two-fund separation cannot hold for all of the distributions in Ross' class whenever the utility function has any strictly convex portion since a portfolio with residual risk in only that region will be preferred. Similarly any strictly convex portion of the probability weighting function will increase the mean of some portfolios with residual risk which again removes the dominance of the index. Ross' two-fund separation theorem can remain valid only for risk aversion and concave weighting functions. Proposition VII shows that this is sufficient as well.

Proposition VII: Two-fund Separation with Concave Probability Weighting. If the returns on all assets are as described in (24), then optimal portfolios for all risk-averse investors are combinations of the risk-free asset and a single portfolio of risky assets if and only if all investors have weakly concave probability weighting functions.

Proof: The necessity of concavity has already been discussed. The proof of sufficiency is given in the Appendix which shows that an increasing concave weighting function preserves the second-order stochastic dominance of the levered index so two-fund separation holds for all risk-averse investors. +

Recall that the decision weight density is the product of the probability density and the derivative of the weighting function. A concave weighting function has a decreasing derivative so it emphasizes the probabilities of the low payoffs relative to those of high payoffs; i.e., $\mathrm{E}_{\omega}[\tilde{\varepsilon}] < 0$ so the mean-preserving spread, ε, adds risk and reduces the expectation. This ensures that objectively dominated prospects remain dominated under probability weighting.

The challenge to two-fund separation under CPT is that some increased risk is liked either because utility is locally convex or because it induces an increased subjective mean through probability weighting. To maintain two-fund separation, we need to insure that such good risk is balanced by bad risk; that is, we need some symmetry in the distributions and weighting function.

To preserve two-fund separation under an inverse-S weighting functions as utilized in CPT rather than just concave weighting functions, we must preclude objective-mean preserving spreads that increase the subjective mean. One way to accomplish this is to assume enough symmetry so that any mean-increasing alterations in one tail have offsetting mean-reducing alterations in the other tail. This requires symmetric distributions and "no better than symmetric" probability weighting adjustments.

Proposition VIII: Two-Fund Separation under Inverse-S Probability Weighting. Sufficient conditions for two-fund separation under risk aversion and probability weighting are: (i) returns satisfy the Ross conditions for two-fund separation as given in (24); (ii) the distributions of $\tilde{y}$ and all asset returns, $\tilde{r}_i$, are symmetric; and (iii) the probability weighting function for the cumulative distribution F has the form $\Omega(F) \equiv \Upsilon(\Xi(F))$ where $\Xi(\cdot)$ [0,1] → [0,1] is increasing and concave below ½ with $\Xi(1 - F) = 1 - \Xi(F)$, and $\Upsilon(\cdot)$ [0,1] → [0,1] is increasing and concave.[36]

Proof: Let F and G be the cumulative distributions of $r_f + \tilde{y}$ and $r_f + b_i\tilde{y} + \tilde{\varepsilon}_i$. Because F and G are the distributions of symmetric random variables, the transformations $\Xi(F)$ and $\Xi(G)$ preserve the riskiness ordering as shown in Lemma 3 in the Appendix. Therefore, $\Xi(G)$ remains riskier than $\Xi(F)$ in a Rothschild-Stiglitz sense. Now applying Proposition VII, we see that the increasing concave transformation Υ preserves the second-order stochastic dominance. +

Unfortunately the assumption in Proposition VIII does not apply to the KT weighting function for any parameter value nor to many of the other probability weighting functions used in CPT. The second derivative of $\Upsilon(\Xi(F))$ is $(\Xi')^2\Upsilon'' + \Xi''\Upsilon'$. Since Υ'' is assumed to be negative, the inflection point in the probability weighting function can only occur when Ξ'' is positive which, by assumption, must be for $F \geq ½$. However, the inflection point for the KT probability weighting function occurs at a probability less than ½ for all values of δ and occurs at $1/e$ for all values of α for Prelec's preferred single parameter function (β = 1).[37] In fact, using a non-parametric approach to determine the "least favored" probability, Wu and Gonzalez (1966) have estimated that the inflection point is no higher than 40%. If this is correct, then the probability weighting function cannot be represented as required in this proposition.

Even the assumed symmetry of the distributions in Proposition VIII is insufficient to

[36] Assumption (ii) does not require that the distributions of $\tilde{\varepsilon}$ be symmetric. If the distributions of ε conditional on $\tilde{y} = \bar{y} + a$ and that of $-\tilde{\varepsilon}$ conditional on $\tilde{y} = \bar{y} - a$ are identical, then the asset returns will be symmetric. As noted previously in footnote 7, assuming $\Xi(1 - F) = 1 - \Xi(F)$ is equivalent to using identical weighting functions for losses and gains and requiring $\Xi(½) = ½$. Of course, after the Υ transformation, they will no longer be identical.

[37] The inflection point for Prelec's two-parameter function can be at probabilities above ½ for some parameter values; e.g., α = 0.9, β = 0.7.

guarantee two-fund separation with loss aversion. The symmetry assures that risk in both the upper and lower tails is the same, but to preserve the separation, the bad upper-tail risk has to more than offset the good lower-tail risk. Consider a portfolio which has a small amount of residual risk $(\sigma_\varepsilon^2 \approx 0)$ which is nonzero only at two isolated points of the index y. By symmetry these two points must be at $\bar{y} \pm a$.[38] Since the portfolio and levered index have the same return except at those two points of the index return, the difference between their expected utilities is equal to the probability that the index return is $\bar{y} + a$ (which equals the probability it is $\bar{y} - a$ by symmetry) multiplied by

$$\begin{aligned} &v(r_f + b\bar{y} + ba) + v(r_f + b\bar{y} - ba) - \mathrm{E}_\varepsilon[v(r_f + b\bar{y} + ba + \tilde{\varepsilon}) + v(r_f + b\bar{y} - ba + \tilde{\varepsilon})] \\ &\qquad \approx -\tfrac{1}{2}[v''(r_f + b\bar{y} + ba) + v''(r_f + b\bar{y} - ba)]\sigma_\varepsilon^2 . \end{aligned} \tag{25}$$

The third line follows from a Taylor expansion and E[ε] = 0. Therefore, to guarantee two-fund separation even assuming symmetric distributions that satisfy (24), we also require $v''(x_1) + v''(x_2) \leq 0$ for the relevant values of x_1 and x_2. Since the index has a symmetric distribution with a positive risk premium, only long positions can be optimal. So the only relevant cases have b (plus r_f, $\bar{y}$, and a) all positive implying x_1 is positive and x_2 is less than x_1. If both x_1 and x_2 are positive the condition is trivially true. Thus a necessary condition in addition to symmetry for Ross two-fund separation to hold is $v''(x_1) + v''(x_2) \leq 0 \ \forall \ x_2 \leq 0 \leq x_1$. This condition does not hold for any twice-differentiable increasing utility function with a strictly convex portion.[39]

The problem of extending two-fund separation to CPT still remains. Solving this problem with loss aversion requires an assumption stronger than symmetry. Solving it with probability weighting requires that we be able to compare portfolios with different levels of both systematic and idiosyncratic risk and not just the latter. We need stronger distributional assumptions that make the comparison of all portfolios easy.

One answer to this problem is mean-variance analysis — the same hypothesis that simplifies portfolio comparison under objective probabilities. Two-fund separation and a resulting CAPM equilibrium do hold under probability weighting and in many cases under loss aversion when asset returns have a multi-variate elliptical distribution.

VIII Mean-Variance Analysis under CPT

Elliptical distributions were introduced into portfolio analysis by Chamberlain (1983) and Owen and Rabinovitch (1983). The distributions get their name from the ellipsoidal shape of their isoprobability manifolds. The best-known example of an elliptical distribution is the multivariate normal. Elliptical distributions are characterized by a probability density (if it exists) and a characteristic function for excess returns **x** of

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[38] We may assume $\bar{y} > 0$ with no loss of generality since $-y$ can equally well serve as the random variable describing the index. In any case investors will hold an index with a symmetric distribution in preference to the risk-free asset only if it has a positive risk premium.

[39] If $v''(x) = c > 0$ at some negative x, then $v''(x) \leq -c$ for all positive x. But if the second derivative is bounded away for zero, the first derivative cannot remain positive as x increases.

$$f(\mathbf{x}) = |\mathbf{\Theta}|^{-1/2}\ g\left((\mathbf{x}-\boldsymbol{\mu})'\mathbf{\Theta}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$
$$\Psi(\mathbf{t}) \equiv \mathrm{E}[e^{i\mathbf{t}'\mathbf{x}}] = e^{i\mathbf{t}'\boldsymbol{\mu}}\psi(\mathbf{t}'\mathbf{\Theta}\mathbf{t}) \tag{26}$$

where $\boldsymbol{\mu}$ is the vector of means and $\mathbf{\Theta}$ is the covariance matrix.[40] We assume that Θ is non-singular so there are no risk-free arbitrages available. If there is a risk-free asset, it can be added in the usual fashion. The function g can be essentially any nonnegative function for which the density can be normalized to unit mass over the relevant domain. For example, the multivariate normal has $g(z) = (2\pi)^{-1/2}\exp(-z/2)$.

The property of elliptical distributions important for portfolio analysis is that all linear combinations of elliptical returns have the same distribution apart from location and scale; i.e., g and Ψ remain the same.[41] In particular, a portfolio investing the fraction of wealth α_i in each of the assets will have a marginal elliptical return with density $\sigma_p^{-1} g\left((x_p - \mu_p)^2/\sigma_p^2\right)$ where $\mu_p \equiv \boldsymbol{\alpha}'\boldsymbol{\mu}$ and $\sigma_p^2 \equiv \boldsymbol{\alpha}'\mathbf{\Theta}\boldsymbol{\alpha}$ are the portfolio's mean and variance. As with the normal distribution any portfolio's return can therefore be expressed as a translated and scaled standardized variable $\tilde{x}_p = \mu_p + \sigma_p\tilde{\rho}$ where $\tilde{\rho}$ is an elliptical variable of the same type with zero mean, unit variance, and a particular cumulative distribution $F(\rho)$.

Proposition IX: Portfolio Separation with Elliptical Distributions under CPT If the returns on all assets are elliptically distributed and investors have homogeneous objective beliefs, then all CPT (and risk-averse) investors will hold objectively mean-variance efficient portfolios. Provided an equilibrium exists, two-fund separation and the CAPM relation between the objective means and covariances will obtain.

Proof: The usual mean-variance mathematics applies to the objective distribution independent of any utility assumptions. The feasible portfolios in mean standard deviation space are bounded by a minimum-variance hyperbola with a tangent borrowing-lending line if a risk-free asset exists. The set of objective minimum-variance is the same for all investors since they have identical objective beliefs.

The expected decision-weighted utility of a portfolio for any utility function is completely determined by the portfolio's objective mean and standard deviation, the cumulative distribution function F (which is the same for all portfolios), and the individual probability weighting function. Define the derived objective mean-variance utility function for a particular weighting function, V_ω, as

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[40] Elliptical distribution can be fat-tailed with undefined means or variances, for example the multi-variate Cauchy distribution. In this case, $\boldsymbol{\mu}$ and $\mathbf{\Theta}$ are the vector of medians and a general co-dispersion matrix. The discussion below remains valid for such elliptical distributions provided, of course, that expected utility is defined. See Chamberlain (1983), Owen and Rabinovitch (1983), and Ingersoll (1987) for more details about elliptical distributions and their applicability to the mean-variance portfolio problem. If different investors have different zero-utility reference points, $\hat{x}$, they will use different mean vectors $\boldsymbol{\mu}$; however, the reference level applies equally to all returns including the interest rate so it will not differentially affect the risk premia or alter any of the trade-offs discussed.

[41] This can be verified immediately from (26). Using the vector $\mathbf{t} \equiv t\boldsymbol{\alpha}$, the characteristic function of the return on any portfolio $\boldsymbol{\alpha}$ with mean, $\mu_p = \boldsymbol{\alpha}'\boldsymbol{\mu}$, and variance, $\sigma_p^2 = \boldsymbol{\alpha}'\mathbf{\Theta}\boldsymbol{\alpha}$, is $\mathrm{E}[\exp(it\boldsymbol{\alpha}'\mathbf{r})] = \exp(it\mu_p)\psi(t^2\sigma_p^2)$. So all portfolios have the same distribution apart from location and scaling.

$$V_\omega(\mu,\sigma) \equiv \mathrm{E}_\omega[v(x_p)] = \int_{-\infty}^{\infty} v(\mu+\sigma\rho)\,d\Omega(F(\rho))\,. \tag{27}$$

Since v is strictly increasing, V_ω must be strictly increasing in μ and every investor likes objective mean. Therefore, all investors will hold portfolios on the upper portion of the objective minimum variance hyperbola or its tangent borrowing-lending line. In either case the set of optimal portfolios is spanned by any two portfolios it includes.[42] If every investor's demand for the tangency portfolio is finite, then an equilibrium exists and the market portfolio is on the minimum variance frontier or tangency line. In either case, the relevant objective CAPM equilibrium will result as it depends only on the mean variance algebra. +

The existence of an equilibrium has been examined by Barberis and Huang (2008) and Levy, De Giorgi and Hens (2004). Both papers consider only the multivariate normal case with a risk-free asset and use a zero-utility reference return equal to the risk-free rate. The former paper constructs an equilibrium assuming identical investors. The latter paper shows that an equilibrium does not exist for investors with heterogeneous TK loss aversion functions all with $\alpha = \beta$. We know from Proposition I in this paper that demand will be bounded and hence an equilibrium will exist if all CPT investors have a zero-utility reference return less than or equal to the risk-free rate and asymptotic risk avoidance ($\beta > \alpha$ for the TK function). Of course, some investors can be risk averse and/or use objective probabilities as well.

The CAPM equilibrium guaranteed by this proposition need not be the same one that would prevail if all investors used objective probabilities and were risk averse. In particular, probability weighting will tend to increase the market price of risk as it emphasizes the extreme outcomes, and investors will therefore be more reluctant to take on the risk of the market. Loss aversion could have either effect on the market price of risk. The makeup on the market portfolio itself could also change as a different market price of risk will alter the point of tangency of the borrowing-lending line.

Unfortunately, while elliptical returns might be a reasonable description of the returns on the primary assets in an economy, it certainly does not describe the returns on the myriad derivative contracts that could be introduced on those primary assets. In fact, if a set of assets has an elliptical return, then a vanilla call or put option on any one of them *cannot* have a return that falls into the same elliptical class since one tail of the distribution is eliminated. Fortunately, the CAPM equilibrium can still be partially valid even if two-fund separation does not hold.

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[42] We have not shown, nor is it necessarily true, that variance is disliked by CPT investors. However, this is neither necessary nor sufficient to guarantee an equilibrium. What must be true is that the indifference curves in μ-σ space be convex and sufficiently sloped at high σ so that they have a finite tangency with the borrowing-lending line (or mean-variance hyperbola if there is no risk-free asset). Loss aversion can prevent this tangency; probability weighting, by itself, cannot. For a risk-averse utility function, u, the expected utility of a levered position in the tangency portfolio is $\mathrm{E}_\omega[u(r_p)] = \int_{-\infty}^{\infty} u(r_f + b(\mu_t - r_f) + b\sigma_t\rho)\,d\Omega(F(\rho))$, which is a concave function of leverage, b, $\partial^2 \mathrm{E}_\omega[u(r_p)]/\partial b^2 = \int_{-\infty}^{\infty} u''(\cdot)(\mu_t - r_f + \sigma_t\rho)^2\,d\Omega(F(\rho)) < 0.$ Therefore, risk-averse investors who use probability weighting will never have infinite demand if there is a risk-free asset. Since this is true regardless of the interest rate assumed (provided it is less than the rate of return on the global minimum variance portfolio so a tangency exists, the indifference curves must also be convex relative to the minimum-variance hyperbola supporting all possible borrowing lending lines.

Dybvig and Ingersoll (1982) have shown that the linear relation between risk premia and beta holds for the set of primary assets described by an elliptical distribution even in the presence of non-elliptical derivative contracts provided the market is complete or effectively so. The same analysis applies here to the objective moments so only a summary of the reasoning is given. The elliptical distributions fall within the class of Ross' separating distributions so amongst just the primary assets any portfolio that is not mean-variance efficient is second-order stochastically dominated. From Proposition VI, the market portfolio is objectively risk-averse efficient within a broader class so it cannot be stochastically dominated, and it must, therefore, be mean-variance efficient within just the primary assets.

This reasoning does not mean that the CAPM can be used to price derivatives. Only the primary (elliptically distributed) assets need have a linear relation between their risk premia and market betas.

IX Multiperiod CPT and First-Order Stochastic Dominance

All of our analysis thus far has been in the context of single period models. In this section of the paper we address two issues that arise in multi-period problems, probability updating and dynamic portfolio allocation. Both of these issues create complications for CPT causing violations of first-order stochastic dominance.

In multi-period models in finance, probabilities of certain events must be updated. This is true both in intertemporal portfolio problems and in information models where agents take actions after receiving signals. The subproportionality property of probability weighting means that Bayes' Law will not hold in general for decision weights. There might be an alternative rule that applies in CPT, but the following simple example illustrates the problem and demonstrates that such a rule likely does not exist. The example uses a linear utility function, so the problem illustrated is not due to loss aversion or even risk aversion. It also uses only losses and therefore a single weighting function so the differential weighting of gains and losses is not an issue nor is the subcertainty of the decision weighs summing to less than unity.

There are four states with losses ranging from 400 to 100 as illustrated in Table IV below. The probability weighting function is that proposed by Tversky and Kahneman (7) with a parameter of $\delta_{-} = 0.833921$.[43] The expected loss is $E_{\pi}[x] = -260$; while the decision-weighted "expected" loss is $E_{\omega}[x] = -255.88$. This latter number may be thought of as the certainty equivalent for the gamble, but in this case, the difference between the certainty equivalent and the expected value is due not to risk preferences but to the probability weighting. At this point, however, there is nothing to distinguish this form of probability weighting from simple heterogeneous beliefs.

Now suppose that the investor receives a binary signal. The signal is the partitioning of the states: $A = \{a, c\}$ or $B = \{b, d\}$ so that after receiving the signal, the investor knows that only two of the four original states remain possible with the original relative likelihoods. The probability of

[43] This parameter value equates the two post-signal expected values as shown. For other parameter values a similar problem arises, but not so obviously.

signal A is 40% (10% + 30%). Once that signal has been observed, the conditional probabilities of states a and c are ¼ and ¾, respectively. On the other hand if signal B is observed (with probability 60%), then only states b and d are possible, and their conditional probabilities are $^5/_6$ and $^1/_6$. The expected values conditional on the signals are $E_\pi[x \mid A] = -250$ and $E_\pi[x \mid B] = -266\ ^2/_3$.

Bayes' Law relates the conditional objective probabilities of these events. That law does not work with the decision weights, but we might hope that some similar rule could be applied. Unfortunately, there does not seem to be any way to accomplish this as the table illustrates. Applying the weighting function to the conditional probabilities, we see that after observing signal A, $\omega_{a|A} = 28.03\%$ and $\omega_{c|A} = 71.97\%$ while after observing signal B, $\omega_{b|B} = 78.03\%$ and $\omega_{d|B} = 21.97\%$. The conditional (decision-weighted) expected losses, $E_\omega[x \mid A] = E_\omega[x \mid B] = -256.06$, are equal. Clearly no updating rule like Bayes' Law that simply weights the conditional expectations can lead to any original unconditional expected value other than –256.06.

An investor who applied this CPT probability weighting myopically would, before the receiving the signal, willingly take on this risk rather than face a sure loss of 255.9. But after receiving the signal, he would always willingly trade the remaining risk for a sure loss of 256. That is, the investor would willingly give up 0.1 for sure while acting myopically in an "optimal" fashion.

Loss aversion has a similar, though probably less severe problem. Suppose the investor's two-period loss-averse utility function is defined as follows[44]

$$V(x_1, x_2) = \rho v(x_1) + \rho^2 v(x_2) \tag{28}$$

$\rho \le 1$ is a discount factor. We assume that v has the same properties as in the single-period problem.

Each period there are S states defined over the assets' returns. The market is complete and the set of states is the same for each period with the same state prices, p_s, and state probabilities. The investor's portfolio problem is

$$\begin{aligned} &\underset{x_{1s}, x_{2s}}{\text{Max}} \quad \rho \sum \pi_s v(x_{1s}) + \rho^2 \sum \pi_s v(x_{2s}) \\ &\text{subject to} \quad \sum p_s x_{1s} = \sum p_s x_{2s} = B \quad x_{1s} \ge \underline{x} \quad x_{2s} \ge \underline{x}\,. \end{aligned} \tag{29}$$

Typically, dynamic programming is used to determine the second portfolio conditional on the realization in the first period, but for a complete-market problem, both optimal portfolios can be determined *ab initio*. Forming the Lagrangian and maximizing gives

[44] For simplicity this example uses loss aversion defined over the percentage gain or loss each period to keep the optimal portfolio the same in each period. Similar examples can be constructed for loss aversion defined over the dollar gain or loss. Since the example illustrates a problem with loss aversion, probability weighting is ignored.

$$L = \rho \sum \pi_s v(x_{1s}) + \rho^2 \sum \pi_s v(x_{2s}) + \eta_1 \left(B - \sum p_s x_{1s}\right) + \eta_2 \left(B - \sum p_s x_{2s}\right)$$

$$0 \geq \frac{\partial L}{\partial x_{ts}} = \pi_s \rho^t v'(x_{ts}) - \eta_t p_s \qquad 0 = (x_{ts} - \underline{x}) \frac{\partial L}{\partial x_{ts}} = (x_{ts} - \underline{x}) \left[\pi_s \rho^t v'(x_{ts}) - \eta_t p_s \right] \tag{30}$$

$$0 = \frac{\partial L}{\partial \eta_t} = B - \sum p_s x_{ts}$$

If $x_{ts} > \underline{x}$, then the first of these conditions holds as an equality. It is clear by inspection that the optimal portfolio will be the same in both periods.[45]

As a specific example assume the investor has TK loss aversion with $\alpha = 0.8$ $\beta = 0.9$, and $\lambda = 2.25$. Each period there are three sates with $\pi_a = 0.2$, $\pi_b = \pi_c = 0.4$ and $p_a = 0.1$, $p_b = 0.3$, $p_c = 0.6$. The optimal single-period portfolio is $\mathbf{x} \approx (1.271, 0.167, -0.296)$. The portfolio returns are inversely aligned with the price-probability ratio, $\boldsymbol{\theta} = (0.50, 0.75, 1.5)$, so we know it is also optimal for some risk-averse agent (as well as our loss-averse investor) and therefore is not stochastically dominated.

Table V shows the state-by-state returns of holding this portfolio for two periods. They range from a high of 5.160 to a low of 0.496. Note, however, that the total returns are no longer ordered inversely to the two-period price-probability ratios. In particular, $\theta_{ac} = p_a p_c / \pi_a \pi_c = 0.75 > \theta_{bb} = p_b^2 / \pi_b^2 = 9/16$ while the return earned in outcome *bb* is smaller than that earned for outcomes *ca* and *ac*. As shown in the table, a two-period portfolio that switched these returns earning 1.363 in states *ac* and *ca* and 1.600 in state *bb* has the same probability distribution but a lower cost. So this second portfolio first-order stochastically dominates the “optimal” portfolio.

Prospect Theory was reformulated as CPT to preclude first-order stochastic dominance, but these example demonstrate that it remains in a dynamic setting with either probability weighting or loss aversion. CPT requires further reformulation if it is to applied in a multiperiod model without first-order stochastic dominance. Of course, one might choose to ignore stochastic dominance instead, but the assumption that more is preferred to less seems much more basic than CPT.

[45] The multipliers will satisfy $\eta_2 = \rho\eta_1$.

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Appendix

This appendix collects some technical results.

Lemma 1: Second-order Stochastic Dominance. The following relations are equivalent ways to express second order stochastic dominance of the random variable *x* with cumulative distribution *F* over the random variable *y* with cumulative distribution *G*:

$$\mathrm{E}[u(x)] \geq \mathrm{E}[u(y)] \quad \forall u \text{ with } u' \geq 0, u'' \leq 0 \tag{A1a}$$

$$\int_{-\infty}^{T} [F(t) - G(t)]dt \leq 0 \quad \forall T \tag{A1b}$$

$$\int_{0}^{P} [G^{-1}(p) - F^{-1}(p)]dp \leq 0 \quad \forall P. \tag{A1c}$$

Proof: It is well known that the first two relations are equivalent. The equivalence of relation (A1c) was proved by Levy and Kroll (1979). It can be demonstrated by Figure 1. Both the integrals of $F(x) - G(x)$ and $G^{-1}(p) - F^{-1}(p)$ give the signed area between the two curves; therefore the two integrals up to any crossover point like (T_i, P_i) must be equal. Consider any point like P' where $G^{-1}(p) < F^{-1}(p)$, clearly

$$\int_{0}^{P'} [G^{-1}(p) - F^{-1}(p)]dp < \int_{0}^{P_{i-1}} [G^{-1}(p) - F^{-1}(p)]dp = \int_{-\infty}^{T_{i-1}} [F(t) - G(t)]dt \leq 0. \tag{A2}$$

The inequality follows because the first integral includes an extra portion where the integrand is negative. Similarly, consider any point like P'' where $G^{-1}(p) > F^{-1}(p)$, again

$$\int_{0}^{P''} [G^{-1}(p) - F^{-1}(p)]dp < \int_{0}^{P_{i+1}} [G^{-1}(p) - F^{-1}(p)]dp = \int_{-\infty}^{T_{i+1}} [F(t) - G(t)]dt \leq 0. \tag{A3}$$

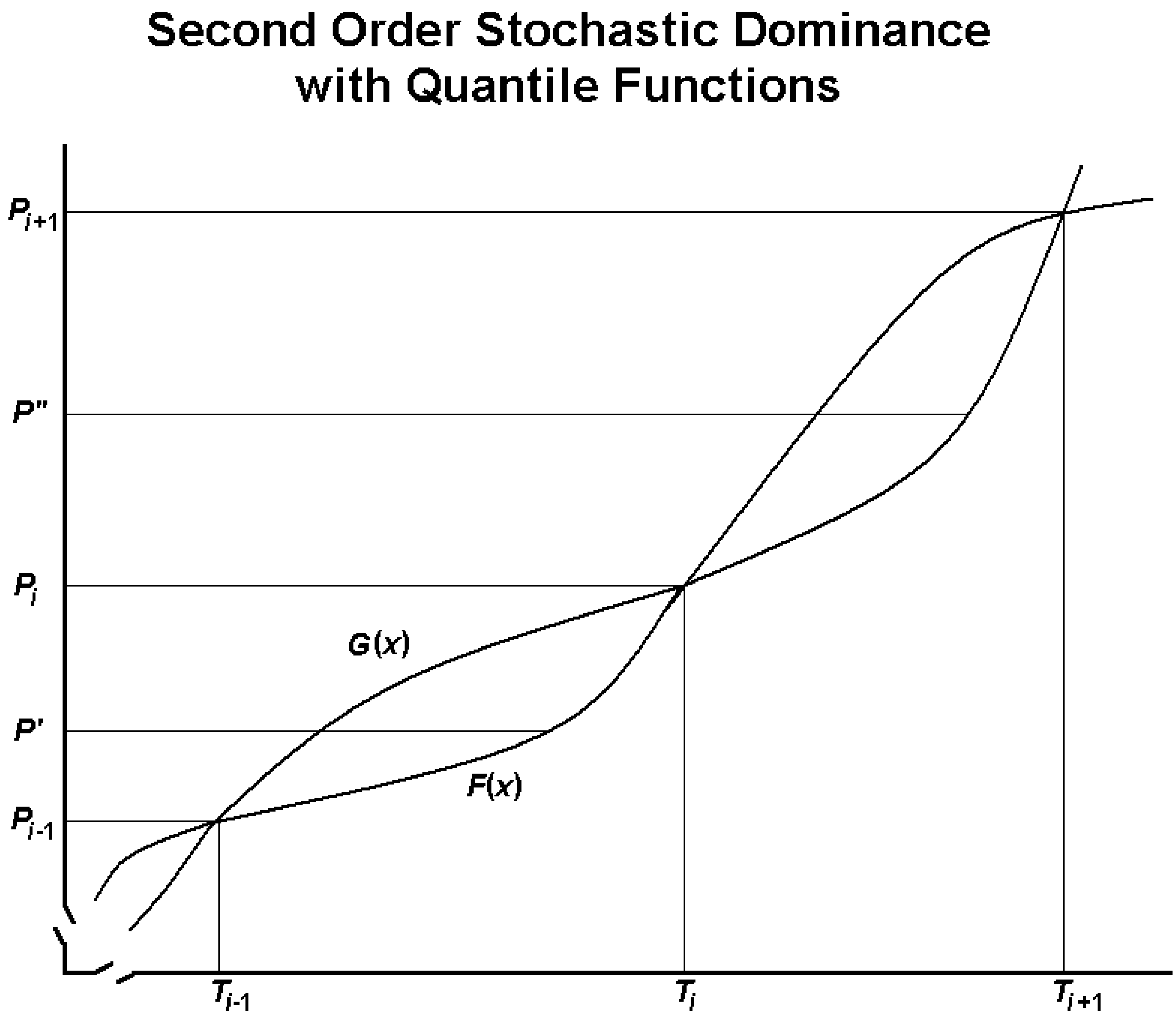
In this case, the first integral excludes a portion where the integrand is positive. +

Proof of Proposition V: The conditions in equation (24) are sufficient for two-fund separation with probability weighting if the weighting function, Ω, preserves second-order stochastic dominance. As given in (A1b) and (A1c), second-order stochastic dominance will be preserved under the probability weighting function, Ω, if and only if

$$\int_{-\infty}^{T} [\Omega(F(t)) - \Omega(G(t))]dt \leq 0 \;\; \forall T \quad \text{or} \quad \int_{0}^{P} [F^{-1}(\Omega^{-1}(p)) - G^{-1}(\Omega^{-1}(p))] \geq 0 dp \;\; \forall P \tag{A4}$$

Using the change in variable, $p \equiv \Omega(q)$, the second integral in (A4) can be re-expressed as

$$\int_{0}^{\Omega^{-1}(P)} [F^{-1}(q) - G^{-1}(q)]\Omega'(q)dq\,. \tag{A5}$$



Now define $h(q) \equiv F^{-1}(q) - G^{-1}(q)$ and $H(Q) \equiv \int_0^Q h(q)dq$. Then integrating (A5) by parts we have

$$\int_0^{\Omega^{-1}(P)} h(q)\Omega'(q)dq = H(q)\Omega'(q)\Big|_0^{\Omega^{-1}(P)} - \int_0^{\Omega^{-1}(P)} H(q)\Omega''(q)dq\,. \tag{A6}$$

The first term is nonnegative since $H(0) = 0$ and H and Ω' are nonnegative elsewhere. The integral is nonpositive as H is nonnegative everywhere and Ω'' is nonpositive everywhere. Therefore the integral in (A5) and equivalently the second in (A4) is nonnegative. +

Lemma 2: Increasing Risk for Variables with Symmetric Distributions and the Same Mean. If two random variables x and y with the same mean have symmetric cumulative distributions F and G, then y is riskier than x in the sense of Rothschild and Stiglitz if and only if

$$\int_0^P [G^{-1}(p) - F^{-1}(p)]dt \le 0 \quad \forall P \le \tfrac{1}{2}\,. \tag{A7}$$

Proof: When *x* and *y* have identical means, then being riskier than and being second-order stochastically dominated are the same. Clearly if (A1c) is valid then (A7) is true as well. Conversely, suppose (A7) is valid. Then clearly (A1c) is valid for all $P \leq ½$. Consider now some $P > ½$,

$$\begin{aligned}
&\int_0^P [G^{-1}(p) - F^{-1}(p)]dp \\
&\quad = \int_0^{1-P} [G^{-1}(p) - F^{-1}(p)]dp + \int_{1-P}^{½} [G^{-1}(p) - F^{-1}(p)]dp + \int_{½}^{P} [G^{-1}(p) - F^{-1}(p)]dp \\
&\quad = \int_0^{1-P} [G^{-1}(p) - F^{-1}(p)]dp \leq 0 \quad \forall P > ½.
\end{aligned} \tag{A8}$$

Since the means are equal, the integral of $G^{-1} - F^{-1}$ from 0 to 1 is zero, and by the symmetry of the two distributions, integrals over corresponding ranges on either side of ½ are equal in magnitude and opposite in sign. Therefore, the last integrals in the second line cancel. The remaining integral is nonpositive by assumption. +

Lemma 3: Symmetry Preservation of Increasing Risk. Consider two random variables *x* and *y* with the same mean and symmetric cumulative distributions *F* and *G* with *y* riskier than *x* in the sense of Rothschild and Stiglitz. Let $\Xi(\cdot)$ be an anti-symmetric probability weighting function (i.e, $\Xi(F) + \Xi(1 - F) = 1$) with Ξ increasing and concave below ½ and increasing and convex above ½. Then $\Xi(G)$ is riskier than $\Xi(F)$.

Proof: The transformation Ξ preserves the symmetry of *F* and *G*. So by Lemma 2, $\Xi(G)$ is riskier than $\Xi(F)$ if

$$\int_0^P [G^{-1}(\Xi^{-1}(p)) - F^{-1}(\Xi^{-1}(p))]dt \leq 0 \quad \forall P \leq \tfrac{1}{2}. \tag{A9}$$

Using the change in variable, $p \equiv \Xi(q)$, this integral can be re-expressed as

$$\int_0^{\Xi^{-1}(P)} [F^{-1}(q) - G^{-1}(q)]\Xi'(q)dq \qquad \forall P \leq \tfrac{1}{2}. \tag{A10}$$

By the symmetry property $\Xi^{-1}(½) = ½$ so we need evaluate this integral only up to $F = G = ½$.. But in this region Ξ is concave so the integral is nonpositive just like that in (A5). +

Table I: Optimal Portfolio with Probability Weighting: Illustrating Portfolio Skewing

This table presents a three-state portfolio problem for an investor with a utility function $u(x) = x^{0.6}$ who uses a Tversky-Kahneman probability weighting function with parameter $\delta = 0.7$. The two sections of the table demonstrate that either assumed ordering of the state's returns (*a*, *b*, *c*) or (*b*, *a*, *c*) leads to a contradiction where the first-order conditions for the optimal portfolio would have the opposite ordering. Therefore, the true optimal portfolio must constrain the outcomes in states *a* and *b* to be equal.

Expected-Utility Maximizing Portfolio

state	π	p	p/π	x^*	$\pi \cdot u(x)$
a	20%	0.3	1.5	0.328	0.102
b	30%	0.3	1.0	0.903	0.282
c	50%	0.4	0.8	1.577	0.657
				$E_\pi[u(\cdot)] = 1.042$	

Optimal Decision-Weighted Portfolio

			Assumed order (*a,b,c*)			Corrected (*b,a,c*)		Constrained		
state	π	p	$\omega(a,b,c)$	x^*	$\omega \cdot u(x)$	$\omega(b,a,c)$	$\omega \cdot u(x)$	ω	x^*	$\omega \cdot u(x)$
a	20%	0.3	25.60%	0.575	0.184	12.93%	0.093	45.74%	0.437	0.278
b	30%	0.3	20.13%	0.315	0.101	32.81%	0.164			
c	50%	0.4	54.26%	1.832	0.780	54.26%	0.780	54.26%	1.845	0.784
				"E_ω"$[u(\cdot)] = 1.065$			$E_\omega[u(\cdot)] = 1.037$			$E_\omega[u(\cdot)] = 1.062$

			Assumed order (*b,a,c*)			Corrected (*a,b,c*)		Constrained		
state	π	p	$\omega(b,a,c)$	x^*	$\omega \cdot u(x)$	$\omega(a,b,c)$	$\omega \cdot u(x)$	ω	x^*	$\omega \cdot u(x)$
b	30%	0.3	32.81%	0.985	0.325	20.13%	0.210	45.74%	0.437	0.278
a	20%	0.3	12.93%	0.096	0.032	25.60%	0.066			
c	50%	0.4	54.26%	1.689	0.743	54.26%	0.780	54.26%	1.845	0.784
				"E_ω"$[u(\cdot)] = 1.100$			$E_\omega[u(\cdot)] = 1.006$			$E_\omega[u(\cdot)] = 1.062$

Table II: Optimal Portfolio with Probability Weighting: Illustrating Midrange Flattening

This table presents a four-state problem for an investor with a von-Neumann Morgenstern utility function who uses an inverted S shaped probability weighting function with the properties: Ω (0.3) = 0.32, Ω(0.5) = 0.51, Ω(0.7) = 0.68, Ω(1) = 1. The middle section of the table demonstrates that a portfolio whose returns are ordered inversely to the likelihood ratio, p/π, will induce a probability weighting likelihood ratio, p/ω, which reverses the order of the middle two states, *b* and *c*. However, as shown in the last section, a portfolio whose returns are ordered inversely to the likelihood ratio, p/ω, will induce a new decision-weighted likelihood ratio, p/ω', which again switches the order of the middle two states. Since both assumed orderings lead to contradictions, the true optimal portfolio must constrain the outcomes in states *b* and *c* to be equal.

state	π	p	p/π	state order	Ω	ω	p/ω	state order	Ω'	ω'	p/ω'
a	30%	0.40	1.33	*a*	32%	32%	1.250	*a*	32%	32%	1.250
b	20%	0.21	1.05	*b*	51%	19%	1.105	*c*	51%	19%	1.000
c	20%	0.19	0.95	*c*	68%	17%	1.118	*b*	68%	17%	1.235
d	30%	0.20	0.67	*d*	100%	32%	0.625	*d*	100%	32%	0.625

**Table III: Optimal Portfolio with Probability Weighting:
Illustrating Non-Monotonic Response**

This table presents a four-state problem for an investor with a von-Neumann Morgenstern utility function who uses an inverted S shaped probability weighting function with the properties: Ω (0.2) = 0.3, $\Omega(0.4) = 0.5$, $\Omega(0.6) = 0.56$, $\Omega(0.8) = 0.75$, $\Omega(1) = 1$. The middle section of the table demonstrates that a portfolio whose returns are ordered inversely to the price-probability ratio, p/π, will induce a probability weighting ratio, p/ω, which reverses the order of the middle two states, *b* and *c*. Furthermore, a portfolio whose returns are ordered inversely to the ratio, p/ω, keeps the same decision-weighted ratio ordering. Therefore, the optimal portfolio will have $x_a < x_c < x_b < x_d$ which is not monotonic in the price-probability ratio p/π.

state	π	p	p/π	state order	Ω	ω	p/ω	state order	Ω'	ω'	p/ω'
a	20%	40%	2.00	*a*	30%	30%	1.33	*a*	30%	30%	1.33
b	20%	20%	1.00	*b*	50%	20%	1.00	*c*	56%	26%	1.15
c	40%	30%	0.55	*c*	75%	25%	1.20	*b*	75%	19%	1.05
d	20%	10%	0.50	*d*	100%	25%	0.40	*d*	100%	25%	0.40

Table IV: Updating Decision Weights

This table demonstrates the intertemporal inconsistency of the updating of the decision weights used in Cumulative Prospect Theory. The post-signal "expected value" is independent of the signal received, but the pre-signal "expected value" is different. The agent forms his decision weights with the Tversky and Kahneman formula (7) with a parameter of δ = 0.833921.

state	x	π	$\pi \cdot x$	Ω	ω	$\omega \cdot x$
Before receiving signal:						
a	–400	10%	–40	13.63%	13.63%	–54.52
b	–300	50%	–150	57.08%	43.45%	–130.35
c	–200	30%	–60	85.17%	28.09%	–56.17
d	–100	10%	–10	100%	14.83%	–14.83
		$E_\pi[x]$ =	–260		$E_\omega[x]$ =	–255.88
After receiving signal *A*:						
a	–400	$^1/_4$	–100	28.03%	28.03%	–112.12
c	–200	$^3/_4$	–150	100%	71.97%	–14.94
		$E_\pi[x]$ =	–250		$E_\omega[x]$ =	–256.06
After receiving signal *B*:						
b	–300	$^5/_6$	–250	78.03%	78.03%	–234.09
d	–100	$^1/_6$	–16.67	100%	21.97%	–21.97
		$E_\pi[x]$ =	–266.67		$E_\omega[x]$ =	–256.06

Table V: Dominated Multi-Period Loss-Averse Portfolio

This table demonstrates that an optimal single-period loss-averse portfolio can be first-order stochastically dominated over two periods. The loss-averse function is $v(x_1) + \rho v(x_2)$ where v is the TK function with $\alpha = 0.8$ $\beta = 0.9$, $\lambda = 2.25$. The zero-utility reference point is at an excess return of 0. The economy is identical each period and the single-period state probabilities and prices are $\boldsymbol{\pi} = (0.2, 0.4, .04)$, $\mathbf{p} = (0.1, 0.3, 0.6)$. The optimal single-period total portfolio returns are $\mathbf{1} + \mathbf{x} = (2.271, 1.167, 0.704)$.

state	prob	price	Loss-Averse Portfolio		Dominating Portfolio	
	$\pi_1 \cdot \pi_2$	$p_1 \cdot p_2$	return	cost	Return	cost
a a	4%	1%	5.160	0.052	5.160	0.052
a b	8%	3%	2.652	0.080	2.652	0.080
a c	8%	6%	1.600	0.096	1.363	0.082
b a	8%	3%	2.652	0.080	2.652	0.080
b b	16%	9%	1.363	0.123	1.600	0.144
b c	16%	18%	0.822	0.148	0.822	0.148
c a	8%	6%	1.600	0.096	1.363	0.082
c b	16%	18%	0.822	0.148	0.822	0.148
c c	16%	36%	0.496	0.179	0.496	0.179
				1.000		0.993