Modeling Tail Risk: from **B**lack **S**choles to **B**lack **S**wans ... and back

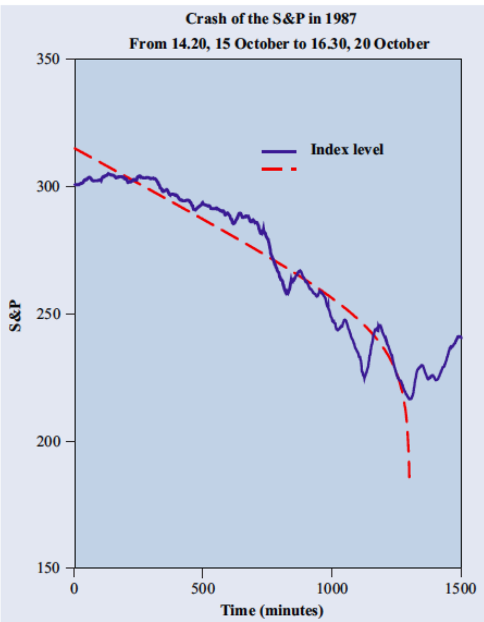
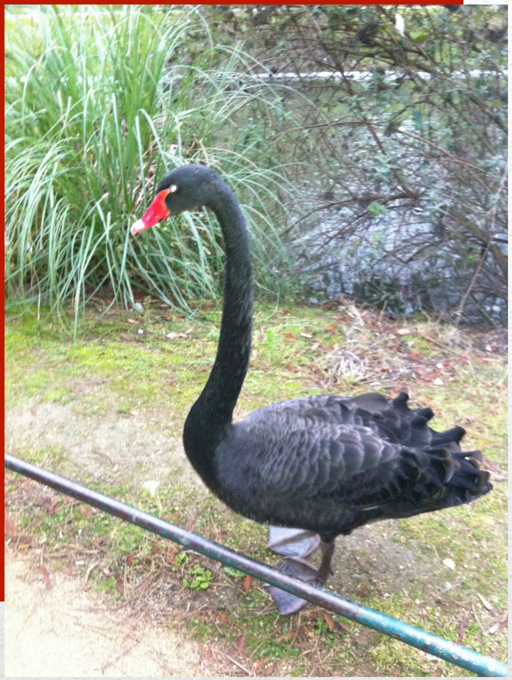


Figure 5. The 1987 crash: evolution of the S&P500 index.



Rama CONT

Imperial College London

The Black-Scholes-Merton model: a powerful workhorse for continuous-time finance

- 40 years after its inception, the Black-Scholes-Merton model still constitutes a powerful framework for analyzing a wide variety of problems in asset pricing and risk management in finance.
- At the heart of this model lies
  - a statistical modeling of market risk in terms of risk factors
  -the possibility of continuous and frictionless trading of securities
- The Black-Scholes-Merton model has inspired many extensions, most of which have focused on alternative specification of dynamics of risk factors, mostly to incorporate
  -statistically observed features of financial data : stochastic volatility, jumps, volatility clustering,….
  - specific features of a given market or asset class: FX, commodities, interest rates, credit,…
  - joint behavior of multiple risk factors: correlation and dependence

Risk Management failures: a challenge to the theory?

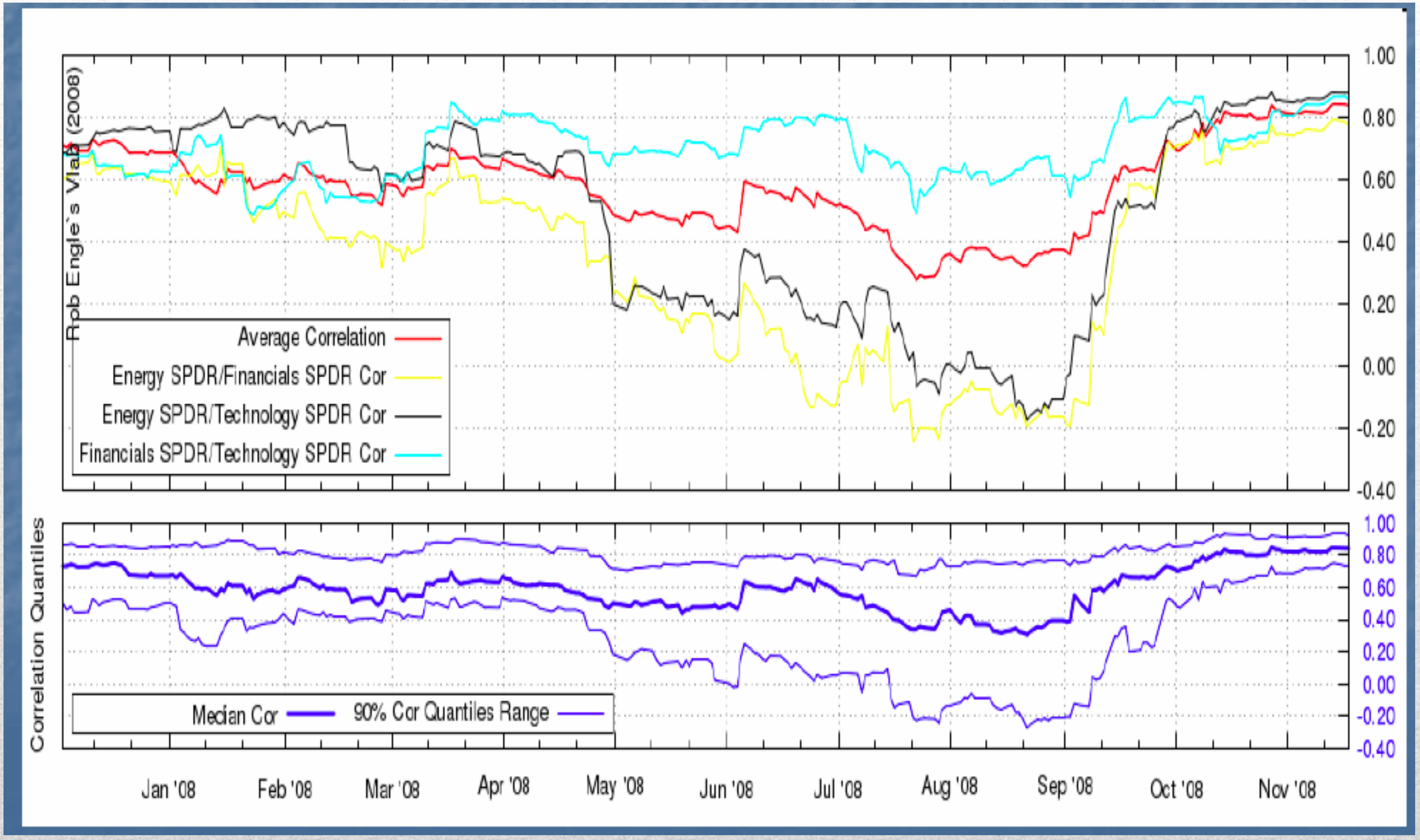
- These 40 years have also witnessed a certain number of (spectacular) risk management failures in which quantitative risk models where blamed, rightly or not, for huge losses in financial institutions.
- Most such examples can be traced back to **‘tail events**’ i.e. extreme moves, often simultaneously occurring in many risk factors.
- These failures are routinely mentioned in risk management textbooks as a cautionary note (usually in the ending chapter…).
- These failures have also been used by opponents of Quantitative Risk Management to criticize this framework as a whole, going as far as declaring the ‘impossibility’ of quantifying or managing financial risk.
- Perhaps the more important question is whether the (theory and practice of) Quantitative Risk Management has drawn lessons from these failures, in terms of modeling and methodology.

Discontinuity and concentration in financial risk

- Financial time series are well known to exhibit *heavy tails, discontinuities, extreme variations* and volatility clustering, documented as early as (Mandelbrot 1963) and confirmed by >50 years of econometric research across markets and periods.
- For a typical financial portfolio, risk is **concentrated** in time in a few large losses: one large loss across a single day can wipe out 20% or more of a portfolio's annual gain
- Thus, one can argue that extreme losses should be the **FOCUS** of risk models instead of a sideline.
- These arguments have led to the development of models with sudden price jumps, starting with (Merton 1974) and leading to a range of extensions of the BSM model incorporating jumps, stochastic volatility, regime changes and various other effects.

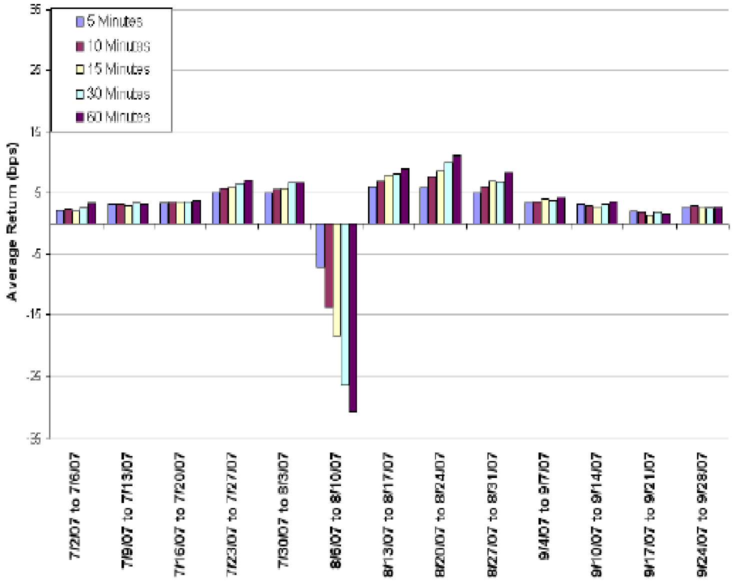
Do statistically sophisticated extensions of BSM capture the risk of catastrophic losses?

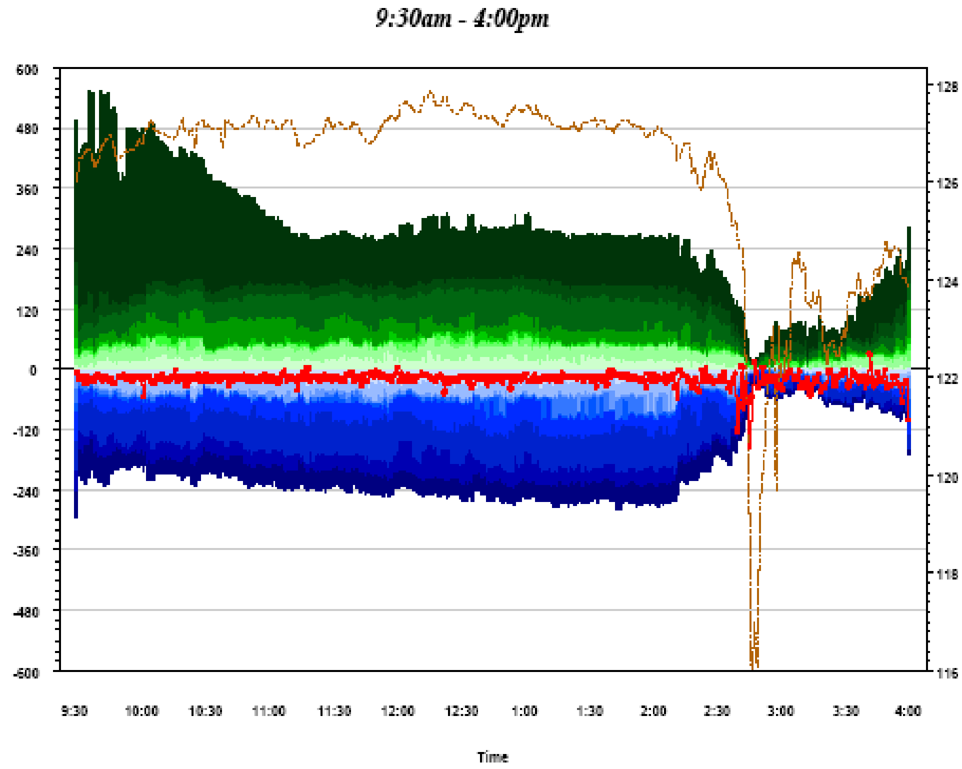
- Extensions of the BSM model with various statistical "bells and whistles" can go very far in modeling statistical features of loss distributions and option prices, in particular 'tail risk'.
- This approach does not account for many events associated with spectacular losses still remain outliers even under such heavy-tailed models: Oct 1987, the magnitude of losses associated with the subprime crisis, post-Lehman market turbulence, quant crash of 2007, …
- Also, in "statistical" extensions of the BSM model, large losses are generated by large 'random' moves (tail events) in risk factors, driven typically by various independent random sources: this makes *large co-movements (*"joint tail events*") very very unlikely*, and certainly not within the range of commonly used risk measures.
- In fact these market turbulence episodes are associated with systematic large shifts of parameters such as *volatility and correlations*.



Correlations among US stock returns, 2008

Average return of a long-short market-neutral equity fund (July-Sept 2007). Source: Lo & Khandani (2009)





The Quant crash of August 2007 (Lo & Khandani, 2009).

Flash Crash: Intraday evolution of prices and transaction volume on IBM Shares, May 6, 2010.

$$Y_t = \gamma_0 + \sum_{k=1}^{4} \gamma_{1k} Y_{t-k} + \gamma_{2k} I_{2006}\ ABX_{t-k} + \gamma_{3k} I_{2007}\ ABX_{t-k} + \epsilon_{it}$$

Y	ABX	γ_{21}	γ_{22}	γ_{23}	γ_{24}	γ_{31}	γ_{32}	γ_{33}	γ_{34}	$p(\gamma_2 = 0)$	$p(\gamma_3 = 0)$	Adj. R^2
One-Year Treasury	AAA	−0.36	−0.21	−0.11	−0.16	2.68	2.71	3.50	0.44	0.988	0.000	0.249
	AA	0.48	0.06	0.09	0.10	3.94	1.43	5.12	−0.73	0.993	0.000	0.322
	A	−0.53	0.49	−0.84	−0.67	0.47	0.77	6.47	−0.21	0.760	0.000	0.364
	BBB	−0.16	0.14	−0.20	−0.06	−0.05	−0.47	5.57	1.35	0.999	0.000	0.303
	BBB−	−0.08	0.41	−0.50	−0.06	0.22	−1.11	5.43	2.60	0.983	0.000	0.323
Ten-Year Treasury	AAA	0.25	−0.20	0.17	0.69	0.69	2.60	0.79	1.57	0.887	0.009	0.081
	AA	1.51	1.58	0.68	0.50	1.44	1.30	2.59	1.13	0.337	0.003	0.135
	A	−0.48	0.45	−1.07	−0.88	0.75	0.66	3.00	0.97	0.638	0.002	0.131
	BBB	−0.37	0.05	−0.77	0.31	0.63	0.47	3.48	0.56	0.931	0.000	0.155
	BBB−	−0.13	0.91	−1.26	0.29	1.37	0.33	2.61	1.46	0.716	0.001	0.152
S&P 500 Financials	AAA	−0.90	−0.09	0.26	0.26	3.17	−1.86	1.12	1.92	0.905	0.007	0.287
	AA	−0.90	−0.13	0.19	0.08	3.67	−2.61	3.48	1.00	0.903	0.000	0.363
	A	−1.28	−0.93	−0.70	−0.44	1.86	−0.50	3.76	0.71	0.462	0.000	0.359
	BBB	−0.50	−0.76	−0.94	−0.11	1.76	0.79	3.58	1.18	0.756	0.000	0.386
	BBB−	−0.34	−0.75	−0.44	0.32	2.04	1.84	2.35	0.06	0.861	0.000	0.355

Correlation between subprime ABX index returns and SP500 returns: negative before 2006, positive after 2007!

Longstaff (2009): The subprime credit crisis and contagion in financial markets.

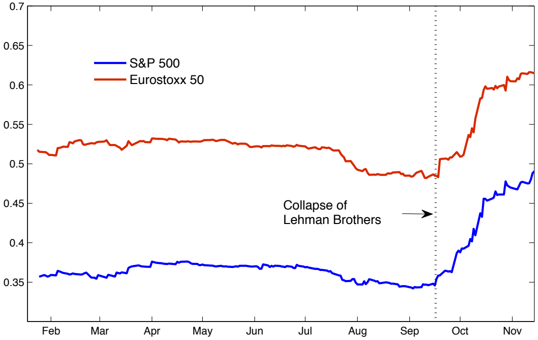


Figure: EWMA average correlation among SPDRs and Eurostoxx 50

Black Swans: a convenient (un)truth?

(with apologies to N Taleb and A Gore)

- Such scenarios cannot be simply brushed aside as ‘outliers’: umbrellas which work only when it doesn’t rain are not a great sell, and risk models need to incorporate them in some way.
- The inability of statistical risk models to capture the risk associated with such large loss scenarios has prompted some to treat them as totally unforeseeable events, or *Black Swans* and claim that, since ‘Black Swans’ are ‘impossible to model’ in a statistical framework, they invalidate in a fundamental way the whole statistical/stochastic modeling approach used in Quantitative Risk Management.
- This viewpoint is *convenient* since it absolves the responsibility of risk managers, quants, regulators,… who cannot be blamed for finding themselves in the middle of a ‘perfect storm’.
- We claim that, on the contrary, simple extensions of the BSM model can go a very long way in modeling, explaining and quantifying such ‘Black Swans’ through well-known economic mechanisms.

Demystifying the Black Swan: endogenous risk

- In fact, many failures of risk management models have been associated with
- **deleveraging/ liquidation of large portfolios (« whale scenario »)**

Ex: 'hedging' of large CDS/CDO positions by JP Morgan (London Whale)

Ex: liquidation of a large market-neutral equity portfolio by GSAM in Aug 2007

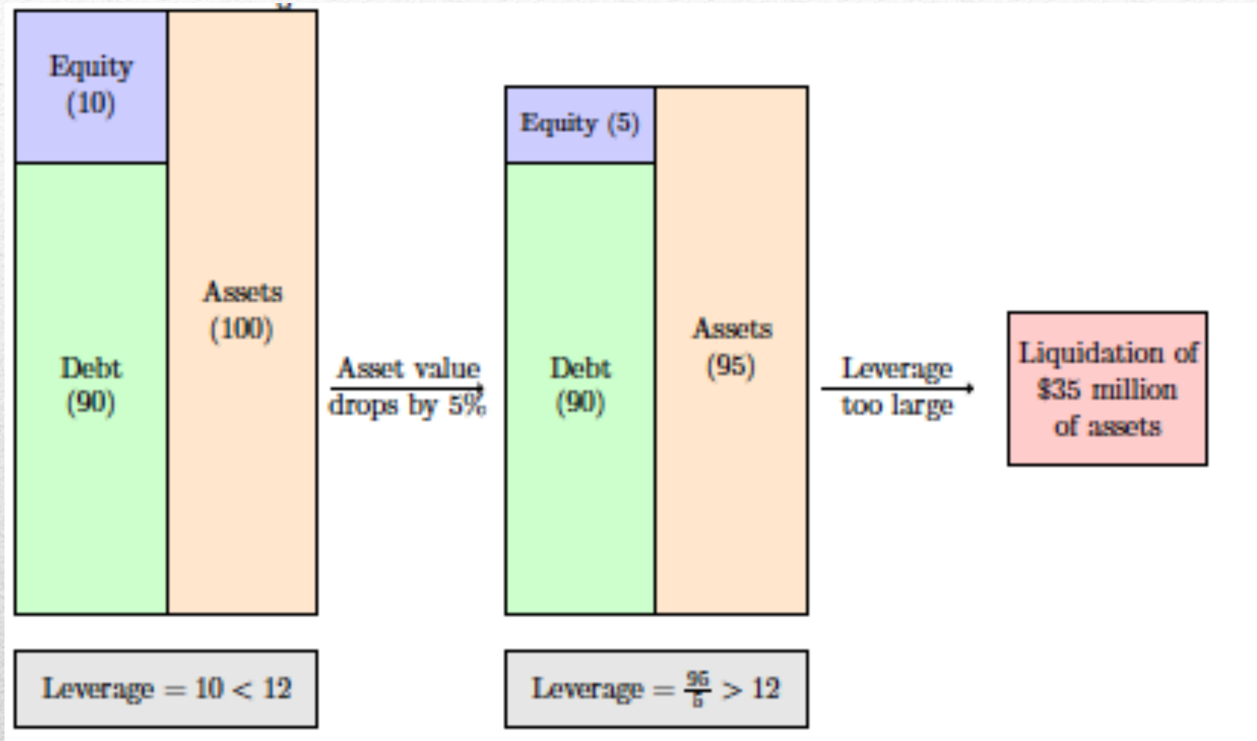
- **synchronized deleveraging / trading activity of many market participants (« run for the exit » scenario)**
- Ex: the role of portfolio insurance in the crash of 1987
- Ex: simultaneous deleveraging of large bank portfolios in Fall 2008
- Both cases lead to a *large systematic component in supply/demand* which,

-far from being random, is triggered by a reaction to recent market moves

-generates in turn an *impact* on market prices (market impact)

- These two elements in conjunction may lead to a feedback loop which can generate an endogenous price instability which may result in trading-induced tail events and large losses, through a non-random amplification mechanism and even in absence of large external shocks/ jumps

A typical example of non-random trades triggered by price moves: asset sales induced by requirements/ thresholds on capital or leverage ratios



Other important examples are given by

-hedging strategies used for options

-other rule based trading strategies: portfolio insurance, trend-following

A simple add-on ingredient: Price Impact

- The ***price impact*** of trading on prices has been long recognized by practitioners, as exemplified in the substantial literature on optimal trade execution and its use by brokers in the design of execution algorithms
- Substantial empirical evidence from intraday study of limit order markets
- Simplest setting: linear price impact model (Kyle 1981): an order of size X

(X>0 for buyer-initiated, X<0 for seller-initiated trade) moves the price according to

$$\frac{\Delta S(t)}{S(t)} = r(t) + I\left(\frac{X}{\lambda}\right)$$

$r(t)$: "fundamental" return,
$I(.)$: price impact function/ price elasticity of demand
λ measures the market depth.
A large trade by a market participant (or many synchronized trades by small ones) may result in a cumulative market impact that substantially modifies price dynamics.

Market impact

Empirical studies (Obizhaeva & Wang 2007, Obizhaeva 2008, ..) on price impact of institutional investors show that

- linear impact is a good approximation for a wide range of trading volumes
- the impact on returns of a volume V is proportional to

$$\frac{V \times \text{daily volatility}}{\text{average daily volume}}$$

  with proportionality constant $\simeq$ 0.1 - 3 in these units

So: a good measure of the size of a position is its ratio to average daily volume.

A stylized model of fire sales

Consider a large institutional investor with positions $_i$ in asset (classes), with prices

$$\frac{\Delta S_i(t)}{S_i(t)} = \sum_j A_{ij}\epsilon_j(t)$$

A_{ij} reflect 'historical'/normal volatilities/correlations . Fund value $V(t) = \sum \alpha_i S_i(t))$
We want to model how fire sales in such a portfolio impact price dynamics, realized volatility and realized correlations of assets liquidated by the fund.

A simple model for endogenous risk

- Consider a leveraged fund holding α_i (≥ 0) units of asset i between dates $t = 0$ and T
- Between t_k and t_{k+1}, price moves due to exogenous economic factors move the value of the fund from $V_k = \sum_{i=1}^{n} \alpha_i S_k^i$ to
$$V_{k+1}^* = \sum_{i=1}^{n} \alpha_i (S_{k+1}^i)^* = V_k + \sum_{i=1}^{n} \alpha_i S_k^i \sqrt{\tau} \xi_{k+1}^i.$$
- As long as the fund is performing well, it holds on to its position in the short term. However, when the fund undergoes a large loss in value, it may start exiting its positions, either due to shortage of liquidity (cash), capital requirements or investors redeeming their positions.

A stylized model of fire sales

- "Market stress": Fund value drops to $V^*(t) < V(t)$
- If loss $V(t) - V^*(t)$ exceeds a threshold (linked to capital requirements, liquidity ratios or performance with respect to a benchmark) fund liquidates positions over horizon T.
- Liquidation strategy: $X(t) = (X_1(t), ..., X_n(t))$ with $X_i(0) = \alpha_i, \; X_i(T) = 0$
- Market impact of liquidation:

$$\frac{\Delta S_i(t)}{S_i(t)} = \sum_j A_{ij}\epsilon_j(t) + \overbrace{I\left(\frac{\Delta X_i(t)}{\lambda_i}\right)}^{\text{Market impact}}$$

$\epsilon_i(t)$: risk factors, A_{ij}= factor loadings, λ_i market depth.

Deleveraging schedule

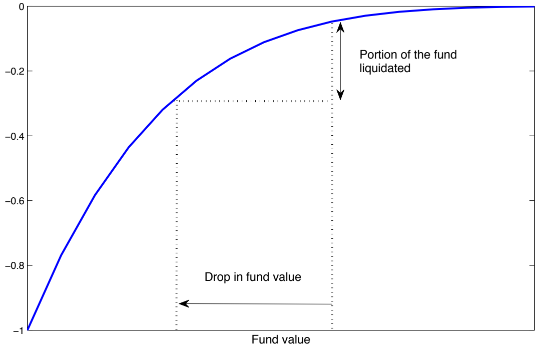


Figure: As fund value drops, manager/investors exit their positions: this is modeled by a 'liquidation schedule' $f(.)$: $X_i(t) = \alpha_i f(V(t)/V(0))$

Price impact of distressed selling

- Distressed selling activity impacts prices: market impact on asset i's return is equal to $\frac{\alpha_i}{\lambda_i}(f(\frac{V^*_{k+1}}{V_0}) - f(\frac{V_k}{V_0}))$
- λ_i represent the *depth* of the market in asset i: a net demand of $\frac{\lambda_i}{100}$ shares for security i moves i's price by one percent.
- This impact is not 'random': it happens precisely when the fund has large losses.
- How does the price impact of distressed selling translate into volatility, correlation and portfolio risk ?

Death spiral: endogenous collapse

Consider a large fund invested in 3 asset classes, whose returns are 'fundamentally uncorrelated': fundmental covariances are assumed ZERO.
The fund quickly reduced by 20% its positions over a one month period, generating a volume which is 10% of market depth during this period.
What is the impact on market correlation/ volatility of the assets?
What is the volatility/correlation experienced by the fund during its liquidation period?

'Fundamentally uncorrelated' assets can correlate during liquidation!

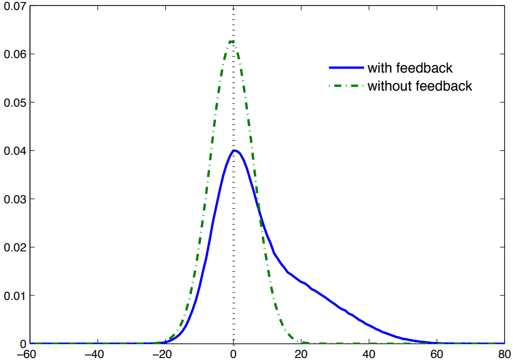
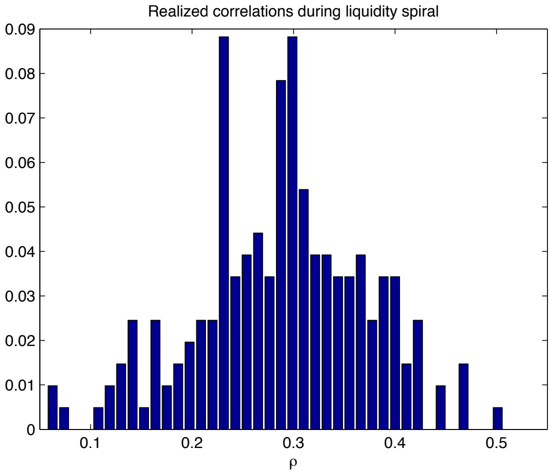


Figure: Distribution of realized correlation between the two securities

Simulation example: 3 uncorrelated asset classes



Fund volatility

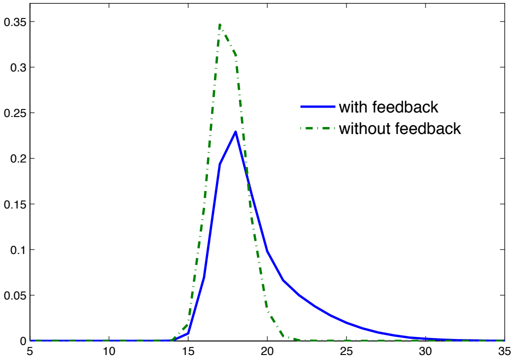


Figure: Distribution of realized volatility of the fund's portfolio with and without feedback effects

Diffusion limit

Theorem

Under the assumption that $f \in C_b^3$ such that $\sup |xf'(x)| < \min \frac{\lambda_i}{\alpha_i}$ and that $\mathbb{E}(|\xi|^4) < \infty$, $S^{(\tau)}_{\lfloor t/\tau \rfloor}$ converges weakly towards a diffusion $P_t = (P_t^1, ...P_t^n)^t$ when τ goes to 0 where

$$\frac{dP_t^i}{P_t^i} = \mu_i(P_t)dt + (\sigma(P_t)dW_t)_i \qquad 1 \leq i \leq n$$

$$\mu_i(P_t) = \frac{\alpha_i}{2\lambda_i} f''(\frac{V_t}{V_0}) \frac{< \pi_t, \Sigma \pi_t >}{V_0^2}; \sigma_{i,j}(P_t) = A_{i,j} + \frac{\alpha_i}{\lambda_i} f'(\frac{V_t}{V_0}) \frac{(A^t \pi_t)_j}{V_0}$$

- $\pi_t = (\alpha_1 P_t^1, ..., \alpha_n P_t^n)^t$ *is the (dollar) allocation of the fund*
- $V_t = \sum \alpha_i P_t^i$ *is the value of the fund*

Realized covariance

Proposition

The realized covariance between securities i and j between 0 and t is equal to $\frac{1}{t}\int_0^t C_s^{i,j}\,\mathrm{d}s$, where $C_s^{i,j}$, the instantaneous covariance between i and j, is given by:

$$C_s^{i,j} = \Sigma_{i,j} + \frac{\alpha_j}{\lambda_j} f^{'}(\frac{V_s}{V_0})\frac{(\Sigma\pi_s)_i}{V_0} + \frac{\alpha_i}{\lambda_i} f^{'}(\frac{V_s}{V_0})\frac{(\Sigma\pi_s)_j}{V_0}$$

$$+\frac{\alpha_i\alpha_j}{\lambda_i\lambda_j}(f^{'})^2(\frac{V_s}{V_0})\frac{<\pi_s,\Sigma\pi_s>}{V_0^2}$$

with $\pi_s = (\alpha_1 P_s^1, ..., \alpha_n P_s^n)^t$.

Realized covariance is *path-dependent*. It is the sum of a fundamental covariance and a *liquidity-dependent excess covariance term*. The impact of the liquidation of a fund is

Excess correlation is exacerbated by illiquidity

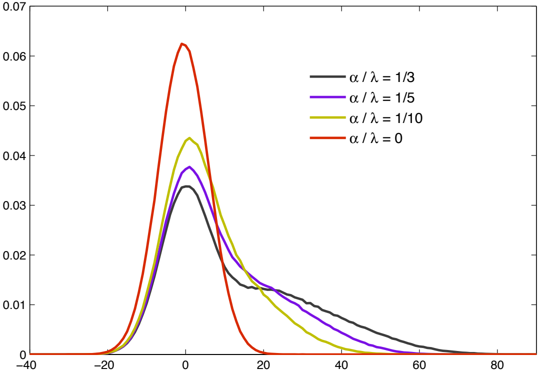


Figure: Distribution of realized correlation for different values of $\frac{\alpha}{\lambda}$

Impact on fund variance

Proposition

The fund's realized variance between 0 and t is equal to $\frac{1}{t}\int_0^t \Gamma_s \, \mathrm{d}s$ where Γ_s, the instantaneous variance of the fund, is given by:

$$\Gamma_s V_s^2 =< \pi_s, \Sigma\pi_s > + \frac{2}{V_0} f'(\frac{V_s}{V_0}) < \pi_s, \Sigma\pi_s >< \Lambda, \pi_s >$$

$$+ \frac{1}{V_0^2} (f'(\frac{V_s}{V_0}))^2 < \pi_s, \Sigma\pi_s > (< \Lambda, \pi_s >)^2$$

- $\pi_t = (\alpha_1 P_t^1, ..., \alpha_n P_t^n)^t$ = (dollar) holdings of the fund,
- $\Lambda = (\frac{\alpha_1}{\lambda_1}, ..., \frac{\alpha_n}{\lambda_n})^t$: positions as a fraction of market depth.

Spillover effects

Consider now a small fund with (dollar) positions $\pi_t^\mu = (\mu_t^1 P_t^1, ..., \mu_t^n P_t^n)^t$.

Proposition

When a large fund with positions π^α liquidates its positions, the small fund experiences a volatility given by

$$< \pi_t^\mu, \Sigma \pi_t^\mu > + \frac{2}{V_0} f'(\frac{V_t}{V_0}) < \pi_t^\mu, \Sigma \pi_t^\alpha >< \Lambda, \pi_t^\mu >$$

$$+ \frac{1}{V_0^2} (f'(\frac{V_t}{V_0}))^2 < \pi_t^\alpha, \Sigma \pi_t^\alpha > (< \Lambda, \pi_t^\mu >)^2$$

where $\Lambda = (\frac{\alpha_1}{\lambda_1}, ..., \frac{\alpha_n}{\lambda_n})^t$ represents the positions of the fund in each market as a fraction of the respective market depth.

Portfolio overlaps as a factor for contagion

Excess volatility generated by price-mediated contagion is driven by the **overlap** between portfolios, weighted by market depth:

$$< \Lambda, \pi_t^{\mu} >= \sum_i \mu_i \alpha_i \frac{P^i}{\lambda_i}$$

In particular, under the 'orthogonality' condition:

$$< \Lambda, \pi_t^{\mu} >= \sum_{1 \le i \le n} \frac{\alpha_i}{\lambda_i} \mu_t^i P_t^i = 0$$

distressed selling in fund α *does not affect* fund μ's variance!
If all assets equally 'liquid' $\rightarrow$ a dollar neutrality condition.
On the contrary, contagion is maximized when allocations have large overlap.

When diversification backfires

These results show that

- Distressed selling can result in exacerbated fund volatility and spikes in correlations when the fund experiences difficulty, reducing the benefit of diversification exactly in scenarios where the fund needs it most.
- This can occur without liquidity drying up (λ constant).
- Predatory trading by short sellers can amplify this but is not needed to generate it.
- In a market where portfolios are all diversified (but in similar ways..), the liquidation of a large diversified portfolio is likely to have a larger spillover effect on other portfolios than in a segmented market: diversification can increase systemic risk if followed as a general rule!

Empirical results

Cont & Wagalath (2012) Fire sales forensics: measuring endogenous risk, to appear in ***Mathematical Finance.***

Main empirical findings:

-The Great Deleveraging of Fall 2008:

The sustained plateau of high equity correlation and volatility can be explained in tems of the deleveraging of portfolios and the corresponding aggregate portfolio may be estimated from observations of price returns

-The Quant Crash of Aug 2007:

'anomalous losses' in long-short market neutral equity funds in Aug 2007 can be quantitatively explained as due to fire-sale liquidation of a large market neutral equity fund (subsequently revealed to be GSAM)

Portfolio estimated from observation of price returns turns out be to "market neutral" and 'orthogonal' to SP500 in terms of liqudity-weighted overlap

The Great Deleveraging: Oct-Dec 2008

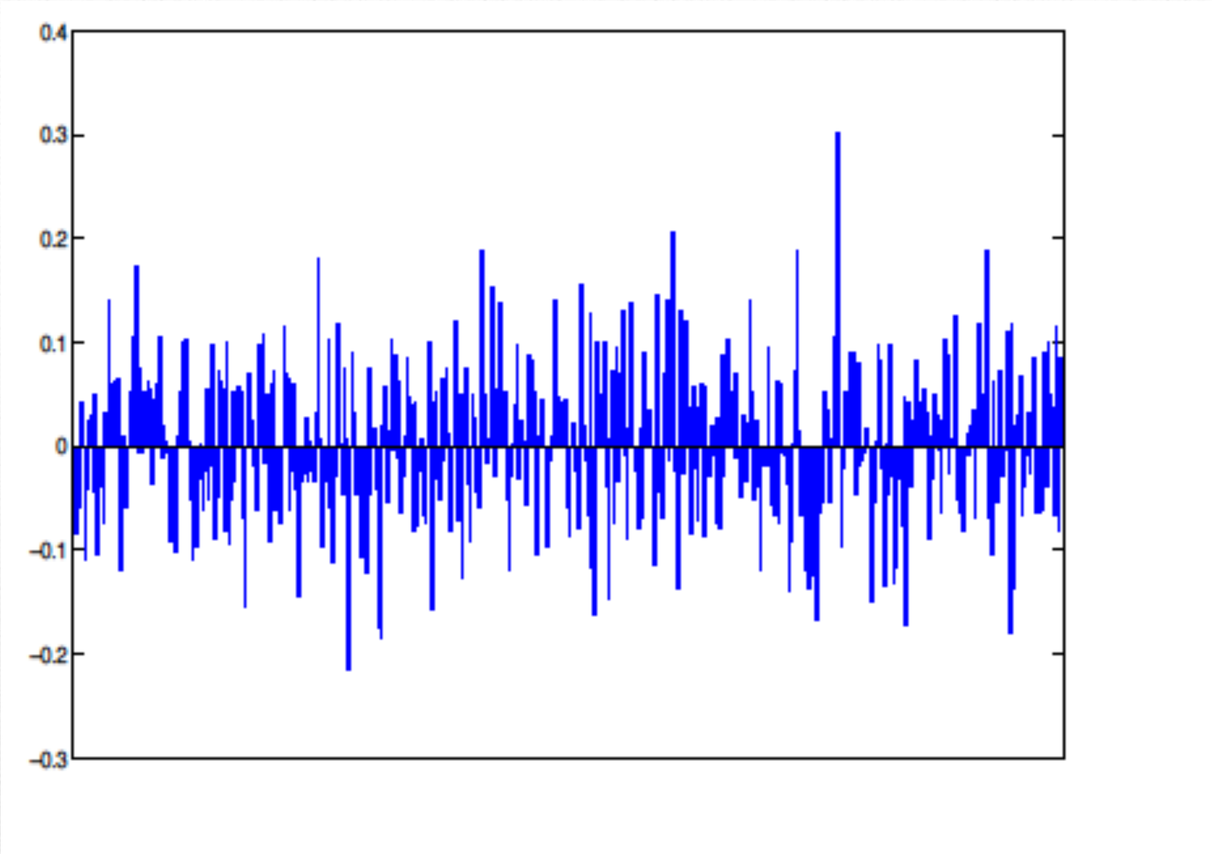
Sector SPDR	Weight
Financials	78%
Consumer Discretionary	0%
Consumer Staples	2.5%
Energy	4%
Health Care	0%
Industrials	0%
Materials	2.5%
Technology	10%
Utilities	3%

Table: Proportions of fire sales between September 15^{th}, 2008 and December, 31^{st}, 2008 associated to the principal eigenvector of M

Strategy crowding: the example of August 2007

- Investors exiting a large market-neutral long short fund lead to high losses/excess volatility for *similar* long short funds.
- However, *index funds*, being *orthogonal* to the reference fund, were unaffected.
- This can happen *without liquidity drying up* ($\neq$ explanation of Khandani and Lo).
- *Crowding* was a major risk factor in this market.

Our framework allows us to quantify *strategy crowding* risk.



Empirical results: fire sales across US stocks reconstituted from Aug 2007 quant crash.

Empirically we find

$$\frac{\hat{\Lambda}.\pi_t^{\hat{\mu}}}{\|\hat{\Lambda}\|\|\pi_t^{\hat{\mu}}\|} = \frac{\sum_{i=1}^{n} \frac{\alpha_i}{\lambda_i}\mu_t^i P_t^i}{\|\hat{\Lambda}\|\|\pi_t^{\hat{\mu}}\|} = 0.0958$$

which corresponds to an angle of 0.47π between the vectors, very close to orthogonality: quantitive explanation for absence of spillover on indices, without having to assume a liquidity dry-up as in Khandani & Lo.

- Market impact of rule based trading strategies or fire sales can be incorporated in the BSM framework
- Even a simple implementation of this idea can go a long way in explaining/modeling the observed

-systematic increase in correlations across asset classes in deleveraging scenarios and the associated spikes in vol/ correlations

-price mediated contagion and associated systemic risk

Analysis of price impact effects in the Black Scholes Merton model can be carried out *analytically* and provide

-tools for monitoring of strategy crowding and concentration risk through monitoring of portfolio overlaps across major financial institutions

- tools for designing stress tests which acount for fire sales and associated feedback effects in a multi-asset setting
- A starting point for quantitative cost-benefit analysis of the impact of microprudential measures on financial stability

Towards 'Next-Generation' risk models?

- Next-generation models should focus on blending statistical/econometric methods with economic mechanisms involving capital flows and supply/demand patterns known to market participants.
- Such models need to incorporate not just risk factors based on returns but "indices" which capture strategy crowding –concentration of market capital in different corners of strategy space.
- Existing work offers proof of concept (Cont & Wagalath 2011, 2012) for quantitative models of this type which allow to compute a portfolio's **exposure to crowding** in a given strategy.

References (click for download)

- R Cont, L Wagalath (2011)
  Running for the Exit: Distressed Selling and Endogenous Correlation in Financial Markets, to appear in **Mathematical Finance**, http://ssrn.com/abstract=1722508.
- R Cont, L Wagalath (2012)
  Fire sale forensics: measuring endogenous risk, http://ssrn.com/abstract=2051013.