Implied Volatility Correlations

Robert Engle, Stephen Figlewski
and Amrut Nashikkar

Date: May 18, 2007

Derivatives Research Conference, NYU

IMPLIED VOLATILITY

- Implied volatilities from market traded options vary across strike and maturity in well studied ways.
- Implied volatilities across underlying assets are not well studied.
- We will investigate how at-the-money Implied Volatilities change over time and across assets.

MOTIVATION

- RISK MANAGEMENT OF A BOOK OF OPTIONS
- OPTION PORTFOLIO SELECTION
- SHORT RUN OPTION TRADING STRATEGIES
- OPTION PRICING IN INCOMPLETE MARKETS WHEN SOME OF THE RISK IS SYSTEMATIC

Risk Management of options portfolios

- Risk for an options book is reduced if all options are delta hedged. It is further reducable by gamma and vega hedging although this is often costly.
- Delta Gamma hedged positions have remaining vega risk and this will be correlated across names.
- We commonly decompose option price changes , dp as depending on dt, ds, ds^2, dv

$$dp = \Theta dt + \Delta ds + \frac{1}{2}\Gamma ds^2 + \Lambda dv$$

VARIANCE OF HEDGE POSITIONS

- Hence a delta-gamma hedged position has a dollar variance of:

$$V\left(\Lambda dv\right) = v^2\Lambda^2 V\left(dv/v\right) \approx v^2\Lambda^2 V\left(d\log\left(v\right)\right)$$

- And the covariances are given by

$$Cov\left(\Lambda_i dv_i, \Lambda_j dv_j\right) = v_i\Lambda_i v_j\Lambda_j Cov\left(d\log\left(v_i\right), d\log\left(v_j\right)\right)$$

- And the correlations depend only on the shocks

$$\rho_{i,j} = Corr\left(d\log\left(v_i\right), d\log\left(v_j\right)\right)$$

PORTFOLIO VARIANCE

- A portfolio of options with a vector of weights w on different underlyings has a dollar value ξ_t .
- The conditional variance of this option portfolio exposed to only vega risk is therefore given by

$$Var_{t-1}\left(\xi_t\right) = w'_{t-1}\,\Lambda_{t-1}D_{t-1}\Psi_{t/t-1}D_{t-1}\Lambda'_{t-1}\,w_{t-1}$$

- Here Ψ denotes the forecast of the variance-covariance matrix of log innovations volatility. Λ denotes the diagonal matrix of vegas of the options and D is a diagonal matrix with the vols on the diagonal.

DYNAMIC CORRELATIONS

- In a dynamic context, *dv* should be interpreted as the innovation to implied volatility.
- The correlations and covariances can be conditioned on time t-1 information

MODELING VOLS

- Let *v* be the implied volatility
- Let $E_{t-1}(v_t) \equiv \psi_t$
- Then a model that insures non-negative volatilities can be written

$$v_t = \psi_t \varepsilon_t, \quad \varepsilon_t \sim D(1, h_t), \; \varepsilon_t \in (0, \infty),$$

$$hence \; v_t = \psi_t + \psi_t(\varepsilon_t - 1)$$

VARIANCES AND COVARIANCES

- The variances, covariances and correlations of v are given by

$$Var_{t-1}\left(v_t\right)=\psi_t^2 h_t,\; Cov_{t-1}\left(v_{i,t},v_{j,t}\right)=\psi_{i,t}\psi_{j,t}Cov_{t-1}\left(\varepsilon_{i,t},\varepsilon_{j,t}\right)$$

$$Corr_{t-1}\left(v_{i,t},v_{j,t}\right)=Corr_{t-1}\left(\varepsilon_{i,t},\varepsilon_{j,t}\right)$$

- and

$$\log\left(\varepsilon_t\right)=\log\left(1\right)+\left(\varepsilon_t-1\right)-\frac{1}{2}\left(\varepsilon_t-1\right)^2+\ldots$$

$$E_{t-1}\left(\log\left(\varepsilon_t\right)\right)=-h_t/2,$$

$$V_{t-1}\left(\log\left(\varepsilon_t\right)\right)\cong h_t$$

MODELING THE MEAN

- Assume that volatilities are autoregressive and mean reverting in the logs and consider the AR(2) model

$$\psi_t = \exp\left(\beta_0 + \beta_1 \log\left(v_{t-1}\right) + \beta_2 \log\left(v_{t-2}\right)\right)$$

- Hence: $\log\left(v_t\right) = \beta_0 + \beta_1 \log\left(v_{t-1}\right) + \beta_2 \log\left(v_{t-2}\right) + \log\left(\varepsilon_t\right)$

MODELING THE VOLATILITY OF VOLATILITY

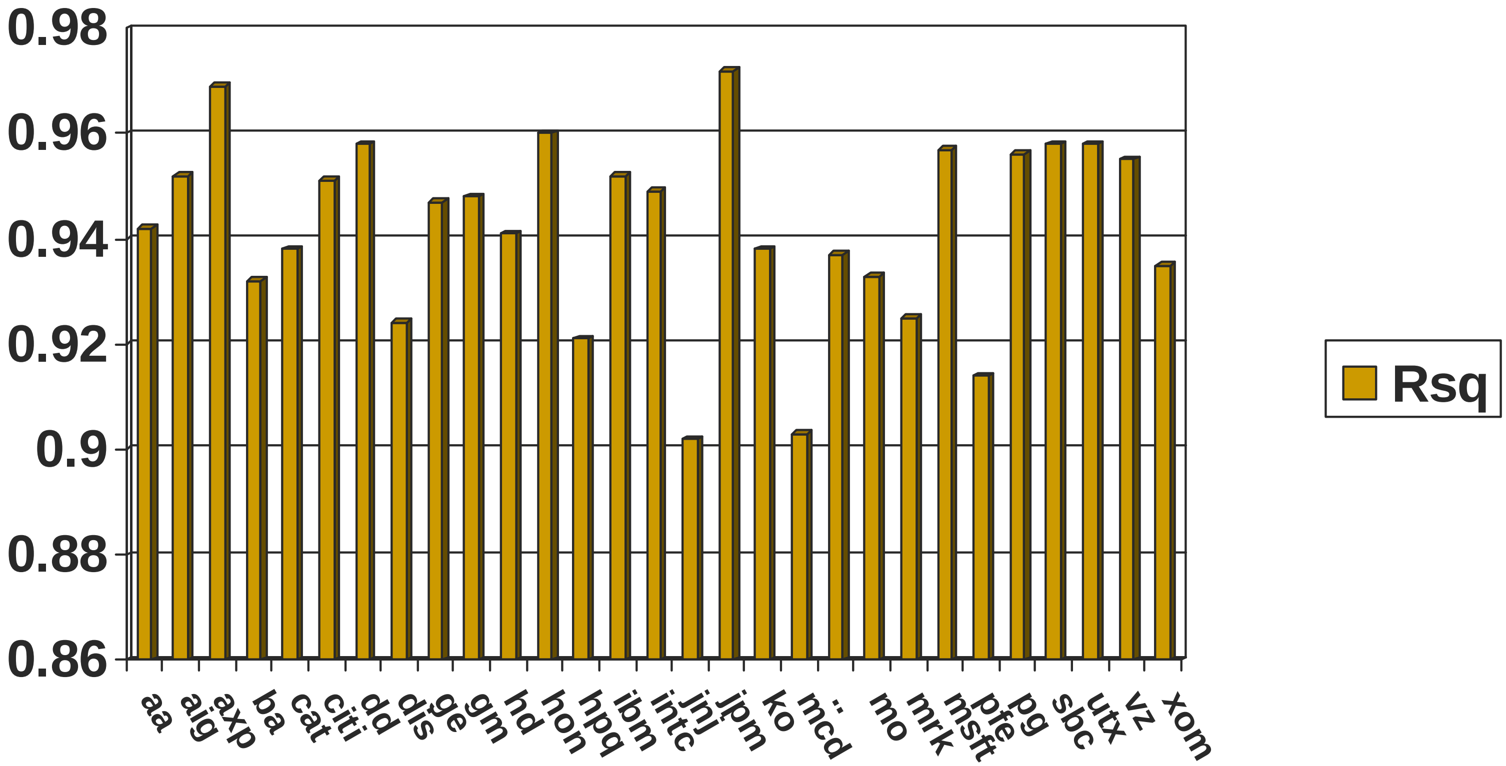
- The variance of the implied volatility is given by:

$$V_{t-1}\left(v_t\right) = \psi_t^2 h_t, \quad V_{t-1}\left(\log\left(\varepsilon_t\right)\right) \cong h_t$$

- Letting *h* be a GARCH(1,1) on the residuals of the log model these expressions are easily calculated.
- The proportional volatility of volatility is just $h^{1/2}$.

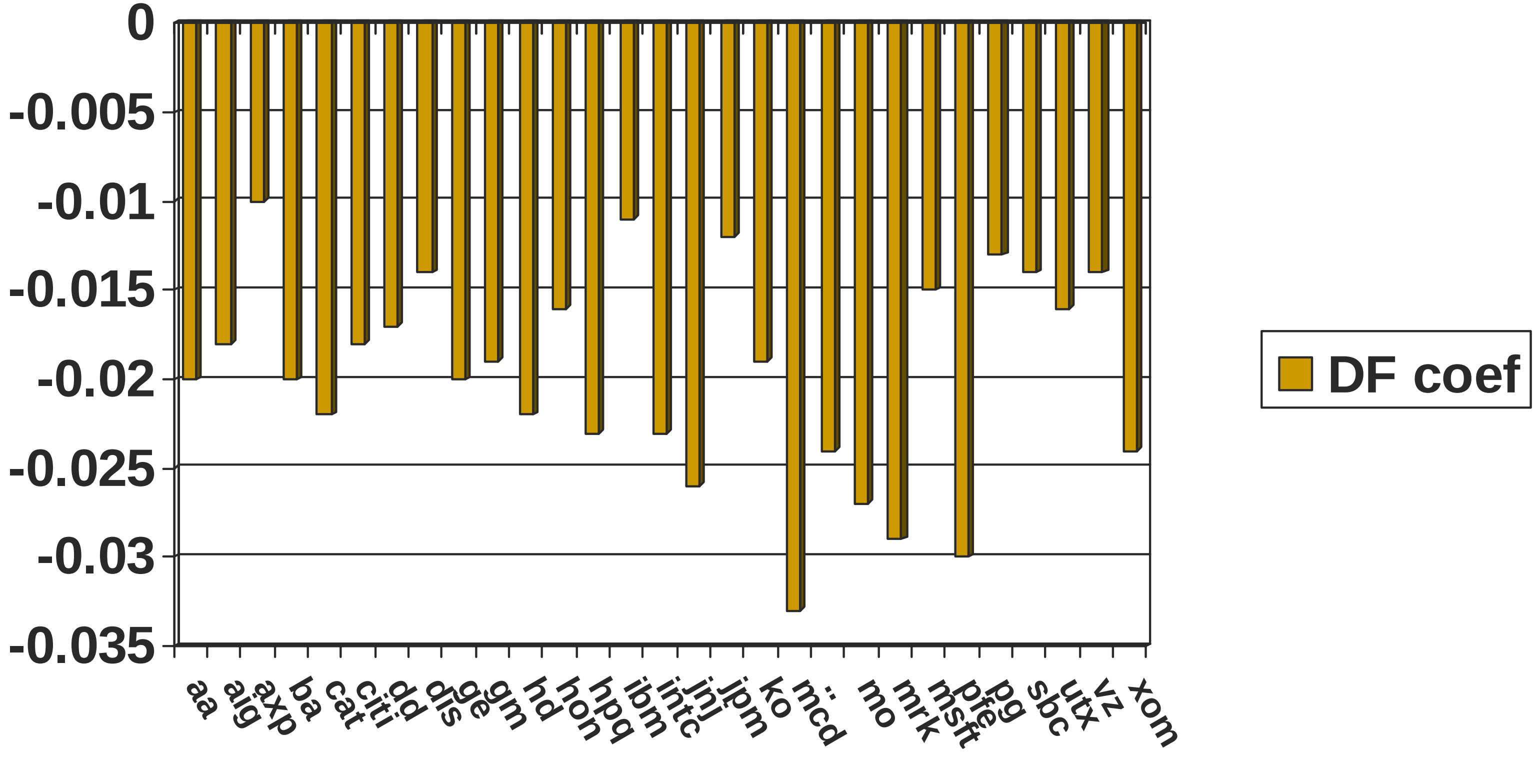
R^2 OF AUTOREGRESSIVE MODEL

$$\log(v) = c + \gamma_1 \log(v_{t-1}) + \gamma_2 \log(v_{t-2}) + e_t$$



Unit Root Test: alpha

$$d\log(v) = c + \alpha\log(v_{t-1}) + \gamma d\log(v_{t-1})$$



FIRST EIGHT NAMES

	aa	aig	axp	ba	cat	citi	dd	dis
AR1	0.699	0.784	0.793	0.680	0.756	0.753	0.735	0.712
(tstat)	(29.9)	(33.8)	(41.7)	(31.1)	(33.7)	(33.4)	(32.6)	(25.6)
AR2	0.281	0.198	0.197	0.301	0.222	0.229	0.248	0.274
(tstat)	(12.0)	(8.58)	(10.3)	(13.7)	(9.90)	(10.2)	(11.0)	(9.60)
Omega	0.001	0.000	0.000	0.000	0.000	0.001	0.001	0.002
(tstat)	(8.01)	(7.24)	(7.07)	(6.30)	(7.82)	(7.39)	(7.45)	(16.6)
Alpha	0.173	0.102	0.077	0.165	0.117	0.135	0.143	0.258
(tstat)	(7.73)	(9.01)	(10.8)	(9.68)	(8.36)	(8.10)	(6.75)	(8.58)
Beta	0.354	0.824	0.885	0.736	0.734	0.643	0.657	0.316
(tstat)	(5.24)	(47.0)	(93.8)	(28.0)	(27.7)	(15.3)	(16.2)	(9.12)
SC	-2.957	-2.992	-2.982	-2.836	-3.042	-2.663	-2.963	-2.469
RSQ	0.942	0.952	0.969	0.932	0.938	0.951	0.958	0.924

MEAN REVERSION

- In AR(2) model, all but one series rejects unit root at the 5% level
- Mean reversion is slow. Sum of roots is between .99 and .96.
- Model explains more than 90% of volatility levels in all cases.
- Volatility of volatility is not very persistent except in a couple of cases.

Unconditional level correlations between a set of implied vols and the VIX

	AA	*AIG*	*AXP*	*BA*	*C*	*CAT*	*DD*	*DIS*	*VIX*
AA	1								
AIG	0.535	1							
AXP	0.742	0.558	1						
BA	0.775	0.526	0.838	1					
C	0.692	0.593	0.87	0.77	1				
CAT	0.795	0.569	0.863	0.822	0.812	1			
DD	0.774	0.589	0.900	0.833	0.837	0.907	1		
DIS	0.778	0.563	0.799	0.839	0.776	0.773	0.769	1	
VIX	0.647	0.557	0.803	0.800	0.854	0.757	0.775	0.789	1

INNOVATION CORRELATIONS

	AA	AIG	AXP	BA	CAT	CITI	DD	DIS
AA	1.000	0.188	0.207	0.165	0.159	0.211	0.240	0.17
AIG	0.188	1.000	0.263	0.229	0.183	0.289	0.233	0.168
AXP	0.207	0.263	1.000	0.206	0.208	0.333	0.267	0.177
BA	0.165	0.229	0.206	1.000	0.178	0.249	0.226	0.196
CAT	0.159	0.183	0.208	0.178	1.000	0.227	0.246	0.158
CITI	0.211	0.289	0.333	0.249	0.227	1.000	0.284	0.208
DD	0.240	0.233	0.267	0.226	0.246	0.284	1.000	0.200
DIS	0.175	0.168	0.177	0.196	0.158	0.208	0.200	1.000

INNOVATION CORRELATIONS VS. LEVEL CORRELATIONS

- Why are these so different?
- The autoregressive model does not explain such differences.
- Only if lags of one volatility predict innovations in another, will the unconditional correlations be systematically bigger than the conditional

Models conditional on the VIX

- Consider a log model of the form

$$d\log\left(v_{i,t}\right)=\beta_1\log\left(v_{i,t-1}\right)+\beta_2 d\log\left(v_{i,t-1}\right)+\beta_3\log\left(Vix_{t-1}\right)+\beta_4 d\log\left(Vix_{t-1}\right)+\varepsilon_t$$

- This is equivalent to the error correction model:

$$d\log\left(v_{i,t}\right)=\beta_1\left[\log\left(v_{i,t-1}\right)-\delta\log\left(Vix_{t-1}\right)\right]+$$
$$+\beta_2 d\log\left(v_{i,t-1}\right)+\beta_4 d\log\left(Vix_{t-1}\right)+\beta_0+\varepsilon_t$$

- δ is a long run elasticity given by

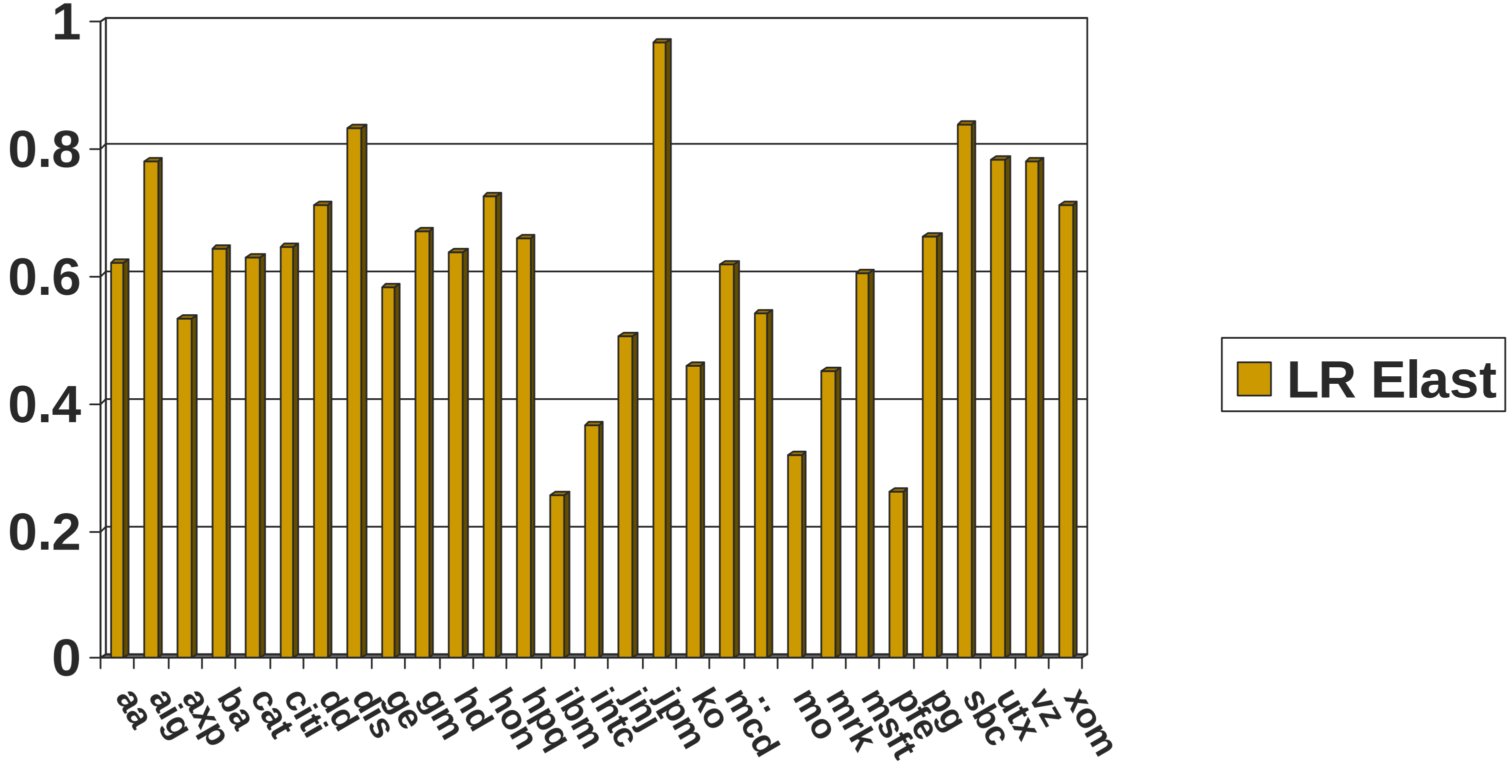
$$\delta=-\beta_3/\beta_1$$

RESULTS

	aa	aig	axp	ba	cat	citi	dd
V(-1)	-0.035	-0.043	-0.014	-0.046	-0.052	-0.032	-0.034
(tstat)	(-5.8)	(-7.5)	(-2.4)	(-4.9)	(-6.6)	(-3.8)	(-4.5)
D(V(-1)	-0.298	-0.229	-0.234	-0.311	-0.226	-0.282	-0.271
(tstat)	(-12.)	(-9.8)	(-11.)	(-13.)	(-9.8)	(-11.)	(-11.)
vix(-1)	0.022	0.033	0.008	0.030	0.033	0.021	0.024
(tstat)	(3.95)	(5.32)	(1.10)	(3.66)	(4.87)	(2.15)	(3.09)
d(vix(-1)	0.079	0.104	0.103	0.072	0.046	0.160	0.096
(tstat)	(3.29)	(5.12)	(4.91)	(3.17)	(2.17)	(5.86)	(4.50)
Omega	0.002	0.000	0.000	0.000	0.000	0.001	0.001
(tstat)	(8.28)	(7.03)	(6.50)	(6.40)	(7.54)	(7.19)	(6.67)
Alpha	0.177	0.099	0.078	0.159	0.110	0.137	0.131
(tstat)	(7.66)	(9.19)	(10.1)	(9.38)	(8.28)	(8.22)	(6.50)
Beta	0.320	0.837	0.879	0.737	0.747	0.650	0.646
(tstat)	(4.60)	(52.0)	(76.7)	(27.4)	(28.9)	(16.0)	(13.8)
SCHWARZ	-2.964	-3.009	-2.987	-2.840	-3.049	-2.676	-2.971
RSQ	0.115	0.052	0.064	0.129	0.087	0.072	0.084
DELTA	0.620	0.779	0.532	0.643	0.630	0.645	0.710

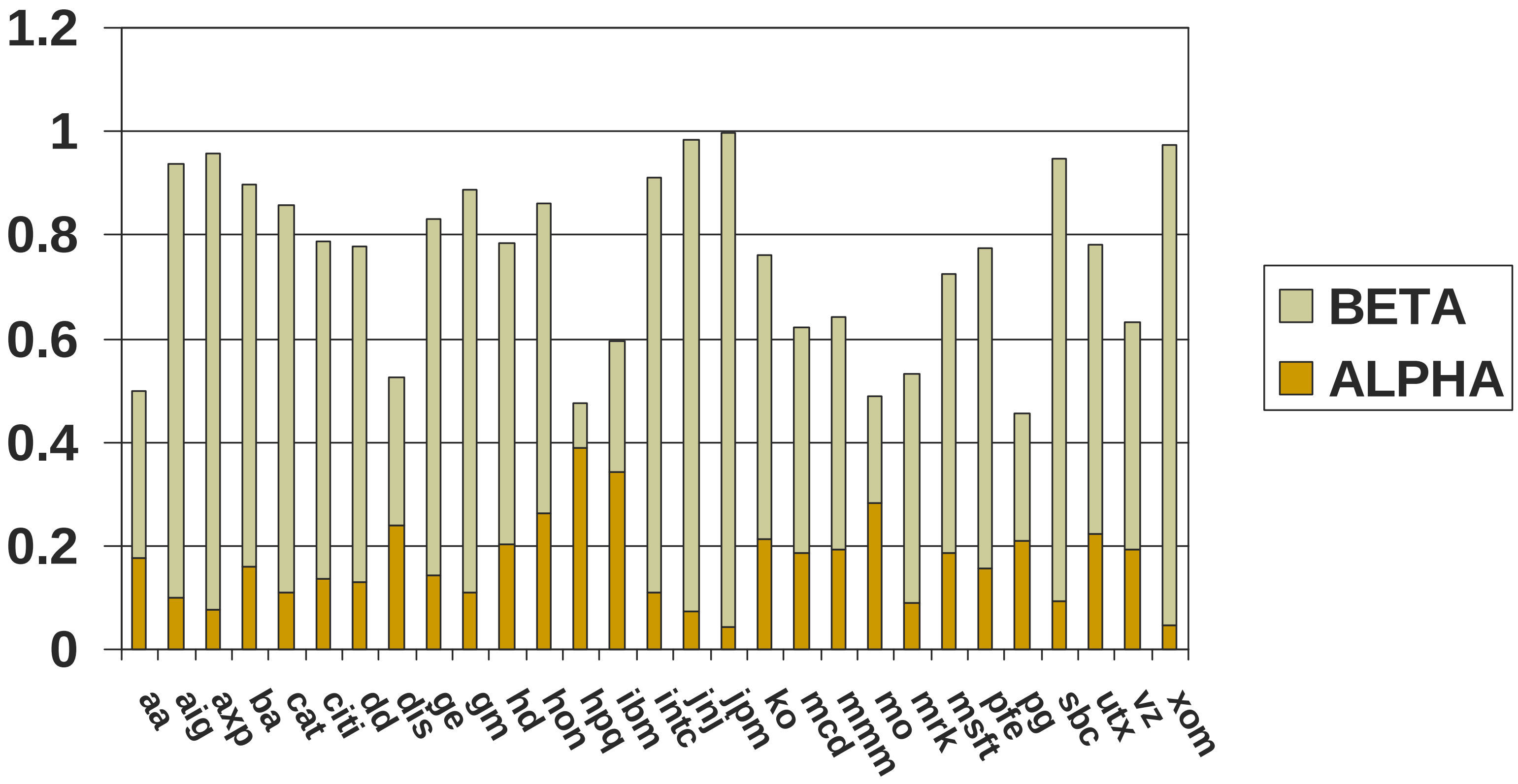
LONG RUN ELASTICITY OF VIX AUTOREGRESSIVE MODEL

$$d\log(v) = c + \beta_1\left[\log(v_{t-1}) - \delta\log\left(vix_{t-1}\right)\right] + \\ + \beta_2 d\log(v_{t-1}) + \beta_3 d\log\left(vix_{t-1}\right) + e_t$$



GARCH PERSISTENCE IN VIX AUTOREGRESSIVE MODEL

$$h_t = \omega + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}$$



Dynamic Conditional Correlation

- DCC is a new type of multivariate GARCH model that is particularly convenient for big systems. See Engle(2002).
- This gives correlations between the innovations.

DCC

1. Estimate volatilities for each innovation and compute the *standardized residuals* or *volatility adjusted returns*.
2. Estimate the time varying covariances between these using a maximum likelihood criterion and one of several models for the correlations.
3. Form the correlation matrix and covariance matrix. They are guaranteed to be positive definite.

HOW TO UPDATE CORRELATIONS

- When two assets move in the same direction, the correlation is increased slightly. When they move in the opposite direction it is decreased.

- The correlations often are assumed to only temporarily deviate from a long run mean

CORRELATIONS UPDATE LIKE GARCH

- Approximately,

$$\rho_{1,2t} = \omega_{1,2} + \alpha\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \beta\rho_{1,2,t-1}$$

$$\bar{\rho}_{1,2} = \frac{\omega_{1,2}}{1-\alpha-\beta}, \ or \ \ \omega_{1,2} = \bar{\rho}_{1,2}\left(1-\alpha-\beta\right)$$

- And the parameters alpha and beta are assumed the same for all pairs. Consequently there are only 2 parameters to estimate, no matter how many assets there are!

The DCC Model
more precisely in matrix terms.

$$V_{t-1}\left(r_t\right) = D_t R_t D_t, \quad D_t \sim Diagonal,\ R_t \sim Correlation\ Matrix$$

$$\varepsilon_t = D_t^{-1} r_t$$

$$R_t = \left\{diag\left(Q_t\right)\right\}^{-1/2} Q_t \left\{diag\left(Q_t\right)\right\}^{-1/2}$$

$$Q_t = \Omega + a\varepsilon_{t-1}\varepsilon_{t-1}' + bQ_{t-1}$$

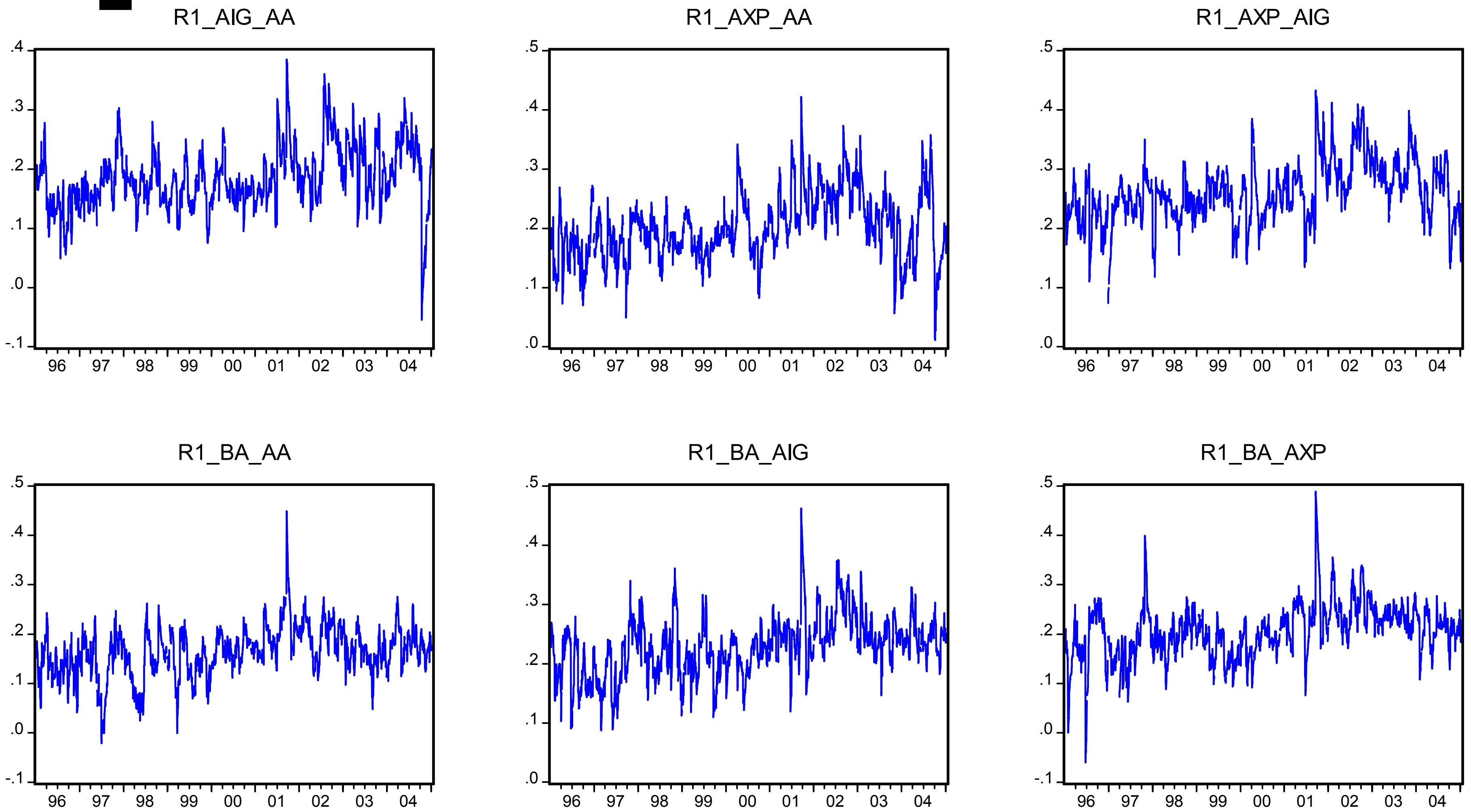
$$\Omega = \bar{R}\left(1-a-b\right)$$

ESTIMATION IS BY "MACGYVER" METHOD

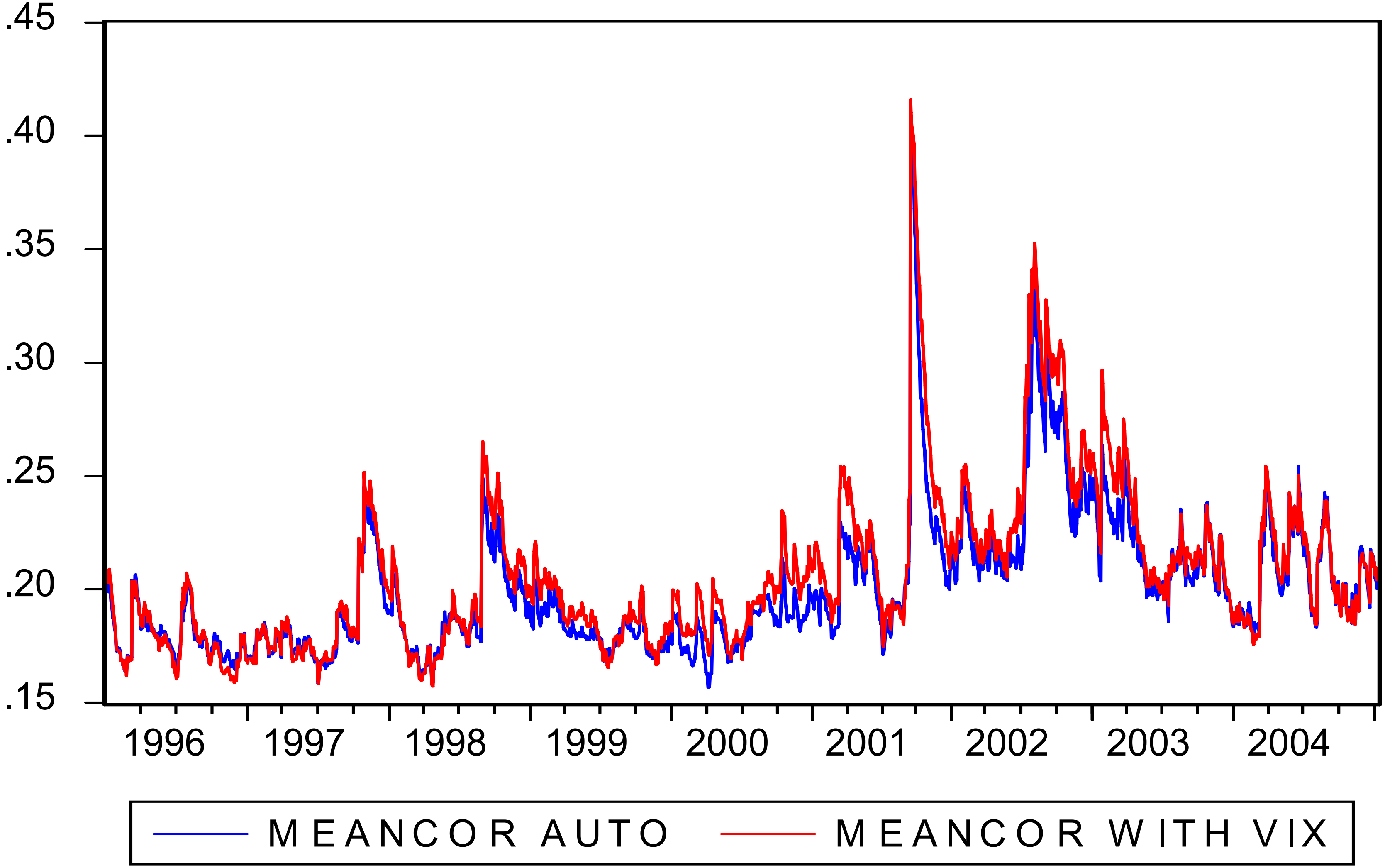
- 400 bivariate DCC models
- ALPHA MEDIAN =.0153
- BETAMEDIAN=.935

- THEN RECALCULATE ALL CORRELATIONS USING THESE PARAMETERS

AA, AIG, AXP, BA CORRELATIONS



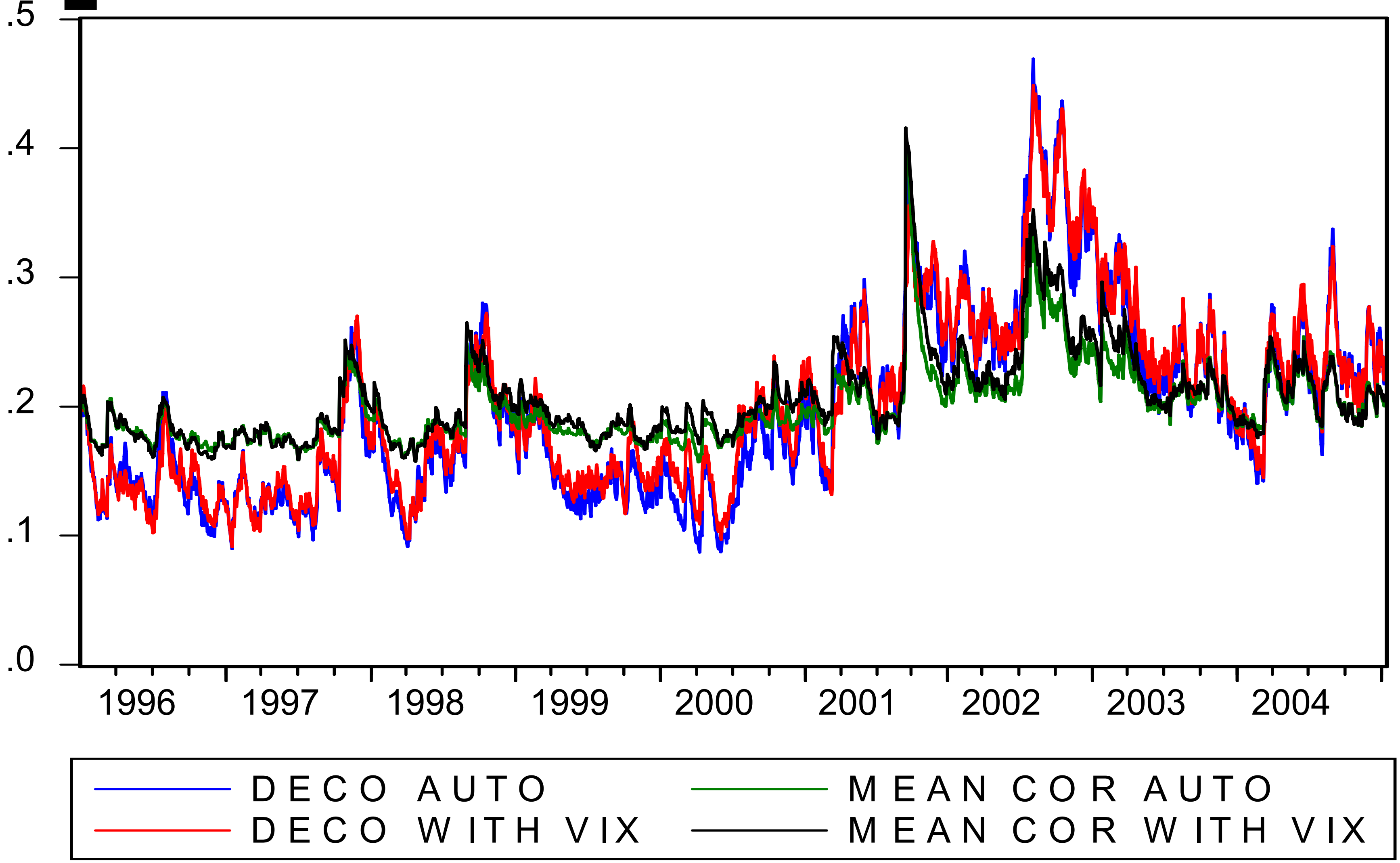
AVERAGE CORRELATION



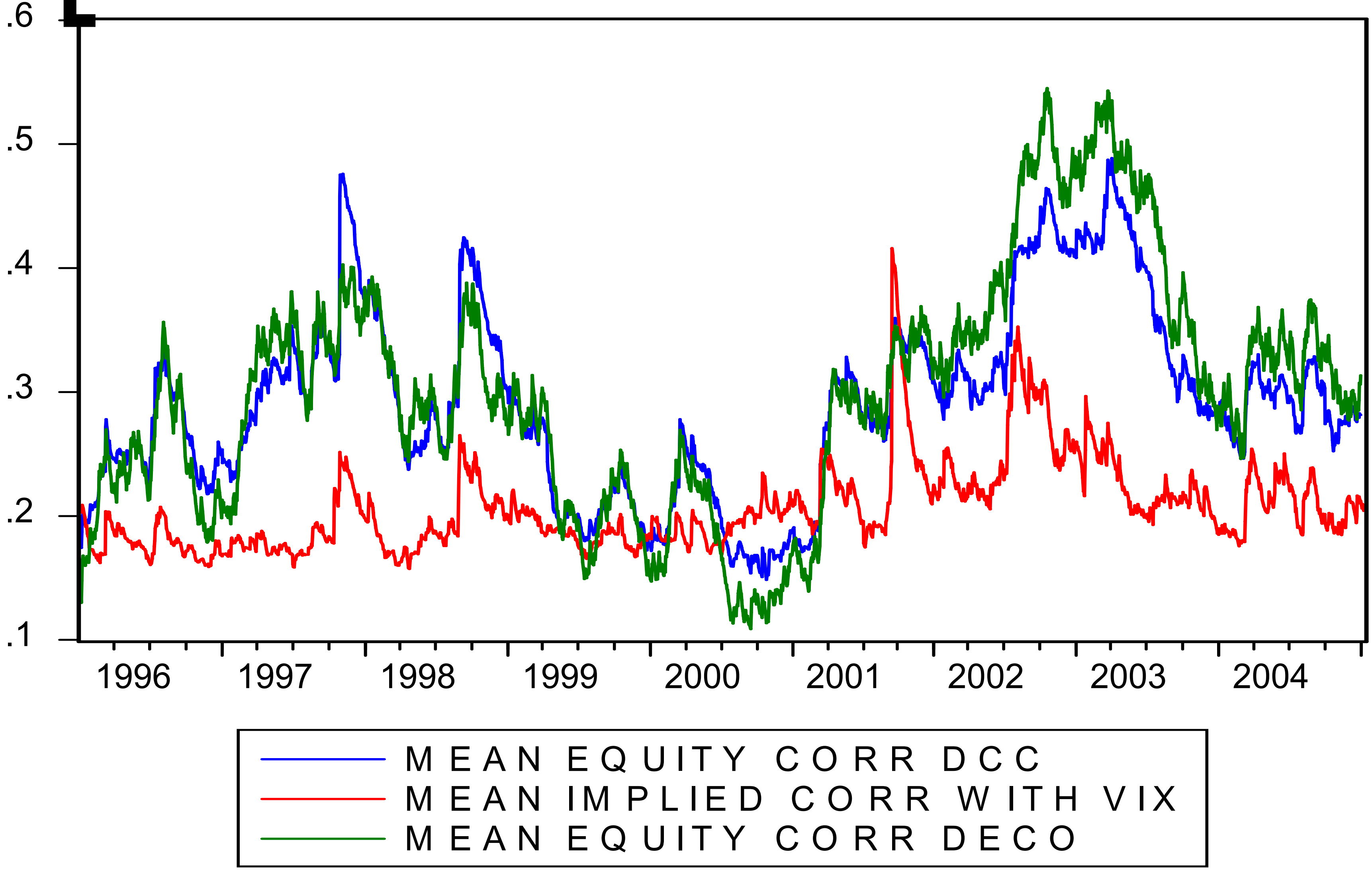
DYNAMIC EQUICORRELATION

- All correlations are equal on a day but they change from one day to the next.
- Estimate like a GARCH no matter how large a set of data.
- Recent work with Bryan Kelly

DECO CORRELATIONS



EQUITY CORRELATIONS



ANALYSIS OF CORRELATIONS BETWEEN OPTIONS

- Individual options are examined for the same sample period 1996 to 2006
- Four names: AA AIG AXP BA, six correlations
- Open interest>100, Bid price>.50, dT<5
- Calculate delta, gamma, theta hedged returns.

$$\pi_{i,t} = dc_{i,t} - \Delta ds_{i,t} - \Theta dt - \frac{1}{2}\Gamma\left(ds_{i,t}\right)^2$$

REGRESSION

- Cross product of hedged returns divided by vega and lagged vol should be noisy estimates of covariances

$$\frac{\pi_{i,t}\pi_{j,t}}{v_{i,t-1}\Lambda_i v_{j,t-1}\Lambda_j} = \operatorname{cov}_{t-1}\left(dv_{i,t}/v_{i,t-1}, dv_{j,t}/v_{j,t-1}\right)$$

$$\equiv \tilde{\pi}_{i,t}\tilde{\pi}_{j,t} \cong \operatorname{cov}_{t-1}\left(\varepsilon_{i,t}\varepsilon_{j,t}\right) = \rho_{i,j,t}\sqrt{h_{i,t}h_{j,t}}$$

- Regression

$$\tilde{\pi}_{i,t}\tilde{\pi}_{j,t} = a + b\rho_{i,j,t}\sqrt{h_{i,t}h_{j,t}} + e_{i,j,t}$$

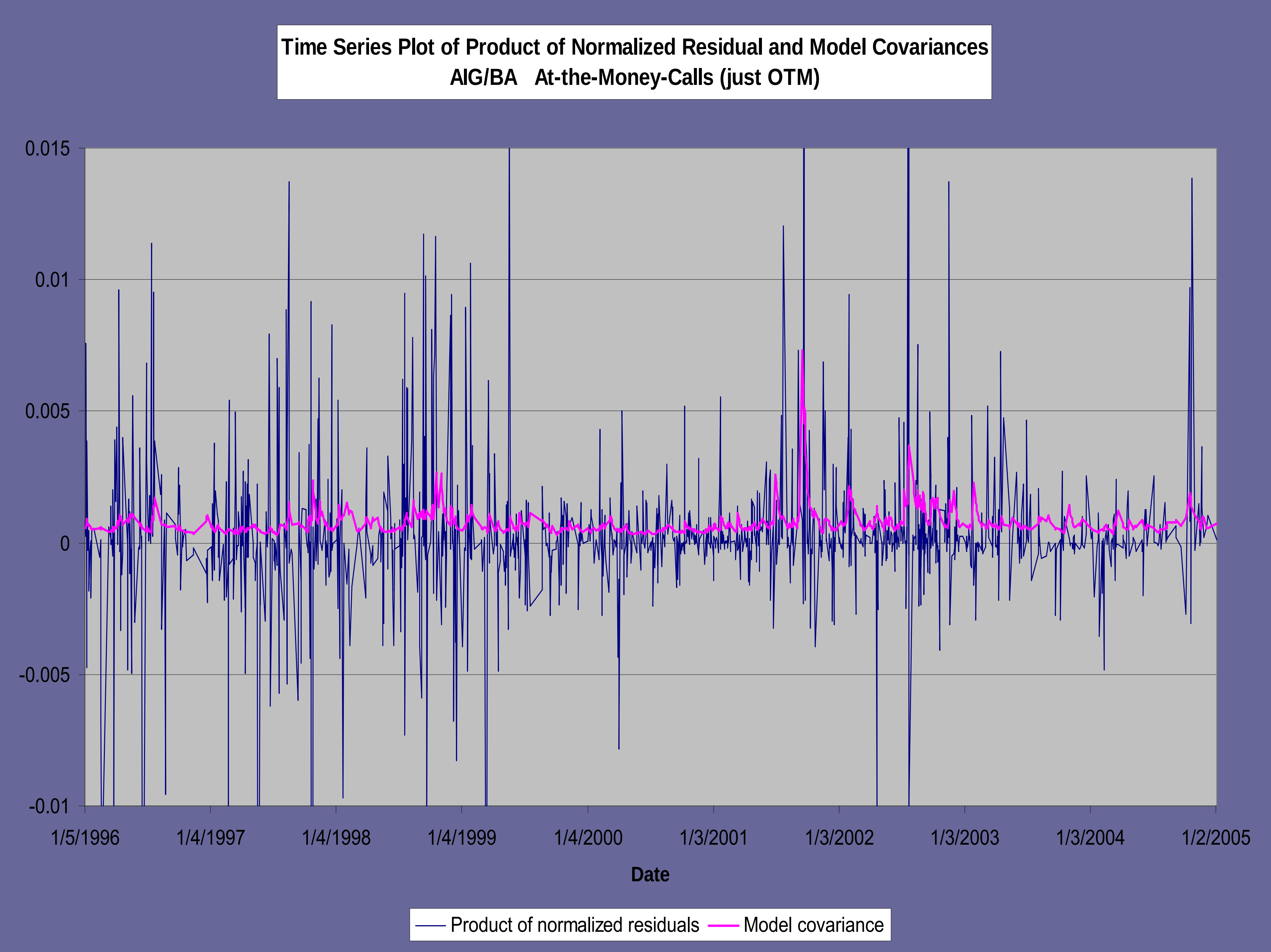
RESULTS AA-AIG

- At-the-money calls 20-45 days maturity 1193 observations with t-statistics in ()

$$\tilde{\pi}_{AA,t}\tilde{\pi}_{AIG,t} = \underset{(-.11)}{-.000016} + \underset{(2.75)}{0.537}\,\text{cov}_{t-1}\left(AA, AIG\right) + e$$

RESULTS OTHER PAIRS

OPTION PAIR	VARIABLE	COEF	STDERR	T-STAT
AA/AIG	cov12	0.53696	0.19492	2.75
AA/AXP	cov12	0.43100	0.24250	1.78
AA/BA	cov12	0.70831	0.34286	2.07
AIG/AXP	cov12	0.50207	0.15287	3.28
AIG/BA	cov12	0.70799	0.20332	3.48
AXP/BA	cov12	-0.02390	0.46090	-0.05
ALL SIX PAIRS	cov12	0.43518	0.11125	3.91


Time Series Plot of Product of Normalized Residual and Model Covariances
AIG/BA At-the-Money-Calls (just OTM)
0.015
0.01
0.005
0
-0.005
-0.01
1/5/1996
1/4/1997
1/4/1998
1/4/1999
1/4/2000
1/3/2001
1/3/2002
1/3/2003
1/3/2004
1/2/2005
Date
Product of normalized residuals
Model covariance

RESULTS OTHER MONEYNESS ALL SIX PAIRS

- CALLS SLIGHTLY IN THE MONEY
  - 0.14325 0.17146 0.84
- CALLS FAR IN THE MONEY
  - 1.27544 1.72522 0.74
- CALLS FAR OUT OF THE MONEY
  - 1.38917 0.16331 8.51
- PUTS SLIGHTLY IN THE MONEY
  - 0.43518 0.11125 3.91
- PUTS FAR IN THE MONEY
  - 0.72257 0.53516 1.35
- PUTS SLIGHTLY OUT OF THE MONEY
  - 0.05602 0.05579 1.00
- PUTS FAR OUT OF THE MONEY
  - 0.42961 0.12331 3.48

RESULTS FOR *just* DELTA HEDGED A-T-M CALLS

AA-AIG	0.73560	0.40835	1.80
AA-AXP	0.48329	0.50848	0.95
AA-BA	1.23431	0.60709	2.03
AIG-AXP	0.36246	0.32355	1.12
AIG-BA	0.21895	0.26579	0.82
AXP-BA	0.00222	1.48422	0.00
ALL SIX	0.38737	0.29871	1.30

CONCLUSIONS

- Model gives reasonable estimates of correlations and covariances.
- These are robust to model specifications
- Correlations are on average .2 rising to .3 or .4 in 2001 and 2002.
- These are roughly matched by delta gamma hedged option positions
- Best results are for at the money calls