# X-CAPM: An Extrapolative Capital Asset Pricing Model†

**Nicholas Barberis*, Robin Greenwood**,**
**Lawrence Jin*, and Andrei Shleifer****

*Yale University and **Harvard University

**Abstract**

Survey evidence suggests that many investors form beliefs about future stock market returns by extrapolating past returns: they expect the stock market to perform well (poorly) in the near future if it performed well (poorly) in the recent past. Such beliefs are hard to reconcile with existing models of the aggregate stock market. We study a consumption-based asset pricing model in which some investors form beliefs about future price changes in the stock market by extrapolating past price changes, while other investors hold fully rational beliefs. We find that the model captures many features of actual prices and returns, but is also consistent with the survey evidence on investor expectations. This suggests that the survey evidence does not need to be seen as an inconvenient obstacle to understanding the stock market; on the contrary, it is consistent with the facts about prices and returns, and may be the key to understanding them.

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## 1. Introduction

Recent theoretical work on the behavior of aggregate stock market prices has tried to account for several empirical regularities. These include the excess volatility puzzle of LeRoy and Porter (1981) and Shiller (1981), the equity premium puzzle of Mehra and Prescott (1985), the low correlation of stock returns and consumption growth noted by Hansen and Singleton (1982, 1983), and, most importantly, the evidence on predictability of stock market returns using the aggregate dividend-price ratio (Campbell and Shiller 1988, Fama and French 1988). Both traditional and behavioral models have tried to account for this evidence.

Yet this research has largely neglected another set of relevant data, namely those on actual investor expectations of stock market returns. As recently summarized by Greenwood and Shleifer (2013) using data from multiple investor surveys, many investors hold extrapolative expectations, believing that stock prices will continue rising after they have previously risen, and falling after they have previous fallen.[1] This evidence is inconsistent with the predictions of many of the models used to account for the other facts about aggregate stock market prices. Indeed, in most traditional models, expected returns are low when stock prices are high: in these models, stock prices are high when investors are less risk averse or perceive less risk. Cochrane (2011) finds the survey evidence uncomfortable, and recommends discarding it.

In this paper, we present a new model of aggregate stock market prices which attempts to both incorporate extrapolative expectations held by a significant subset of investors, and address the evidence that other models have sought to explain. The model includes both rational investors and price extrapolators, and examines security prices when both types are active in the market. Moreover, it is a consumption-based asset pricing model with infinitely lived consumers optimizing their decisions in light of their beliefs and market prices. As such, it can be directly compared to some of the existing

[1] Greenwood and Shleifer (2013) analyze data from six different surveys; some are of individual investors, while others cover institutions. Most of the surveys ask about expectations for the next year's stock market performance, but some also include questions about the longer term. The average investor expectations computed from each of the six surveys are highly correlated with one another and are all extrapolative. Earlier studies of stock market investor expectations include Vissing-Jorgensen (2004), Amromin and Sharpe (2008), and Bacchetta, Mertens, and Wincoop (2009).

research. We suggest that our model can reconcile the evidence on expectations with the evidence on volatility and predictability that has animated recent work in this area.

Why is a new model needed? As Table 1 indicates, traditional models of financial markets have been able to address pieces of the existing evidence, but not the data on expectations. The same holds true for preference-based behavioral finance models, as well as for the first generation belief-based behavioral models that focused on random noise traders without imposing a specific structure on beliefs. Several papers listed in Table 1 have studied extrapolation of fundamentals. However, these models also struggle to match the survey evidence: after good stock market returns, the investors they describe expect higher cash flows, but not higher returns. Finally, a small literature, starting with Cutler, Poterba, and Summers (1990) and DeLong et al. (1990b), focuses on models in which some investors extrapolate prices. Our goal is to write down a more "modern" model that includes infinite horizon investors, some of whom are fully rational, who make optimal consumption decisions given their beliefs, so that the predictions can be directly compared to those of the more traditional models.

Our infinite horizon continuous-time economy contains two assets: a risk-free asset with a fixed return; and a risky asset, the stock market, which is a claim to a stream of dividends and whose price is determined in equilibrium. There are two types of traders. Both types maximize expected lifetime consumption utility. They differ only in their expectations about the future. Traders of the first type, "extrapolators," believe that the expected price change of the stock market is a weighted average of past price changes, where more recent price changes are weighted more heavily. Traders of the second type, "rational traders," are fully rational: they know how the extrapolators form their beliefs and trade accordingly. The model is simple enough to allow for a closed-form solution.

We first use the model to understand how extrapolators and rational traders interact. Suppose that, at time *t*, there is a positive shock to dividends. The stock market goes up in response to this good cash-flow news. However, the extrapolators cause the price jump to be amplified: since their expectations are based on past price changes, the stock price increase generated by the good cash-flow news leads them to forecast a higher *future* price change on the stock market; this, in turn, causes them to push the time $t$ stock price even higher.

More interesting is rational traders' response to this development. We find that the rational traders do *not* aggressively counteract the overvaluation caused by the extrapolators. This is because they reason as follows. The rise in the stock market caused by the good cash-flow news -- and by extrapolators' reaction to it -- means that, in the near future, extrapolators will continue to have bullish expectations for the stock market: after all, their expectations are based on past price changes, which, in our example, are high. As a consequence, they will continue to exhibit strong demand for the stock market in the near term. This means that, even though the stock market is overvalued at time $t$, its returns in the near future will not be particularly low – they will be bolstered by the ongoing demand from extrapolators. Recognizing this, the rational traders do not sharply decrease their demand at time $t$; they only mildly reduce their demand. Put differently, they only partially counteract the overpricing caused by the extrapolators.

Using a combination of formal propositions and numerical simulations, we then examine our model's predictions about prices and returns. We find that these predictions are consistent with several of the key facts about the aggregate market and, in particular, with the basic fact that when prices are high (low) relative to dividends, the stock market subsequently performs poorly (well). When good cash-flow news is released, the stock price in our model jumps up more than it would in an economy made up of rational investors alone: as described above, the price jump caused by the good cash-flow news feeds into extrapolators' expectations, which, in turn, generates an additional price increase. At this point, the stock market is overvalued and prices are high relative to dividends. Since, subsequent to the overvaluation, the stock market performs poorly on average, the level of prices relative to dividends predicts subsequent price changes in our model, just as it does in actual data. The same mechanism also generates excess volatility -- stock market prices are more volatile than can be explained by rational forecasts of future cash flows – as well as negative autocorrelations in price changes at all horizons, capturing the negative autocorrelations we see at longer horizons in actual data.

The model also matches some empirical facts that, thus far, have been taken as evidence for other models. For example, in actual data, surplus consumption, a measure of consumption relative to past consumption, is correlated with the value of the stock market; moreover, it predicts the market's subsequent performance. These facts have been taken as support for habit-based models. However, they also emerge naturally in our framework.

Our numerical analysis allows us to quantify the effects described above. Specifically, we use the survey data studied by Greenwood and Shleifer (2013) and others to parameterize the functional form of extrapolation in our model. For this parameterization, we find, for example, that if 50% of investors are extrapolators while 50% are rational traders, the standard deviation of annual price changes is 30% higher than in an economy consisting of rational traders alone.

There are aspects of the data that our model does not address. For example, even though some of the investors in the economy are price extrapolators, the model does not predict momentum in price changes: the presence of fully rational traders means that price changes are negatively autocorrelated at all lags. Also, there is no mechanism in our model, other than high risk aversion, that can generate a large equity premium. And while the presence of extrapolators reduces the correlation of consumption changes and price changes, this correlation is still much higher in our model than in actual data.

In summary, our analysis suggests that, simply by introducing some extrapolative investors into an otherwise traditional consumption-based model of asset prices, we can make sense not only of some important facts about prices and returns, but also, by construction, of the available evidence on the expectations of real-world investors. This suggests that we do not need to think of the survey evidence as a nuisance, or as an impediment to understanding the facts about prices and returns. On the contrary, the extrapolation that is present in the survey data is perfectly consistent with the facts about prices and returns, and may be the key to understanding them.

In Section 2, we present our model and its solution, and discuss some of the basic insights that emerge from it. In Section 3, we assign values to the model parameters. In Section 4, we show analytically that the model reproduces several key features of stock prices. Our focus here is on quantities defined in terms of differences – price changes, for example; given the structure of the model, these are the natural objects of study. In Section 5, we use simulations to document the model's predictions for ratio-based quantities, such as the price-dividend ratio, that are commonly studied by empiricists. Section 6 concludes. All proofs, as well as some discussion of technical issues, are in the Appendix.

## 2. The Model

In this section, we propose a heterogeneous-agent, consumption-based model in which some investors extrapolate past price changes when making forecasts about future price changes. Constructing such a model presents significant challenges, both because of the heterogeneity across agents, but also because it is the change in *price*, an endogenous quantity, that is being extrapolated. By contrast, constructing a model based on extrapolation of exogenous fundamentals is somewhat simpler. To prevent our model from becoming too complex to interpret, we make some simplifying assumptions – about the dividend process (a random walk in levels), about investor preferences (exponential utility), and about the risk-free rate (an exogenous constant). We expect the intuitions of the model to carry over to more complex formulations.

We consider an economy with two assets: a risk-free asset in perfectly elastic supply with a constant interest rate $r$; and a risky asset, which we think of as the aggregate stock market, and which has a fixed per-capita supply of $Q$.[2] The risky asset is a claim to a continuous dividend stream whose level per unit time evolves as an arithmetic Brownian motion

$$dD_t = g_D dt + \sigma_D d\omega, \tag{1}$$

where $g_D$ and $\sigma_D$ are the expected value and standard deviation of dividend changes, respectively, and where $\omega$ is a standard one-dimensional Wiener process. Both $g_D$ and $\sigma_D$ are constant in our model. We denote the value of the stock market at time $t$ by $P_t$.

There are two types of infinitely-lived traders in the economy: "extrapolators" and "rational traders." Both types maximize expected lifetime consumption utility. The only difference between them is that one type has correct beliefs about the expected return of the risky asset, while the other type does not.

[2] We discuss the constant interest rate assumption at the end of Section 2.

The modeling of extrapolators is motivated by the survey evidence analyzed by Vissing-Jorgensen (2004), Amromin and Sharpe (2008), Bacchetta, Mertens, and Wincoop (2009), and Greenwood and Shleifer (2013). These investors form beliefs about the future price change of the stock market by extrapolating the market's past price changes. To formalize this, we introduce a measure of "sentiment," defined as:

$$S_t = \beta \int_{-\infty}^{t} e^{-\beta(t-s)} dP_{s-dt}, \qquad \beta > 0, \tag{2}$$

where $s$ is the running variable for the integral. $S_t$ is simply a weighted average of past price changes on the stock market where the weights decrease exponentially the further back we go into the past. The definition of $S_t$ includes even the most recent price change, $dP_{t-dt} \equiv P_t - P_{t-dt}$. The parameter $\beta$ plays an important role in our model. When it is high, sentiment is determined primarily by the most recent price changes; when it is low, even price changes in the distant past have a significant effect on current sentiment. In Section 3, we use survey data to estimate $\beta$.

We assume that extrapolators' expected price change, per unit time, in the value of the stock market, is

$$g^e_{P,t} \equiv \mathbb{E}^e_t[dP_t]/dt = \lambda_0 + \lambda_1 S_t, \tag{3}$$

where the superscript "$e$" is an abbreviation for "extrapolator," and where, for now, the only requirement we impose on the constant parameters $\lambda_0$ and $\lambda_1$ is that $\lambda_1 > 0$. Taken together, equations (2) and (3) capture the essence of the survey results in Greenwood and Shleifer (2013): after good stock market returns, extrapolators expect the stock market to continue to perform well; and after poor stock market returns, they expect continued weak performance. While we leave $\lambda_0$ and $\lambda_1$ unspecified for now, natural values are $\lambda_0 = 0$ and $\lambda_1 = 1$, and these are indeed the values that we use later.

We do not take a strong stand on the underlying source of the extrapolative expectations in (3). However, one possible source is the representativeness heuristic, or the closely-related belief in the law of small numbers (Barberis, Shleifer, and Vishny 1998; Rabin 2002). For example, under the law of small numbers, people think that even short samples will resemble the parent population from which they are drawn. As a result, when they see good recent returns in the stock market, they infer that the stock market must currently have a high average return and that it will therefore continue to do well.[3]

The second type of investor, the rational trader, has correct beliefs about the evolution of future stock prices. By correctly conjecturing the equilibrium price process, the rational investors take full account of extrapolators' endogenous responses to price movements at all future times.

There is a continuum of both rational traders and extrapolators in the economy. Each investor, whether a rational trader or an extrapolator, takes the risky asset price as given when making his trading decision, and has CARA preferences with absolute risk aversion $\gamma$ and time discount factor $\delta$.[4] At time 0, each extrapolator maximizes

$$\mathbb{E}_0^e\left[-\int_0^\infty \frac{e^{-\delta t-\gamma C_t^e}}{\gamma}dt\right] \tag{4}$$

subject to his budget constraint

$$\begin{aligned} dW_t^e \equiv W_{t+dt}^e - W_t^e &= (W_t^e - C_t^e dt - N_t^e P_t)(1+rdt) + N_t^e D_t dt + N_t^e P_{t+dt} - W_t^e \\ &= rW_t^e dt - C_t^e dt - rN_t^e P_t dt + N_t^e dP_t + N_t^e D_t dt, \end{aligned} \tag{5}$$

where $N_t^e$ is the per-capita number of shares he invests in the risky asset at time $t$. Similarly, at time 0, each rational trader maximizes

$$\mathbb{E}_0^r\left[-\int_0^\infty \frac{e^{-\delta t-\gamma C_t^r}}{\gamma}dt\right] \tag{6}$$

subject to his budget constraint

$$\begin{aligned} dW_t^r \equiv W_{t+dt}^r - W_t^r &= (W_t^r - C_t^r dt - N_t^r P_t)(1+rdt) + N_t^r D_t dt + N_t^r P_{t+dt} - W_t^r \\ &= rW_t^r dt - C_t^r dt - rN_t^r P_t dt + N_t^r dP_t + N_t^r D_t dt, \end{aligned} \tag{7}$$

where $N_t^r$ is the per-capita number of shares he invests in the risky asset at time $t$, and where the superscript "$r$" is an abbreviation for "rational trader." Since rational traders correctly conjecture the price process $P_t$, their expectation is consistent with that of an outside econometrician.

[3] We think of the effect that we are modeling as distinct from the experience effect reported by Malmendier and Nagel (2011). Some evidence that these are indeed distinct effects is that, as we show later, the investor expectations documented in surveys depend only on recent past returns, not the distant past returns that play a role in Malmendier and Nagel's (2011) results.

[4] The model remains analytically tractable even if the two types of investors have different values of $\gamma$ or $\delta$.

We assume that rational traders make up a fraction μ, and extrapolators 1 − μ, of the total investor population. The market clearing condition that must hold at each time is:

$$\mu N_t^r + (1-\mu) N_t^e = Q , \tag{8}$$

where $Q$ is the per-capita supply of the risky asset.

We assume that both extrapolators and rational traders observe $D_t$ and $P_t$ on a continuous basis. Moreover, they know the values of μ and $Q$; and traders of one type understand how traders of the other type form beliefs about the future.[5]

Using the stochastic dynamic programming approach developed in Merton (1971), we obtain the following proposition.

**Proposition 1 (Model solution).** In the heterogeneous-agent model described above, the equilibrium price of the risky asset is

$$P_t = A + BS_t + \frac{D_t}{r} . \tag{9}$$

The price of the risky asset $P_t$ and the sentiment variable $S_t$ evolve according to

$$dP_t = \left( -\frac{\beta B}{1-\beta B} S_t + \frac{g_D}{(1-\beta B) r} \right) dt + \frac{\sigma_D}{(1-\beta B) r} d\omega, \tag{10}$$

$$dS_t = -\frac{\beta}{1-\beta B} \left( S_t - \frac{g_D}{r} \right) dt + \frac{\beta \sigma_D}{(1-\beta B) r} d\omega. \tag{11}$$

At time $t$, the value functions for the extrapolators and the rational traders are

$$J^e(W_t^e, S_t, t) \equiv \max_{\{C_s^e, N_s^e\}_{s \ge t}} \mathbb{E}_t^e \left[ -\int_t^\infty \frac{e^{-\delta s - \gamma C_s^e}}{\gamma} ds \right] = -\exp\left[ -\delta t - r\gamma W_t^e + a^e S_t^2 + b^e S_t + c^e \right],$$

$$J^r(W_t^r, S_t, t) \equiv \max_{\{C_s^r, N_s^r\}_{s \ge t}} \mathbb{E}_t^r \left[ -\int_t^\infty \frac{e^{-\delta s - \gamma C_s^r}}{\gamma} ds \right] = -\exp\left[ -\delta t - r\gamma W_t^r + a^r S_t^2 + b^r S_t + c^r \right]. \tag{12}$$

The optimal per-capita share demands for the risky asset from the extrapolators and from the rational traders are

[5] As in any framework with less than fully rational traders, the extrapolators could, in principle, come to learn that their beliefs about the future are inaccurate. We do not study this learning process; rather, we study the behavior of asset prices when extrapolators are unaware of the bias in their beliefs.

$$N_t^e = \eta_0^e + \eta_1^e S_t, \qquad N_t^r = \frac{Q}{\mu} - \frac{1-\mu}{\mu} N_t^e, \tag{13}$$

and the optimal consumption flows of the two types are

$$\begin{aligned} C_t^e &= rW_t^e - \frac{1}{\gamma}\left(a^e S_t^2 + b^e S_t + c^e\right) - \frac{\log(r\gamma)}{\gamma}, \\ C_t^r &= rW_t^r - \frac{1}{\gamma}\left(a^r S_t^2 + b^r S_t + c^r\right) - \frac{\log(r\gamma)}{\gamma}, \end{aligned} \tag{14}$$

where the optimal wealth levels, $W_t^e$ and $W_t^r$, evolve as in (5) and (7), respectively. The coefficients $A$, $B$, $a^e$, $b^e$, $c^e$, $a^r$, $b^r$, $c^r$, $\eta_0^e$ and $\eta_1^e$ are determined through a system of simultaneous equations. ■

To understand the role that extrapolators play in our model, we compare the model's predictions to those of a benchmark "rational" economy, in other words, an economy where all traders are of the fully rational type, so that $\mu = 1$.[6]

**Corollary 1 (Rational benchmark).** If all traders in the economy are rational ($\mu = 1$), the equilibrium price of the risky asset satisfies

$$P_t = -\frac{\gamma\sigma_D^2}{r^2} Q + \frac{g_D}{r^2} + \frac{D_t}{r}. \tag{15}$$

It therefore evolves according to

$$dP_t = \frac{g_D}{r} dt + \frac{\sigma_D}{r} d\omega. \tag{16}$$

The value function for the rational traders is

$$J^r(W_t^r, t) = -\frac{1}{r\gamma} \exp\left[-\delta t - r\gamma W_t^r + \frac{1}{r}\left(r - \delta - \frac{1}{2}\gamma^2\sigma_D^2 Q^2\right)\right]. \tag{17}$$

The optimal consumption flow is

$$C_t = rW_t^r - \frac{r-\delta}{r\gamma} + \frac{\gamma\sigma_D^2 Q^2}{2r}, \tag{18}$$

[6] Another way of reducing our model to a fully rational economy is to set $\lambda_0$ and $\lambda_1$, the parameters in (3), to $g_D/r$ and 0, respectively. In this case, both the rational traders and the extrapolators have the same, correct beliefs about the expected per unit time price change of the risky asset.

where the optimal wealth level, $W_t^r$, evolves as

$$dW_t^r = \left( \frac{r-\delta}{r\gamma} + \frac{\gamma\sigma_D^2 Q^2}{2r} \right) dt + \frac{Q\sigma_D}{r} d\omega. \qquad (19)$$

■

### 2.1. Discussion

In Sections 4 and 5, we discuss the model's implications in detail. However, the closed-form solution in Proposition 1 already makes apparent the basic properties of our framework.

From equation (11), we see that the sentiment level $S_t$ follows a mean-reverting process with long-run mean $g_D/r$. Equation (9) shows that, when sentiment is high, stock market prices are pushed up -- the coefficient $B$ is positive for all values of the basic parameters that we have considered. Intuitively, if the sentiment level is high, indicating that past price changes have been high, extrapolators expect the stock market to perform well in the future and therefore push its current price higher. While extrapolators' beliefs are, by definition, extrapolative, rational traders' beliefs are contrarian: their beliefs are based on the true price process (10) whose drift depends negatively on $S_t$.

Comparing equations (10) and (16), we also see that, as noted in the Introduction, the presence of extrapolators amplifies the volatility of price changes – specifically, by a factor of $1/(1-\beta B) > 1$. And while in an economy made up of rational investors alone, price changes are not predictable -- see equation (16) -- equation (10) shows that they *are* predictable in the presence of extrapolators. Specifically, if the stock market has recently experienced good returns, so that the sentiment variable $S_t$ has a high value, the subsequent stock market return is low on average: the coefficient on $S_t$ in equation (10) is negative. In short, high valuations in the stock market are followed by low returns, and low valuations are followed by high returns. This anticipates some of our results on stock market predictability in Sections 4 and 5.

In the analysis we conduct later, we find that, for reasonable values of the basic model parameters, the derived parameter $\eta_1^e$ in equation (13) is positive. In other words, after a period of good stock market performance, one that generates a high level of sentiment $S_t$, extrapolators increase the number of shares of the stock market that they hold. With a fixed supply of these shares, this automatically means that the share demand of rational traders varies negatively with the sentiment variable $S_t$: rational traders absorb the shocks in extrapolators' demand.[7]

We also find that, for reasonable values of the basic model parameters, the derived parameters $a^e$, $a^r$, $b^e$, and $b^r$ in equation (14) typically satisfy $a^e < 0$, $a^r < 0$, and $b^e < b^r$. The fact that $b^e < b^r$ indicates that extrapolators increase their consumption more than rational traders do after strong stock market returns. After strong returns, extrapolators expect the stock market to continue to rise; an income effect therefore leads them to consume more. Rational traders, recognizing that extrapolators' ebullience has caused the stock market to become overvalued, correctly perceive low future returns; they therefore do not raise their consumption as much.

Equations (9) and (11) indicate that the mispricing created by extrapolators is eventually corrected, and more quickly so for high values of β. To understand this – in other words, to understand why, in our framework, bubbles eventually burst – recall that an overpricing occurs when good cash-flow news generates a price increase that then feeds into extrapolators' beliefs, leading them to push prices still higher. The form of extrapolation in equation (2), however, means that as time passes, the price increase caused by the good cash-flow news plays a smaller and smaller role in determining extrapolators' beliefs. As a result, these investors become less bullish over time, and the bubble deflates. This happens more rapidly when β is high because, in this case, extrapolators quickly "forget" all but the most recent price changes.

Since extrapolators have incorrect beliefs about future price changes, it is likely that, in the long run, their wealth will decline relative to that of rational traders. However, the price process in (10) is unaffected by the relative wealth of the two trader types: under exponential utility, the share demand of each type, and hence also prices, are independent

[7] Since the supply of the risky asset is fixed and there are only two groups of traders, the share demand of rational traders must vary negatively with the sentiment level. In a stripped-down version of our framework, we have also analyzed what happens when there are three types of traders: the two types we examine here, but also a group of partially-rational investors who buy (sell) the risky asset when its price is low (high) relative to fundamentals. We find that, in this economy, the share demand of the fully rational traders is *positively* related to the sentiment level. In other words, consistent with the findings of Brunnermeier and Nagel (2004), these traders "ride the bubble" generated by extrapolators.

of wealth. In other words, the exponential utility assumption allows us to abstract from the effect of "survival" on prices, and to focus on what happens when both types of trader play a role in setting prices.

At the heart of our model is an amplification mechanism: if good cash-flow news pushes the stock market up, this price increase feeds into extrapolators' expectations about future price changes, which then leads them to push current prices up even higher. However, this then further increases extrapolators' expectations about future price changes, leading them to push the current price still higher, and so on. Given this infinite feedback loop, it is important to ask whether the heterogeneous agent equilibrium we described above exists. The following corollary provides a condition for existence of equilibrium.

**Corollary 2 (Existence of equilibrium).** The equilibrium described in Proposition 1 exists if and only if $1 - \beta|B| > 0$. When $\mu = 0$ (all investors are extrapolators), the equilibrium described in Proposition 1 exists if and only if

$$\lambda_1 \beta < r, \tag{20}$$

assuming that $\lambda_1 < 2$. ■

Corollary 2 shows that, when all investors in the economy are extrapolators, there may be no equilibrium even for reasonable parameter values; loosely put, the feedback loop described above may fail to converge. For example, if $\lambda_1 = 1$ and $\beta = 0.5$, there is no equilibrium in the case of $\mu = 0$ if the interest rate is less than 50%. However, if even a small fraction of investors are rational traders, the equilibrium is very likely to exist. Indeed, for $\mu \geq 0.05$, we have found an equilibrium for all the parameter values we have tried.

One of the assumptions of our model is that the risk-free rate is constant. To evaluate this assumption, we compute the aggregate demand for the risk-free asset across the two types of trader. We find that this aggregate demand is very stable over time and, in particular, that it is uncorrelated with the sentiment level $S_t$. This is because the demand for the risk-free asset from one type of trader is largely offset by the demand from the other type: when sentiment $S_t$ is high, rational traders increase their demand for the risk-free asset (and move out of the stock market), while extrapolators reduce their demand for the risk-free asset (and move into the stock market). When sentiment is low, the reverse occurs. This suggests that, even if the risk-free rate were endogenously determined, it would not fluctuate wildly, nor would its fluctuations significantly attenuate the effects we describe here.

## 3. Parameter Values

In this section, we assign benchmark values to the basic model parameters. We use these values in the numerical simulations of Section 5. However, we also use them in Section 4. While the core of that section consists of analytical propositions, we can get more out of the propositions by evaluating the expressions they contain for reasonable parameter values.

For easy reference, we list the model parameters in Table 2. The asset-level parameters are the risk-free rate $r$; the initial level of the dividend $D_0$; the mean $g_D$ and standard deviation $\sigma_D$ of dividend changes; and the risky asset supply $Q$. The investor-level parameters are the initial wealth levels for the two types of agents, $W_0^e$ and $W_0^r$; absolute risk aversion $\gamma$ and the time discount rate $\delta$; the proportion $\mu$ of rational traders in the economy; $\beta$, which governs the relative weighting of recent and distant past price changes in the definition of the sentiment variable; and, finally, $\lambda_0$ and $\lambda_1$, which govern the relationship between the sentiment variable and extrapolators' beliefs.[8]

We set $r = 2.5\%$, consistent with the low historical risk-free rate. We set the initial dividend level $D_0$ to 10, and given this, we choose $\sigma_D = 0.25$; in other words, we choose a volatility of dividend changes small enough to ensure that we only rarely encounter negative dividends and prices in the simulations we conduct later. We set $g_D = 0.05$ to match, approximately, the empirical ratio of $g_D/\sigma_D$ in the data. Finally, we set the risky asset supply to $Q = 5$.

[8] The variables $D_0$, $W_0^e$, and $W_0^r$ are listed at the bottom of Table 2 because, for much of the analysis, we do not need to specify their values; their values are needed only for the simulations in Section 5.

We now turn to the investor-level parameters. We set the initial wealth levels to $W_0^e = W_0^r = 5000$. Given this, we set risk aversion $\gamma$ equal to 0.1 so that relative risk aversion, computed from the value function as $RRA = -\frac{WJ_{WW}}{J_W} = r\gamma W$, is 12.5 at the initial wealth levels. We choose a low time discount rate of $\delta = 1.5\%$, consistent with most other asset pricing frameworks; and, as noted earlier, we set $\lambda_0$ and $\lambda_1$ in equation (3) to $\lambda_0 = 0$ and $\lambda_1 = 1$. These values imply that, while extrapolators overestimate the subsequent price change of the stock market after good past price changes and underestimate it after poor past price changes, the errors in their forecasts of future price changes over any finite horizon will, in the long run, average out to zero.

This leaves just two parameters: $\mu$, the fraction of rational investors in the economy; and $\beta$, the weighting parameter in equation (2). Because these two parameters play an important role in our framework, we consider a range of values for each one in the numerical analysis that follows. Specifically, we consider four different values of $\mu$: 1 (an economy where all investors are fully rational), 0.75, 0.5, and 0.25. We do not consider the case of $\mu = 0$ because Corollary 2 indicates that, when all investors are extrapolators, the equilibrium does not exist for reasonable values of $\beta$ and $\lambda_1$. For $\beta$, we consider three possible values: 0.05, 0.5, and 0.75. Recall that, for higher values of $\beta$, extrapolators weigh recent returns more heavily when forecasting future returns. For example, when $\beta = 0.05$, the realized annual price change on the stock market starting four years ago is weighted 86% as much as the most recent annual price change; when $\beta = 0.5$, it is weighted 22% as much; and when $\beta = 0.75$, it is weighted only 11% as much.

While we consider four different values of $\mu$, we focus on the lowest of the four values, namely 0.25. The fact that the average investor in the surveys studied by Greenwood and Shleifer (2013) – surveys that include both sophisticated and less sophisticated respondents – exhibits extrapolative expectations suggests that many if not most investors in actual financial markets may be extrapolators.

We can also use the survey evidence to get a better sense of a reasonable value of $\beta$. The idea is simple: if investors' expectations of future stock market returns depend primarily on very recent market returns – on returns over the past year, or past two years

– then β is high. Conversely, if investors' expectations of future returns depend to a significant extent on returns in the distant past, then this points to a lower value for β. In the Appendix, we describe in detail how we use the survey data to estimate β. The estimation makes use of Proposition 2 below, and specifically, equation (22), which describes the change in the value of the stock market expected by extrapolators over any future horizon.

**Proposition 2 (Price change expectations of rational traders and extrapolators)**. Conditional on an initial sentiment level $S_0 = s$, rational traders' expectation of the price change in the stock market over the finite time horizon $(0, t_1)$ is:

$$\mathbb{E}_0^r\left[P_{t_1} - P_0 \middle| S_0 = s\right] = B\left(1 - \mathrm{e}^{-kt_1}\right)\left(\frac{g_D}{r} - s\right) + \frac{g_D t_1}{r}, \tag{21}$$

while extrapolators' expectation of the same quantity is:

$$\mathbb{E}_0^e\left[P_{t_1} - P_0 \middle| S_0 = s\right] = \left(\lambda_0 + \lambda_1 s\right)t_1 + \left(\beta\lambda_0 - ms\right)\frac{\lambda_1(mt_1 + e^{-mt_1} - 1)}{m^2}, \tag{22}$$

where $k = \dfrac{\beta}{1-\beta B}$ and $m = \beta(1 - \lambda_1)$. When $\lambda_0 = 0$ and $\lambda_1 = 1$, (22) reduces to

$$\mathbb{E}_0^e\left[P_{t_1} - P_0 \middle| S_0 = s\right] = st_1 \,. \tag{23}$$

■

Equations (21) and (22) confirm that the expectations of extrapolators load positively on the sentiment level, while the expectations of rational traders load negatively.

When we use the procedure described in the Appendix to estimate β from the survey data, we obtain a value of approximately 0.5. Consequently, while we present results for three different values of β, we pay most attention to the case of β = 0.5.[9]

For a given set of values of the basic parameters in Table 2, we use the procedure outlined in the Appendix to compute the "derived" parameters: $\eta_0^e$ and $\eta_1^e$, which

[9] When we estimate β from the survey data, we assume, for simplicity, that all the surveyed investors are extrapolators. If we instead allowed some of them to fall into the category of rational investors – in other words, if we instead tried to estimate both β *and* μ from the survey data -- we might obtain a different value of β. However, we would not expect the estimate of β to change very much.

determine extrapolators', and hence rational traders', optimal share demand (see equation (13)); $a^e$, $b^e$, $c^e$, $a^r$, $b^r$ and $c^r$, which determine investors' optimal consumption policies (see equation (14)); *A* and *B*, which specify how the price level *P* depends on the level of the sentiment *S* and the level of the dividend *D* (see equation (9)); and finally $\sigma_P$, the volatility of price changes in the stock market (see equation (10)). For example, if $\mu$ = 0.25, $\beta$ = 0.5, and the other basic parameters have the values shown in Table 2, the values of the derived parameters are:

$$\begin{gathered}\eta_0^e = 1.54, \eta_1^e = 0.51, \sigma_P = 19.75, A = -117.04, B = 0.99, \\ a^e = -1.22\times 10^{-3}, a^r = -1.28\times 10^{-3}, b^e = -7.31\times 10^{-3}, b^r = 0.042, \\ c^e = 1.63, c^r = -3.47.\end{gathered} \tag{24}$$

## 4. Empirical Implications

In this section, we present a detailed analysis of the empirical predictions of the model. Under the assumptions that the dividend level follows an arithmetic Brownian motion and that investors have exponential utility, it is more natural, in our analysis, to work with quantities defined in terms of *differences* rather than ratios – for example, to work with price changes $P_t - P_0$ rather than returns; and with the "price-dividend difference" $P - D/r$ rather than the price-dividend ratio. For example, Corollary 1 shows that, in the benchmark rational economy, it is $P - D/r$ that is constant over time, not $P/D$. In this section, then, we study the predictions of price extrapolation for these difference-based quantities. In Section 5, we also consider the ratio-based quantities.

We study the implications of the model for the difference-based quantities with the help of formal propositions. For example, if we are interested in the autocorrelation of price changes, we first compute this autocorrelation analytically, and then report its value for the parameter values in Table 2. For two crucial parameters, $\mu$ and $\beta$, we consider a range of possible values. Recall that $\mu$ is the fraction of rational traders in the overall investor population, while $\beta$ controls the relative weighting of near-past and distant-past price changes in extrapolators' forecast of future price changes.

We are interested in how the presence of extrapolators in the economy affects the behavior of the stock market. To understand this more clearly, in the results that we present below, we always include, as a benchmark, the case of $\mu = 1$, in other words, the case where the economy consists entirely of rational traders.

### 4.1. Predictive power of $D/r - P$ for future price changes

A basic fact about the stock market is that the dividend-price ratio of the stock market predicts subsequent returns, but not subsequent cash flows. It is helpful to express this fact in the more structured way suggested by Cochrane (2011), among others. If we run three univariate regressions – a regression of future returns on the current dividend-price ratio; a regression of future dividend growth on the current dividend-price ratio; and a regression of the future dividend-price ratio on the current dividend-price ratio – then, as a matter of accounting, the three regression coefficients must (approximately) sum to one. Empirically – and this is the basic fact that needs to be explained – the three regression coefficients are roughly 1, 0, and 0, respectively, at long horizons. In other words, at long horizons, the dividend-price ratio forecasts future returns – not future cash flows, and not its own future value.

We can express this point in a way that fits more naturally with our model, using quantities defined as differences, rather than ratios. Given the accounting identity

$$\frac{D_0}{r} - P_0 = \left(P_t - P_0\right) - \left(\frac{D_t}{r} - \frac{D_0}{r}\right) + \left(\frac{D_t}{r} - P_t\right), \tag{25}$$

it is immediate that if we run three regressions – of the future price change, the (negative) future dividend change, and the future dividend-price difference, on the current dividend-price difference – the three coefficients we obtain must sum to one, at any horizon. To match the empirical facts, our model needs to predict a regression coefficient in the first regression that is close to one, particularly at long horizons. The next proposition shows that this is exactly the case.

**Proposition 3 (The predictive power of $D/r - P$).** Consider a regression of the price change in the stock market over some time horizon $(0, t_1)$ on the level of $D/r - P$ at the start of the horizon. In population, the coefficient on the independent variable in the regression is[10]

$$\beta_{DP}(t_1) \equiv \frac{\operatorname{cov}\left(D_0/r - P_0, P_{t_1} - P_0\right)}{\operatorname{var}\left(D_0/r - P_0\right)} = 1 - e^{-kt_1}, \tag{26}$$

where $k = \dfrac{\beta}{1-\beta B}$. ■

Table 3 reports the value of the regression coefficient in Proposition 3 for various values of μ and β, and for five different time horizons: a quarter, a year, two years, three years, and four years. The table shows that, consistent with the empirical facts, $D/r - P$ does indeed predict future price changes with a positive sign. Moreover, the regression coefficient, $\beta_{DP}(t_1)$, is increasing in the length of the time horizon $t_1$. For long time horizons $t_1$, the regression coefficient converges to 1, as it does empirically in a regression of long-horizon returns on the dividend yield. In the benchmark rational economy, the quantity $D/r - P$ is constant; the regression coefficient we compute in Proposition 3 is therefore undefined.

The intuition for why $D/r - P$ predicts subsequent price changes is straightforward. A sequence of good cash flow news pushes up stock prices, which then raises extrapolators' expectations about the future price change of the stock market and causes them to push stock prices even higher. At this point, the stock market is overvalued; the value of $D/r - P$ is therefore low. Precisely because the stock market is overvalued, the subsequent price change is low, on average. The quantity $D/r - P$ therefore forecasts price changes with a positive sign.

The table shows that, for a fixed horizon, the predictive power of $D/r - P$ is stronger for low μ: since the predictability of price changes is driven by the presence of extrapolators, it is natural that this predictability is stronger when there are more extrapolators in the economy. The predictive power of $D/r - P$ is also weaker for low β: when β is low, extrapolators' beliefs are more persistent; as a result, it takes longer for an

[10] The expectations that we compute in the propositions in Section 4 are taken over the steady-state distribution of the sentiment level $S$. Ergodicity of the stochastic process $S_t$ guarantees that time-series averages will converge to our analytical results in the long run.

overvaluation to correct, reducing the predictive power of $D/r - P$ for price changes at any fixed horizon.

### 4.2. Autocorrelations of *P* – *D*/*r*

In the data, price-dividend ratios are highly autocorrelated at short lags, and we would like to know if our model can capture this. The natural analog of the price-dividend ratio in our model is the difference-based quantity $P - D/r$. We therefore examine the autocorrelation structure of this quantity.

In our discussion of the accounting identity in equation (25), we noted that, if we run regressions of the future change in the stock price, the future change in dividends, and the future dividend-price difference on the current dividend-price difference, then the three regression coefficients we obtain must sum to one. Since dividends follow a random walk in our model, we know that the coefficient in the second regression is zero. We also know, from Proposition 3, that the coefficient in the first regression is $1-e^{-kt_1}$. The coefficient in the third regression, which is also the autocorrelation of the dividend-price difference $D/r - P$, must therefore equal $e^{-kt_1}$. The next proposition confirms this.

**Proposition 4 (Autocorrelations of *P* – *D*/*r*)**. In population, the autocorrelation of $P - D/r$ at a time lag of $t_1$ is

$$\rho_{PD}(t_1) \equiv \text{corr}\left(P_0 - \frac{D_0}{r}, P_{t_1} - \frac{D_{t_1}}{r}\right) = e^{-kt_1}, \tag{27}$$

and $k = \dfrac{\beta}{1-\beta B}$. ■

In Table 4, we compute the autocorrelations in Proposition 4 for several pairs of values of μ and β, and for lags of one quarter, one year, two years, three years, and four years. The table shows that, in our model, and consistent with the empirical facts, the price-dividend difference is highly persistent at short horizons, while at long horizons, the autocorrelation drops to zero: at long horizons, the price-dividend difference forecasts price changes, not its own future value. The table shows that the autocorrelations are higher for low values of β: when β is low, extrapolators' beliefs are very persistent, which, in turn, imparts persistence to the price-dividend difference.

### 4.3. Volatility of price changes and of *P – D/r*

Empirically observed stock market returns and price-dividend ratios are thought to exhibit "excess volatility," in other words, to be more volatile than can be explained purely by fluctuations in rational expectations about future cash flows. We now show that, in our model, price changes and the price-dividend difference – the natural analogs of returns and of the price-dividend ratio in our framework – also exhibit such excess volatility. In particular, they are more volatile than in the benchmark rational economy described in Corollary 1, an economy where prices are set only by rational forecasts of future cash flows.

**Proposition 5 (Excess volatility).** In the economy of Section 2, the volatility of price changes over a finite time horizon $(0, t_1)$ is

$$\sigma_{\Delta P}(t_1) \equiv \sqrt{\operatorname{var}\left(P_{t_1} - P_0\right)} = \sqrt{\frac{\sigma_S B}{k}\left(\sigma_S B + \frac{2\sigma_D}{r}\right)\left(1 - e^{-kt_1}\right) + \frac{\sigma_D^2}{r^2} t_1} \ , \qquad (28)$$

while the volatility of $P - D/r$ over $(0, t_1)$ is

$$\sigma_{\Delta PD}(t_1) \equiv \sqrt{\operatorname{var}\left( (P_{t_1} - \frac{D_{t_1}}{r}) - (P_0 - \frac{D_0}{r}) \right)} = \sigma_S B \sqrt{\frac{1 - e^{-kt_1}}{k}} \ , \qquad (29)$$

where $k = \dfrac{\beta}{1 - \beta B}$ and $\sigma_S = \dfrac{\beta \sigma_D}{(1 - \beta B) r}$. ■

Table 5 reports the standard deviation of annual price changes and of the annual price-dividend difference $P - D/r$ for several $(\mu, \beta)$ pairs. Panel A shows that, in the fully rational economy ($\mu = 1$), the standard deviation of annual price changes is 10, in other words, $\sigma_D / r$. When extrapolators are present, however, the standard deviation is considerably higher: 30% higher when there are an equal number of extrapolators and rational traders in the economy, a figure that, as we explain below, depends little on the parameter β. Similarly, while in the fully rational economy the price-dividend difference is constant, in the presence of extrapolators, it varies significantly.

The results in Proposition 5 and in Table 5 confirm the intuition we described in the Introduction, namely that the presence of extrapolators amplifies the volatility of stock prices. A good cash flow shock pushes stock prices up. However, this increase in stock prices immediately leads extrapolators to expect higher *future* price changes in the stock market, which, in turn, leads them to push the value of the stock market up even further. Rational investors counteract this overvaluation, but only mildly so: since they understand how extrapolators form beliefs, they know that extrapolators will continue to have optimistic beliefs about the stock market in the near future, which, in turn, means that subsequent price changes, while lower than average, will not be *very* low. As a consequence, rational investors do not push back strongly against the overvaluation caused by the extrapolators. Put differently, even if the fraction of extrapolators in the overall population is low, this can be enough to significantly amplify the volatility of the stock market.

The table shows that, as expected, the greater the fraction of extrapolators in the economy, the more "excess volatility" there is in price changes and in the price-dividend difference. More interesting, it also shows that the amount of excess volatility is largely insensitive to the parameter $\beta$. This may seem surprising at first: since extrapolators' beliefs are more variable when $\beta$ is high, one might have thought that a higher $\beta$ would correspond to higher price volatility. However, there is another force that pushes in the opposite direction: rational traders know that, precisely because extrapolators change their beliefs more quickly when $\beta$ is high, any mispricing caused by the extrapolators will correct more quickly in this case. As a result, when $\beta$ is high, rational traders trade more aggressively against the extrapolators, dampening volatility. Overall, then, $\beta$ has little effect on volatility.

Does the higher price volatility generated by extrapolators leave the rational traders worse off? It does not. Specifically, we find that, if we start with an economy consisting of only rational traders and then gradually add more extrapolators while keeping the per-capita supply of the risky asset constant, the value function of the rational trader *increases* in value. In other words, while the higher price volatility lowers rational traders' utility, this is more than compensated for by the higher profits they make by exploiting the extrapolators.

### 4.4. Autocorrelations of price changes

Empirically, returns on the stock market are positively autocorrelated at short lags; at longer lags, they are negatively autocorrelated. We now examine what our model predicts about the autocorrelation structure of the analogous quantity to returns in our framework, namely price changes.

**Proposition 6 (Autocorrelations of price changes)**. In population, the autocorrelation of price changes between (0, $t_1$) and ($t_2$, $t_3$), where $t_2 \geq t_1$, is

$$\rho_{\Delta P}(t_1, t_2, t_3) \equiv \operatorname{corr}\left(P_{t_1} - P_0, P_{t_3} - P_{t_2}\right) \equiv \frac{\operatorname{cov}\left(P_{t_1} - P_0, P_{t_3} - P_{t_2}\right)}{\sqrt{\operatorname{var}\left(P_{t_1} - P_0\right) \operatorname{var}\left(P_{t_3} - P_{t_2}\right)}}, \tag{30}$$

where

$$\begin{aligned}
\operatorname{cov}\left(P_{t_1} - P_0, P_{t_3} - P_{t_2}\right) &= \frac{\sigma_S B}{2k}\left(\sigma_S B + \frac{2\sigma_D}{r}\right)\left(e^{-kt_3} - e^{-kt_2}\right)\left(e^{kt_1} - 1\right), \\
\operatorname{var}\left(P_{t_1} - P_0\right) &= \frac{\sigma_S B}{k}\left(\sigma_S B + \frac{2\sigma_D}{r}\right)\left(1 - e^{-kt_1}\right) + \frac{\sigma_D^2}{r^2} t_1, \\
\operatorname{var}\left(P_{t_3} - P_{t_2}\right) &= \frac{\sigma_S B}{k}\left(\sigma_S B + \frac{2\sigma_D}{r}\right)\left(1 - e^{-k(t_3 - t_2)}\right) + \frac{\sigma_D^2}{r^2}(t_3 - t_2),
\end{aligned} \tag{31}$$

and $k = \dfrac{\beta}{1 - \beta B}$. ■

In Table 6, we use Proposition 6 to compute the autocorrelation of price changes for several pairs of values of $\mu$ and $\beta$, and at lags of one, two, three, four, eight, and twelve quarters. The table shows that price changes are negatively autocorrelated at *all* lags, with the autocorrelation tending to zero at long lags. To see why, suppose that there is good cash flow news at time $t$. The stock market goes up in response to this news; but since this price rise causes extrapolators to expect higher future price changes, the stock market is pushed even further up. Now that the stock market is overvalued, the price change is lower, on average, going forward. In other words, past price changes have negative predictive power for future price changes.

Negative autocorrelations are also observed in the data, at several lags; to some extent, then, our model matches the data. However, there is also a way in which our model does not match the data: actual returns are *positively* autocorrelated at the first quarterly lag, while the price changes generated by our model are not.

Some earlier models of return extrapolation – for example, Cutler, Poterba, and Summers (1990), De Long et al. (1990b), Hong and Stein (1999), and Barberis and Shleifer (2003) – do generate positive short-term autocorrelation, or "momentum," for short. Barberis and Shleifer (2003), for example, consider an economy with two groups of investors. The first group's demand for the risky asset at time $t$ depends on the asset's past price changes up to time $t-1$; the second group buys (sells) the risky asset when its price is low (high) relative to fundamentals, but does not know the exact structure of extrapolator demand. This model generates positive short-term autocorrelation and negative long-term autocorrelation in price changes. In De Long et al. (1990b) and Hong and Stein (1999), the time $t$ risky asset demand of some investors depends positively on the price change between time $t-2$ and time $t-1$; these frameworks also generate positive short-term autocorrelation in price changes, and negative long-term autocorrelation.

Given that these earlier extrapolation-based models generate momentum, why does our model not do so? There are two differences between the earlier models and the current one, each of which, taken alone, suffices to remove any momentum. First, in contrast to the earlier models, our economy contains fully rational investors who understand how extrapolators form beliefs. Second, in the current model, extrapolators' demand for the risky asset depends on even the most recent price change $P_t - P_{t-dt}$, while in the earlier models, it depends only on price changes up to, but not including, the most recent price change. This last assumption is important for generating momentum: if extrapolators' demand at time $t$ depends on the price change between time $t-2$ and time $t-1$, a positive price change between $t-2$ and $t-1$ is likely to generate a positive price change between $t-1$ and $t$.[11]

[11] Our claim that *either* the presence of rational investors *or* an extrapolator demand function that depends on the most recent price change is enough to remove momentum is based on a re-examination of the Barberis and Shleifer (2003) model. We find that, if we either replace the partially rational investors in that model with fully rational investors, or make extrapolator demand a function of the most recent price change, the model no longer generates momentum.

### 4.5. Correlation of consumption changes and price changes

Another quantity of interest is the correlation of consumption growth and returns. In the data, this correlation is low. We now look at what our model predicts about the analogous quantity: the correlation of consumption changes and price changes.

**Proposition 7 (Correlation between consumption changes and price changes).** In population, the correlation between the change in consumption and the change in price over a finite time horizon $(0, t_1)$ is

$$\operatorname{corr}\left(C_{t_1} - C_0, P_{t_1} - P_0\right) = \frac{\operatorname{cov}(C_{t_1} - C_0, P_{t_1} - P_0)}{\sqrt{\operatorname{var}(C_{t_1} - C_0)\operatorname{var}(P_{t_1} - P_0)}}, \tag{32}$$

where

$$\begin{aligned}
&\operatorname{cov}(C_{t_1} - C_0, P_{t_1} - P_0) \\
&= Br(1 - e^{-kt_1})\frac{\sigma_W \sigma_S}{k} - B\gamma^{-1}\left(\frac{2ag_D\sigma_S^2}{kr}(1 - e^{-kt_1}) + (1 - e^{-kt_1})\frac{b\sigma_S^2}{k}\right) \\
&+\sigma_D a_W \frac{2g_D\sigma_S}{rk}\left[t_1 - \frac{1 - e^{-kt_1}}{k}\right] + \sigma_D b_W \frac{\sigma_S}{k}\left[t_1 - \frac{1 - e^{-kt_1}}{k}\right] \\
&+\sigma_D\sigma_W t_1 - \gamma^{-1}a\sigma_D\frac{2g_D\sigma_S}{r}\frac{1 - e^{-kt_1}}{rk} - \gamma^{-1}b\sigma_D(1 - e^{-kt_1})\frac{\sigma_S}{rk},
\end{aligned} \tag{33}$$

$$\operatorname{var}(P_{t_1} - P_0) = \frac{\sigma_S B}{k}\left(\sigma_S B + \frac{2\sigma_D}{r}\right)(1 - e^{-kt_1}) + \frac{\sigma_D^2}{r^2}t_1, \tag{34}$$

and

$$
\begin{aligned}
&\operatorname{var}(C_{t_1} - C_0) = \\
&r^2 a_W^2 \left( \frac{4 g_D^2 \sigma_S^2}{r^2 k^2} (t_1 - \frac{1 - e^{-kt_1}}{k}) + \frac{\sigma_S^4}{4k^3} (2t_1 - \frac{1 - e^{-2kt_1}}{k}) \right) + \frac{r^2 b_W^2 \sigma_S^2}{k^2} (t_1 - \frac{1 - e^{-kt_1}}{k}) \\
&+ \frac{r a_W b_W \sigma_S^2 g_D}{k^2} (3t_1 + 2\frac{e^{-2kt_1} - 1}{k} + e^{-2kt_1} t_1) + r^2 \sigma_W^2 t_1 + \gamma^{-2} a^2 \left( \frac{4 g_D^2 \sigma_S^2}{r^2} \frac{1 - e^{-kt_1}}{k} + \frac{\sigma_S^4 (1 - e^{-2kt_1})}{k^2} \right) \\
&+ \frac{\gamma^{-2} b^2 \sigma_S^2}{k} (1 - e^{-kt_1}) + \frac{4\gamma^{-2} a b \sigma_S^2 g_D}{kr} (1 - e^{-kt_1}) + 2r^2 a_W \sigma_W \left( \frac{2 g_D \sigma_S}{rk} (t_1 - \frac{1 - e^{-kt_1}}{k}) \right) \\
&+ 2r^2 b_W \sigma_W \frac{\sigma_S}{k} (t_1 - \frac{1 - e^{-kt_1}}{k}) - 2\gamma^{-1} b_W a \frac{\sigma_S^2 g_D}{k} (\frac{4e^{-kt_1} - 1 - 3e^{-2kt_1}}{2k} - t_1 e^{-2kt_1}) \\
&- \frac{4\gamma^{-1} a \sigma_W g_D \sigma_S}{k} (1 - e^{-kt_1}) - 2r\gamma^{-1} \sigma_W b (1 - e^{-kt_1}) \frac{\sigma_S}{k}.
\end{aligned}
\tag{35}
$$

Note that $a = \mu a^r + (1-\mu) a^e$, $b = \mu b^r + (1-\mu) b^e$, $a_W = a/\gamma$, $b_W = \frac{b}{\gamma} - \frac{\beta BQ}{1-\beta B} - rBQ$, $\sigma_W = \frac{\sigma_D Q}{(1-\beta B) r}$, $k \equiv \frac{\beta}{1-\beta B}$ and $\sigma_S \equiv \frac{\beta \sigma_D}{(1-\beta B) r}$. ■

Panels A and B of Table 7 use Proposition 7 to compute the correlation of consumption changes and price changes at a quarterly and annual frequency, respectively, and for several ($\mu$, $\beta$) pairs. The two panels show that, while the presence of extrapolators slightly reduces this correlation relative to its value in the fully rational economy, the correlation is nonetheless high. As is the case for virtually all consumption-based pricing models, then, our model fails to match the low correlation of consumption growth and returns in the data.

### 4.6. Predictive power of the surplus consumption ratio

Prior empirical research has shown that a variable called the "surplus consumption ratio" – a measure of consumption relative to past consumption levels, is contemporaneously correlated with the price-dividend ratio on the overall stock market; and furthermore, that it predicts subsequent returns with a negative sign. These findings have been taken as support for habit-based models of the aggregate stock market. We show, however, that these patterns also emerge from our model.

As we have done throughout this section, we look at a surplus consumption *difference* rather than a surplus consumption ratio; moreover, we focus on the simplest possible surplus consumption difference, namely the current level of aggregate consumption minus the level of aggregate consumption at some point in the past. Proposition 8 computes the correlation between this variable and the contemporaneous price-dividend difference $P - D/r$.

**Proposition 8**: **(Correlation between consumption change and *P* – *D*/*r*).** In population, the correlation between the change in consumption over a finite time horizon $(0, t_1)$ and $P - D/r$ is

$$\operatorname{corr}(C_{t_1} - C_0, P_{t_1} - \frac{D_{t_1}}{r}) = \frac{\operatorname{cov}(C_{t_1} - C_0, P_{t_1} - \frac{D_{t_1}}{r})}{\sqrt{\operatorname{var}(C_{t_1} - C_0)\operatorname{var}(P_{t_1} - \frac{D_{t_1}}{r})}}, \tag{36}$$

where

$$\operatorname{cov}(C_{t_1} - C_0, P_{t_1} - \frac{D_{t_1}}{r}) = (1 - e^{-kt_1})B\left(\frac{a_W g_D \sigma_S^2}{k^2} + \frac{rb_W \sigma_S^2}{2k^2} + \frac{r\sigma_W \sigma_S}{k} - \frac{a g_D \sigma_S^2}{kr\gamma} - \frac{b\sigma_S^2}{2k\gamma}\right), \tag{37}$$

$\operatorname{var}(P_{t_1} - \frac{D_{t_1}}{r}) = B^2 \frac{\sigma_S^2}{2k}$, and $\operatorname{var}(C_{t_1} - C_0)$ is as in (35). Also, $a = \mu a^r + (1-\mu)a^e$,

$b = \mu b^r + (1-\mu)b^e$, $a_W = a/\gamma$, $b_W = \frac{b}{\gamma} - \frac{\beta BQ}{1-\beta B} - rBQ$, $\sigma_W = \frac{\sigma_D Q}{(1-\beta B)r}$, $k \equiv \frac{\beta}{1-\beta B}$ and

$\sigma_S \equiv \frac{\beta \sigma_D}{(1-\beta B)r}$. ■

Proposition 9 examines whether the surplus consumption difference can predict future price changes.

**Proposition 9 (The predictive power of changes in consumption).** Consider a regression of the price change in the stock market from $t_1$ to $t_2$ on the change in consumption over the finite time horizon (0, $t_1$). In population, the coefficient on the independent variable is

$$\beta_{\Delta C}(t_1,t_2) \equiv \frac{\operatorname{cov}\left(C_{t_1}-C_0, P_{t_2}-P_{t_1}\right)}{\operatorname{var}\left(C_{t_1}-C_0\right)}, \tag{38}$$

where

$$\begin{aligned}&\operatorname{cov}(C_{t_1}-C_0, P_{t_2}-P_{t_1})\\ &= (e^{-k(t_2-t_1)}-1)(1-e^{-kt_1})B\left(\frac{a_W g_D \sigma_S^2}{k^2}+\frac{rb_W\sigma_S^2}{2k^2}+\frac{r\sigma_W\sigma_S}{k}-\frac{ag_D\sigma_S^2}{kr\gamma}-\frac{b\sigma_S^2}{2k\gamma}\right),\end{aligned} \tag{39}$$

and $\operatorname{var}(C_{t_1}-C_0)$ is as in (35). Also, $a=\mu a^r+(1-\mu)a^e$, $b=\mu b^r+(1-\mu)b^e$, $a_W=\dfrac{a}{\gamma}$, $b_W=\dfrac{b}{\gamma}-\dfrac{\beta BQ}{1-\beta B}-rBQ$, $\sigma_W=\dfrac{\sigma_D Q}{(1-\beta B)r}$, $k=\dfrac{\beta}{1-\beta B}$, and $\sigma_S=\dfrac{\beta\sigma_D}{(1-\beta B)r}$. ■

Panel C of Table 7 uses Proposition 8 to compute, for several ($\mu$, $\beta$) pairs, the correlation between the surplus consumption difference and the price-dividend difference. Here, the surplus consumption difference is computed as the current level of aggregate consumption minus the level of aggregate consumption a quarter ago. The panel shows that the two quantities are significantly correlated. Table 8 uses Proposition 9 to compute the coefficient on the independent variable in a regression of the price change in the stock market over some horizon – one quarter, one year, two years, three years, or four years – on the surplus consumption difference. It shows that the surplus consumption difference has significant negative predictive power for price changes, and that the predictive power is particularly strong for low $\mu$ and high $\beta$. Taken together, then, Panel C of Table 7 and Table 8 show that the surplus consumption difference can be correlated with the valuation level of the stock market and with the subsequent stock price change even in a framework that does not involve habit-type preferences in any way.

The intuition for these results is straightforward. After a sequence of good cash flow news, extrapolators cause the stock market to become overvalued and hence the price-dividend difference to be high. However, at the same time, extrapolators' optimistic beliefs about the future lead them to raise their consumption; while the rational traders do not raise their consumption as much, aggregate consumption nonetheless increases overall, pushing the surplus consumption difference up. This generates a positive correlation between the price-dividend difference and the surplus consumption difference. Since the stock market is overvalued at this point, the subsequent price change in the stock market is low, on average. As a consequence, the surplus consumption difference predicts future price changes with a negative sign.

### 4.7. Equity premia and Sharpe ratios

Proposition 10 below computes the equity premium and Sharpe ratio of the stock market.

**Proposition 10 (Equity Premium and Sharpe Ratio).** In the economy of Section 2, the equity premium, defined as the per unit time expectation of the excess price change and dividend, can be written as

$$\frac{1}{dt}\mathbb{E}\left[dP_t + D_t dt - rP_t dt\right] = (1 - rB)\frac{g_D}{r} - rA. \tag{40}$$

The Sharpe ratio is

$$\begin{aligned}&\frac{\mathbb{E}\left[\frac{1}{dt}(dP_t + D_t dt - rP_t dt)\right]}{\sqrt{\mathrm{Var}\left[\frac{1}{dt}(dP_t + D_t dt - rP_t dt)\right]}} \\ &= \left[\frac{\sigma_D^2}{(1-\beta B)^2 r^2} + \left(\frac{\beta B}{1-\beta B} + rB\right)^2 \frac{\beta\sigma_D^2}{2(1-\beta B)r^2}\right]^{-1/2} \left((1-rB)\frac{g_D}{r} - rA\right).\end{aligned} \tag{41}$$

■

Panel A of Table 9 uses the proposition to compute the average excess price change and dividend at an annual horizon for several (μ, β) pairs. The panel shows that the equity premium rises as the fraction of extrapolators in the economy goes up: the more extrapolators there are, the more volatile the stock market is; the equity premium therefore needs to be higher to compensate for the higher risk. Panel B of the table shows that it is not just the equity premium that goes up as μ falls, but also the Sharpe ratio.

## 5. Further Analysis: Ratio-based Quantities

In Section 4, we focused on quantities defined in terms of differences: on price changes, and on the price-dividend difference $P - D/r$. Given the additive structure of our model, these are the natural quantities to study. However, most empirical research works with ratio-based quantities such as returns and price-dividend ratios. While these are not the most natural quantities to look at in the context of our model, we can nonetheless examine what our model predicts about them. This is what we do in this section.

Since analytical results are not available for ratio-based quantities, we use numerical simulations to study their properties. In Section 5.1, we explain the methodology behind these simulations. In Section 5.2, we present our results. In brief, the results for the ratio-based quantities are broadly consistent with those for the difference-based quantities in Section 4. However, we also interpret these results cautiously: precisely because they are not the natural objects of study in our model, the ratio-based quantities are not as well-behaved as the difference-based quantities we examined in Section 4.

### 5.1. Simulation Methodology

To conduct the simulations, we first discretize the model. In this discretized version, we use a time-step of $\Delta t = 1/4$, in other words, of one quarter. As indicated in Section 3, the initial level of the dividend is $D_0 = 10$ and the initial wealth levels are $W_0^e = W_0^r = 5000.$ We further set the initial sentiment level, $S_0$, to the steady-state mean of $\frac{g_D}{r}$.

We know from Proposition 1 that, at time 0,

$$N_0^e = \eta_0^e + \eta_1^e S_0, \quad P_0 = A + BS_0 + \frac{D_0}{r},$$
$$C_0^i = rW_0^i - \frac{1}{\gamma}\left(a^i S_0^2 + b^i S_0 + c^i\right) - \frac{1}{\gamma}\log(r\gamma), \qquad i \in \{e, r\}. \tag{42}$$

The proposition also tells us that, from time $n\Delta t$ to $(n+1)\Delta t$, we have:

$$
\begin{aligned}
D_{(n+1)\Delta t} &= D_{n\Delta t} + g_D \Delta t + \sigma_D \sqrt{\Delta t}\,\varepsilon_{(n+1)\Delta t}, \\
S_{(n+1)\Delta t} &= S_{n\Delta t} - \frac{\beta}{1-\beta B}\left( S_{n\Delta t} - \frac{g_D}{r} \right)\Delta t + \frac{\sigma_D \beta \sqrt{\Delta t}}{(1-\beta B) r}\varepsilon_{(n+1)\Delta t}, \\
P_{(n+1)\Delta t} &= A + B S_{(n+1)\Delta t} + \frac{D_{(n+1)\Delta t}}{r}, \\
N^e_{(n+1)\Delta} &= \eta^e_0 + \eta^e_1 S_{(n+1)\Delta t}, \qquad N^r_{(n+1)\Delta t} = \frac{Q-(1-\mu)N^e_{(n+1)\Delta t}}{\mu}, \\
W^i_{(n+1)\Delta t} &= W^i_{n\Delta t} + N^i_{n\Delta t}(P_{(n+1)\Delta t} - P_{n\Delta t}) - C^i_{n\Delta t}\Delta t \\
&\quad + r W^i_{n\Delta t}\Delta t - r N^i_{n\Delta t} P_{n\Delta t}\Delta t + N^i_{n\Delta t} D_{(n+1)\Delta t}\Delta t, \\
C^i_{(n+1)\Delta t} &= r W^i_{(n+1)\Delta t} - \frac{1}{\gamma}\left( a^i S^2_{(n+1)\Delta t} + b^i S_{(n+1)\Delta t} + c^i \right) - \frac{1}{\gamma}\log(r\gamma),
\end{aligned} \tag{43}
$$

with $i \in \{e, r\}$, and where $\{\varepsilon_{(n+1)\Delta t}, n \geq 1\}$ are *i.i.d.* standard normal random variables with mean 0 and a standard deviation of 1. We make the conventional assumptions that the level of the consumption stream for the period between ($n\Delta t$, $(n+1)\Delta t$) is determined at the beginning of the period; and that the level of the dividend paid over this period is determined at the end of the period.

For a given set of values of the basic model parameters in Table 2, we use the procedure described in the proof of Proposition 1 to compute the parameters that determine the optimal portfolio holdings and consumption choice – variables such as $\eta^e_1$, for example.[12] We then use the above equations to simulate a sample path for our economy that is 200 periods long, in other words, 50 years long. We compute quantities of interest from this 200-period time series – the autocorrelation of stock market returns, say. We then repeat this process 10,000 times. In the next section, we report the *average* return autocorrelation that we obtain across these 10,000 simulated paths.

[12] Here, we are assuming that the values of the derived parameters, such as $\eta^e_1$, that determine investors' optimal policies in the continuous-time framework are a good approximation to the values of these parameters in the discrete-time analog of our model. One indication that this is a reasonable assumption is that our numerical results are robust to changing $\Delta t$ from 1/4 to 1/48, say.

### 5.2. Results

Table 10 presents the model's predictions for ratio-based quantities for $\mu$ = 0.25 and for three different values of $\beta$. For each ($\mu$, $\beta$) pair, we simulate 10,000 paths, each of which is 200 periods long. For each of the 10,000 paths, we compute various quantities of interest – specifically, the quantities listed in the left column of Table 10. The table reports the average value of each quantity across the 10,000 paths. The last column of the table reports the empirical value of each quantity over the post-war period from 1947 to 2011.[13]

We now discuss each of these quantities in turn. Most of them are simply the ratio-based analogs of the quantities we studied in Section 4: for example, instead of computing the standard deviation of price changes, we compute the standard deviation of returns. However, we are also able to address some questions that we did not discuss in any form in Section 4, such as whether the consumption-wealth ratio or more complex formulations of the surplus consumption ratio have any predictive power for returns.

**Row 1:** We report the coefficient on the independent variable in a regression of total log excess returns measured over a one-year horizon on the log dividend-price ratio at the start of the year. To be clear, as described above, we run this regression in each of the 10,000 paths we simulate; the table reports the average coefficient across all paths, as well as the average R-squared, in parentheses. Consistent with the findings of Section 4.1, the table shows that the dividend-price ratio predicts subsequent returns with a positive sign.

**Row 2:** We report the autocorrelation of the price-dividend ratio at a one-year lag. Consistent with the results of Section 4.2, the ratio is highly persistent.

**Row 3:** We compute the excess volatility of returns -- specifically, the standard deviation of stock returns in the heterogeneous-agent economy relative to the standard deviation of returns in the rational benchmark economy. Consistent with the findings of Section 4.3, we see that stock returns exhibit excess volatility.

[13] For the nondurable consumption data, the sample period starts in 1952. Returns are based on the CRSP value-weighted index. For the consumption-wealth ratio, wealth is computed in two different ways: the first way uses the market capitalization of the CRSP stock market, and the second uses aggregate household wealth from the *Flow of Funds* accounts, following Lettau and Ludvigson (2001).

**Row 4:** We compute the excess volatility of price-dividend ratios: the standard deviation of the price-dividend ratio in the heterogeneous-agent economy relative to its standard deviation in the rational benchmark economy. Consistent with Section 4.3, the standard deviation of the price-dividend ratio goes up in the presence of extrapolators.

**Row 5:** We compute the autocorrelation of quarterly log excess stock returns at lags of one quarter and two years. As in Section 4.4, returns are negatively autocorrelated.

**Row 6:** We compute the correlation of annual log excess stock returns with annual changes in quarterly log consumption. As in Section 4.5, this correlation is higher than the correlation observed in the data.

**Row 7:** We compute the correlation between the surplus consumption ratio and the price-dividend ratio, where both quantities are measured at a quarterly frequency. Given the greater flexibility afforded by numerical simulations, we use a more sophisticated definition of surplus consumption than in Section 4.6, one that is still simpler than in Campbell and Cochrane (1999) but that nonetheless preserves the spirit of their calculation. Specifically, we define the surplus consumption ratio as:

$$S_t^a = \frac{C_t^a - X_t}{C_t^a}, \tag{44}$$

where the superscript "*a*" stands for "aggregate," and where the habit level $X_t$ adjusts slowly to changes in consumption:

$$X_t = \sum_{k=1}^{n} w(k;n,\xi) C_{t-k}, \quad \text{where } w(k;n,\xi) \equiv \frac{e^{-\xi k \Delta t}}{\sum_{j=1}^{n} e^{-\xi j \Delta t}}. \tag{45}$$

In simple terms, $X_t$ is a weighted sum of past consumption levels, where recent consumption levels are weighted more heavily. For a given $\xi$, we choose $n$ so that $\frac{\sum_{j=1}^{n} e^{-\xi j \Delta t}}{\sum_{j=1}^{\infty} e^{-\xi j \Delta t}} = \frac{e^{-\xi \Delta t} - e^{-\xi(n+1)\Delta t}}{e^{-\xi \Delta t}} > 90\%;$ that is, we choose $n$ so that even consumption changes in the distant past receive at least some weight in the computation of the habit level. In our calculations, we set $\xi = 0.95$ and $n = 12$.[14]

Row 7 of Table 10 shows that, as in Section 4.6, the surplus consumption ratio and price-dividend ratio are positively correlated, consistent with the actual data.

**Row 8:** We report the coefficient on the independent variable in a regression of total log excess returns over a year on the surplus consumption ratio at the start of the year. Consistent with our results in Section 4.6 using a simpler measure of surplus consumption, the surplus consumption ratio predicts subsequent returns with a negative sign, as it does in actual data.

**Row 9:** Empirically, the consumption-wealth ratio has predictive power for subsequent returns. Here, we examine whether our model can generate this pattern. We compute the coefficient on the independent variable in a regression of total log excess returns over a year on the log consumption-wealth ratio at the start of the year. The table shows that the ratio does indeed have some predictive power.

What is the intuition for this predictive power? After a sequence of good cash flow news, extrapolators cause the stock market to become overvalued. This, in turn, increases aggregate wealth in the economy; it also increases aggregate consumption, but not to the same extent: rational traders, in particular, do not increase their consumption very much because they realize that future returns on the stock market are likely to be low. Overall, the consumption-wealth ratio falls. Since the stock market is overvalued, its subsequent return is lower than average. The consumption-wealth ratio therefore predicts subsequent returns with a positive sign.

**Row 10:** We compute the equity premium and Sharpe ratio in our economy.

In summary, while it is natural, in our framework, to study difference-based quantities rather than ratio-based quantities, Table 10 shows that the ratio-based quantities exhibit patterns that are broadly similar to those that we obtained in Section 4 for the difference-based quantities.

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[14] When $\xi = 0.95$, quarterly consumption one year ago is weighted about 40% as much as current consumption.

## 6. Conclusion

Survey evidence suggests that many investors form beliefs about future stock market returns by extrapolating past returns: they expect the stock market to perform well (poorly) in the near future if it has recently performed well (poorly). Such beliefs are hard to reconcile with existing models of the aggregate stock market. We study a heterogeneous-agent model in which some investors form beliefs about future stock market price changes by extrapolating past price changes, while other investors have fully rational beliefs. We find that the model captures many features of actual returns and prices. Importantly, however, it is also consistent with the survey evidence on investor expectations. This suggests that the survey evidence does not need to be seen as a nuisance; on the contrary, it is consistent with the facts about prices and returns and may be the key to understanding them.

# Appendices

## A. Proof of Proposition 1

In order to solve the stochastic dynamic programming problem, we need the differential forms for the evolution of the state variables. From the definition of the sentiment variable, $S_t = \beta\int_{-\infty}^{t} e^{-\beta(t-s)} dP_{s-dt}$, its differential form is

$$dS_t = -\beta S_t dt + \beta dP_t \,. \tag{A1}$$

The term $-\beta S_t dt$ captures the fact that, when we move from time $t$ to time $t + dt$, all the earlier price changes that contribute to $S_t$ need to be associated with smaller weights since they are further away from time $t + dt$ than they were from time $t$; the term $\beta dP_t$ captures the fact that the latest price change pushes $S_t$ in the same direction; and the parameter $\beta$ captures the stickiness of this belief updating rule. Also, the wealth of each type of trader evolves as

$$\begin{aligned} W_{t+dt}^i &= (W_t^i - C_t^i dt - N_t^i P_t)(1 + rdt) + N_t^i D_t dt + N_t^i P_{t+dt} \\ \Rightarrow dW_t^i &= rW_t^i dt - C_t^i dt - rN_t^i P_t dt + N_t^i dP_t + N_t^i D_t dt, \qquad i \in \{e, r\}\,, \end{aligned} \tag{A2}$$

consistent with the budget constraints in (5) and (7).

As noted in the main text, the derived value functions for the extrapolators and the rational traders are

$$J^i(W_t^i, S_t, t) \equiv \max_{\{C_s^i, N_s^i\}_{s\geq t}} \mathbb{E}_t^i\left[-\int_t^{\infty} \frac{e^{-\delta s - \gamma C_s^i}}{\gamma} ds\right], \qquad i \in \{e, r\}\,. \tag{A3}$$

The assumptions that traders have CARA preferences, that $D_t$ follows an arithmetic Brownian motion, that $S_t$ evolves in a Markovian fashion as in (A1), and that extrapolators' biased beliefs in (3) are linearly related to $S_t$ jointly guarantee that the derived value functions are only functions of time, of the level of wealth, and of the level of sentiment, but of nothing else (such as $D_t$ or $P_t$). We verify this and discuss it further after solving the model.

If we define

$$\phi^i(C^i, N^i; W^i, S, t) \equiv -\frac{e^{-\delta t - \gamma C^i}}{\gamma} + \frac{1}{dt}\mathbb{E}_t^i[dJ^i]\,, \qquad i \in \{e, r\}\,, \tag{A4}$$

then, from the theory of stochastic control, we have that[15]

$$0 = \max_{\{C^i, N^i\}} \phi^i(C^i, N^i; W^i, S, t)\,, \qquad i \in \{e, r\}\,. \tag{A5}$$

By Ito's lemma, (A5) leads to the stochastic Bellman equations which state that, along the optimal path of consumption and asset allocation,

$$\begin{aligned} 0 = &-\frac{e^{-\delta t - \gamma C^i}}{\gamma} + J_t^i + J_W^i\left(rW^i - C^i - rN^i P + N^i g_P^i + N^i D\right) + \frac{1}{2} J_{WW}^i \sigma_P^2 (N^i)^2 \\ &+ J_S^i\left(-\beta S + \beta g_P^i\right) + \frac{1}{2}\beta^2 J_{SS}^i \sigma_P^2 + \beta J_{WS}^i N^i \sigma_P^2\,, \qquad i \in \{e, r\}\,, \end{aligned} \tag{A6}$$

[15] See Kushner (1967) for a detailed discussion of this topic.

where $g_P^e$ and $g_P^r$ are the per unit time price change of the stock market expected by extrapolators and rational traders, respectively, and where $\sigma_P$ is the per unit time volatility of the stock price. Note that, as stated in (3), $g_P^e = \lambda_0 + \lambda_1 S$, and that $g_P^r$ comes from rational traders' conjecture about the stock price process, which is yet to be determined. Note also that, in continuous time, the volatility $\sigma_P$ is essentially observable by computing the quadratic variation; as a result, the two types of traders agree on its value. We assume, and later verify, that $\sigma_P$ is an endogenously determined constant that does not depend on *S* or *t*. Finally, from the evolution of *S* in (A1), we know that $dW^i$ and *S* are locally perfectly correlated for both types of trader.

Since the infinite-horizon model is perpetual, and since, as verified later, the evolutions of $W^e$ and $W^r$ do not depend explicitly on the level of the dividend or the stock price, we know that the passage of time only affects the value functions through time discounting. We can therefore write, for $i \in \{e, r\}$,

$$J^i(W_t^i, S_t, t) = e^{-\delta t} I^i(W_t^i, S_t), \quad \text{where } I^i(W_t^i, S_t) \equiv \max_{\{C_s^i, N_s^i\}_{s\geq t}} \mathbb{E}_t^i \left[ -\int_t^\infty \frac{e^{-\delta(s-t)-\gamma C_s^i}}{\gamma} ds \right]. \tag{A7}$$

Substituting (A7) into (A6) gives the reduced Bellman equations

$$\begin{aligned} 0 = &-\frac{e^{-\gamma C^i}}{\gamma} - \delta I^i + I_W^i \left( rW^i - C^i - rN^i P + N^i g_P^i + N^i D \right) + \frac{1}{2} I_{WW}^i \sigma_P^2 (N^i)^2 \\ &+ I_S^i \left( -\beta S + \beta g_P^i \right) + \frac{1}{2} \beta^2 I_{SS}^i \sigma_P^2 + \beta I_{WS}^i N^i \sigma_P^2, \qquad i \in \{e, r\}. \end{aligned} \tag{A8}$$

The first-order conditions of (A8) with respect to $C^i$ and $W^i$ are

$$I_W^i = e^{-\gamma C^i} \tag{A9}$$

and

$$N^i = -\frac{I_W^i}{I_{WW}^i} \frac{g_P^i - rP + D}{\sigma_P^2} - \frac{\beta I_{WS}^i}{I_{WW}^i}, \qquad i \in \{e, r\}. \tag{A10}$$

The first term on the right-hand side of (A10) is the share demand due to mean-variance considerations; the second term is the hedging demand due to sentiment-related risk.

We now conjecture, and later verify, that the true equilibrium stock price satisfies

$$P_t = A + BS_t + \frac{D_t}{r}. \tag{A11}$$

The coefficients *A* and *B* are yet to be determined. Assuming that the rational traders know this price equation and the true process for $D_t$, they can obtain the true evolution of the stock price as

$$dP_t = \left( -\frac{\beta B}{1-\beta B} S_t + \frac{g_D}{(1-\beta B) r} \right) dt + \frac{\sigma_D}{(1-\beta B) r} d\omega \tag{A12}$$

by combining (1), (A1), and (A11). Substituting (A12) into (A1) yields

$$dS_t = -\frac{\beta}{1-\beta B} \left( S_t - \frac{g_D}{r} \right) dt + \frac{\beta \sigma_D}{(1-\beta B) r} d\omega. \tag{A13}$$

From (A12) and (A13) it is clear that when $B < \beta^{-1}$, the sentiment variable $S_t$ follows an Ornstein-Uhlenbeck process with a steady-state distribution that is Normal with mean $\frac{g_D}{r}$ and variance $\frac{\sigma_D^2\beta}{2r^2(1-\beta B)}$, and that the expected per unit time price change, $\frac{\mathbb{E}_t^r[dP_t]}{dt}$, also fluctuates around its long-run mean of $\frac{g_D}{r}$ with long-run variance of $\frac{\sigma_D^2\beta^3 B^2}{2r^2(1-\beta B)^3}$. In addition

$$g_P^r = -\frac{\beta B}{1-\beta B}S_t + \frac{g_D}{(1-\beta B)r}, \qquad \sigma_P = \frac{\sigma_D}{(1-\beta B)r}. \tag{A14}$$

That is, rational traders' future expected price change is *negatively* and linearly related to the sentiment level, and $\sigma_P$ is a constant if the conjecture in (A11) is valid.

Given the imposed belief structure that $g_P^e = \lambda_0 + \lambda_1 S$, the extrapolators subjectively believe that the stock price evolves as

$$dP_t = \left(\lambda_0 + \lambda_1 S_t\right)dt + \frac{\sigma_D}{(1-\beta B)r}d\omega^e, \tag{A15}$$

where $d\omega^e$ is extrapolators' *perceived* innovation term from the dividend process, which itself follows

$$dD_t = g_D^e dt + \sigma_D d\omega^e, \tag{A16}$$

where $g_D^e$ is extrapolators' perceived expected per unit time dividend change.[16]

Differentiating (A12) and substituting in (A1) and (A16), extrapolators obtain

$$dP_t = \left(-\frac{\beta B}{1-\beta B}S_t + \frac{g_D^e}{(1-\beta B)r}\right)dt + \frac{\sigma_D}{(1-\beta B)r}d\omega^e, \tag{A17}$$

in contrast with the price process (A12) obtained by the rational traders. Comparing (A15) and (A17) suggests that

$$g_D^e(S_t) = \lambda_0 r(1-\beta B) + \left[\lambda_1 r(1-\beta B) + r\beta B\right]S_t. \tag{A18}$$

That is, extrapolators' perceived expected dividend change per unit time depends explicitly on $S_t$. (We note that this is quite different from directly extrapolating past dividend changes.)

Price-agreement across the two types of traders, in other words,

$$dP = g_P^r dt + \sigma_P d\omega = g_P^e dt + \sigma_P d\omega^e \tag{A19}$$

prevents extrapolators from seeing, through retrospection, that their belief structure is biased, and provides a direct relation between $d\omega$ and $d\omega^e$. Equations (A12), (A17), and (A19) jointly confirm dividend-agreement across traders:

$$dD = g_D dt + \sigma_D d\omega = g_D^e dt + \sigma_D d\omega^e. \tag{A20}$$

[16] If instead, the extrapolators know the true process of $D_t$, they will believe that $dP_t = (\lambda_0 + \lambda_1 S_t)dt + \sigma_P d\omega$, a price process that, given that $-\beta B/(1-\beta B) < 0 < \lambda_1$, clearly deviates from the true process in (A12). In other words, even after a time interval of length $dt$, extrapolators will, in principle, be able to learn that their beliefs are wrong.

We guess that the solutions of $I^e(W^e, S)$ and $I^r(W^r, S)$ are

$$I^i(W^i, S) = -\exp\left[-r\gamma W^i + a^i S^2 + b^i S + c^i\right], \qquad i \in \{e, r\}. \tag{A21}$$

Substituting (A21) into the optimal consumption rule in (A9) and the optimal share demand of the stock in (A10) yields

$$C^i = rW^i - \frac{1}{\gamma}\left(a^i S^2 + b^i S + c^i\right) - \frac{1}{\gamma}\log(r\gamma), \tag{A22}$$

and

$$N^i = \frac{g_P^i - rP + D}{r\gamma\sigma_P^2} + \frac{\beta(2a^i S + b^i)}{r\gamma}, \qquad i \in \{e, r\}. \tag{A23}$$

For the extrapolators, substituting $g_P^e = \lambda_0 + \lambda_1 S$ and the price equation (A11) into (A23) gives

$$N^e = \eta_0^e + \eta_1^e S, \quad \text{where } \eta_0^e \equiv \frac{\lambda_0 - rA + b^e\beta\sigma_P^2}{r\gamma\sigma_P^2} \text{ and } \eta_1^e \equiv \frac{\lambda_1 - rB + 2a^e\beta\sigma_P^2}{r\gamma\sigma_P^2}. \tag{A24}$$

Substituting the price equation (A11), the form of $I^e$ in (A21), the optimal consumption $C^e$ in (A22), and the optimal share demand $N^e$ in (A24) into the reduced Bellman equation (A8) for the extrapolators, we obtain the following quadratic equation in $S$:

$$\begin{aligned}
0 = (r-\delta) - r\gamma\Bigg[&\frac{(\lambda_0 - rA + b^e\beta\sigma_P^2) + (\lambda_1 - rB + 2a^e\beta\sigma_P^2)S}{r\gamma\sigma_P^2}(\lambda_0 + \lambda_1 S - rA - rBS) \\
&+ \frac{a^e S^2 + b^e S + c^e + \log(r\gamma)}{\gamma}\Bigg] + \frac{[(\lambda_0 - rA + b^e\beta\sigma_P^2) + (\lambda_1 - rB + 2a^e\beta\sigma_P^2)S]^2}{2\sigma_P^2} \\
&+ (2a^e S + b^e)[-\beta S + \beta(\lambda_0 + \lambda_1 S)] + a^e\beta^2\sigma_P^2 + \frac{1}{2}\beta^2\sigma_P^2(2a^e S + b^e)^2 \\
&- \beta(2a^e S + b^e)[(\lambda_0 - rA + b^e\beta\sigma_P^2) + (\lambda_1 - rB + 2a^e\beta\sigma_P^2)S],
\end{aligned} \tag{A25}$$

which is equivalent to three simultaneous equations:

$$0 = -\frac{(\lambda_1 - rB + 2a^e\beta\sigma_P^2)^2}{2\sigma_P^2} - ra^e + 2a^e\beta(\lambda_1 - 1) + 2(a^e\beta\sigma_P)^2, \tag{A26}$$

$$0 = -\frac{(\lambda_1 - rB + 2a^e\beta\sigma_P^2)(\lambda_0 - rA + b^e\beta\sigma_P^2)}{\sigma_P^2} - rb^e + 2a^e\beta\lambda_0 + b^e\beta(\lambda_1 - 1) + 2\beta^2\sigma_P^2 a^e b^e, \tag{A27}$$

$$0 = (r-\delta) - \frac{(\lambda_0 - rA + b^e\beta\sigma_P^2)^2}{2\sigma_P^2} - rc^e - r\log(r\gamma) + b^e\beta\lambda_0 + a^e\beta^2\sigma_P^2 + \frac{1}{2}(b^e\beta\sigma_P)^2. \tag{A28}$$

These three equations determine the coefficients $a^e$, $b^e$, $c^e$, $\eta_0^e$, and $\eta_1^e$ as functions of the coefficients $A$ and $B$. If, as we assume, extrapolators know the belief structure of the rational traders as well as the parameters $\mu$ and $Q$, it follows that they can go through the intertemporal maximization problem for the rational investors (specified below) and figure out the price equation (A11). As a result, extrapolators know the coefficients $A$ and

$B$, and through equations (A26), (A27), and (A28), they can solve for their optimal share demand $N^e$, as well as for their value function $J^i$.

We now turn to the rational traders. Using $g_P^r$ and $\sigma_P$ from (A14), the form of $I^r$ in (A21), $N^r$ from (A23), the optimal share demand of the stock from extrapolators in (A24), and the market clearing condition $\mu N^r + (1-\mu)N^e = Q$, we obtain $a^r$ and $b^r$ as functions of $A$ and $B$,

$$\begin{aligned} a^r &= \frac{(1-\beta B)^2 r^2}{2\sigma_D^2\beta}\left(\frac{\beta B}{1-\beta B} + rB - \frac{(1-\mu)\gamma\sigma_D^2\eta_1^e}{r\mu(1-\beta B)^2}\right), \\ b^r &= \frac{(1-\beta B)^2 r^2}{\sigma_D^2\beta}\left(\frac{\gamma\sigma_D^2}{\mu r(1-\beta B)^2}Q + rA - \frac{g_D}{(1-\beta B)r} - \frac{(1-\mu)\gamma\sigma_D^2\eta_0^e}{r\mu(1-\beta B)^2}\right). \end{aligned} \tag{A29}$$

Substituting the price equation (A11), $g_P^r$ from (A14), the form of $I^e$ in (A21), the optimal consumption $C^r$ in (A22), and the optimal share demand $N^r = \frac{Q}{\mu} - \frac{1-\mu}{\mu}(\eta_0^e + \eta_1^e S)$ into the reduced Bellman equation (A8) for the rational traders, we obtain another quadratic equation in $S$

$$\begin{aligned} 0 = (r-\delta) - r\gamma &\left[\frac{Q-(1-\mu)(\eta_0^e + \eta_1^e S)}{\mu}\left(-\frac{\beta B}{1-\beta B}S + \frac{g_D}{(1-\beta B)r} - rA - rBS\right)\right. \\ &\left. + \frac{a^r S^2 + b^r S + c^r + \log(r\gamma)}{\gamma}\right] + \frac{r^2\gamma^2\sigma_P^2[Q-(1-\mu)(\eta_0^e+\eta_1^e S)]^2}{2\mu^2} - \frac{\beta(2a^r S + b^r)}{1-\beta B}\left(S - \frac{g_D}{r}\right) \\ &+ a^r\beta^2\sigma_P^2 + \frac{1}{2}\beta^2\sigma_P^2(2a^r S + b^r)^2 - \frac{r\gamma\beta\sigma_P^2(2a^r S + b^r)[Q-(1-\mu)(\eta_0^e+\eta_1^e S)]}{\mu}, \end{aligned} \tag{A30}$$

which is equivalent to three simultaneous equations:

$$\begin{aligned} 0 = -r\gamma &\left[\frac{(1-\mu)\beta\eta_1^e B}{\mu(1-\beta B)} + \frac{(1-\mu)\eta_1^e rB}{\mu}\right] - ra^r + \frac{[(1-\mu)r\gamma\eta_1^e\sigma_P]^2}{2\mu^2} - \frac{2a^r\beta}{1-\beta B} + 2(a^r\beta\sigma_P)^2 \\ &+ \frac{2a^r\beta(1-\mu)r\gamma\eta_1^e\sigma_P^2}{\mu}, \end{aligned} \tag{A31}$$

$$\begin{aligned} 0 = -r\gamma &\left[\frac{Q-(1-\mu)\eta_0^e}{\mu}\left(-\frac{\beta B}{1-\beta B} - rB\right) - \frac{(1-\mu)\eta_1^e}{\mu}\left(\frac{g_D}{(1-\beta B)r} - rA\right)\right] - rb^r \\ &- \frac{r^2\gamma^2\sigma_P^2(1-\mu)\eta_1^e[Q-(1-\mu)\eta_0^e]}{\mu^2} - \frac{b^r\beta}{1-\beta B} + \frac{2\beta a^r g_D}{(1-\beta B)r} + 2\beta^2\sigma_P^2 a^r b^r \\ &- \frac{r\gamma\beta\sigma_P^2}{\mu}\left(2a^r[Q-(1-\mu)\eta_0^e] - b^r(1-\mu)\eta_1^e\right), \end{aligned} \tag{A32}$$

$$
\begin{aligned}
0 &= (r-\delta) - \frac{r\gamma[Q-(1-\mu)\eta_0^e]}{\mu}\left(\frac{g_D}{(1-\beta B)r} - rA\right) - rc^r - r\log(r\gamma) \\
&+ \frac{r^2\gamma^2\sigma_P^2[Q-(1-\mu)\eta_0^e]^2}{2\mu^2} + \frac{\beta b^r g_D}{(1-\beta B)r} + a^r\beta^2\sigma_P^2 + \frac{1}{2}(b^r\beta\sigma_P)^2 - \frac{\beta b^r r\gamma\sigma_P^2[Q-(1-\mu)\eta_0^e]}{\mu}.
\end{aligned}
\tag{A33}
$$

These three equations determine the coefficients $A$, $B$, and $c^r$. Equations (A26)-(A28) and (A31)-(A33) are the mathematical characterization of the endogenous interaction between rational traders and the extrapolators. The procedure for solving these simultaneous equations is left to the next section of the Appendix.

The fact that the conjectured forms of $P_t$, $I^e$, and $I^r$ in (A11) and (A21) satisfy the Bellman equations in (A8) for all $W_t$ and $S_t$ verifies these conjectures, conditional on the validity of the assumption that $W_t$ and $S_t$ are the only two stochastic state variables. To verify the latter, note that the price equation in (A11), the optimal consumption rules in (A22), and the fact that the solutions of $N_t^e$ and $N_t^r$ are linearly related to $S_t$ jointly guarantee that the evolutions of $W_t^e$ and $W_t^r$ in (A2) depend explicitly only on $S_t$. Lastly, the derived evolution of the stock price in (A12) verifies the assumption that $\sigma_P$ is an endogenously determined parameter. This completes the verification procedure.

Equations (A11), (A12) and (A13), (A24), and (A22) confirm equations (9), (10), (11), (13), and (14) in the main text, respectively, and equations (A7) and (A21) together confirm (12). This completes the proof of Proposition 1. ■

## B. Solving the Simultaneous Equations

To solve equations (A26), (A27), (A28), (A31), (A32), and (A33), we group them into three pairs of equations and solve each pair in sequence. First, we use (A26) and (A31) to determine $a^e$ and $B$, where, in turn, we use (A14), (A24), and (A29) to express $\sigma_P$, $\eta_1^e$, and $a^r$ as functions of $a^e$ and $B$. Second, we use (A27) and (A32) to determine $b^e$ and $A$, where, in turn, we use (A24) and (A29) to express $\eta_0^e$ and $b^r$ as functions of $b^e$, $A$, and $B$. Lastly, we solve each of (A28) and (A33) to obtain $c^e$ and $c^r$, respectively. The fact that the value function $J^i(W,S,t)$ is multiplicatively separable in $W$, $S$, and $t$ simplifies the model and ensures tractability. For instance, our model has the feature that the discount factor $\delta$ only affects optimal consumption and optimal wealth, but not the equilibrium price: for both types of investor, optimal share demand is unrelated to $\delta$.

## C. Proof of Corollary 1

When all the traders in the economy are fully rational, (A21) reduces to

$$
I^r(W^r) = -e^{-r\gamma W^r}K, \tag{A34}
$$

where $K$ is a constant to be determined. Substituting (A34) into (A10) and using $N^r = Q$, we know that the equilibrium stock price is

$$P_t = -\frac{\gamma\sigma_D^2}{r^2}Q + \frac{g_D}{r^2} + \frac{D_t}{r}. \quad \text{(A35)}$$

This third term on the right-hand side of this equation shows that $P_t$ is pegged to the current level of the dividend; the other two terms capture dividend growth and compensation for risk. Substituting (A34) and (A35) into the Bellman equation (A8) determines the coefficient $K$ as

$$K = \frac{1}{r\gamma}\exp\left(\frac{r-\delta}{r} - \frac{\gamma^2\sigma_D^2 Q^2}{2r}\right). \quad \text{(A36)}$$

From (A9), optimal consumption is

$$C = rW - \frac{1}{\gamma}\log(r\gamma K) = rW - \frac{r-\delta}{r\gamma} + \frac{\gamma\sigma_D^2 Q^2}{2r}. \quad \text{(A37)}$$

From (A2), (A35), and (A37), optimal wealth evolves according to

$$dW_t^r = \left(\frac{r-\delta}{r\gamma} + \frac{\gamma\sigma_D^2 Q^2}{2r}\right)dt + \frac{Q\sigma_D}{r}d\omega. \quad \text{(A38)}$$

This completes the proof of Corollary 1. ■

**D: Proof of Corollary 2**

Differentiating both sides of (A11) gives

$$dP_t = \beta B(-Sdt + dP_t) + \frac{g_D dt + \sigma_D d\omega}{r}. \quad \text{(A39)}$$

If there is positive cash-flow news that increases the stock price by $\Delta$, then, from (A39), the presence of sentiment in the equilibrium price will push the price up by a further amount $\beta B\Delta$, and then by a further amount $\beta^2 B^2\Delta$, and so on. The total price increase due to a shock of size $\Delta$ is therefore

$$\Delta + \beta B\Delta + \beta^2 B^2\Delta + \cdots = \Delta(1 + \beta B + \beta^2 B^2 + \cdots) = \frac{1}{1-\beta B}\Delta. \quad \text{(A40)}$$

This geometric series converges if and only if $1 - |\beta B| > 0$. That is, the price equation (A11) is an "equilibrium" price equation if and only if $1 - |\beta B| > 0$.

When all investors are extrapolators ($\mu = 0$), the market clearing condition implies

$$\eta_0^e \equiv \frac{\lambda_0 - rA + b^e\beta\sigma_P^2}{r\gamma\sigma_P^2} = Q, \quad \text{(A41)}$$

$$\eta_1^e \equiv \frac{\lambda_1 - rB + 2a^e\beta\sigma_P^2}{r\gamma\sigma_P^2} = 0 \Rightarrow \lambda_1 - rB + 2a^e\beta\sigma_P^2 = 0. \quad \text{(A42)}$$

Substituting (A42) into (A26), we obtain

$$0 = a^e[-r + 2\beta(\lambda_1 - 1) + 2a^e\beta^2\sigma_P^2] = a^e[-r(1-\beta B) - \beta(2-\lambda_1)]. \quad \text{(A43)}$$

Under the condition that $\lambda_1 < 2$, (A43) implies $a^e = 0$. Given this, (A42) then implies that

$$B = \lambda_1 / r. \quad \text{(A44)}$$

Since the necessary and sufficient condition for existence of the conjectured equilibrium is $1 - |\beta B| > 0$, (A44) now means that a necessary condition for existence is

$$\lambda_1\beta < r. \tag{A45}$$

We have not yet shown the sufficiency of this condition; to do so, we need to check (A27) and (A28) to see whether we can determine $A$, $b^e$, and $c^e$. Substituting $a^e = 0$ and (A44) into (A27), we obtain

$$b^e = 0 \qquad \text{unless } r = \beta(\lambda_1 - 1). \tag{A46}$$

With $b^e = 0$, we then obtain, from (A41), that

$$A = \frac{\lambda_0 - Qr\gamma\sigma_P^2}{r} = \frac{\lambda_0 - Q\gamma\sigma_D^2 r^{-1}(1-\beta\lambda_1 r^{-1})^{-2}}{r}. \tag{A47}$$

Now substituting $a^e = 0, b^e = 0$, (A41), (A44), and $\sigma_P = \dfrac{\sigma_D}{(1-\beta B)r}$ into (A28) gives

$$c^e = \frac{r-\delta}{r} - \frac{\gamma^2\sigma_D^2 Q^2}{2(1-\beta\lambda_1 r^{-1})^2 r} - \log(r\gamma). \tag{A48}$$

Quite generally, then, we can solve for $A$, $b^e$, and $c^e$ if condition (A45) holds. Therefore, we can claim that (A45) is both a necessary and sufficient condition.

We note that this proof does not rule out any nonlinear equilibria. ■

## E: Proofs of Propositions 2 to 10

The statistical properties of the sentiment process $S_t$ can be derived by studying a related process, $Z_t \equiv e^{kt}S_t$, which evolves according to

$$dZ_t = \frac{ke^{kt}g_D}{r}dt + e^{kt}\sigma_S d\omega. \tag{A49}$$

Unlike the sentiment process, the $Z_t$ process has a non-stochastic drift term, and is therefore easier to analyze. We use this process repeatedly in our proofs of Propositions 2 to 6.

### E.1. Proof of Proposition 2

It is straightforward to calculate the price change expectations of rational traders. Combining extrapolators' belief about the instantaneous price change, (A15), and the differential definition of the sentiment variable, (A1), we find that extrapolators' subjective belief about the evolution of $S_t$ is

$$dS_t = \beta[\lambda_0 + (\lambda_1 - 1)S_t]dt + \sigma_S d\omega^e. \tag{A50}$$

Extrapolators believe that $\omega^e$ is a standard Wiener process. This means that, from the perspective of extrapolators, the evolution of $Z_t^e \equiv e^{mt}S_t$, where $m = \beta(1 - \lambda_1)$, is

$$dZ_t^e = e^{mt}\beta\lambda_0 dt + e^{mt}\sigma_S d\omega^e. \tag{A51}$$

Using the statistical properties of the $Z_t^e$ process, as perceived by extrapolators, we obtain (22). When $m = 0$, applying L'Hôpital's rule to (22) gives (23). ■

### E.2. Proof of Proposition 3

From (A11), we know that

$$\operatorname{cov}\left(D_0/r-P_0,P_{t_1}-P_0\right)=-B^2\operatorname{cov}\left(S_0,S_{t_1}-S_0\right)-Br^{-1}\operatorname{cov}\left(S_0,D_{t_1}-D_0\right). \tag{A52}$$

It is obvious that $\operatorname{cov}\left(S_0,D_{t_1}-D_0\right)=0$. Using the properties of the $Z$ process, we can show

$$\operatorname{cov}\left(S_0,S_{t_1}-S_0\right)=-\frac{(1-e^{-kt_1})\sigma_S^2}{2k}. \tag{A53}$$

We also obtain

$$\operatorname{var}\left(D_0/r-P_0\right)=B^2\operatorname{var}\left(S_0\right)=\frac{\sigma_S^2B^2}{2k}. \tag{A54}$$

Equations (A52), (A53), and (A54) then jointly give (26). ■

### E.3. Proof of Proposition 4

For the autocorrelation structure of $P-D/r$, we know from (A11) that

$$\rho_{PD}(t_1)=\operatorname{corr}\left(S_0,S_{t_1}\right). \tag{A55}$$

We can show that

$$\begin{aligned}\operatorname{cov}\left(S_0,S_{t_1}\right)&=\mathbb{E}_s\left[\left(\mathbb{E}_0\left[S_0\middle|s\right]-\mathbb{E}_0\left[S_0\right]\right)\left(\mathbb{E}_0\left[S_{t_1}\middle|s\right]-\mathbb{E}_0\left[S_{t_1}\right]\right)\right]\\&=\mathbb{E}_s\left[e^{-kt_1}\left(s-\frac{g_D}{r}\right)^2\right]=\frac{e^{-kt_1}\sigma_S^2}{2k}.\end{aligned} \tag{A56}$$

It is straightforward to show that $\operatorname{var}(S_0)=\operatorname{var}(S_{t_1})=\sigma_S^2/2k$. Putting these results together, we obtain equation (27) in the main text. ■

### E.4. Proof of Proposition 5

From the price equation (A11), we know that the variance of price changes is given by

$$\operatorname{var}\left(P_{t_1}-P_0\right)=B^2\operatorname{var}\left(S_{t_1}-S_0\right)+2Br^{-1}\operatorname{cov}\left(S_{t_1}-S_0,D_{t_1}-D_0\right)+r^{-2}\operatorname{var}\left(D_{t_1}-D_0\right). \tag{A57}$$

The quantity $\operatorname{var}\left(S_{t_1}-S_0\right)$ can be expressed as

$$\operatorname{var}\left(S_{t_1}-S_0\right)=\mathbb{E}_s\left[\operatorname{var}\left(S_{t_1}-S_0\middle|s\right)\right]+\mathbb{E}_s\left[\left(\mathbb{E}\left[S_{t_1}-S_0\middle|s\right]-\mathbb{E}\left[S_{t_1}-S_0\right]\right)^2\right], \tag{A58}$$

where the subscript $s$ means that we are conditioning on $S_0=s$. We can show

$$\operatorname{var}\left(S_{t_1}-S_0\middle|s\right)=e^{-2kt_1}\operatorname{var}\left(Z_{t_1}-Z_0\middle|s\right)=e^{-2kt_1}\int_0^{t_1}e^{2kt}\sigma_S^2dt=\frac{(1-e^{-2kt_1})\sigma_S^2}{2k}. \tag{A59}$$

Using the properties of the $Z$ process, we also find that

$$\mathbb{E}_0\left[S_{t_1}-S_0\middle|s\right]=(1-e^{-kt_1})\left(\frac{g_D}{r}-s\right), \tag{A60}$$

and

$$\operatorname{cov}\left(S_{t_1}-S_0, D_{t_1}-D_0\right)=\frac{(1-e^{-kt_1})\sigma_S\sigma_D}{k}. \tag{A61}$$

Substituting (A59) and (A60) into (A58) gives

$$\operatorname{var}\left(S_{t_1}-S_0\right)=\frac{(1-e^{-kt_1})\sigma_S^2}{k}. \tag{A62}$$

Substituting (A61), (A62), and $\operatorname{var}(D_{t_1}-D_0)=\sigma_D^2 t_1$ into (A57) gives equation (28) in the main text. Combining the price equation (A11) with (A62) leads to (29). ■

**E.5. Proof of Proposition 6**

From (A11), we know that

$$\begin{aligned}\operatorname{cov}\left(P_{t_1}-P_0, P_{t_3}-P_{t_2}\right) &= B^2\operatorname{cov}\left(S_{t_1}-S_0, S_{t_3}-S_{t_2}\right)+r^{-2}\operatorname{cov}\left(D_{t_1}-D_0, D_{t_3}-D_{t_2}\right)\\ &\quad +Br^{-1}\operatorname{cov}\left(S_{t_1}-S_0, D_{t_3}-D_{t_2}\right)+Br^{-1}\operatorname{cov}\left(S_{t_3}-S_{t_2}, D_{t_1}-D_0\right).\end{aligned} \tag{A63}$$

Using the properties of the $Z$ process, we obtain

$$\operatorname{cov}\left(S_{t_1}-S_0, S_{t_3}-S_{t_2}\right)=\frac{\sigma_S^2}{2k}\left(e^{-kt_3}-e^{-kt_2}\right)\left(e^{kt_1}-1\right), \tag{A64}$$

and

$$\operatorname{cov}\left(D_{t_1}-D_0, S_{t_3}-S_{t_2}\right)=\frac{\sigma_S\sigma_D}{k}\left(e^{-kt_3}-e^{-kt_2}\right)\left(e^{kt_1}-1\right). \tag{A65}$$

In addition, since the increments in future dividends are independent of any random variable that is measurable with respect to the information set at the current time,

$$\operatorname{cov}\left(D_{t_1}-D_0, D_{t_3}-D_{t_2}\right)=\operatorname{cov}\left(S_{t_1}-S_0, D_{t_3}-D_{t_2}\right)=0. \tag{A66}$$

Substituting (A64), (A65), and (A66) into (A63) yields the first equation in (31). The second equation in (31) is derived in Proposition 5, and the third equation can be derived in a similar way. ■

**E.6. Proof of Propositions 7 to 9**

From the budget constraints (A2), the price equation (A11), and the optimal consumptions (A22), we know that aggregate wealth evolves as

$$dW=(a_W S_t^2+b_W S_t+c_W)dt+\sigma_W d\omega. \tag{A67}$$

Substituting this into (A22) yields

$$\begin{aligned}C_{t_1}-C_0 &= r(W_{t_1}-W_0)-\gamma^{-1}[a(S_{t_1}^2-S_0^2)+b(S_{t_1}-S_0)]\\ &= r\int_0^{t_1}(a_W S_t^2+b_W S_t+c_W)dt+r\sigma_W\int_0^{t_1}d\omega-\gamma^{-1}[a(S_{t_1}^2-S_0^2)+b(S_{t_1}-S_0)].\end{aligned} \tag{A68}$$

To compute $\operatorname{cov}(C_{t_1} - C_0, P_{t_1} - P_0)$ and $\operatorname{var}(C_{t_1} - C_0)$, we first need to compute the covariance of every combination of two terms in the last line of (A68). For example, one of these covariances is[17]

$$\begin{aligned}\operatorname{cov}\left(\int_0^{t_1} S_t dt, S_{t_1}\right) &= \int_0^{t_1} \mathbb{E}_s\left[\mathbb{E}_0\left[S_t S_{t_1} \middle| s\right]\right] dt - \mathbb{E}_s\left[\mathbb{E}_0\left[S_{t_1} \middle| s\right]\right] \int_0^{t_1} \mathbb{E}_s\left[\mathbb{E}_0\left[S_t \middle| s\right]\right] dt \\ &= \frac{(1-e^{-kt_1})\sigma_S^2}{2k}.\end{aligned} \tag{A69}$$

The other covariance terms can be computed in a similar way. Rearranging and simplifying terms, we obtain (33), (35), (37), and (39). Equation (34) has been derived in Proposition 5. ■

### E.7. Proof of Propositions 10

Substituting the equilibrium price equation (A11) and its evolution (A12) into our definition of the equity premium, $\frac{1}{dt}\mathbb{E}\left[dP_t + D_t dt - rP_t dt\right]$, gives (40) in the main text. For the Sharpe ratio, by the law of total variance,

$$\begin{aligned}&\operatorname{Var}\left[\tfrac{1}{dt}(dP_t + D_t dt - rP_t dt)\right] \\ &= \mathbb{E}_s\left[\operatorname{Var}\left[\tfrac{1}{dt}(dP_t + D_t dt - rP_t dt)\middle| s = S_t\right]\right] + \operatorname{Var}_s\left[\mathbb{E}\left[\tfrac{1}{dt}(dP_t + D_t dt - rP_t dt)\middle| s = S_t\right]\right] \\ &= \frac{\sigma_D^2}{(1-\beta B)^2 r^2} + \left(\frac{\beta B}{1-\beta B} + rB\right)^2 \frac{\beta\sigma_D^2}{2(1-\beta B) r^2}.\end{aligned} \tag{A70}$$

Combining (40) with (A70) gives (41). ■

## F: Estimating β

*Estimating Equations*

Our objective is to estimate the model parameters $\beta$, $\lambda_0$, and $\lambda_1$ using the survey data.

Suppose we have a time-series of aggregate stock market prices with sample frequency $\Delta t$ (we use $\Delta t$ = ¼ for quarterly data). Then, at time $t$, the proper discretization of (A1) is

$$S_t(\beta, n) = \sum_{j=0}^{n-1} w(j;\beta,n)(P_j - P_{j-\Delta t}), \tag{A71}$$

where $w(j;\beta,n) = \dfrac{e^{-\beta j\Delta t}}{\sum_{k=0}^{n-1} e^{-\beta k\Delta t}}$. Here the weighting functions are parameterized by $\beta$ and by $n$, which measures how far back investors look when forming their beliefs. These weights must sum to 1.

---

[17] The derivation of (A69) makes use of Fubini's theorem. We have checked that the conditions that allow the use of Fubini's theorem hold in our context. For more on these conditions, see Theorem 1.9 in Liptser and Shiryaev (2001).

The key assumption of our model is that extrapolators' expected price change (not expected return) is

$$\mathbb{E}_t^e[dP_t]/dt = \lambda_0 + \lambda_1 S_t(\beta). \tag{A72}$$

The expectation in (A72) is computed over the next instant of time, from $t$ to $t + dt$, not over a finite time horizon. In the surveys, however, investors are typically asked to state their beliefs about stock market performance over the next year. It is therefore not fully correct to estimate ($\beta$, $\lambda_0$, $\lambda_1$) using (A72). We must instead compute what the model implies for the price change extrapolators expect over a finite horizon. We do this in Proposition 2 of the paper, and find:

$$\mathbb{E}_t^e[P_{t_1} - P_t \mid S_t = s] = (\lambda_0 + \lambda_1 s)(t_1 - t) + \lambda_1(\beta\lambda_0 - ms)\frac{m(t_1 - t) + e^{-m(t_1 - t)} - 1}{m^2}, \tag{A73}$$

where $m = \beta(1 - \lambda_1)$.

The first term on the right-hand side of (A73) is extrapolators' expected price change at time $t$, $\lambda_0 + \lambda_1 s$, multiplied by the time horizon, $t_1 - t$. (For example, $t_1 - t = 0.5$ for a six-month horizon). The second term captures extrapolators' subjective beliefs about how the sentiment level will evolve over the time horizon, $t_1 - t$, The parameters ($\beta$, $\lambda_0$, $\lambda_1$) enter here in a non-linear fashion.

To determine ($\beta$, $\lambda_0$, $\lambda_1$), we therefore estimate both

$$\mathbb{E}_t^e[P_{t_1} - P_t] = [\hat{\lambda}_0 + \hat{\lambda}_1 S_t(\hat{\beta})](t_1 - t) + \varepsilon_{t_1}, \tag{A74}$$

and

$$\mathbb{E}_t^e[P_{t_1} - P_t] = [\hat{\lambda}_0 + \hat{\lambda}_1 S_t(\hat{\beta})](t_1 - t) + \hat{\lambda}_1(\hat{\beta}\hat{\lambda}_0 - mS_t(\hat{\beta}))\frac{m(t_1 - t) + e^{-m(t_1 - t)} - 1}{m^2} + \varepsilon_{t_1}, \tag{A75}$$

with $m(\hat{\beta}, \hat{\lambda}_1) = \hat{\beta}(1 - \hat{\lambda}_1)$ and $S_t(\hat{\beta})$ constructed as described above. We also estimate equation (A75) for the special case where $\lambda_1$ is fixed at 1. In this case, equation (A75) becomes:

$$\mathbb{E}_t^e[P_{t_1} - P_t] = [\hat{\lambda}_0 + S_t(\hat{\beta})](t_1 - t) + \frac{\hat{\beta}\hat{\lambda}_0 (t_1 - t)^2}{2} + \varepsilon_{t_1}. \tag{A76}$$

*Survey Data*

We estimate equations (A74), (A75), and (A76) using the Gallup survey data studied by Greenwood and Shleifer (2013) and others. We start with the "rescaled" version of the series described in that paper. After the rescaling, the reported expectations are in units of percentage expected returns on the aggregate stock market over the following 12 months. We then convert this series into expected *price* changes by multiplying by the level of the S&P 500 price index at the end of the month in which participants have been surveyed. That is,

$$\mathbb{E}_t^e[P_{t_1} - P_t] = \underbrace{\mathbb{E}_t^e[\frac{P_{t_1} - P_t}{P_t}]}_{Survey} \cdot P_t \tag{A77}$$

The resulting Gallup series comprises 135 datapoints between October 1996 and November 2011. The data are monthly but there are also some gaps.

We estimate equations (A74), (A75), and (A76) using nonlinear least squares regression. We use 60 quarters of lagged price changes in the S&P 500 price index when constructing *S* above. We report coefficients and Newey West standard errors using a lag length of 6 months.

| Coefficient | Equation (A74) | Equation (A75) | Equation (A76) |
|---|---|---|---|
| $\beta$ | 0.49 | 0.44 | 0.68 |
| [*t*-stat] | [6.50] | [5.77] | [10.73] |
| $\lambda_0$ | 0.09 | 0.07 | 0.07 |
| [*t*-stat] | [30.24] | [35.41] | [36.18] |
| $\lambda_1$ | 1.35 | 1.32 | |
| [*t*-stat] | [8.70] | [9.48] | |
| R-squared | 0.77 | 0.74 | 0.75 |

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## Table 1: Selected Models of the Aggregate Stock Market

| | | Model allows for intermediate consumption | D/P predicts returns | Accounts for volatility | Accounts for equity premium | Accounts for survey evidence |
|---|---|---|---|---|---|---|
| **TRADITIONAL** | | | | | | |
| Habit | Campbell and Cochrane (1999) | Yes | Yes | Yes | Yes | No |
| Long-run risk | Bansal and Yaron (2004) | Yes | No | Yes | Yes | No |
| | Bansal, Kiku, and Yaron (2012) | Yes | Yes | Yes | Yes | No |
| Rare disasters | Rietz (1988), Barro (2006) | Yes | No | No | Yes | No |
| | Gabaix (2012), Wachter (2013) | Yes | Yes | Yes | Yes | No |
| **LEARNING** | Timmerman (1993) | Yes | Yes | Yes | No | No |
| **BEHAVIORAL** | | | | | | |
| **Preference-based** | | | | | | |
| Prospect theory | Barberis, Huang, and Santos (2001) | Yes | Yes | Yes | Yes | No |
| Ambiguity aversion | Ju and Miao (2012) | Yes | Yes | Yes | Yes | No |
| **Belief-based** | | | | | | |
| Noise trader risk | De Long et al. (1990a) | Yes | Yes | Yes | No | No |
| Extrapolation of fundamentals | Barberis, Shleifer, Vishny (1998) | No | Yes | Yes | No | No |
| | Choi (2006) | Yes | Yes | Yes | No | No |
| | Fuster, Herbert, Laibson (2011) | Yes | Yes | Yes | No | No |
| | Alti and Tetlock (2013) | No | Yes | Yes | No | No |
| | Hirshleifer and Yu (2013) | Yes | Yes | Yes | No | No |
| Extrapolation of returns | Cutler, Poterba, Summers (1989) | No | Yes | Yes | No | Yes |
| | De Long et al. (1990b) | No | Yes | Yes | No | Yes |
| | Hong and Stein (1999) | No | Yes | Yes | No | Yes |
| | Barberis and Shleifer (2003) | No | Yes | Yes | No | Yes |
| | Barberis, Greenwood, Jin, and Shleifer (2013) | Yes | Yes | Yes | No | Yes |

## Table 2: Parameter Values

The table reports the values we assign to the risk-free rate $r$; the per unit time mean $g_D$ and standard deviation $\sigma_D$ of dividend changes; the risky asset per-capita supply $Q$; absolute risk aversion $\gamma$; the discount rate $\delta$; the proportion $\mu$ of rational traders in the economy; the parameters $\beta$, $\lambda_0$ and $\lambda_1$ which govern the beliefs of extrapolators; the initial level of the dividend $D_0$; and the initial wealth levels, $W_0^e$ and $W_0^r$, of extrapolators and rational traders, respectively.

| Parameter | Value |
|---|---|
| $r$ | 2.50% |
| $g_D$ | 0.05 |
| $\sigma_D$ | 0.25 |
| $Q$ | 5 |
| $\gamma$ | 0.1 |
| $\delta$ | 1.50% |
| $\mu$ | {0.25, 0.5, 0.75, 1} |
| $\beta$ | {0.05, 0.5, 0.75} |
| $\lambda_0$ | 0 |
| $\lambda_1$ | 1 |
| $D_0$ | 10 |
| $W_0^e$ | 5000 |
| $W_0^r$ | 5000 |

## Table 3: Predictive Power of *D*/*r* − *P* for Future Stock Price Changes

The table reports the population estimate of the regression coefficient when regressing the price change from time *t* to time *t* + *k* (in quarters) on the time *t* level of *D*/*r* − *P* for *k* =1, 4, 8, 12, and 16, and for various pairs of values of the parameters μ and β:

$$P_{t+k} - P_t = a + b(D_t / r - P_t) + \varepsilon_{t+k} .$$

The calculations make use of Proposition 3 in the main text.

| | | μ | | | |
|---|---|---|---|---|---|
| β | *k* | 1 | 0.75 | 0.5 | 0.25 |
| 0.05 | 1 | - | 0.014 | 0.016 | 0.022 |
| | 4 | - | 0.055 | 0.064 | 0.085 |
| | 8 | - | 0.106 | 0.124 | 0.162 |
| | 12 | - | 0.155 | 0.180 | 0.233 |
| | 16 | - | 0.201 | 0.233 | 0.298 |
| 0.5 | 1 | - | 0.134 | 0.161 | 0.219 |
| | 4 | - | 0.438 | 0.504 | 0.628 |
| | 8 | - | 0.684 | 0.754 | 0.861 |
| | 12 | - | 0.822 | 0.878 | 0.948 |
| | 16 | - | 0.900 | 0.940 | 0.981 |
| 0.75 | 1 | - | 0.194 | 0.232 | 0.311 |
| | 4 | - | 0.579 | 0.652 | 0.774 |
| | 8 | - | 0.822 | 0.879 | 0.949 |
| | 12 | - | 0.925 | 0.958 | 0.988 |
| | 16 | - | 0.968 | 0.985 | 0.997 |

## Table 4: Autocorrelations of $P - D/r$

The table reports the autocorrelation of $P - D/r$ at various lags $k$ (in quarters) and for various pairs of values of the parameters $\mu$ and $\beta$. The calculations make use of Proposition 4 in the main text.

| | | $\mu$ | | | |
|---|---|---|---|---|---|
| $\beta$ | $k$ | 1 | 0.75 | 0.5 | 0.25 |
| 0.05 | 1 | - | 0.986 | 0.984 | 0.978 |
| | 4 | - | 0.945 | 0.936 | 0.915 |
| | 8 | - | 0.894 | 0.876 | 0.838 |
| | 12 | - | 0.845 | 0.820 | 0.767 |
| | 16 | - | 0.799 | 0.767 | 0.702 |
| 0.5 | 1 | - | 0.866 | 0.839 | 0.781 |
| | 4 | - | 0.562 | 0.496 | 0.372 |
| | 8 | - | 0.316 | 0.246 | 0.139 |
| | 12 | - | 0.178 | 0.122 | 0.052 |
| | 16 | - | 0.100 | 0.060 | 0.019 |
| 0.75 | 1 | - | 0.806 | 0.768 | 0.689 |
| | 4 | - | 0.421 | 0.348 | 0.226 |
| | 8 | - | 0.178 | 0.121 | 0.051 |
| | 12 | - | 0.075 | 0.042 | 0.012 |
| | 16 | - | 0.032 | 0.015 | 0.003 |

## Table 5: Volatility of Price Changes and Volatility of $P - D/r$

Panel A reports the volatility of annual price changes for various pairs of values of the parameters μ and β; Panel B reports the volatility of $P - D/r$, measured at an annual frequency, for various pairs of μ and β. The calculations make use of Proposition 5 in the main text.

Panel A: Volatility of Annual Price Changes

| | μ | | | |
|---|---|---|---|---|
| β | 1 | 0.75 | 0.5 | 0.25 |
| 0.05 | 10 | 11.20 | 13.15 | 17.43 |
| 0.5 | 10 | 11.17 | 13.03 | 16.86 |
| 0.75 | 10 | 11.04 | 12.67 | 15.90 |

Panel B: Volatility of Annual $P - D/r$

| | μ | | | |
|---|---|---|---|---|
| β | 1 | 0.75 | 0.5 | 0.25 |
| 0.05 | 0 | 1.21 | 3.19 | 7.53 |
| 0.5 | 0 | 1.32 | 3.42 | 7.77 |
| 0.75 | 0 | 1.25 | 3.20 | 7.09 |

## Table 6: Autocorrelations of Price Changes

The table reports the autocorrelations of quarterly and cumulative stock price changes at various lags *k* (in quarters) and for various pairs of values of the parameters μ and β. The calculations make use of Proposition 6 in the main text.

| | | Autocorrelations at Horizon *k* | | | | Cumulative Autocorrelations to Horizon *k* | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | μ | | | | μ | | | |
| β | *k* | 1 | 0.75 | 0.5 | 0.25 | 1 | 0.75 | 0.5 | 0.25 |
| 0.05 | 1 | 0 | -0.001 | -0.003 | -0.007 | 0 | -0.001 | -0.003 | -0.007 |
| | 2 | 0 | -0.001 | -0.003 | -0.007 | 0 | -0.002 | -0.005 | -0.010 |
| | 3 | 0 | -0.001 | -0.003 | -0.007 | 0 | -0.002 | -0.006 | -0.013 |
| | 4 | 0 | -0.001 | -0.003 | -0.007 | 0 | -0.003 | -0.007 | -0.015 |
| | 8 | 0 | -0.001 | -0.003 | -0.006 | 0 | -0.004 | -0.010 | -0.020 |
| | 12 | 0 | -0.001 | -0.003 | -0.006 | 0 | -0.005 | -0.011 | -0.024 |
| 0.5 | 1 | 0 | -0.016 | -0.038 | -0.079 | 0 | -0.016 | -0.038 | -0.079 |
| | 2 | 0 | -0.013 | -0.032 | -0.062 | 0 | -0.021 | -0.050 | -0.103 |
| | 3 | 0 | -0.012 | -0.027 | -0.048 | 0 | -0.024 | -0.058 | -0.118 |
| | 4 | 0 | -0.010 | -0.022 | -0.038 | 0 | -0.026 | -0.063 | -0.127 |
| | 8 | 0 | -0.006 | -0.011 | -0.014 | 0 | -0.029 | -0.070 | -0.138 |
| | 12 | 0 | -0.003 | -0.006 | -0.005 | 0 | -0.029 | -0.069 | -0.134 |
| 0.75 | 1 | 0 | -0.022 | -0.054 | -0.110 | 0 | -0.022 | -0.054 | -0.110 |
| | 2 | 0 | -0.018 | -0.041 | -0.076 | 0 | -0.029 | -0.069 | -0.140 |
| | 3 | 0 | -0.014 | -0.032 | -0.053 | 0 | -0.032 | -0.077 | -0.154 |
| | 4 | 0 | -0.012 | -0.024 | -0.036 | 0 | -0.034 | -0.081 | -0.161 |
| | 8 | 0 | -0.005 | -0.008 | -0.008 | 0 | -0.035 | -0.083 | -0.159 |
| | 12 | 0 | -0.002 | -0.003 | -0.002 | 0 | -0.033 | -0.077 | -0.147 |

## Table 7: Consumption, *P* – *D*/*r*, and Price Changes

Panel A shows the correlation between quarterly changes in consumption and quarterly changes in price; Panel B shows the correlation between annual changes in consumption and annual changes in price; Panel C shows the correlation between the annual change in consumption and *P* – *D*/*r*. The calculations make use of Propositions 7 and 8 in the main text.

Panel A: Correlation between quarterly consumption changes and quarterly price changes

| | μ | | | |
|---|---|---|---|---|
| β | 1 | 0.75 | 0.5 | 0.25 |
| 0.05 | 1 | 0.994 | 0.985 | 0.984 |
| 0.5 | 1 | 0.929 | 0.842 | 0.840 |
| 0.75 | 1 | 0.903 | 0.794 | 0.792 |

Panel B: Correlation between annual consumption changes and annual price changes

| | μ | | | |
|---|---|---|---|---|
| β | 1 | 0.75 | 0.5 | 0.25 |
| 0.05 | 1 | 0.994 | 0.985 | 0.984 |
| 0.5 | 1 | 0.947 | 0.878 | 0.876 |
| 0.75 | 1 | 0.935 | 0.853 | 0.849 |

Panel C: Correlation between quarterly consumption changes and *P* – *D*/*r*

| | μ | | | |
|---|---|---|---|---|
| β | 1 | 0.75 | 0.5 | 0.25 |
| 0.05 | - | 0.152 | 0.148 | 0.148 |
| 0.5 | - | 0.436 | 0.398 | 0.409 |
| 0.75 | - | 0.504 | 0.446 | 0.456 |

## Table 8: Predictive Power of Changes in Consumption for Future Price Changes

The table reports the population estimate of the regression coefficient when regressing the price change from time $t$ to time $t + k$ (in quarters) on the most recent quarterly consumption change for $k$ = 1, 4, 8, 12, 16, and for various pairs of values of the parameters $\mu$ and $\beta$:

$$P_{t+k} - P_t = a + b(C_t - C_{t-1}) + \varepsilon_{t+k}\,.$$

The calculations make use of Proposition 9 in the main text.

| | | μ | | | |
|---|---|---|---|---|---|
| β | *k* | 1 | 0.75 | 0.5 | 0.25 |
| 0.05 | 1 | 0 | -0.011 | -0.026 | -0.053 |
| | 4 | 0 | -0.043 | -0.101 | -0.205 |
| | 8 | 0 | -0.084 | -0.195 | -0.393 |
| | 12 | 0 | -0.123 | -0.284 | -0.565 |
| | 16 | 0 | -0.159 | -0.366 | -0.722 |
| 0.5 | 1 | 0 | -0.107 | -0.214 | -0.442 |
| | 4 | 0 | -0.350 | -0.671 | -1.266 |
| | 8 | 0 | -0.547 | -1.003 | -1.738 |
| | 12 | 0 | -0.658 | -1.168 | -1.914 |
| | 16 | 0 | -0.720 | -1.250 | -1.979 |
| 0.75 | 1 | 0 | -0.144 | -0.270 | -0.552 |
| | 4 | 0 | -0.429 | -0.759 | -1.375 |
| | 8 | 0 | -0.609 | -1.023 | -1.686 |
| | 12 | 0 | -0.686 | -1.115 | -1.756 |
| | 16 | 0 | -0.718 | -1.147 | -1.772 |

## Table 9: Equity Premia and Sharpe Ratios

Panel A reports equity premia for various pairs of values of the parameters μ and β; Panel B reports Sharpe ratios for various pairs of μ and β. The calculations make use of Proposition 10 in the main text.

Panel A: Equity Premia

| | μ | | | |
|---|---|---|---|---|
| β | 1 | 0.75 | 0.5 | 0.25 |
| 0.05 | 1.25 | 1.58 | 2.19 | 3.91 |
| 0.5 | 1.25 | 1.65 | 2.46 | 4.88 |
| 0.75 | 1.25 | 1.66 | 2.48 | 4.92 |

Panel B: Sharpe Ratios

| | μ | | | |
|---|---|---|---|---|
| β | 1 | 0.75 | 0.5 | 0.25 |
| 0.05 | 0.125 | 0.140 | 0.165 | 0.220 |
| 0.5 | 0.125 | 0.143 | 0.173 | 0.233 |
| 0.75 | 0.125 | 0.143 | 0.172 | 0.227 |

## Table 10: Model Predictions for Ratio-based Quantities

The table summarizes the model's predictions for ratio-based quantities. A full description of these quantities can be found in Section 5.2 of the main text. The values of the basic model parameters are given in Table 2, and μ (the fraction of rational traders) is 0.25. For β = 0.05, 0.5, and 0.75, we report estimates of each quantity averaged over 10,000 simulated paths. In rows (1), (8), and (9), we report both a regression coefficient and, in parentheses, an R-squared. The right column shows the empirical estimates for the post-war period from 1947-2011 (1952-2011 for consumption-related quantities because nondurable consumption data are available only from 1952).

| quantity of interest | β | | | post-war U.S. stock market data | |
|---|---|---|---|---|---|
| | 0.05 | 0.5 | 0.75 | | |
| (1) predictive power of log($D/P$) | 0.29 (0.20) | 0.46 (0.22) | 0.45 (0.21) | 0.11 (0.08) | |
| (2) autocorrelation of $P/D$ | 0.93 | 0.84 | 0.85 | 0.94 | |
| (3) excess volatility of returns | 2.32 | 2.91 | 2.97 | - | |
| (4) excess volatility of $P/D$ | 7.21 | 4.85 | 4.55 | - | |
| (5) autocorrelation of log excess return ($k$ = 1) | -0.01 | -0.09 | -0.14 | 0.11 | |
| autocorrelation of log excess return ($k$ = 8) | -0.01 | -0.01 | -0.01 | -0.02 | |
| (6) correlation between $\Delta_4 c$ and $r_t^x$ | 0.72 | 0.54 | 0.47 | 0.32 | |
| (7) correlation between surplus consumption and $P/D$ | 0.26 | 0.33 | 0.27 | 0.10 | |
| (8) predictive power of the surplus consumption ratio | -0.27 (0.15) | -0.89 (0.18) | -0.77 (0.17) | -0.77 (0.09) | |
| (9) predictive power of log($C/W$) | 0.51 (0.15) | 0.12 (0.15) | 0.01 (0.15) | 0.02 (0.03) | 0.33 (0.05) |
| (10) equity premium | 1.16% | 1.62% | 1.64% | 7.97% | |
| Sharpe ratio | 0.12 | 0.14 | 0.13 | 0.44 | |