Estimating the Probability of Bankruptcy:

A Statistical Approach

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While many of the highly regarded bankruptcy prediction models in past literature are effective in classifying companies as bankrupt or healthy, they share the same limitation – the inability to estimate the probability of bankruptcy. This probability has countless applications, including the valuation of different types of assets and liabilities and investment decisions. Using Altman's 1968 Z-score model as a foundation, this paper explores the model re-estimation process with consideration to industry characteristics and changing macroeconomic conditions. Using a sample of telecom companies, the paper illustrates how to employ both discriminant analysis and logistic regression to derive the probability of bankruptcy.

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In an environment tarnished by recent scandals involving Enron and Global Crossing, bankruptcy prediction has become a major concern. The ability to forecast such debacles benefits all stakeholders, including shareholders, managers, employees, lenders, suppliers, clients, the community, and the government. Bankruptcy prediction models can help decision makers evaluate firms in problems of credit analysis, investment analysis, and going-concern evaluation. Since the Great Depression, academics and practitioners have created a variety of prediction models, ranging from Beaver's (1967) univariate analysis of financial ratios to the application of rough sets by, among others, Slowinski and Zopounidis (1995), Dimitras (1995), and Greco et al. (1997). The most influential, however, remains to be Altman's 1968 Z-score model, the first bankruptcy classification model to apply the technique known as discriminant analysis. This paper discusses the limitations of Altman's renowned model. Using his model as a foundation, I will examine methods to estimate models that may have more usefulness in today's ever-changing financial environment.

Limitations of Altman's Z-score Model

Altman's 1968 Z-score model remains the most common tool for evaluating the financial health of companies. Not only is the model extremely easy to use, having only five simple financial ratios as its inputs, it is fairly accurate in predicting bankruptcy up to five years before bankruptcy. Altman was the first to employ a statistical technique known as discriminant analysis. This assumes that, for two populations, the independent variables are distributed with each group according to a multivariate normal distribution with different means but equal dispersion matrices. For his model, the two groups were bankrupt and non-bankrupt companies, and the independent variables were five common financial ratios that could be obtained by publicly available financial statements. Discriminant analysis obtains a linear combination of the independent variables that maximizes the variance between the populations relative to within group variance. His resulting discriminant function was as follows:

$$Z = 1.2X_1 + 1.4X_2 + 3.3X_3 + 0.6X_4 + 0.999X_5,$$

where

X_1 = Working Capital/Total Assets (WC/TA)

X_2 = Retained Earnings/Total Assets (RE/TA)

X_3 = Earnings before Interest and Taxes/Total Assets (EBIT/TA)

X_4 = Market Value of Equity/Book Value of Total Liabilities (MVE/TL)

X_5 = Sales/Total Assets (S/TA).

After establishing the optimal z-score cutoff for bankrupt and non-bankrupt companies, any company can be classified with fairly high accuracy.

Altman's 1968 model was estimated using an initial sample composed of 66 companies with 33 firms in each of the two groups. These companies were all manufacturing firms from the period between 1946 and 1965. Using data collected from one financial statement prior to bankruptcy, the discriminant model misclassified only three of the 66 companies. For two statements prior to bankruptcy, the model was 83% effective. Despite the obvious upward bias from having the same sample as both the estimation and testing samples, the model's results are very encouraging. However, Grice and Ingram (2001) indicate that the model's accuracy is significantly lower in recent periods than that reported in Altman's study. They point out that researchers often mistakenly assume that their models are stable across economic conditions that change over time, such as inflation, interest rates, and credit availability. The business environment of the mid-1900s, from which Altman's model was estimated, was drastically different from today's environment. Intuitively, it would make sense that his model is outdated and would not be accurate in classifying today's firms. Mensah (1984) developed four models using samples from the 1972-1973, 1974-1975, 1976-1977, and 1978-1980 periods, each period representing a different economic environment. He found that the accuracy and structure of the models changed over the four time periods. Given Mensah's findings that models can change in such short subsequent time periods as two years, we would expect dramatic differences from Altman's model, which was derived from a sample that included companies from up to 50 years ago.

Another limitation is that Altman's sample consisted of only manufacturing firms. Platt and Platt (1991) demonstrated that a model developed using firms from one set of industries may not be highly accurate in predicting bankruptcy for firms in other industries. Specific industries have different characteristics so it would not be feasible to apply a general model for all industries. A more accurate model can be achieved by re-estimating the model's coefficients using estimation samples from specific industries and from periods close to the periods for which one would like to predict. Using the telecommunications industry as the focus of my study, I will attempt to devise a new model that accurately predicts bankruptcy.

The discriminant model serves as a great early warning system by providing the decision maker with a dichotomous classification of companies. Though important, this classification does not provide any estimate of the associated risk of bankruptcy. It would be preferable to classify firms in more than two classes according to the level of risk. To take that concept one step further, I will discuss methods to estimate models that yield probabilities of bankruptcy, a statistic that has more practical uses than a simple classification.

Sample selection and data

All the telecommunications firms that declared bankruptcy from May 2000 to January 2002 as indicated by BankruptcyData.com were included in the initial sample. Firms that did not have financial statements for at least two years were removed from the sample because they would not be able to be tested by Altman's model. This, however, created a bias against younger firms. I view this bias as a benefit because it eliminates skewed data that might have resulted during the buildup of the technology bubble, in which numerous telecom companies went public and continued to raise capital despite questionable earnings potential. Under more normal economic conditions, these companies probably would not have been able to go public. Since I believe that the buildup and the bursting of the technology bubble was an extremely rare event, I eliminated these so-called crash-and-burn companies in order to better isolate the business and financial performances of the firms from the unusual economic conditions.

The resulting sample includes 30 bankrupt firms, which I attempted to match with non-bankrupt firms in the same sector within the telecom industry and having comparable asset size. This was merely a best efforts attempt since the telecommunications industry is characterized by firms that provide all sorts of combinations of services, ranging from wireless data to undersea fiber to international long distance. Furthermore, some firms, such as the RBOCs (regional Bell operating companies) and the ILECs (incumbent local exchange carriers), have been less susceptible to bankruptcy than other sectors, notably the CLECs (competitive local exchange carriers). In fact, some particular telecom sectors have been so decimated that it was virtually impossible to match the bankrupt firms with similar firms that have survived. Likewise, some sectors have not been affected by bankruptcy at all. For these reasons, our total sample of 60 telecom companies is not a collection of perfectly paired bankrupt and non-bankrupt companies. A list of the firms can be found in Exhibit 1 of the Appendix.

Variable selection

Altman initially selected 22 financial ratios on the basis of their popularity in academic literature and their potential relevancy to bankruptcy prediction. After evaluating the discriminant powers of the variables in an iterative process, he selected five as doing the best overall job.

- Working Capital/Total Assets (WC/TA). Working capital is defined as the difference between current assets and current liabilities. The WC/TA ratio is a measure of liquidity in relation to total capitalization. Firms headed towards bankruptcy would be expected to have a shrinking WC/TA ratio.
- Retained Earnings/Total Assets (RE/TA). This ratio, a measure of cumulative profitability over time, is an indicator of the firm's age. A young company is less likely to have been able to build up its retained earnings since it would have to reinvest much, if not all, of its earnings to stimulate

growth. More mature and stable companies would have a higher RE/TA ratio. The younger firm is somewhat discriminated against by this ratio, but it turns out that this is precisely the situation in the real world. Altman (1993) shows that the frequency of bankruptcy is much higher in a firm's earlier years.

- Earnings before Interest and Taxes/Total Assets (EBIT/TA). This ratio is a measure of the productivity of the firm's assets, which is a fundamental element in the survival of a firm.
- Market Value of Equity/Book Value of Total Liabilities (MVE/TL). This ratio is the reciprocal of the debt-to-equity ratio, which measures financial leverage. Altman explains that the MVE/TL ratio shows how much a firm's assets can decline in value before the liabilities exceed the assets and the firm becomes insolvent.
- Sales/Total Assets (S/TA). This ratio, the standard capital turnover ratio, indicates the sales generating ability of the firm's assets. It is also a measure of management's ability in dealing with competitive conditions. Altman shows that this is the least significant ratio on an individual basis, but it has the second highest discriminating ability due to its unique relationship with the other variables.

Instead of using the market value of equity, I decided to use the book value. Today's financial markets are much more volatile than in the past. The market value of equity can be extraordinarily high and then suddenly collapse within a matter of months. This would distort and shorten the predictive ability of any model based on market values. For instance, after accounting for the reverse stock splits made shortly before declaring bankruptcy, the shares of Exodus Communications traded in the $50 to $70 range less than two years ago. If we used the data from the financial statements prior to bankruptcy (i.e. the statements ending December 2000), the high stock price would be reflected in the MVE/TL ratio, and Altman's model probably would have predicted that the company was healthy. Because stock prices can be so high less than one year before bankruptcy, the MVE/TL ratio does not have much predictive power. Using the book value of equity somewhat eliminates the wild investor sentiment during the buildup and collapse of the technology bubble. The same argument against using the market value of equity can apply to the notion of using the book-to-market ratio, which has gained prominence as a strong indicator of financial distress in recent finance literature.

The empirical data for each of the companies in the sample were collected from the financial statements provided by Disclosure. The data for firms in this sample were supplemented by information from Marketguide.com and Bloomberg.com. For the bankrupt companies, I collected data from the past two financial statements prior to bankruptcy. For the non-bankrupt companies, I collected data from their last two financial statements. I then tested Altman's original model with the telecom sample, but before discussing the results, let's clarify a few questions and concerns that arose.

Clarification

Ideally, we would have preferred to examine the financial ratios of firms in one period in order to make predictions about other firms in the subsequent period. Unfortunately, this was not possible due to the limited number of bankrupt telecom companies in the past couple of years. Given such a small sample size, making a distinction between the estimation sample and the validation, or test, sample presents a dilemma. If we keep the estimation and validation samples as they are, the shortage of data may render the estimation model inadequate. In this case, we may end up using an inadequate model to test the validation sample. This would have no value. Another option would be to include the validation sample in the estimation sample. The additional information would result in a better estimation model, but there would be no sample with which to test the model.

Another dilemma is the choice of the time period from which to choose the sample of firms. We could increase the size of the sample by extending the time period to include the past five or ten years. This, however, sacrifices the timeliness of the estimation model. For example, Altman selected his sample from the time period between 1946 and 1965. The resulting estimation model would not have much use in future years given the ever-changing business landscape. Ideally, we would like to have a large estimation sample of firms from a specific industry and from a relatively short time period just prior to the period from which the validation sample is chosen. This, of course, is very difficult for our telecom study so I decided to "pick my poison" and resort to only having an estimation sample. Because our sample is so small, we must realize that analyzing the viability of the estimation model is not as precise as we would like. Even though the results do hold some valuable meaning and illustrative power, we must keep in mind their limitations and focus our attention more on the process than on the results.

Testing Altman's Model

To get a better understanding of its durability and effectiveness, I tested Altman's 1968 model with the current sample of 60 telecom companies. Keeping the same coefficients and replacing the market value of equity by the book value for one of the variables, I calculated z-scores for each company. I then adjusted the cutoff z-score to 0.5, the point that best discriminates between the bankrupt and non-bankrupt groups. There are two possible types of errors. A Type I error occurs when the model predicts that a company is not bankrupt when it is actually bankrupt. A Type II error occurs when the model predicts that a company is bankrupt when it is actually not bankrupt. The chosen optimal cutoff z-score assumes that the cost of a Type I error equals the cost of a Type II error. This does not hold true in the real world, where Type I errors are substantially more costly than Type II errors. In practical situations, we would need to consider this cost differential when setting the appropriate z-score. Since our purposes are to illustrate the process and to assess the classification ability of the model, we maintain the assumption of equal costs. The results are as follows:

Classification Results, One Statement Prior to Bankruptcy

	Number Correct	Percent Correct	Percent Error	n	Actual	Predicted Bankrupt	Predicted Not
					Bankrupt	27	3
					Not	2	28
Type I	27	90	10	30			
Type II	28	93	7	30			
Total	55	92	8	60			

Classification Results, Two Statements Prior to Bankruptcy

	Number Correct	Percent Correct	Percent Error	n	Actual	Predicted Bankrupt	Predicted Not
					Bankrupt	24	6
					Not	3	27
Type I	24	80	20	30			
Type II	27	90	10	30			
Total	51	85	15	60			

The results show that the model is extremely accurate in classifying 92% of the total sample correctly when using data from one financial statement prior to bankruptcy. The model is still fairly accurate when using data from two statements prior to bankruptcy, correctly classifying 85% of the companies. The results show that Altman's model still holds very strong predictive power, indicating his wise choice of independent variables.

We must remember that the coefficients of Altman's model were estimated based on a sample of manufacturing companies from the mid-1900s. The relationships between the independent variables have probably changed so we would expect different coefficients for a new model, especially since we changed one of the variables and applied our focus on a particular industry. Altman's model also fails to provide an estimate of the associated risk of bankruptcy. This probability of bankruptcy is, however, inherent in discriminant analysis. The imputed z-scores appear to be aligned along a spectrum, the lower z-scores representing a higher probability of bankruptcy than for higher z-scores. In the next section, I re-estimate the parameters of the model and show how the estimation model can provide a probability of bankruptcy.

Re-estimating the Model

Using the Discriminant Analysis function of the statistical software package MINITAB, I derived a linear discriminant function for each of the 31 combinations of the five financial ratios. The results are based on classification functions. Each model has two classification functions, one for bankrupt companies and the other for non-bankrupt companies. For each company, the values of the ratios are entered into the classification function, and the function with the highest value denotes the group to which the company belongs. This process is repeated for each of the 31 combinations of the five ratios, and the results are presented in Panel A of Exhibit 2 in the Appendix.

Despite the limitations of using a relatively small sample and the possibility of data-dredging, the methodology of using classification functions gives us a rough idea of which models are most effective. The results show that several combinations of the variables accurately classify the companies into the correct group. The high accuracy is to be expected due to the fact that each of the models was created to best fit the data. Altman (1993) explains that when the firms used to determine the coefficients of the model are reclassified, the model's accuracy is biased upward by sampling errors in the original sample and search bias. In essence, we are using the sample to estimate the model and then using the same sample to test its accuracy. We would, therefore, expect very high accuracy. While the models may be effective for the given sample, there is no guarantee that it will be effective for the entire population of companies. Ideally, we would like to have a much larger estimation sample, as well as a validation sample with which to test the model. Due to data constraints, we must resort to having only an estimation sample. Fortunately, MINITAB allows us to use cross-validation, which mimics the estimation and validation processes. The program drops one company from the sample, constructs a discriminant model based on the rest of the data, and then classifies the omitted company. This process is repeated for each of the companies in the sample. Panel B of Exhibit 2 shows the summary of classification with cross-validation.

Discriminant analysis is only valid for two populations when the independent variables are distributed within each group according to a multivariate normal distribution with different means but equal dispersion matrices. To determine whether these assumptions are met, we examine the stem-and-leaf plots for each of the variables, which are presented in Exhibit 3 of the Appendix. The stem-and-leaf plots indicate that the RE/TA and EBIT/TA ratios are closest to having normal distributions and different means for the bankrupt and non-bankrupt groups. It seems that these ratios have strong discriminant power, as evident in the classification table. A model constructed by using only the RE/TA or EBIT/TA ratio will correctly classify 81.7% and 70.0% of the companies, respectively. This suggests that we should select the model with only these two variables. Because of our small sample size, we cannot be sure that the other variables are not normally distributed. Therefore, I will select the following three combinations of variables for further analysis:

- WC/TA, RE/TA, EBIT/TA, BVE/TL, and S/TA
- RE/TA, EBIT/TA, and BVE/TL
- RE/TA and EBIT/TA

I chose the combination that includes all of the variables because I would like to compare the estimated model to that of Altman. I chose the second combination because it includes the three variables that are closest to having normal distributions and different means for each group of companies. This combination also yielded the highest classification accuracy.

Using the Eigen Analysis function of MINITAB, I determined linear discriminant functions for each of the combinations. This yielded a simple formula with which to calculate z-scores. For each of the functions, the optimal cutoff z-score is zero. Once again, we assume that the costs of Type I and Type II errors are equal. Because we adjusted the cutoff score, the results are slightly more accurate compared to the results obtained when using the two classification functions. For each combination, the results for one and two statements prior to bankruptcy are presented below.

Classification Results, One Statement Prior to Bankruptcy
$Z = 0.0761X_1 + 0.0881X_2 + 0.4153X_3 + 0.0356X_4 + 0.0182X_5$

	Number Correct	Percent Correct	Percent Error	n	Actual	Predicted Bankrupt	Predicted Not
					Bankrupt	28	2
					Not	1	29
Type I	28	93	7	30			
Type II	29	97	3	30			
Total	57	95	5	60			

Classification Results, Two Statements Prior to Bankruptcy
$Z = 0.0761X_1 + 0.0881X_2 + 0.4153X_3 + 0.0356X_4 + 0.0182X_5$

	Number Correct	Percent Correct	Percent Error	n	Actual	Predicted Bankrupt	Predicted Not
					Bankrupt	21	9
					Not	2	28
Type I	21	70	30	30			
Type II	28	93	7	30			
Total	49	82	18	60			

Classification Results, One Statement Prior to Bankruptcy
$Z = 0.0989X_2 + 0.3887X_3 + 0.0404X_4$

	Number Correct	Percent Correct	Percent Error	n	Actual	Predicted Bankrupt	Predicted Not
					Bankrupt	28	2
					Not	2	28
Type I	28	93	7	30			
Type II	28	93	7	30			
Total	56	93	7	60			

Classification Results, Two Statements Prior to Bankruptcy
$Z = 0.0989X_2 + 0.3887X_3 + 0.0404X_4$

	Number Correct	Percent Correct	Percent Error	n	Actual	Predicted Bankrupt	Predicted Not
					Bankrupt	26	4
					Not	3	27
Type I	26	87	13	30			
Type II	27	90	10	30			
Total	53	88	12	60			

Classification Results, One Statement Prior to Bankruptcy

$Z = 0.1159X_2 + 0.3952X_3$

	Number Correct	Percent Correct	Percent Error	n	Actual	Predicted Bankrupt	Predicted Not
					Bankrupt	30	0
					Not	8	22
Type I	30	100	0	30			
Type II	22	73	27	30			
Total	52	87	13	60			

Classification Results, Two Statements Prior to Bankruptcy

$Z = 0.1159X_2 + 0.3952X_3$

	Number Correct	Percent Correct	Percent Error	n	Actual	Predicted Bankrupt	Predicted Not
					Bankrupt	28	2
					Not	12	18
Type I	30	100	0	30			
Type II	22	73	27	30			
Total	52	87	13	60			

Comparing the classification accuracy of the models with that of Altman, we can see that the effectiveness is very similar for both one and two statements prior to bankruptcy. The three-variable model slightly outperformed Altman's model for both time periods, while the five-variable model outperformed Altman's model for one statement prior to bankruptcy. The relationships between the coefficients of the linear function, however, are drastically different. This is expected since we decided to use the book value of equity instead of the market value. This change would affect all of the relationships between the variables. Comparing both models, it appears that the EBIT/TA ratio has more weight (i.e. the coefficient is relatively higher) in the telecom model than in Altman's model, while the S/TA ratio has much less weight. In other words, a telecom company's survival depends more on EBIT and less on sales, when evaluated in dollar-for-dollar terms. There seems to be a much higher premium on EBIT than on sales. Put in yet another way, a firm would require less EBIT and more sales to achieve the same specific z-score. Given the nature of the telecom industry in the past couple of years, this finding is not surprising. There has been an overemphasis on top-line growth for these young companies, while concern for the bottom-line has been deferred until future years. Former high-growth companies, such as Global Crossing and Winstar Communications, once encountered highly receptive capital markets on the basis of their high growth stories alone. Investors were not overly concerned that some of these companies had not yet become EBITDA-positive. These capital infusions kept many telecom companies afloat despite their questionable earnings and cash flow potential.

Another explanation for the differences in the parameters of Altman's model and the re-estimated telecom model is the changing accounting environment. Several accounting rule changes have been made in the areas of capitalized leases, goodwill and intangibles, research and development costs, deferred charges, and revenue recognition, among others. Many of these changes influence the accounting for total assets, which is the denominator in four of the models' variables. The relevance of accounting changes is just another reason why models must be re-estimated using industry-specific samples. Most industries have accounting practices that are unique to that particular industry, and we would want our prediction models to capture those unique practices. For example, all network infrastructure providers such as Level 3 Communications and Global Crossing treat indefeasible rights of use (IRUs) in a similar fashion that may appear to be aggressive accounting compared to other industries. IRUs are a type of long-tern lease of capacity on another company's fiber optic network. Global Crossing can sell IRUs to another company and simultaneously pay an equal amount to buy IRUs from the same company, resulting in an even trade. However, generally accepted accounting principles (at least for now) allow the company to recognize revenues immediately and to depreciate the lease over its life, making revenues look larger even though no money changes hands. This illustrates the importance of industry-specific models and also provides yet another reason for the change in weighting of the S/TA ratio.

The variables clearly hold different levels of significance for today's telecom companies compared to general manufacturing companies of the past. The three-variable model (RE/TA, EBIT/TA, and BVE/TL) is the most effective prediction model, followed closely by the five-variable model. In the next section, we use both of the models as examples to show how we can derive the probability of bankruptcy.

Deriving the Probability of Bankruptcy Using Discriminant Analysis

In this study, I followed a retroactive sampling scheme, in which I saw the final state of the business first and then chose the sample. Because I chose an equal number of bankrupt and non-bankrupt companies instead of randomly sampling companies from the entire population, we must make an adjustment for prior probabilities. MINITAB allows us to input this adjustment and then calculates the classification functions. I chose a prior probability of 10%, meaning that about 10% of the companies in the population will declare bankruptcy within one year. This is a highly conservative estimate, which would yield higher probabilities of bankruptcy and, hopefully, earlier warning signs. Assuming that the average telecom company has a B bond rating, our estimate is understandable since the one-year default rates for companies with B1, B2, and B3 ratings have been 3.5%, 6.9%, and 12.2% over the period between 1983 and 1999 (Moody's 2000). We would expect these default rates to be substantially higher for the past couple of years. We can now derive the probability of bankruptcy by using the values of the adjusted classification functions through a logistic model:

$$P(Y=1) = \frac{e^{CF_1}}{e^{CF_1} + e^{CF_2}}$$

where

CF_1 = the classification function for bankruptcy

CF_2 = the classification function for non-bankruptcy.

The derived probabilities of bankruptcy for each of the telecom companies in the sample are presented in Exhibit 1 of the Appendix. If we adjust the cutoff to 0.10, our estimate of prior probability, the results are fairly accurate in classifying bankrupt and non-bankrupt companies. Both of our models correctly classified all of the non-bankrupt companies. The five-variable model misclassified seven bankrupt companies, while the three-variable model misclassified six.

Although discriminant analysis allows us to derive a probability of bankruptcy, this statistic may be somewhat misleading. The major concern is that the independent variables have not met the assumptions necessary for discriminant analysis. We saw earlier that the distribution of the variables for each group was not perfectly normal and the means were not very distinct from one another. This creates a level of ambiguity in the model, especially with such a small sample size. Discriminant analysis does not provide many statistics that allow us to test the significance of each independent variable. For these reasons, we must turn to the next best alternative, logistic regression.

Deriving the Probability of Bankruptcy Using Logistic Regression

Unlike discriminant analysis, logistic regression does not assume multivariate normality and provides several statistics that indicate the significance of each variable. It also handles relatively smaller sample sizes better than discriminant analysis, but we must keep in mind that the sample size in our study still warrants caution. Using a dichotomous dependent variable (1=bankrupt, 0=non-bankrupt), I used MINITAB to generate the best fitting logistic models for each of the 31 combinations of independent variables. Exhibit 4 in the Appendix presents the summary statistics for each combination. The results suggest that the models with the following combinations of variables are the most significant:

- WC/TA, RE/TA, and EBIT/TA
- RE/TA, EBIT/TA, and BVE/TL
- EBIT/TA and BVE/TL

With the null hypothesis being that the model fits, each model passes the Hosmer-Lemeshow goodness-of-fit test, having p-values in excess of 0.945. Each of the p-values for the coefficients suggests statistical significance for the model as a whole, but this may be due to the upward bias of having a small sample size. We should be hesitant to conclude with any certainty that the models are the best fitting, especially when most of the p-values for the coefficients are on the borderline between statistical significance and insignificance. A larger sample would yield much more comprehensive and convincing results. Nonetheless, to illustrate the process of deriving the probability of bankruptcy, we proceed with these three models.

We can calculate the estimated probability of bankruptcy by using the formula below:

$$P(Y=1) = \frac{e^{\ln(Y/N) + \alpha + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_\kappa X_k}}{1 + e^{\ln(Y/N) + \alpha + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_\kappa X_k}}$$

where

α = constant

B_n = coefficient for variable n

Y = prior probability that a company of the population will go bankrupt

N = prior probability that a company of the population will not go bankrupt.

The adjustment for prior probabilities is necessary because we are using a retroactive sample that is not representative of the entire population. Once again, I have chosen a prior probability of 10% based on the estimated one-year default rates for B-rated companies during the past two years. The derived probabilities of bankruptcy for each of the telecom companies in the sample are presented in Exhibit 1 of the Appendix. After setting the cutoff probability to the prior probability of 10%, we find that the logistic model's predictive power is as strong as that of the discriminant function. The classification results for the logistic model are presented below.

Classification Results

$P(Y=1) = (e^{\ln(0.1/0.9) - 0.8935 - 7.91X_1 - 4.773X_2 - 15.975X_3})/(1+ e^{\ln(0.1/0.9) + 0.8935 - 7.91X_1 - 4.773X_2 - 15.975 X_3})$

	Number Correct	Percent Correct	Percent Error	n	Actual	Predicted Bankrupt	Predicted Not
					Bankrupt	26	4
					Not	3	27
Type I	26	87	13	30			
Type II	27	90	10	30			
Total	53	88	12	60			

Classification Results

$P(Y=1) = (e^{\ln(0.1/0.9) - 0.333 - 5.286X_2 - 17.113X_3 - 1.5957X_4})/(1+ e^{\ln(0.1/0.9) - 0.333 - 5.286X_2 - 17.113X_3 - 1.5957X_4})$

	Number Correct	Percent Correct	Percent Error	n	Actual	Predicted Bankrupt	Predicted Not
					Bankrupt	27	3
					Not	2	28
Type I	27	90	10	30			
Type II	28	93	7	30			
Total	55	92	8	60			

Classification Results

$P(Y=1) = (e^{\ln(0.1/0.9) + 1.6181 - 26.281X_3 - 2.4867X_4})/(1+ e^{\ln(0.1/0.9) + 1.6181 - 26.281X_3 - 2.4867X_4})$

	Number Correct	Percent Correct	Percent Error	n	Actual	Predicted Bankrupt	Predicted Not
					Bankrupt	27	3
					Not	3	27
Type I	27	90	10	30			
Type II	27	90	10	30			
Total	54	90	10	60			

If we compare the probabilities of bankruptcy derived from the three logistic models and the two discriminant functions, we will notice a fairly significant disparity. This can be explained by the small sample size from which we developed the various models. We would expect much more consistent results with a larger sample. Nonetheless, we can still observe patterns across the different models, which indicate future bankruptcy for some of the non-bankrupt companies. For example, the probabilities of bankruptcy for Leap Wireless and Allegiance Telecom suggest that troubles may be ahead for the companies. Because there are major limitations to making claims based on the different prediction models, we first turn to an examination of alternative methods for comparison purposes and then move onto a discussion of the ideal model formation process and practical application of the logistic and discriminant models.

Alternative Methods

Countless academic studies have attempted to devise bankruptcy prediction models. Zopounidis and Dimitras (1998) review most of the methods and models introduced in past literature. They find that the majority of the prediction models are essentially classification models that offer no measure for the probability of bankruptcy. They discuss such complex techniques as recursive partitioning algorithm, survival analysis (a type of proportional hazard model), expert systems, and neural networks. Bankruptcy prediction models have also been heavily explored in the professional field, especially among credit risk agencies, insurance companies, investment banks, and other financial institutions. For example, KMV, a leading provider of market-based quantitative credit risk products for credit risk investors, utilizes a model that extends the Black-Scholes-Merton option-pricing framework (2001). Their Vasicek-Kealhofer (VK) model measures the probability of default during the forthcoming year. Because most of these academic and proprietary methods demand heavy computational effort and somewhat mask the intuition behind the models, we will focus on simpler methods.

Damodaran (2002) discusses three simple methods to estimate the probability of bankruptcy – probit analysis, reverse engineering (i.e. backing out the probability from the prices of corporate bonds), and using historical default rates by bond rating. Probit analysis is very similar to logistic regression.

The main difference between them is that the probit function assumes a cumulative standard normal distribution, whereas the logistic function assumes a binomial distribution. Both methods employ maximum likelihood estimation and should produce very similar results, especially with large sample sizes. However, Gloubos and Grammatikos (1998) believe that the scarcity of probit analysis in literature, compared to logistic regression, is due to the higher computational effort required.

The second method assumes that the prices of bonds accurately reflect the expected cash flows on the bond (i.e. the principal and the coupon payments), discounted back at the cost of debt. Knowing that the probability of bankruptcy affects the bond's expected cash flow, we can write the following formula:

$$\text{Bond Price} = \sum_{t=1}^{t=N} \frac{C^*[1-P(Y=1)]^t}{(1+r_F)^t} + \frac{FV^*[1-P(Y=1)]^N}{(1+r_F)^N}$$

where

N = years to maturity

C = coupon payment

r_F = risk-free rate

FV = face value of the bond

Given the price of the bond, its coupon rate and the years to maturity, we can back out the probability of bankruptcy. This method assumes that bond markets are efficient and will only work for straight bonds. Another method is to use the historical probabilities of bankruptcy associated with particular classes of bond ratings. These two methods are quick-and-dirty ways to estimate probabilities and should be used with a bit of caution. They are, however, very useful in substantiating other more data-intensive methods, such as the probit and logistic analyses.

An Extension of the Model Formation Process

Earlier we discussed the many dilemmas encountered when deciding how to form the prediction model using either discriminant analysis or logistic regression. The ideal process would include a large sample size of both bankrupt and non-bankrupt companies chosen randomly from the overall population of companies. The ratio of bankrupt to non-bankrupt companies in the sample should reflect the ratio observed in the overall population. This would eliminate errors resulting from the prior probabilities adjustment we employed for both the discriminant and logistic functions.

Another major problem concerns the time period from which to select the companies for the estimation sample. As others have done before, we have also shown that Altman's 1968 model, which was formed based on a broad selection of companies over a 20-year period, does not retain its effectiveness over time and across different industries. Business and economic conditions change over short periods of time so the model's coefficients will also require adjustments. The same philosophy applies to all other bankruptcy prediction models. In order to form an accurate model, one must consider the current macroeconomic conditions, including growth rates, inflation, and interest rates. The estimation sample should include companies from the same industry and from macroeconomic environments that resemble the current environment. This cross-sectional sample selection aggregates these "slices of history" into one model. This notion of highly customized bankruptcy models is an intriguing area for future study.

Applications

Why is deriving the probability of bankruptcy so important? Bondholders and other lenders would want to know the probability of receiving payments. The probability of bankruptcy would help suppliers in the valuation of their accounts receivable. Customers would want to know whether they could rely on particular company to meet their demand for goods and services in the future. Shareholders would want to know the possibility that there is any residual value remaining after bondholders are paid. The probability of bankruptcy is obviously an important measure for all stakeholders.

Altman (1993) discusses several specific applications of bankruptcy prediction models. As the number and size of bankrupt companies increase, the market for distressed securities has exploded. Distressed securities can be defined in the broad sense as equity or debt securities of companies in or facing a bankruptcy, reorganization, or other troubled situation. Investors purchase these securities at low prices with the hopes that they appreciate when the company emerges from the distressed situation. One investment strategy would be to employ a bankruptcy prediction model to select the securities of companies that have lower probabilities of bankruptcy. Altman shows that security selection based on the ZETA credit evaluation system (another member of Altman's z-score family of models) outperforms the overall market of distressed securities. Altman also discusses the usefulness of models in the valuation of corporate loans and the management of a financial turnaround. For the latter application, models whose independent variables are financial ratios can serve a tool for recovery. Since the ratios are indicators of the financial health of the company, a wise manager would target those activities that improve the ratios. The output of the model would serve as a good indicator of the progress achieved during the turnaround. An effective financial turnaround would result in a downward trend for the computed probability of bankruptcy.

Another important application is the valuation of equity. Damodaran (2002) argues that traditional valuation techniques fail to capture all of the effects of financial distress because they assume unconstrained access to capital markets and often fail to adjust expected cash flows and discount rates to incorporate the possibility of bankruptcy. He proposes methods to incorporate distress in relative valuation as well as discounted cash flow valuation, including simulation analysis, modified DCF models with adjusted expected cash flows and discount rates, separate going-concern and distress valuations, and adjusted present value (APV) models. For many of these methods, the probability of bankruptcy is a required input.

Conclusion

The probability of bankruptcy is an essential statistic for valuation and other types of financial analysis. However, deriving this probability using an intuitive, non-complex methodology has presented a major challenge for academics and professionals. Using Altman's 1968 Z-score model as a foundation and backbone, this study discusses its many limitations and explores statistical techniques to better estimate the probability of bankruptcy. With a sample of telecom companies, I re-estimate Altman's model to show that the parameters are not stable across specific industries and changing macroeconomic conditions. I then illustrate the relatively simple process of converting linear discriminant and logistic regression functions to derive the probability of bankruptcy. By emphasizing the importance and intricacies of proper sample selection and model selection, this paper hopes to provide some guidance in creating accurate bankruptcy prediction models using both discriminant analysis and logistic regression.

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