Lévy Processes and Option Pricing

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Outline

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Diffusion, jumps, & Lévy processes

- A Lévy process is a continuous-time process that generates stationary, independent increments ...
- Think of return innovations (ε) in discrete time: $R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1}$.
  - Normal return innovation — diffusion
  - Non-normal return innovation — jumps
- Traditional Lévy specifications:
  - either a Brownian motion (Black-Scholes)
  - or a compound Poisson process with normal jump size (Merton).

  $\Rightarrow$ The return innovation distribution is either normal or mixture of normals.

Lévy processes and return innovations

- Lévy processes greatly expand our continuous-time choices of iid return innovation distributions via the Lévy triplet $(\mu, \sigma, \pi(x))$. ($\pi(x)$–Lévy density).
- The Lévy-Khintchine Theorem:

$$\begin{aligned} \phi_{X_t}(u) &\equiv \mathbb{E}\left[e^{iuX_t}\right] = e^{-t\psi(u)}, \\ \psi(u) &= -iu\mu + \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}_0}\left(1 - e^{iux} + iux1_{|x|<1}\right)\pi(x)dx, \end{aligned}$$

Innovation distribution
$\leftrightarrow$ characteristic exponent $\psi(u)$
$\leftrightarrow$ Lévy triplet $(\mu, \sigma, \pi(x))$

  - Constraint: $\int_0^1 x^2\pi(x)dx < \infty$ (finite quadratic variation).

Tractable examples

- Brownian motion ($\mu t + \sigma W_t$): normal shocks.
- Merton's compound Poisson jumps: Large but rare events.

$$\pi(x) = \lambda \frac{1}{\sqrt{2\pi v_J}} \exp\left(-\frac{(x-\mu_J)^2}{2v_J}\right).$$

- Dampened power law (DPL):

$$\pi(x) = \begin{cases} \lambda \exp\left(-\beta_+ x\right) x^{-\alpha-1}, & x > 0, & \lambda, \beta_\pm > 0, \\ \lambda \exp\left(-\beta_- |x|\right) |x|^{-\alpha-1}, & x < 0, & \alpha \in [-1, 2) \end{cases}$$

  - Finite activity when $\alpha < 0$: $\int_{\mathbb{R}^0} \pi(x) dx < \infty$. Large and rare events.
  - Infinite activity when $\alpha \geq 0$: Both small and large jumps. Jump frequency increase with declining jump size, and approaches infinity as $x \to 0$.
  - Infinite variation when $\alpha \geq 1$: many small jumps.

Analytical characteristic exponents

- Diffusion: $\psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2$.
- Merton's compound Poisson jumps:

$$\psi(u) = \lambda\left(1 - e^{iu\mu_J - \frac{1}{2}u^2 v_J}\right).$$

- Dampened power law: (for $\alpha \neq 0, 1$)
  $\psi(u) = -\lambda\Gamma(-\alpha)\left[(\beta_+ - iu)^\alpha - \beta_+^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha\right]$
  - When $\alpha \to 2$, smooth transition to diffusion (quadratic function of u).
  - When $\alpha = 0$ (Variance-gamma by Madan et al):

$$\psi(u) = \lambda \ln\left(1 - iu/\beta_+\right)\left(1 + iu/\beta_-\right).$$

  - When $\alpha = 1$, exponentially dampened Cauchy, Wu (06):
    $\psi(u) = -\lambda\left((\beta_+ - iu)\ln\left(\beta_+ - iu\right)/\beta_+ + \lambda\left(\beta_- + iu\right)\ln\left(\beta_- + iu\right)/\beta_-\right).$

Other Lévy examples

- The normal inverse Gaussian (NIG) process of Barndorff-Nielsen (1998)
- The generalized hyperbolic process (Eberlein, Keller, Prause (1998))
- The Meixner process (Schoutens (2003))
- All tractable in terms of the characteristic exponents $\psi(u)$.
- We can use FFT to generate the density function of the innovation (for model estimation).
- We can also use FFT to compute option values.

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Run Brownian motions on a business clock

- Clark (1973): If one runs a Brownian motion on a business clock, the resulting process matches financial time series better.
- The possibility that business clock may not move while calendar time marches forward is important ...
  - A standard Poisson process $\Rightarrow$ the resulting process is a compound Poisson process with normal jump sizes.
  - A compound Poisson process with exponentially distributed jump size $\Rightarrow$ double-exponential compound Poisson process. (DPL with $\alpha = -1$)
  - A gamma process $\Rightarrow$ variance gamma (DPL with $\alpha = 0$).
  - A continuous clock $\Rightarrow$ a continuous process.

General evidence on Lévy return innovations

- Credit risk: (compound) Poisson process
  - The whole intensity-based credit modeling literature...
- Market risk: Infinite-activity jumps
  - Evidence from stock returns (CGMY (2002)): The α estimates for DPL on most stock return series are greater than zero.
  - Evidence from options: Models with infinite-activity return innovations price equity index options better (Carr and Wu (2003), Huang and Wu (2004))
  - Li, Wells, and Yu (2006): Infinite-activity jumps cannot be approximated by finite-activity jumps.

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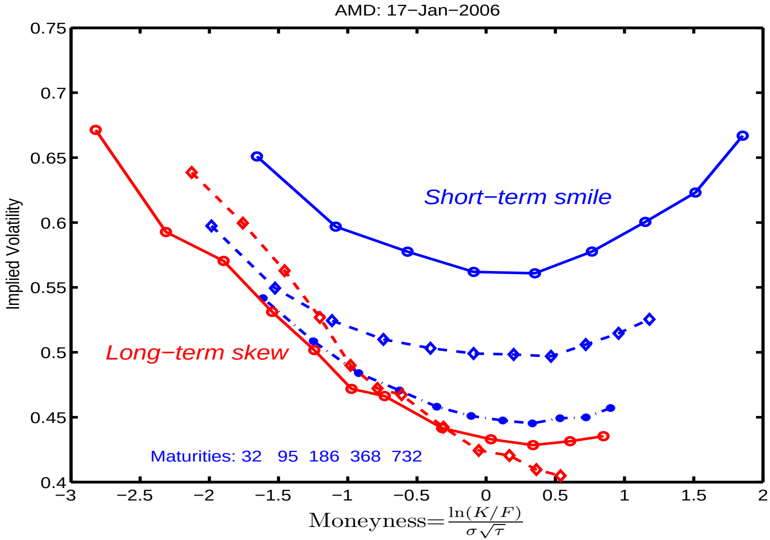
Implied volatility smiles & skews on a stock



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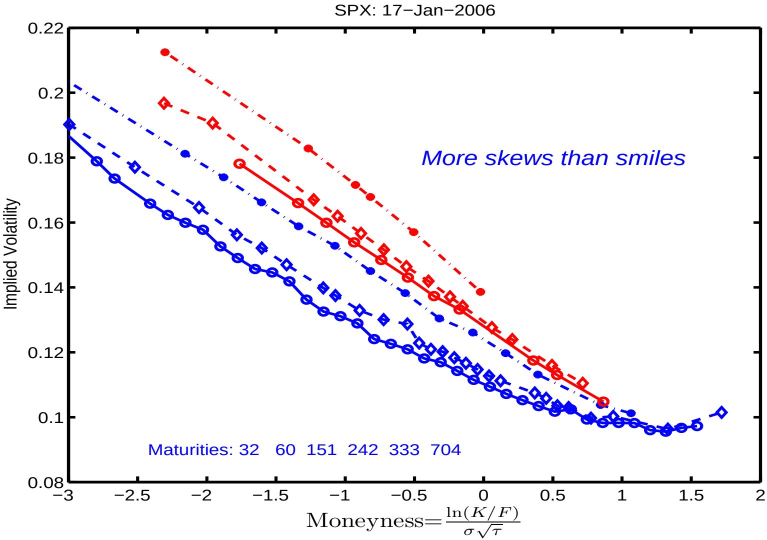
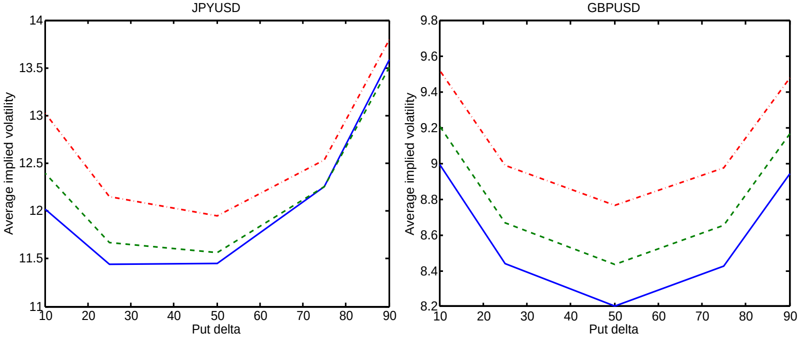
Implied volatility skews on SPX



Average implied volatility smiles on currencies



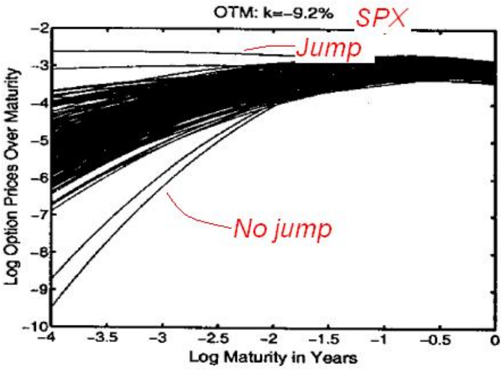
Maturities: 1m (solid), 3m (dashed), 1y (dash-dotted)

Implied volatility smiles at short maturities

- Implied volatility smiles/skews $\leftrightarrow$ non-normality/asymmetry for the underlying asset return risk-neutral distribution.
- Both jumps and stochastic volatility can generate return normalities, through different mechanisms.
  $R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1}$
  - Jumps generate non-normality through the innovation distribution (ε).
  - Stochastic volatility generates non-normality through mixing over multiple periods.
- Over short maturities (1 period), only jumps contribute to return non-normalities.

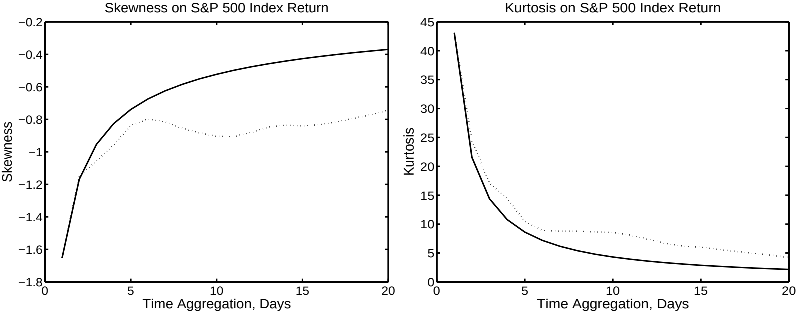
Time decay of short-term OTM options

- As option maturity ↓ zero, OTM option value ↓ zero.
- The speed of decay is exponential $\mathcal{O}(e^{-c/T})$ under pure diffusion, but linear $\mathcal{O}(T)$ in the presence of jumps.
- Term decay plot (Carr&Wu,2003): $\ln(T) \sim \ln(OTM/T)$:



Central Limit Theorem (CLT) at long horizons

- CLT: As option maturity increases, the smile should flatten.
- Evidence: The skew does not flatten, but steepens!
- FMLS: Maximum negatively skewed α-stable Lévy process.
  - Return variance is infinite. Hence, CLT does not apply.
  - All price moments are finite. Option has finite value.
- *But CLT seems to hold fine statistically:*



Reconcile $\mathbb{P}$ with $\mathbb{Q}$ via DPL

- Model return innovations under $\mathbb{P}$ by DPL:

$$\pi(x) = \begin{cases} \lambda \exp\left(-\beta_+ x\right) x^{-\alpha-1}, & x > 0, \\ \lambda \exp\left(-\beta_- |x|\right) |x|^{-\alpha-1}, & x < 0. \end{cases}$$

  All return moments are finite with $\beta_\pm > 0$. *CLT applies*.

- Apply different market prices for up and down jumps: $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t = \exp(-\gamma^+ J^+ - \gamma^- J^- + \text{convexity adjustment})$
- The return innovation process remains DPL under $\mathbb{Q}$:

$$\pi(x) = \begin{cases} \lambda \exp\left(-\left(\beta_+ + \gamma^+\right) x\right) x^{-\alpha-1}, & x > 0, \\ \lambda \exp\left(-\left(\beta_- - \gamma^-\right) |x|\right) |x|^{-\alpha-1}, & x < 0. \end{cases}$$

- To break CLT under $\mathbb{Q}$, set $\gamma^- = \beta_-$ so that $\beta_-^{\mathbb{Q}} = 0$.
- Reconciling $\mathbb{P}$ with $\mathbb{Q}$: *Investors charge maximum allowed market price on down jumps.*

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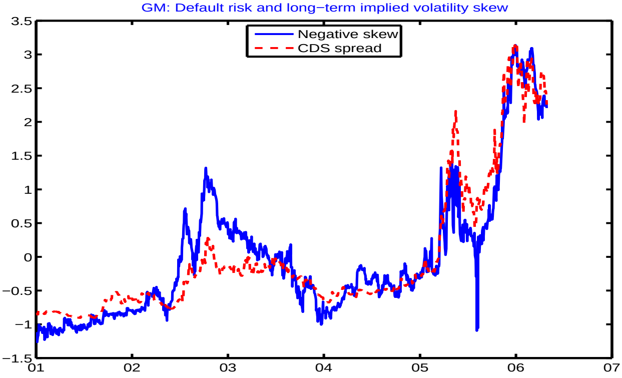
Default risk & long-term implied volatility skews

- When a company defaults, its stock value jumps to zero.
- It generates a steep skew in long-term stock options.
  - Default is really a first-moment effect: The pre-default risk-neutral drift is $r - q + \lambda_t$. CLT does not apply.
  - Using the second moment (implied vol) to capture the first-moment effect will generate large skews.



Capture Implied volatility smiles & skews with three (jump) components

I. Market risk (FMLS under $\mathbb{Q}$, DPL under $\mathbb{P}$)

II. Idiosyncratic risk (DPL under both $\mathbb{P}$ and $\mathbb{Q}$)

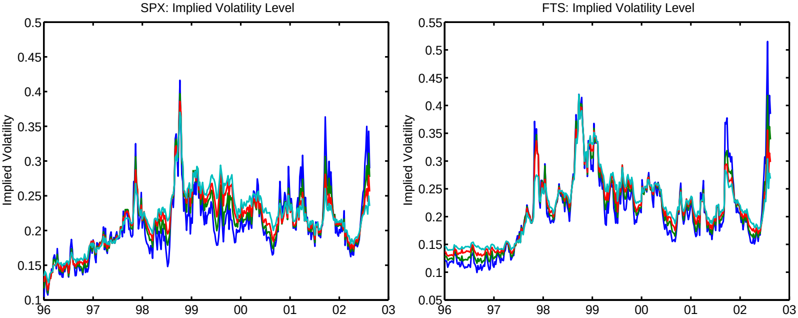
III. Default risk (Poisson arrival, jumps to zero).

- Remarks:
  - Long-term implied volatilities are more correlated cross-sectionally than stock returns are.
  - Market risk (I) is important. Identify (I) from SPX or QQQQ options.
  - Default risk (III) is important for companies with low credit ratings (GM).
  - Identify the credit risk component from the CDS market.
- Currency: The difference of two market risks.

Beyond Lévy processes

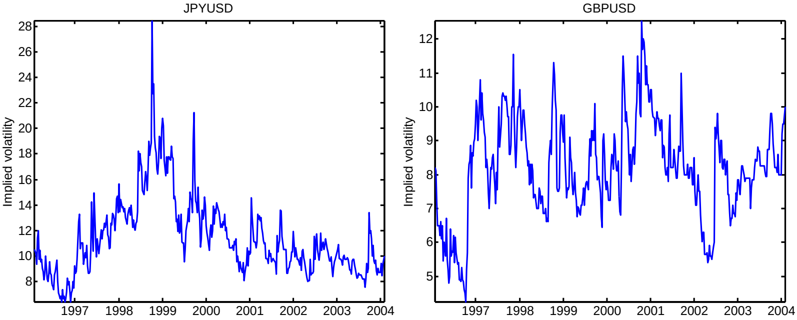
- Lévy processes can be used to generate different iid return innovation distributions.
- Yet, return distribution is iid, but varies stochastically over time.
- We need to go beyond Lévy processes to capture the stochastic nature of the return distribution.

Stochastic volatility on stock indexes



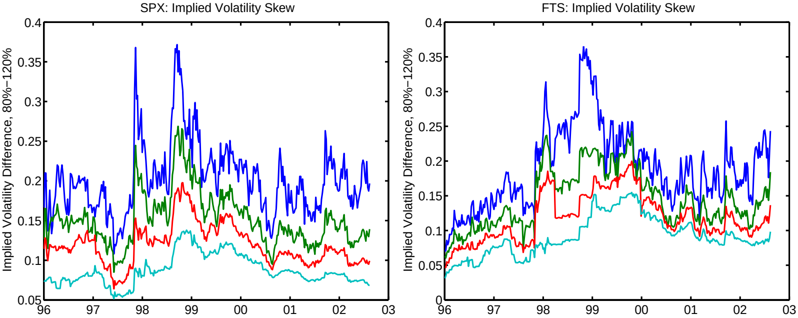
At-the-money implied volatilities at fixed time-to-maturities from 1 month to 5 years.

Stochastic volatility on currencies



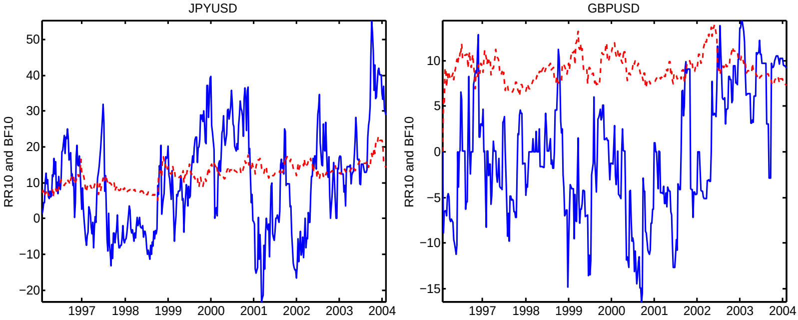
Three-month delta-neutral straddle implied volatility.

Stochastic skewness on stock indexes



Implied volatility spread between 80% and 120% strikes at fixed time-to-maturities from 1 month to 5 years.

Stochastic skewness on currencies



Three-month 10-delta risk reversal (blue lines) and butterfly spread (red lines).

Stochastically time-changed Lévy processes

- Discrete-time analog again: $R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1}$
  - ε_{t+1} is an iid return innovation $\leftrightarrow$ Lévy process.
  - (μ_t, σ_t) can be time-varying, stochastic...
- If we start with a Lévy process, $(\mu, \sigma, \lambda\nu(x))$,

$$\begin{aligned} \phi(u) &\equiv \mathbb{E}\left[e^{iuX_t}\right] = e^{-t\psi(u)}, \\ \psi(u) &= = -iu\mu + \frac{1}{2}u^2\sigma^2 + \lambda \int_{\mathbb{R}_0} \left(1 - e^{iux} + iux1_{|x|<1}\right) \nu(x)dx, \end{aligned}$$

- The drift μ, the diffusion variance σ^2, and the arrival rate λ are all proportional to time t.
- We can randomize the time $t \to \mathcal{T}_t$ instead of randomizing (μ, σ^2, λ), for the same result.
- We define $\mathcal{T}_t \equiv \int_0^t v_{s-} ds$ as the (stochastic) time change, with v_t being the instantaneous activity rate.

Model financial security returns for option pricing

- Start with the risk-neutral ($\mathbb{Q}$) process — That's where tractability is needed the most dearly.
  - Identify the economic risk sources, model innovation on each source with a Lévy process (X_t^k for $k = 1, \cdots, K$)
  - Apply separate time changes: $X_t^k \to X_{\mathcal{T}_t}^k$ to capture stochastic responses of financial security returns to economic shocks.

$$\ln S_t/S_0 = (r-q)t + \sum_{k=1}^{K} \left(b^k X_{\mathcal{T}_t^k}^k - \varphi_{x^k}(b^k)\mathcal{T}_t^k \right),$$

- The framework makes model design more intuitive, parsimonious, and economically sensible.
  - Each Lévy component captures shocks from one economic source.
  - Time change captures the time-varying intensity of its impact.

Economic implications of using jumps

- Black-Scholes (one-factor diffusion):
  - The market is complete with a bond and a stock.
  - If you can estimate the statistical dynamics of the stock, you can price options on that stock.
  - Utility-free option pricing. Option prices are redundant. Options market reveals no extra information.
- Heston (two-factor diffusion): We can still complete the market with one extra option.
- In the presence of jumps of random sizes,
  - The market is inherently incomplete (with stocks alone).
  - Need all options (+ model) to complete the market.
  - Options market is informative/useful:
    - Cross-sectional behavior of options $(K, T) \Leftrightarrow \mathbb{Q}$ dynamics.
    - Time-series behavior of stocks/options $(t) \Leftrightarrow \mathbb{P}$ dynamics.
    - The difference $\mathbb{Q}/\mathbb{P} \Leftrightarrow$ market prices of economic risks.

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Bottom line

- Different types of jumps affect option pricing at both short and long maturities.
  - Implied volatility smiles at very short maturities can only be accommodated by a jump component.
  - Implied volatility skews at very long maturities ask for a jump process that generates infinite variance.
  - Credit risk exposure may also help explain the long-term skew on single name stock options.
- The choice of jump types depends on the modeled events:
  - Infinite-activity jumps $\Leftrightarrow$ frequent market order arrival.
  - Finite-activity Poisson jumps $\Leftrightarrow$ rare events (credit).
- Applying stochastic time changes to the Lévy processes
  - generates stochastic responses to each economic shock.
  - generates stochastic volatility, skewness, ...
- The presence of jumps of random sizes have important and practical applications for hedging...